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EQUILIBRIUM STATES OF ENDOMORPHISMS OF \mathbb{P}^k II: SPECTRAL STABILITY AND LIMIT THEOREMS

FABRIZIO BIANCHI AND TIEN-CUONG DINH

ABSTRACT. We establish the existence of a spectral gap for the transfer operator induced on $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ by a generic holomorphic endomorphism and a suitable continuous weight and its perturbations on various functional spaces, which is new even in dimension one. The main issue to overcome is the rigidity of the complex objects, since the transfer operator is a non-holomorphic perturbation of the operator f_* . The system is moreover non-uniformly hyperbolic and one may have critical points on the Julia set. The construction of our norm requires the introduction and study of several intermediate new norms, and a careful combination of ideas from pluripotential and interpolation theory. As far as we know, this is the first time that pluripotential methods have been applied to solve a mixed real-complex problem.

Thanks to the spectral gap, we establish an exponential speed of convergence for the equidistribution of the backward orbits of points towards the conformal measure. Moreover, we obtain a full list of statistical properties for the equilibrium states: exponential mixing, CLT, Berry-Esseen theorem, local CLT, ASIP, LIL, LDP, almost sure CLT. Many of these properties are new even in dimension one, some even in the case of zero weight function (i.e., for the measure of maximal entropy).

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Notation. Throughout the paper, \mathbb{P}^k denotes the complex projective space of dimension k endowed with the standard Fubini-Study form ω_{FS} . This is a Kähler $(1, 1)$ -form normalized so that ω_{FS}^k is a probability measure. We will use the metric and distance $\text{dist}(\cdot, \cdot)$ on \mathbb{P}^k induced by ω_{FS} and the standard ones on \mathbb{C}^k when we work on open subsets of \mathbb{C}^k . We denote by $\mathbb{B}_{\mathbb{P}^k}(a, r)$ (resp. $\mathbb{B}_r^k, \mathbb{D}(a, r), \mathbb{D}_r$) the ball of center a and radius r in \mathbb{P}^k (resp. the ball of center 0 and radius r in \mathbb{C}^k , the disc of center a and radius r in \mathbb{C} , and the disc of center 0 and radius r in \mathbb{C}). Leb denotes the standard Lebesgue measure on a Euclidean space or on a sphere.

The pairing $\langle \cdot, \cdot \rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. If S and R are two $(1, 1)$ -currents, we will write $|R| \leq S$ when $\Re(\xi R) \leq S$ for every function $\xi: \mathbb{P}^k \rightarrow \mathbb{C}$ with $|\xi| \leq 1$, i.e., all currents $S - \Re(\xi R)$ with ξ as before are positive. Notice that this forces S to be real and positive. We also write other inequalities such as $|R| \leq |R_1| + |R_2|$ if $|R| \leq S_1 + S_2$ whenever $|R_1| \leq S_1$ and $|R_2| \leq S_2$. Recall that $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ and $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. The notations \lesssim and \gtrsim stand for inequalities up to a multiplicative constant. The function identically equal to 1 is denoted by $\mathbb{1}$. We also use the function $\log^*(\cdot) := 1 + |\log(\cdot)|$.

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Consider a holomorphic endomorphism $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$ of algebraic degree $d \geq 2$ satisfying the Assumption **(A)** in the Introduction. Denote respectively by T , $\mu = T^k$, $\text{supp}(\mu)$ the Green $(1, 1)$ -current, the measure of maximal entropy (also called the Green measure or the equilibrium measure), and the small Julia set of f . If S is a positive closed $(1, 1)$ -current on \mathbb{P}^k , its dynamical potential is denoted by u_S and is defined in Section 2.3.

We also consider a weight ϕ which is a continuous function on \mathbb{P}^k . We often assume that ϕ is real. The transfer operator (Perron-Frobenius operator) $\mathcal{L} = \mathcal{L}_\phi$ is introduced in the Introduction together with the scaling ratio $\lambda = \lambda_\phi$, the conformal measure m_ϕ , the density function $\rho = \rho_\phi$, the equilibrium state $\mu_\phi = \rho m_\phi$, the pressure $P(\phi)$. The measures m_ϕ and μ_ϕ are probability measures. The operator L is a suitable modification of \mathcal{L} and is introduced at the beginning of Section 5.

The oscillation $\Omega(\cdot)$, the modulus of continuity $m(\cdot, \cdot)$, the semi-norms $\|\cdot\|_{\log^p}$ and $\|\cdot\|_*$ of a function are defined in Section 2.1. Other norms and semi-norms $\|\cdot\|_p$, $\|\cdot\|_{p,\alpha}$, $\|\cdot\|_{(p,\alpha)}$, $\|\cdot\|_{(p,\alpha),\gamma}$ for $(1, 1)$ -currents and functions are introduced in Section 3 and the norms $\|\cdot\|_{\mathfrak{o}_1}$, $\|\cdot\|_{\mathfrak{o}_2}$ in Section 4.4. The semi-norms we consider are almost norms: they vanish only on constant functions. It is easy to make them norms by adding a suitable functional such as $g \mapsto |\langle m_\phi, g \rangle|$. However, for simplicity, it is more convenient to work directly with these semi-norms. The versions of these semi-norms for currents are actually norms. The positive numbers q_0, q_1, q_2 are defined in Lemmas 3.6, 3.9, 3.13 and the families of weights $\mathcal{P}(q, M, \Omega)$ and \mathcal{Q}_0 are introduced in Sections 4.1 and 4.3.

1. INTRODUCTION AND RESULTS

Let $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$ be a holomorphic endomorphism of the complex projective space $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$, with $k \geq 1$, of algebraic degree $d \geq 2$. Denote by μ the unique measure of maximal entropy for the dynamical system (\mathbb{P}^k, f) [Lyu83; BD09; DS10a; BM01]. The support $\text{supp}(\mu)$ of μ is called *the small Julia set* of f . Given a *weight*, i.e., a real-valued continuous function, ϕ as above, we define the *pressure* of ϕ as

$$P(\phi) := \sup \{ \text{Ent}_f(\nu) + \langle \nu, \phi \rangle \},$$

where the supremum is taken over all Borel f -invariant probability measures ν and $\text{Ent}_f(\nu)$ denotes the metric entropy of ν . An *equilibrium state* for ϕ is then an invariant probability measure μ_ϕ realizing a maximum in the above formula, that is,

$$P(\phi) = \text{Ent}_f(\mu_\phi) + \langle \mu_\phi, \phi \rangle.$$

Define also the *Perron-Frobenius* (or *transfer*) operator \mathcal{L} with weight ϕ as (we often drop the index ϕ for simplicity)

$$(1.1) \quad \mathcal{L}g(y) := \mathcal{L}_\phi g(y) := \sum_{x \in f^{-1}(y)} e^{\phi(x)} g(x),$$

where $g: \mathbb{P}^k \rightarrow \mathbb{R}$ is a continuous test function and the points x in the sum are counted with multiplicity. A *conformal measure* is an eigenvector for the dual operator \mathcal{L}^* acting on positive measures.

The following result was obtained in [BD20]. We refer to that paper for references to earlier related results.

Theorem 1.1. *Let f be an endomorphism of \mathbb{P}^k of algebraic degree $d \geq 2$ and satisfying the Assumption **(A)** below. Let ϕ be a real-valued \log^q -continuous function on \mathbb{P}^k , for some $q > 2$, such that $\Omega(\phi) := \max \phi - \min \phi < \log d$. Then ϕ admits a unique equilibrium state μ_ϕ , whose support is equal to the small Julia set of f . This measure μ_ϕ is K -mixing and mixing of all orders, and repelling periodic points of period n (suitably weighted) are equidistributed with respect to μ_ϕ as n goes to infinity. Moreover, there is a unique conformal measure m_ϕ associated to ϕ . We have $\mu_\phi = \rho m_\phi$ for some strictly positive continuous function ρ on \mathbb{P}^k and the preimages of points by f^n (suitably weighted) are equidistributed with respect to m_ϕ as n goes to infinity.*

Recall that a function is \log^q -continuous if its oscillation on a ball of radius r is bounded by a constant times $(\log^* r)^{-q}$, see Section 2.1 for details. We made use of the following technical assumption for f :

(A) the local degree of the iterate $f^n := f \circ \dots \circ f$ (n times) satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \max_{a \in \mathbb{P}^k} \deg(f^n, a) = 0.$$

Here, $\deg(f^n, a)$ is the multiplicity of a as a solution of the equation $f^n(z) = f^n(a)$. Note that generic endomorphisms of \mathbb{P}^k satisfy this condition, see [DS10b]. We assume (A) also throughout all the current paper.

A reformulation of Theorem 1.1 is the following: given ϕ as in the statement, there exist a number $\lambda > 0$ and a continuous function $\rho = \rho_\phi: \mathbb{P}^k \rightarrow \mathbb{R}$ such that, for every continuous function $g: \mathbb{P}^k \rightarrow \mathbb{R}$, the following uniform convergence holds:

$$(1.2) \quad \lambda^{-n} \mathcal{L}^n g(y) \rightarrow c_g \rho$$

for some constant c_g depending on g . By duality, this is equivalent to the convergence, uniform on probability measures ν ,

$$(1.3) \quad \lambda^{-n} (\mathcal{L}^*)^n \nu \rightarrow m_\phi,$$

where m_ϕ is a conformal measure associated to the weight ϕ . The equilibrium state μ_ϕ is then given by $\mu_\phi = \rho m_\phi$, and we have $c_g = \langle m_\phi, g \rangle$.

We aim here at establishing an exponential speed of convergence in (1.2), when g satisfies necessary regularity properties. This requires to build a suitable (semi-)norm for (or equivalently, a suitable functional space on) which the operator $\lambda^{-1} \mathcal{L}$ turns out to be a contraction. Observe that condition (A) is necessary for the uniform convergence.

Establishing the following statement is then our main goal in the current paper. As far as we know, this is the first time that the existence of a spectral gap for the perturbed Perron-Frobenius operator is proved in this context even in dimension 1, except for hyperbolic endomorphisms or for weights with ad-hoc conditions (see for instance [Rue92; MS00]). This is one of the most desirable properties in dynamics.

Theorem 1.2. *Let f, q, ϕ, ρ, m_ϕ be as in Theorem 1.1 and \mathcal{L}, λ the above Perron-Frobenius operator and scaling factor associated to ϕ . Let $A > 0$ and $0 < \Omega < \log d$ be two constants. Then, for every constant $0 < \gamma \leq 1$, there exist two explicit equivalent norms for functions on \mathbb{P}^k : $\|\cdot\|_{\diamond_1}$, depending on f, γ, q and independent of ϕ , and $\|\cdot\|_{\diamond_2}$, depending on f, ϕ, γ, q , such that*

$$\|\cdot\|_\infty + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \simeq \|\cdot\|_{\diamond_2} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}.$$

Moreover, there are positive constants $c = c(f, \gamma, q, A, \Omega)$ and $\beta = \beta(f, \gamma, q, A, \Omega)$ with $\beta < 1$, both independent of ϕ and n , such that when $\|\phi\|_{\diamond_1} \leq A$ and $\Omega(\phi) \leq \Omega$ we have

$$\|\lambda^{-n} \mathcal{L}^n\|_{\diamond_1} \leq c, \quad \|\rho\|_{\diamond_1} \leq c, \quad \|1/\rho\|_{\diamond_1} \leq c, \quad \text{and} \quad \|\lambda^{-1} \mathcal{L}g\|_{\diamond_2} \leq \beta \|g\|_{\diamond_2}$$

for every function $g: \mathbb{P}^k \rightarrow \mathbb{R}$ with $\langle m_\phi, g \rangle = 0$. Furthermore, given any constant $0 < \delta < d^{\gamma/(2\gamma+2)}$, when A is small enough, the norm $\|\cdot\|_{\diamond_2}$ can be chosen so that we can take $\beta = 1/\delta$.

The construction of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ is quite involved. We use here ideas from the theory of interpolation between Banach spaces [Tri95] combined with techniques from pluripotential theory and complex dynamics. Roughly speaking, an idea from interpolation theory allows us to reduce the problem to the case where $\gamma = 1$. The definition of the above norms in this case requires a control of the derivatives of g (in the distributional sense), and this is where we use techniques from pluripotential theory. This also explains why these norms are bounded by the \mathcal{C}^1 norm. Note that we should be able to bound the derivatives of $\mathcal{L}g$ in a similar way. A quick expansion of the derivatives of $\mathcal{L}g$ using (1.1) gives an idea of the difficulties that one faces. The existence of these norms is still surprising to us.

Let us highlight two among these difficulties. First, the objects from complex analysis and geometry are too rigid for perturbations with a non-constant weight: none of the operators f_* , d , and dd^c commutes with the operator \mathcal{L} . In particular, the dd^c -method developed by the second author and Sibony (see for instance [DS10a]) cannot be applied in this context, even for small perturbations of the weight $\phi = 0$. Moreover, we have critical points on the support of the measure, which cause a loss in the regularity of functions under the operators f_* and \mathcal{L} (see Section 3). Notice that we do not assume that our potential degenerates at the critical points.

Our solution to these problems is to define a new invariant functional space and norm in this mixed real-complex setting, that we call the *dynamical Sobolev space* and *semi-norm*, taking into account both the regularity of the function (to cope with the rigidity of the complex objects) and the action of f (to take into account the critical dynamics), see Definitions 3.8 and 3.11. The construction of this norm requires the definition of several intermediate semi-norms and the precise study of the action of the operator f_* with respect to them, and is carried out in Section 3. Some of the intermediate estimates already give new or more precise convergence properties for the operator f_* and the equilibrium measure μ , see for instance Theorem 3.2.

A spectral gap for the Perron-Frobenius operator and its perturbations is one of the most desirable properties in dynamics. It allows us to obtain several statistical properties of the equilibrium state. In the present setting, we have the following result, see Appendix A for the definitions.

Theorem 1.3. *Let $f, \phi, \mu_\phi, m_\phi, \|\cdot\|_{\diamond_1}$ be as in Theorems 1.1 and 1.2, λ the scaling ratio associated to ϕ , and assume that $\|\phi\|_{\diamond_1} < \infty$. Then the equilibrium state μ_ϕ is exponentially mixing for observables with bounded $\|\cdot\|_{\diamond_1}$ norm and the preimages of points by f^n (suitably weighted) equidistribute exponentially fast towards m_ϕ as n goes to infinity. The measure μ_ϕ satisfies the LDP for all observables with finite $\|\cdot\|_{\diamond_1}$ norm, the ASIP, CLT, almost sure CLT, LIL for all observables with finite $\|\cdot\|_{\diamond_1}$ norm which are not coboundaries, and the local CLT for all observables with finite $\|\cdot\|_{\diamond_1}$ norm which are not $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycles. Moreover, the pressure $P(\phi) = \log \lambda$ is analytic in the following sense: for $\|\psi\|_{\diamond_1} < \infty$ and t sufficiently small, the function $t \mapsto P(\phi + t\psi)$ is analytic.*

In particular, all the properties in Theorem 1.3 hold when the weight ϕ and the observable are Hölder continuous and satisfy the necessary coboundary/cocycles requirements. In this assumption, some of the above properties were previously obtained with ad hoc arguments, see [PUZ89; DU91a; DU91b; DPU96; Hay99; DNS07; SUZ15] when $k = 1$, [UZ13; SUZ14] for mixing, CLT, LIL when $k \geq 1$, and [Dup10] for the ASIP when $k \geq 1$ and $\phi = 0$. Our results are more general, sharper, and with better error control. Note that the LDP and the local CLT are new even for $\phi = 0$ (for all $k \geq 1$ and for all $k > 1$, respectively). Our method of proof of Theorem 1.3 exploits the spectral gap established in Theorem 1.2 and is based on the theory of perturbed operators. This approach was first developed by Nagaev [Nag57] in the context of Markov chains, see also [Rou83; Bro96; Gou15], and provides a unified treatment for all the statistical study. Observe in particular that the fine control given for instance by the local CLT is simply impossible to prove using weaker arguments, such as martingales, see, e.g., [Gou15, p. 163].

For the reader's convenience, the above statistic properties (exponential mixing, LDP, ASIP, CLT, almost sure CLT, LIL, local CLT) and the notions of coboundary and multiplicative cocycle will be recalled in Section 5 and Appendix A at the end of the paper.

Outline of the organization of the paper. In Section 2, we introduce some notations and establish comparison principles for currents and potentials that will be the technical key in the construction of our norms. In Section 3, we introduce the main (semi-)norms that we will need, and study the action of the operator f_* with respect to these (semi-)norms. Section 4 is dedicated to the proof of Theorem 1.2. Finally, in Section 5, we develop the statistical study of the equilibrium states. This section contains the proof of Theorem 1.3 and more precise

statements. In Appendix A, we recall statistical properties and criteria in abstract settings that we use to prove results in Section 5.

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2. PRELIMINARIES AND COMPARISON PRINCIPLES

2.1. Some definitions. We collect here some notions that we will use throughout the paper.

Definition 2.1. Given a subset U of \mathbb{P}^k or \mathbb{C}^k and a real-valued function $g: U \rightarrow \mathbb{R}$, define the *oscillation* $\Omega_U(g)$ of g as

$$\Omega_U(g) := \sup g - \inf g$$

and its continuity modulus $m_U(g, r)$ at distance r as

$$m_U(g, r) := \sup_{x, y \in U: \text{dist}(x, y) \leq r} |g(x) - g(y)|.$$

We may drop the index U when there is no possible confusion.

The following semi-norms will be the main building blocks for all the semi-norms that we will construct and study in the sequel.

Definition 2.2. The semi-norm $\|\cdot\|_{\log^p}$ is defined for every $p > 0$ and $g: \mathbb{P}^k \rightarrow \mathbb{R}$ as

$$\|g\|_{\log^p} := \sup_{a, b \in \mathbb{P}^k} |g(a) - g(b)| \cdot (\log^* \text{dist}(a, b))^p = \sup_{r > 0, a \in \mathbb{P}^k} \Omega_{\mathbb{B}_{\mathbb{P}^k}(a, r)}(g) \cdot (1 + |\log r|)^p,$$

where $\mathbb{B}_{\mathbb{P}^k}(a, r)$ denotes the ball of center a and radius r in \mathbb{P}^k . The definition can be extended to functions on any metric space.

Definition 2.3. The norm $\|\cdot\|_*$ of a $(1, 1)$ -current R is given by

$$\|R\|_* := \inf \|S\|$$

where the infimum is taken over all positive closed $(1, 1)$ -currents S such that $|R| \leq S$, see the Notation at the beginning of the paper. When such a current S does not exist, we put $\|R\|_* := +\infty$. The semi-norm $\|\cdot\|_*$ of an integrable function $g: \mathbb{P}^k \rightarrow \mathbb{R}$ is given by

$$\|g\|_* := \|dd^c g\|_*.$$

Note that when R is a real closed $(1, 1)$ -current the above norm is equivalent to the usual one defined as

$$\|R\|_* := \inf(\|S^+\| + \|S^-\|)$$

where the infimum is taken over all positive closed $(1, 1)$ -currents S^\pm on \mathbb{P}^k such that $R = S^+ - S^-$. In particular, for $R = dd^c g$, the currents S^+ and S^- are cohomologous and thus have the same mass, i.e., $\|S^+\| = \|S^-\|$, see [DS10a, App. A.4] for details.

In this paper, we only consider continuous functions g . So the above semi-norms (and the others that we will introduce later) are almost norms: they only vanish when g is constant. In particular, they are norms on the space of functions g satisfying $\langle \nu, g \rangle = 0$ for some fixed probability measure ν . For convenience, we will use later $\nu = m_\phi$ or $\nu = \mu_\phi$ to obtain a spectral gap for the Perron-Frobenius operator and to study the statistical properties of μ_ϕ .

2.2. Approximations for Hölder continuous functions. We will need the following property for Hölder continuous functions, see Definition 3.11 and Remark 3.12.

Lemma 2.4. *Let $0 < \gamma \leq 1$ be a constant. Then, for every \mathcal{C}^γ function $g: \mathbb{P}^k \rightarrow \mathbb{R}$, $s \geq 1$, and $0 < \epsilon \leq 1$, there exist a \mathcal{C}^s function $g_\epsilon^{(1)}$ and a continuous function $g_\epsilon^{(2)}$ such that*

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \quad \|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \leq c \|g\|_\infty (1/\epsilon)^{s/\gamma} \quad \text{and} \quad \|g_\epsilon^{(2)}\|_\infty \leq c \|g\|_{\mathcal{C}^\gamma} \epsilon,$$

where $c = c(\gamma, s)$ is a positive constant independent of g and ϵ .

Proof. Using a partition of unity, we can reduce the problem to the case where g is supported by the unit ball of an affine chart $\mathbb{C}^k \subset \mathbb{P}^k$. Consider a smooth non-negative function χ with support in the unit ball of \mathbb{C}^k whose integral with respect to the Lebesgue measure is 1. For $\nu > 0$, consider the function $\chi_\nu(z) := \nu^{-2k} \chi(z/\nu)$ which has integral 1 and tends to the Dirac mass at 0 when ν tends to 0. Define an approximation of g using the standard convolution operator $g_\nu := g * \chi_\nu$, and define $g_\epsilon^{(1)} := g_\nu$ and $g_\epsilon^{(2)} := g - g_\nu$. We consider $\nu := \epsilon^{1/\gamma}$. It remains to bound $\|g_\epsilon^{(1)}\|_{\mathcal{C}^s}$ and $\|g_\epsilon^{(2)}\|_\infty$.

By standard properties of the convolution we have, for some constant $\kappa > 0$,

$$\|g_\epsilon^{(2)}\|_\infty \lesssim m(g, \kappa\nu) \lesssim \|g\|_{\mathcal{C}^\gamma} (\kappa\nu)^\gamma \lesssim \|g\|_{\mathcal{C}^\gamma} \epsilon.$$

and, by definition of g_ν ,

$$\|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \lesssim \|g\|_\infty \|\chi_\nu\|_{\mathcal{C}^s} \text{Leb}(\mathbb{B}_\nu^k) \lesssim \|g\|_\infty \nu^{-s} \lesssim \|g\|_\infty \epsilon^{-s/\gamma}.$$

The lemma follows. \square

2.3. Dynamical potentials. Let T denote the Green $(1, 1)$ -current of f . It is positive closed and of unit mass. Let S be any positive closed $(1, 1)$ -current of mass m on \mathbb{P}^k . There is a unique function $u_S: \mathbb{P}^k \rightarrow \mathbb{R} \cup \{-\infty\}$ which is p.s.h. modulo mT and such that

$$S = mT + dd^c u_S \quad \text{and} \quad \langle \mu, u_S \rangle = 0.$$

Locally, u_S is the difference between a potential of S and a potential of mT . We call it *the dynamical potential* of S . Observe that the dynamical potential of T is zero, i.e., $u_T = 0$.

Recall that T has Hölder continuous potentials. So, u_S is locally the difference between a p.s.h. function and a Hölder continuous one. We refer the reader to [DS10a; BD20] for details. In this paper, we only need currents S such that u_S is continuous.

2.4. Complex Sobolev functions. In our study, we will be naturally lead to consider currents of the form $i\partial u \wedge \bar{\partial} u$. These currents are always positive. In this section, we study the regularity of u under the assumption that $i\partial u \wedge \bar{\partial} u \leq dd^c v$ for some v of given regularity. Recall that, given a smooth bounded open set $\Omega \subset \mathbb{C}^k$, the Sobolev space $W^{1,2}(\Omega)$ is defined as the space of functions $u: \Omega \rightarrow \mathbb{R}$ such that $\|u\|_{W^{1,2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\partial u\|_{L^2(\Omega)} < \infty$, where the reference measure is the standard Lebesgue measure on Ω . The Poincaré-Wirtinger's inequality implies that $\|u\|_{W^{1,2}(\Omega)} \lesssim \|\partial u\|_{L^2(\Omega)} + \left| \int_\Omega u \, d\text{Leb} \right|$. We will need the following lemmas.

Lemma 2.5. *There is a universal positive constant c such that*

$$\int_K |u| \, d\text{Leb} \leq c \text{Leb}(K) (\log^* \text{Leb}(K))^{1/2}.$$

for every compact set $K \subset \mathbb{D}_1$ with $\text{Leb}(K) > 0$ and function $u: \mathbb{D}_1 \rightarrow \mathbb{R}$ such that $\|u\|_{W^{1,2}(\mathbb{D}_1)} \leq 1$.

Proof. By Trudinger-Moser's inequality [Mos71], there are positive constants c_0 and α such that

$$\int_{\mathbb{D}_1} e^{2\alpha|u|^2} \, d\text{Leb} \leq c_0.$$

Let m denote the restriction of the measure Leb to K multiplied by $1/\text{Leb}(K)$. This is a probability measure. It follows from Cauchy-Schwarz's inequality that

$$\int e^{\alpha|u|^2} dm \leq \left(\int e^{2\alpha|u|^2} dm \right)^{1/2} \lesssim \text{Leb}(K)^{-1/2}.$$

Observe that the function $t \mapsto e^{\alpha t^2}$ is convex on \mathbb{R}^+ and its inverse is the function $t \mapsto \alpha^{-1/2}(\log t)^{1/2}$. By Jensen's inequality, we obtain

$$\int |u| dm \leq \alpha^{-1/2} \left[\log \int e^{\alpha|u|^2} dm \right]^{1/2} \lesssim (\log^* \text{Leb}(K))^{1/2}.$$

The lemma follows. \square

Lemma 2.6. *Let $u: \mathbb{D}_2 \rightarrow \mathbb{R}$ be a continuous function and $\chi: \mathbb{D}_2 \rightarrow \mathbb{R}$ a smooth function with compact support in \mathbb{D}_2 and equal to 1 on $\bar{\mathbb{D}}_1$. Set $\chi_z := \partial\chi/\partial z$. Then we have, for all $0 < r < s < 1$,*

$$(2.1) \quad u(0) - u(r) = \frac{1}{2\pi} \left\langle i\partial u, \chi(s^{-1}z) \frac{r}{z(\bar{z}-r)} d\bar{z} \right\rangle + \frac{1}{2\pi} \left\langle u, \chi_z(s^{-1}z) \frac{r}{s\bar{z}(\bar{z}-r)} idz \wedge d\bar{z} \right\rangle.$$

Proof. Denote by δ_ξ the Dirac mass at $\xi \in \mathbb{C}$. Observe that

$$\frac{i}{2\pi} \partial \frac{d\bar{z}}{\bar{z}-\xi} = dd^c \log |z-\xi| = \delta_\xi,$$

where the equalities are in the sense of currents on \mathbb{C} . Hence, for $|\xi| < s$,

$$\frac{i}{2\pi} \partial \left[\frac{\chi(s^{-1}z) d\bar{z}}{\bar{z}-\xi} \right] = \frac{\chi_z(s^{-1}z) idz \wedge d\bar{z}}{2\pi s(\bar{z}-\xi)} + \chi(s^{-1}z) \delta_\xi = \frac{\chi_z(s^{-1}z) idz \wedge d\bar{z}}{2\pi s(\bar{z}-\xi)} + \delta_\xi.$$

Applying this identity for $\xi = 0$ and $\xi = r$, and since $u(0) - u(r) = \langle u, \delta_0 - \delta_r \rangle$, we obtain

$$\begin{aligned} u(0) - u(r) &= \frac{1}{2\pi} \left\langle u, i\partial \left[\chi(s^{-1}z) \left(\frac{1}{\bar{z}} - \frac{1}{\bar{z}-r} \right) d\bar{z} \right] \right\rangle - \frac{1}{2\pi s} \left\langle u, \chi_z(s^{-1}z) \left(\frac{1}{\bar{z}} - \frac{1}{\bar{z}-r} \right) idz \wedge d\bar{z} \right\rangle \\ &= \frac{1}{2\pi} \left\langle i\partial u, \chi(s^{-1}z) \frac{r}{\bar{z}(\bar{z}-r)} d\bar{z} \right\rangle + \frac{1}{2\pi} \left\langle u, \chi_z(s^{-1}z) \frac{r}{s\bar{z}(\bar{z}-r)} idz \wedge d\bar{z} \right\rangle. \end{aligned}$$

The assertion is proved. \square

The following is a main result in this section. It will be a crucial technical tool in the construction of the norms with respect to which the transfer operator has a spectral gap.

Proposition 2.7. *Let $u: \mathbb{B}_5^k \rightarrow \mathbb{R}$ be continuous and such that $\|\partial u\|_{L^2(\mathbb{B}_5^k)} < \infty$. Assume that $i\partial u \wedge \bar{\partial} u \leq dd^c v$ where $v: \mathbb{B}_5^k \rightarrow \mathbb{R}$ is continuous, p.s.h., and such that*

$$(2.2) \quad \int_0^1 m_{\mathbb{B}_4^k}(v, t) (\log \log^* t)^4 t^{-1} dt < +\infty.$$

Then there is a positive constant c such that, for all $0 < r \leq 1/2$, we have

$$(2.3) \quad m_{\mathbb{B}_1^k}(u, r) \leq c \left(\int_0^{r^{1/2}} m_{\mathbb{B}_4^k}(v, t) (\log \log^* t)^2 t^{-1} dt \right)^{1/2} + c m_{\mathbb{B}_4^k}(v, r)^{1/3} \Omega_{\mathbb{B}_4^k}(v)^{1/6} (\log^* r)^{1/2} + c \Omega_{\mathbb{B}_4^k}(v)^{1/2} r^{1/2} (\log^* r)^{1/2}.$$

Proof. Let $x, y \in \mathbb{B}_1^k$ be such that $\text{dist}(x, y) \leq r \leq 1/2$. We need to bound $|u(x) - u(y)|$ by the RHS of (2.3). We can assume without loss of generality that $\text{dist}(x, y) = r$. By a change of coordinates and restricting to the complex line through x and y we can reduce the problem to the case of dimension 1. More precisely, we can assume that $x = 0$ and $y = r$ in \mathbb{C} and that u, v are defined on \mathbb{D}_4 . By subtracting constants, we can assume that $v(0) = 0$ and $\int_{\mathbb{D}_3} u(z) idz \wedge d\bar{z} = 0$. By multiplying u and v by suitable constants γ and γ^2 , we can assume

that $m_{\mathbb{D}_3}(v, 1) = 1/8$, which implies that $|v| \leq 1$ on \mathbb{D}_3 . In order to establish (2.3) it is enough to show that $|u(0) - u(r)|$ is bounded by a constant times

$$(2.4) \quad \left(\int_0^{r^{1/2}} m_{\mathbb{D}_3}(v, t) (\log \log^* t)^2 t^{-1} dt \right)^{1/2} + m_{\mathbb{D}_3}(v, r)^{1/3} (\log^* r)^{1/2} + r^{1/2} (\log^* r)^{1/2}.$$

Since v is bounded, by Chern-Levine-Nirenberg's inequality [CLN69] the mass of $dd^c v$ on \mathbb{D}_2 is bounded by a constant. Thus, by the hypotheses on u and v , the L^2 -norm of ∂u on \mathbb{D}_2 is bounded by a constant and therefore, by Poincaré-Wirtinger's inequality, $\|u\|_{W^{1,2}(\mathbb{D}_2)}$ is also bounded by a constant.

Fix a smooth function $0 \leq \chi(z) \leq 1$ with compact support in \mathbb{D}_2 and such that $\chi = 1$ on $\bar{\mathbb{D}}_1$. Define $\chi_z := \partial \chi / \partial z$ and set

$$s := r \min \{r^{-1/2}, m_{\mathbb{D}_3}(v, r)^{-1/3}\}.$$

We have

$$(2.5) \quad \sqrt{2}r \leq s \leq r^{1/2} < 1$$

because $m_{\mathbb{D}_3}(v, r) \leq m_{\mathbb{D}_3}(v, 1) = 1/8$ and $0 < r \leq 1/2$. The functions u and χ satisfy the assumptions of Lemma 2.6. Thus, (2.1) holds for the above s and r . The second term in the RHS of (2.1) is an integral over $\mathbb{D}_{2s} \setminus \mathbb{D}_s$ because χ_z has support in $\mathbb{D}_2 \setminus \mathbb{D}_1$. Moreover, for $z \in \mathbb{D}_{2s} \setminus \mathbb{D}_s$, we have $\frac{r}{s\bar{z}(z-r)} = O(rs^{-3})$ because of (2.5). Using that $\|u\|_{W^{1,2}(\mathbb{D}_2)} \lesssim 1$ and that $\text{Leb}(\mathbb{D}_{2s}) \lesssim s^2$, Lemma 2.5 implies that the considered term has modulus bounded by a constant times

$$rs^{-1} (\log^* s)^{1/2} \lesssim \max \{r^{1/2}, m_{\mathbb{D}_3}(v, r)^{1/3}\} (\log^* r)^{1/2}.$$

The last expression is bounded by the sum in (2.4).

In order to conclude, it remains to bound the first term in the RHS of (2.1). Choose a smooth decreasing function $h(t)$ defined for $t > 0$ and such that $h(t) := (-\log t)(\log(-\log t))^2$ for t small enough and $h(t) = 1$ for t large enough. Define $\eta(z) := h(|z|) + h(|z-1|)$. We will also use the function $\tilde{v}(z) := v(z) - r^{-1}v(r)\Re(z)$. This function satisfies $dd^c \tilde{v} = dd^c v$ and $\tilde{v}(0) = \tilde{v}(r) = 0$. By Cauchy-Schwarz's inequality we have for the first term in the RHS of (2.1)

$$\left| \left\langle i\partial u, \chi(s^{-1}z) \frac{r}{\bar{z}(\bar{z}-r)} d\bar{z} \right\rangle \right|^2 \leq \langle i\partial u \wedge \bar{\partial} u, \chi(s^{-1}z) \eta(r^{-1}z) \rangle \int \frac{\chi(s^{-1}z)}{\eta(r^{-1}z)} \frac{r^2}{|z^2(z-r)^2|} idz \wedge d\bar{z}.$$

Using the change of variable $z \mapsto rz$, the fact that $0 \leq \chi \leq 1$, and the definition of η we see that the last integral is bounded by

$$\int_{\mathbb{C}} \frac{idz \wedge d\bar{z}}{[h(|z|) + h(|z-1|)] |z^2(z-1)^2|}.$$

Using polar coordinates for z and for $z-1$ and the definition of h it is not difficult to see that the last integral is finite. Therefore, since $i\partial u \wedge \bar{\partial} u \leq dd^c v = dd^c \tilde{v}$, we get

$$\left| \left\langle i\partial u, \chi(s^{-1}z) \frac{r}{\bar{z}(\bar{z}-r)} d\bar{z} \right\rangle \right|^2 \lesssim \langle dd^c \tilde{v}, \chi(s^{-1}z) \eta(r^{-1}z) \rangle.$$

Define $\hat{v}(z) := \tilde{v}(sz)$. The RHS in the last expression is then equal to

$$\langle dd^c \hat{v}, \chi(z) \eta(r^{-1}sz) \rangle = \langle dd^c \hat{v}, \chi(z) h(r^{-1}s|z|) \rangle + \langle dd^c \hat{v}, \chi(z) h(|r^{-1}sz-1|) \rangle.$$

In order to conclude the proof of the proposition, it is enough to show that each term in the last sum is bounded by a constant times

$$(2.6) \quad \int_0^s m_{\mathbb{D}_3}(v, t) (\log^* |\log t|)^2 t^{-1} dt + m_{\mathbb{D}_3}(v, r)^{2/3} \log^* r.$$

We will only consider the first term. The second term can be treated in a similar way using the coordinate $z' := z - rs^{-1}$. Since h is decreasing, the first term we consider is bounded by

$$\langle dd^c \hat{v}, \chi(z) h(|z|) \rangle.$$

Claim. *We have*

$$(2.7) \quad \langle dd^c \hat{v}, \chi(z)h(|z|) \rangle = \int_{\mathbb{D}_2 \setminus \{0\}} \hat{v}(z) dd^c [\chi(z)h(|z|)].$$

We assume the claim for now and conclude the proof of the proposition. Notice that the assumption (2.2) will be used in the proof of this claim.

Using the definitions of h and χ , we can bound the RHS of (2.7) by a constant times

$$\begin{aligned} \int_{\mathbb{D}_2 \setminus \{0\}} |\hat{v}(z)z^{-2}| (\log \log^* |z|)^2 idz \wedge d\bar{z} &= \int_{\mathbb{D}_{2s} \setminus \{0\}} |\tilde{v}(z)z^{-2}| (\log \log^* |z/s|)^2 idz \wedge d\bar{z} \\ &\lesssim \int_{\mathbb{D}_{2s} \setminus \{0\}} |\tilde{v}(z)z^{-2}| (\log \log^* |z|)^2 idz \wedge d\bar{z}, \end{aligned}$$

where we used the change of variable $z \mapsto sz$ and the fact that $\log^* |z/s| \lesssim \log^* |z|$ for $0 < |z| < 2s < 2$. Moreover, by the definition of \tilde{v} and using that $v(0) = \tilde{v}(0) = 0$, we have for $|z| < 2s$

$$|\tilde{v}(z)| \leq m_{\mathbb{D}_3}(v, |z|) + r^{-1}|v(r)\Re(z)| \leq m_{\mathbb{D}_3}(v, |z|) + m_{\mathbb{D}_3}(v, r)r^{-1}|z|.$$

Therefore, using polar coordinates, we see that the last integral is bounded by a constant times

$$\int_0^{2s} m_{\mathbb{D}_3}(v, t) (\log \log^* t)^2 t^{-1} dt + m_{\mathbb{D}_3}(v, r)r^{-1}s(\log \log^*(2s))^2.$$

The first term in this sum is bounded by a constant times the integral in (2.6) because $m_{\mathbb{D}_3}(v, t') \leq 4m_{\mathbb{D}_3}(v, t)$ for $s/2 \leq t \leq s \leq t' \leq 2s$. The second one is bounded by a constant times the second term in (2.6) by the definition of s and (2.5). The proposition follows. \square

Proof of the claim. Observe that $h(|z|)$ tends to infinity when z tends to 0. Let $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth increasing concave function such that $\vartheta(t) = t$ for $t \leq 0$ and $\vartheta(t) = 1$ for $t \geq 2$. Define $\vartheta_n(t) := \vartheta(t - n) + n$. This is a sequence of smooth functions increasing to the identity. Define $l(z) := \chi(z)h(|z|)$. Using an integration by parts, we see that the LHS of (2.7) is equal to

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle dd^c \hat{v}, \vartheta_n(l(z)) \rangle &= \lim_{n \rightarrow \infty} \int_{\mathbb{D}_3} \hat{v}(z) dd^c \vartheta_n(l(z)) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{D}_3} \hat{v}(z) \vartheta'_n(l(z)) dd^c l(z) + \lim_{n \rightarrow \infty} \int_{\mathbb{D}_3} \hat{v}(z) \vartheta''_n(l(z)) dl(z) \wedge d^c l(z). \end{aligned}$$

The first term in the last sum converges to the RHS of the identity in the claim using Lebesgue's dominated convergence theorem and (2.2). We need to show that the second term tends to 0. Since $\chi(z) = 1$ for z near 0, for n large enough, the considered term has an absolute value bounded by a constant times

$$\lim_{n \rightarrow \infty} \int_{\{h(|z|) > n\}} |\hat{v}(z)| i\partial h(|z|) \wedge \bar{\partial} h(|z|) \lesssim \lim_{n \rightarrow \infty} \int_{\{h(|z|) > n\}} |\hat{v}(z)z^{-2}| (\log \log^* |z|)^4 idz \wedge d\bar{z}.$$

Using the arguments as at the end of the proof of Proposition 2.7 and the assumption (2.2) on v , we see that the last integrand is an integrable function on \mathbb{D}_1 . Since the set $\{h(|z|) > n\}$ decreases to $\{0\}$ when n tends to infinity, the last limit is zero according to Lebesgue's dominated convergence theorem. This ends the proof of the claim. \square

Corollary 2.8. *Let S_0 be a positive closed $(1, 1)$ -current on \mathbb{P}^k of unit mass, whose dynamical potential u_S satisfies $\|u_S\|_{\log^p} \leq 1$ for some $p > 3/2$. Let $\mathcal{F}(S_0)$ denote the set of all continuous functions $g : \mathbb{P}^k \rightarrow \mathbb{R}$ such that $i\partial g \wedge \bar{\partial} g \leq S_0$. Then for any positive number $q < \frac{p}{3} - \frac{1}{2}$ we have $\|g\|_{\log^q} \leq c$ for some positive constant $c = c(p, q)$ independent of S_0 . In particular, the family $\mathcal{F}(S_0)$ is equicontinuous.*

Proof. Notice that (2.2) is satisfied for all v such that $\|v\|_{\log^p} < \infty$ for some $p > 1$. It follows that if u and v are as in Proposition 2.7 and v is \log^p -continuous for some $p > 3/2$ then u is \log^q -continuous on \mathbb{B}_1^k for all q as in the statement, with $\|u\|_{\mathbb{B}_1^k, \log^q} \leq c \|v\|_{\mathbb{B}_5^k, \log^p}^{1/2}$ for some positive constant c independent of u, v . The result is thus deduced from Proposition 2.7 by means of a finite cover of \mathbb{P}^k . \square

3. SOME SEMI-NORMS AND EQUIDISTRIBUTION PROPERTIES

In this section, we consider the action of the operator $(f^n)_*$ on functions and currents. We also introduce the semi-norms which are crucial in our study. Some results and ideas here are of independent interest. Recall that we always assume that f satisfies the Assumption **(A)** in the Introduction.

3.1. Bounds with respect to the semi-norm $\|\cdot\|_{\log^p}$. In this section, we study the action of the operator f_* on functions with bounded semi-norm $\|\cdot\|_{\log^p}$. We first prove that, with respect to this semi-norm, the operator f_* is Lipschitz.

Lemma 3.1. *For every constant $A > 1$, there exists a positive constant $c = c(A)$ such that for every $n \geq 0$, $p > 0$, and continuous function $g: \mathbb{P}^k \rightarrow \mathbb{R}$, we have*

$$\|d^{-kn}(f^n)_*g\|_{\log^p} \leq c^p A^{pn} \|g\|_{\log^p}.$$

Proof. We have

$$\|d^{-kn}(f^n)_*g\|_{\log^p} = \sup_{x, y \in \mathbb{P}^k} d^{-kn} |(f^n)_*g(x) - (f^n)_*g(y)| \cdot (\log^* \text{dist}(x, y))^p.$$

We need to bound the RHS by $c^p A^{pn} \|g\|_{\log^p}$.

Applying [DS10b, Cor. 4.4] inductively to some iterate of f , we see that the Assumption **(A)** implies:

(A') for every constant $\kappa > 1$, there are an integer $n_\kappa \geq 0$ and a constant $c_\kappa > 0$ independent of n such that for all $x, y \in \mathbb{P}^k$ and $n \geq n_\kappa$ we can write $f^{-n}(x) = \{x_1, \dots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \dots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\text{dist}(x_j, y_j) \leq c_\kappa \text{dist}(x, y)^{1/\kappa^n} \quad \text{for } j = 1, \dots, d^{kn}.$$

Fix $\kappa < A$. We have, for $n \geq n_\kappa$,

$$\begin{aligned} d^{-kn} |(f^n)_*g(x) - (f^n)_*g(y)| (\log^* \text{dist}(x, y))^p &\leq \max_j |g(x_j) - g(y_j)| (\log^* \text{dist}(x, y))^p \\ &\leq \max_j \frac{\|g\|_{\log^p}}{(\log^* \text{dist}(x_j, y_j))^p} (\log^* \text{dist}(x, y))^p \\ &= \max_j \left(\frac{\log^* \text{dist}(x, y)}{\kappa^n \log^* \text{dist}(x_j, y_j)} \right)^p \|g\|_{\log^p} \kappa^{pn}. \end{aligned}$$

We need to check that the expression in the last parentheses is bounded by a constant. Fix a large constant $M > 0$. Since $\log^* \text{dist}(x_j, y_j)$ is bounded from below by 1, when $\log^* \text{dist}(x, y)$ is bounded by $2M\kappa^n$ the considered expression is bounded by some constant c as desired. Assume now that $\log^* \text{dist}(x, y) \geq 2M\kappa^n$. Since M is large, we deduce that $\log \text{dist}(x, y) \leq -2M\kappa^n + 1 \leq -M\kappa^n$. Hence, by **(A')**, since M is large, we have

$$\log \text{dist}(x_j, y_j) \leq \log c_\kappa + \kappa^{-n} \log \text{dist}(x, y) \leq \frac{1}{2} \kappa^{-n} \log \text{dist}(x, y).$$

It is now clear that $\kappa^n \log^* \text{dist}(x_j, y_j) \geq \frac{1}{2} \log^* \text{dist}(x, y)$ which implies that the considered expression is bounded, as desired. This implies the lemma for $n \geq n_\kappa$.

As the multiplicity of f^n at a point is at most d^{kn} , we also have (see again [DS10b, Cor. 4.4]):

(**A''**) there is a constant $c_0 > 0$ such that for every $n \geq 0$, for all $x, y \in \mathbb{P}^k$, we can write $f^{-n}(x) = \{x_1, \dots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \dots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\text{dist}(x_j, y_j) \leq c_0 \text{dist}(x, y)^{1/d^{kn}} \quad \text{for } j = 1, \dots, d^{kn}.$$

Hence, when $n \leq n_\kappa$, it is enough to use (**A''**) instead of (**A'**). Since n_κ is fixed it is clear that $\frac{\log^* \text{dist}(x, y)}{\log^* \text{dist}(x_j, y_j)} \lesssim d^{kn_\kappa}$, which is bounded. The proof is complete. \square

We will need the following result which is an improvement of [DS10b, Th. 1.1] in the case where f satisfies the Assumption (**A**). By duality, this result implies an exponential equidistribution of $d^{-kn}(f^n)_* \nu$ towards μ for every probability measure ν . The assumption (**A**) is necessary here to get the estimate in the norm $\|\cdot\|_\infty$.

Theorem 3.2. *Let f be an endomorphism of \mathbb{P}^k of algebraic degree $d \geq 2$ and satisfying the Assumption (**A**). Consider a real number $p > 0$. Let $g: \mathbb{P}^k \rightarrow \mathbb{R}$ be such that $\|dd^c g\|_* \leq 1$, $\langle \mu, g \rangle = 0$ and $\|g\|_{\log^p} \leq 1$. Then, for every constant $\eta > d^{-p/(p+1)}$, there is a positive constant c independent of g such that for every $n \geq 0$*

$$\|d^{-kn}(f^n)_* g\|_\infty \leq c\eta^n.$$

Proof. Set $g_n := d^{-kn}(f^n)_* g$. Recall (see, e.g., [Sko72; DNS10]) that there exists a positive constant c_0 independent of g and n such that

$$(3.1) \quad \int_{\mathbb{P}^k} e^{d^n |g_n|} \leq c_0,$$

where the integral above is taken with respect to the Lebesgue measure associated to the volume form $\omega_{\mathbb{F}_S}^k$ on \mathbb{P}^k .

Fix a constant $A > 1$ such that $\eta > (A/d)^{p/(p+1)}$. Suppose by contradiction that for infinitely many n there exists a point $a_n \in \mathbb{P}^k$ such that $|g_n(a_n)| \geq 3\eta^n$ for some g as above. Choose $r := e^{-c_A A^n \eta^{-n/p}}$ with c_A the constant given by Lemma 3.1 (we write c_A instead of c in order to avoid confusion). By that lemma, when $\text{dist}(z, a_n) < r$, we have

$$|g_n(z)| \geq |g_n(a_n)| - |g_n(z) - g_n(a_n)| \geq 3\eta^n - c_A^p A^{pn} (1 + |\log r|)^{-p} \geq \eta^n.$$

This implies that

$$c_0 \geq \int_{\mathbb{P}^k} e^{d^n |g_n|} \geq \int_{\text{dist}(z, a_n) < r} e^{d^n |g_n(z)|} \gtrsim r^{2k} e^{d^n \eta^n} \gtrsim e^{-2kc_A A^n \eta^{-n/p} + d^n \eta^n}.$$

By the choice of A , the last expression diverges when n tends to infinity. This is a contradiction. The theorem follows. \square

3.2. The semi-norm $\|\cdot\|_p$. In this section, we combine the semi-norms $\|\cdot\|_{\log^p}$ and $\|\cdot\|_*$ to build a new semi-norm $\|\cdot\|_p$ and study its first properties. For our convenience, we will use dynamical potentials of currents, but this is avoidable.

For every positive closed $(1, 1)$ -current S on \mathbb{P}^k we first define

$$\|S\|'_p := \|S\| + \|u_S\|_{\log^p},$$

where u_S is the dynamical potential of S . When R is any $(1, 1)$ -current we define

$$\|R\|_p = \min \|S\|'_p,$$

where the minimum is taken over all positive closed $(1, 1)$ -currents S such that $|R| \leq S$, and we set $\|R\|_p := \infty$ when no such S exists. Finally, for all $g: \mathbb{P}^k \rightarrow \mathbb{R}$, define

$$\|g\|_p := \|dd^c g\|_p.$$

The following lemma shows in particular that the norm $\|\cdot\|_p$ is equivalent to the norm $\|\cdot\|'_p$ when both are defined. We will thus just consider the norm $\|\cdot\|_p$ in the sequel.

Lemma 3.3. *Let S be a positive closed $(1, 1)$ -current on \mathbb{P}^k and let $g: \mathbb{P}^k \rightarrow \mathbb{R}$ be a continuous function. Then*

$$\|u_S\|_p \leq 2\|S\|'_p, \quad \|S\|_p \leq \|S\|'_p \leq c\|S\|_p, \quad \text{and} \quad \|g\|_{\log^p} \leq c\|g\|_p$$

for some positive constant $c = c(p)$ independent of S and g .

Proof. Define $m := \|S\|$. We have $\|S\|'_p \geq m$. Since $u_T = 0$, we have $\|T\|_p = \|T\|'_p = 1$ and

$$\|u_S\|_p = \|dd^c u_S\|_p = \|S - mT\|_p \leq \|S\|_p + m\|T\|_p \leq \|S\|'_p + m \leq 2\|S\|'_p.$$

This proves the first assertion in the lemma.

We prove now the second assertion. The first inequality is true by definition. For the second one, it is enough to prove that if \tilde{S} is also positive closed, and such that $S \leq \tilde{S}$, then $\|S\|'_p \leq c\|\tilde{S}\|'_p$ for some constant c independent of S, \tilde{S} . It is clear that $\|S\| \leq \|\tilde{S}\|$. So we can assume that $\|\tilde{S}\| = 1$ and we only need to check that $\|u_S\|_{\log^p}$ is bounded by $c(1 + \|u_{\tilde{S}}\|_{\log^p})$ for some constant c .

We cover \mathbb{P}^k with a finite family of open sets of the form $\Phi_j(\mathbb{B}_{1/2}^k)$ where Φ_j is an injective holomorphic map from \mathbb{B}_4^k to \mathbb{P}^k . Let h_j denote a potential of T on $\Phi_j(\mathbb{B}_4^k)$. This is a Hölder continuous function. By definition of dynamical potential, $v_j := u_{\tilde{S}} + h_j$ is a potential of \tilde{S} on $\Phi_j(\mathbb{B}_4^k)$. By [BD20, Corollary 2.5], we have that $\Omega(u_S)$ is bounded by a constant. Taking into account the distortion of the maps Φ_j , a comparison principle [BD20, Corollary 2.7] implies that for all r smaller than some constant $r_0 > 0$

$$m_{\mathbb{P}^k}(u_S, r) \lesssim \max_j m_{V_j}(v_j, c\sqrt{r}) + \sqrt{r} \lesssim (1 + \|u_{\tilde{S}}\|_{\log^p}) |\log r|^{-p}.$$

This proves the second assertion.

Finally, let us consider the last inequality in the statement. By linearity, we can assume that there exists a positive closed $(1, 1)$ -current \tilde{S} of mass $\|\tilde{S}\| = 1$ such that $|dd^c g| \leq \tilde{S}$ and prove that $\|g\|_{\log^p} \leq c(1 + \|u_{\tilde{S}}\|_{\log^p})$ for some positive constant c independent of g and \tilde{S} . Observe that $dd^c g + \tilde{S}$ is a positive closed current and $dd^c g + \tilde{S} \leq 2\tilde{S}$. So we can apply the arguments in the previous paragraph to $g + u_{\tilde{S}}, 2u_{\tilde{S}}$ instead of $u_S, u_{\tilde{S}}$. We obtain $\|g + u_{\tilde{S}}\|_{\log^p} \lesssim 1 + \|u_{\tilde{S}}\|_{\log^p}$. This implies the last assertion, and completes the proof of the lemma. \square

The following lemma can be applied to a non-convex \mathcal{C}^2 function χ as it can be written as the difference of two convex functions. We can also apply it to χ Lipschitz and convex because such a function can be approximated by smooth convex functions with bounded first derivatives. The statement for the smooth convex case that we need is however simpler.

Lemma 3.4. *There is a positive constant $c = c(p)$ such that for all continuous functions $g, h: \mathbb{P}^k \rightarrow \mathbb{R}$ with finite $\|\cdot\|_p$ semi-norms and any \mathcal{C}^2 convex function $\chi: \mathbb{R} \rightarrow \mathbb{R}$ we have*

$$\|\partial g \wedge \bar{\partial} h\|_p \leq c(\Omega(g)\|h\|_p + \|g\|_p \Omega(h)) \quad \text{and} \quad \|\chi(g)\|_p \leq c\|\chi'(g)\|_\infty \|g\|_p.$$

Proof. We can write $|dd^c g| \leq S$ with $\|S\| \lesssim \|g\|_p$ and $\|u_S\|_{\log^p} \lesssim \|g\|_p$. We first prove the second inequality. Set $A := \|\chi'(g)\|_\infty$. We have

$$dd^c \chi(g) = \chi'(g)dd^c g + \frac{1}{\pi}\chi''(g)i\partial g \wedge \bar{\partial} g = \left[\chi'(g)dd^c g + AS + \frac{1}{\pi}\chi''(g)i\partial g \wedge \bar{\partial} g \right] - [AS].$$

Write the last expression as $R^+ - R^-$ where R^+ (resp. R^-) is the expression in the first (resp. second) brackets. Using the definition of A , the inequality $|dd^c g| \leq S$, the convexity of χ , and the fact that $i\partial g \wedge \bar{\partial} g$ is always positive, we deduce that both R^+ and R^- are positive currents. Clearly, the current R^- is closed and its mass is $\lesssim A\|g\|_p$. The current R^+ is cohomologous to R^- because $dd^c \chi(g)$ is an exact current. It follows that R^+ is also a positive closed current of mass $\lesssim A\|g\|_p$.

We have $\|u_{R^-}\|_{\log^p} = A\|u_S\|_{\log^p} \lesssim A\|g\|_p$. This and the above estimate for the mass imply that $\|R^-\|_p \lesssim A\|g\|_p$. On the other hand, the above identities imply that $\chi(g) + u_{R^-}$ differs from u_{R^+} by a constant. We deduce that

$$\|u_{R^+}\|_{\log^p} \leq \|u_{R^-}\|_{\log^p} + \|\chi(g)\|_{\log^p} \lesssim \|u_{R^-}\|_{\log^p} + A\|g\|_{\log^p} \lesssim A\|g\|_p.$$

Therefore, we also have $\|R^+\|_p \lesssim A\|g\|_p$. It is now clear that $\|dd^c\chi(g)\|_p \lesssim A\|g\|_p$. Hence, we get the second assertion in the lemma.

For the first assertion, we first consider the case where $g = h$. We can replace g by $g - \min g$ in order to assume that $\min g = 0$ and hence $\|g\|_\infty = \Omega(g)$. The above computation gives

$$0 \leq i\partial g \wedge \bar{\partial} g = \pi(dd^c g^2 - 2gdd^c g) \lesssim dd^c g^2 + 2\|g\|_\infty S.$$

We thus have

$$\|i\partial g \wedge \bar{\partial} g\|_p \lesssim \|dd^c g^2\|_p + \|g\|_\infty \|S\|_p \lesssim \|dd^c g^2\|_p + \|g\|_\infty \|g\|_p.$$

We obtain the desired estimate by applying the second assertion in the lemma to $\|dd^c g^2\|_p$, using the function $\chi(t) := t^2$.

Finally, let us consider the first inequality for g and h . As above, we can assume that $\min h = 0$ and hence $\|h\|_\infty = \Omega(h)$. It follows from Cauchy-Schwarz's inequality that

$$|\partial g \wedge \bar{\partial} h \pm \partial h \wedge \bar{\partial} g| \lesssim \frac{\|h\|_\infty}{\|g\|_\infty} i\partial g \wedge \bar{\partial} g + \frac{\|g\|_\infty}{\|h\|_\infty} i\partial h \wedge \bar{\partial} h.$$

The assertion thus follows from the particular case considered above. This completes the proof of the lemma. \square

3.3. The dynamical norm $\|\cdot\|_{p,\alpha}$. In this section, we define the main norms $\|\cdot\|_{p,\alpha}$ for $(1,1)$ -currents that we will use to quantify the convergence (1.2). Based on the results in the previous sections, we will see later that these norms satisfy the inequalities

$$\|\cdot\|_q \lesssim \|\cdot\|_{p,\alpha} \lesssim \|\cdot\|_p$$

for some explicit q depending on p, α , and d . In particular, the new norms are at the same time weaker than the previous norm $\|\cdot\|_p$, but still inherit the main properties of a similar norm $\|\cdot\|_q$ which are obtained in the previous section.

Definition 3.5. Given a positive closed $(1,1)$ -current S on \mathbb{P}^k and a real number α such that $d^{-1} \leq \alpha < 1$, we define the current S_α by

$$S_\alpha := \sum_{n=0}^{\infty} \alpha^n \frac{(f^n)_*(S)}{d^{(k-1)n}}.$$

For any $(1,1)$ -current R on \mathbb{P}^k and real number $p > 0$, we define

$$(3.2) \quad \|R\|_{p,\alpha} := \inf \left\{ c \in \mathbb{R} : \exists S \text{ positive closed} : \|S\|_p \leq 1, |R| \leq cS_\alpha \right\}$$

and we set $\|R\|_{p,\alpha} := \infty$ if such a number c does not exist.

Note that when $\|R\|_{p,\alpha}$ is finite, by compactness, the infimum in (3.2) is actually a minimum. We have the following lemma where the assumption $d^{-1} \leq \alpha < d^{-1/(p+1)}$ is equivalent to $0 < q_0 \leq p$.

Lemma 3.6. *Let α and p be positive and such that $d^{-1} \leq \alpha < d^{-1/(p+1)}$. Then, for every $0 < q < q_0 := \frac{\lfloor \log \alpha \rfloor}{\log d} (p+1) - 1$, there are positive constants $c_1 = c_1(p, \alpha)$ and $c_2 = c_2(p, \alpha, q)$ such that, for every $(1,1)$ -current R ,*

$$\|R\|_{p,\alpha} \leq c_1 \|R\|_p \quad \text{and} \quad \|R\|_q \leq c_2 \|R\|_{p,\alpha}.$$

Proof. The first inequality holds by the definition of $\|\cdot\|_{p,\alpha}$ and Lemma 3.3. We prove the second inequality. Consider a current R such that $\|R\|_{p,\alpha} = 1$. We have to show that $\|R\|_q$ is bounded by a constant.

From the definition of $\|\cdot\|_{p,\alpha}$, we can find a positive closed current S such that $\|S\|_p = 1$ and $|R| \leq S_\alpha$. By the definition of the norm $\|\cdot\|_q$ and Lemma 3.3 applied to S_α , it is enough to show that $\|S_\alpha\|'_q$ is bounded. Denote by u_α the dynamical potential of S_α . Since the mass of S_α is bounded, we only need to show that $\|u_\alpha\|_{\log^q}$ is bounded. By definition of S_α , we have

$$u_\alpha = \sum_{n=0}^{\infty} \alpha^n \frac{(f^n)_* u_S}{d^{(k-1)n}}.$$

It follows that, for every positive number N ,

$$m(u_\alpha, r) \leq \sum_{n \leq N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_{\log^p} (\log^* r)^{-p} + 2 \sum_{n > N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_\infty.$$

Fix constants $A > 1$ close enough to 1, $\eta > d^{-p/(p+1)}$ close enough to $d^{-p/(p+1)}$ and $\alpha' > \alpha A^p$ close enough to α . In particular, by the assumption on α and the choice of η , we have that $\alpha d \eta$ is close to $\alpha d^{1/(p+1)}$ and smaller than 1. By Lemma 3.1 and Theorem 3.2 we know that $\|(f^n)_* u_S\|_{\log^p} \lesssim d^{kn} A^{pn}$ and $\|(f^n)_* u_S\|_\infty \lesssim d^{kn} \eta^n$. This, the above estimate on $m(u_\alpha, r)$, and the fact that $\alpha' d > \alpha d \geq 1$ imply that

$$m(u_\alpha, r) \lesssim \sum_{n \leq N} \alpha^n d^n A^{pn} (\log^* r)^{-p} + \sum_{n > N} \alpha^n d^n \eta^n \lesssim (\alpha' d)^N (\log^* r)^{-p} + (\alpha d \eta)^N.$$

Finally, choose $N = \frac{p+1}{\log d} \log \log^* r$. Observe that if we replace α' by α and $\alpha d \eta$ by $\alpha d^{1/(p+1)}$, the last sum is equal to $2(\log^* r)^{-q_0}$. So, this sum is bounded by a constant times $(\log^* r)^{-q}$ for $q < q_0$ because α' is chosen close to α and $\alpha d \eta$ is close to $\alpha d^{1/(p+1)}$. This concludes the proof of the lemma. \square

The following shifting property of the norm $\|\cdot\|_{p,\alpha}$ is very useful when we work with the action of f , and is the key property that we need of this norm.

Lemma 3.7. *For every $n \geq 0$ and every $(1, 1)$ -current R on \mathbb{P}^k , we have*

$$\|d^{-kn} (f^n)_* R\|_{p,\alpha} \leq \frac{1}{d^n \alpha^n} \|R\|_{p,\alpha}.$$

Proof. We can assume that $\|R\|_{p,\alpha} = 1$, so that there is a positive closed current S with $\|S\|_p = 1$ and $|R| \leq S_\alpha$, see the Notation at the beginning of the paper. Consider any function $\xi: \mathbb{P}^k \rightarrow \mathbb{C}$ such that $|\xi| \leq 1$ and define $\xi_n := \xi \circ f^n$. Since $|\xi_n| \leq 1$, we have

$$\Re(\xi d^{-kn} (f^n)_* R) = \frac{1}{d^n} \Re\left(\frac{(f^n)_* (\xi_n R)}{d^{(k-1)n}}\right) \leq \frac{1}{d^n \alpha^n} \sum_{j=0}^{\infty} \alpha^{n+j} \frac{(f^n)_*}{d^{(k-1)n}} \frac{(f^j)_*}{d^{(k-1)j}} S \leq \frac{1}{d^n \alpha^n} S_\alpha.$$

The lemma follows. \square

3.4. The dynamical Sobolev semi-norm $\|\cdot\|_{\langle p,\alpha \rangle}$. We can now define the first semi-norm for functions $g: \mathbb{P}^k \rightarrow \mathbb{R}$ with respect to which we will be able to prove the existence of a spectral gap for the transfer operator. We can also define this norm for 1-forms.

Definition 3.8. Let α be a real number such that $d^{-1} \leq \alpha < 1$. For any function $g: \mathbb{P}^k \rightarrow \mathbb{R}$ we set

$$\|g\|_{\langle p,\alpha \rangle} := \|i\partial g \wedge \bar{\partial} g\|_{p,\alpha}^{1/2}.$$

The following two lemmas give the main properties of the semi-norm $\|\cdot\|_{\langle p,\alpha \rangle}$ that we will need in Section 5, together with Lemma 3.7. Recall that q_0 is defined in Lemma 3.6. Note that the hypothesis $p > 3/2$ ensures that $d^{-1} < d^{-5/(2p+2)}$ and the hypothesis on α ensures that $q_0 > 3/2$, and hence that q_1 is positive.

Lemma 3.9. *Let α and p be positive numbers such that $p > 3/2$ and $d^{-1} \leq \alpha < d^{-5/(2p+2)}$. Then, for every $0 < q < q_1 := \frac{q_0}{3} - \frac{1}{2}$, there are positive constants $c_1 = c_1(p, \alpha, q)$ and $c_2 = c_2(p, \alpha)$ such that for every $g: \mathbb{P}^k \rightarrow \mathbb{R}$ we have*

$$\|g\|_{\log^q} \leq c_1 \|g\|_{\langle p, \alpha \rangle}, \quad \|g\|_{\langle p, \alpha \rangle} \leq c_2 \|g\|_p, \quad \text{and} \quad \|g\|_{\langle p, \alpha \rangle} \leq c_2 \|g\|_{\mathcal{C}^1}.$$

Proof. We can assume that $\|g\|_{\langle p, \alpha \rangle} \leq 1$. By the definition of the norm $\|\cdot\|_{\langle p, \alpha \rangle}$ and Lemma 3.6, $\|i\partial g \wedge \bar{\partial} g\|_{q'}$ is bounded by a constant for any $q' < q_0$. Therefore, we have $i\partial g \wedge \bar{\partial} g \leq R$ for some positive closed current R such that $\|R\|$ and $\|u_R\|_{\log^{q'}}$ are bounded by a constant. The first inequality follows from Corollary 2.8. The second assertion follows from Lemmas 3.6 and 3.4. The last assertion follows from Definition 3.8. \square

Lemma 3.10. *Let α and p be positive numbers such that $d^{-1} \leq \alpha < 1$. Then for all functions $g, h: \mathbb{P}^k \rightarrow \mathbb{R}$ we have*

$$\|gh\|_{\langle p, \alpha \rangle} \leq \sqrt{2} (\|g\|_{\langle p, \alpha \rangle} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p, \alpha \rangle}).$$

Proof. Using an expansion of $i\partial(gh) \wedge \bar{\partial}(gh)$ and Cauchy-Schwarz's inequality, we have

$$\begin{aligned} \|i\partial(gh) \wedge \bar{\partial}(gh)\|_{p, \alpha} &\leq \|h\|_\infty^2 \|i\partial g \wedge \bar{\partial} g\|_{p, \alpha} + \|g\|_\infty^2 \|i\partial h \wedge \bar{\partial} h\|_{p, \alpha} \\ &\quad + \|g\|_\infty \|h\|_\infty \|i\partial g \wedge \bar{\partial} h + i\partial h \wedge \bar{\partial} g\|_{p, \alpha} \\ &\leq \|h\|_\infty^2 \|i\partial g \wedge \bar{\partial} g\|_{p, \alpha} + \|g\|_\infty^2 \|i\partial h \wedge \bar{\partial} h\|_{p, \alpha} \\ &\quad + \|g\|_\infty \|h\|_\infty \left(\frac{\|h\|_\infty}{\|g\|_\infty} \|i\partial g \wedge \bar{\partial} g\|_{p, \alpha} + \frac{\|g\|_\infty}{\|h\|_\infty} \|i\partial h \wedge \bar{\partial} h\|_{p, \alpha} \right) \\ &\leq 2 \|h\|_\infty^2 \|i\partial g \wedge \bar{\partial} g\|_{p, \alpha} + 2 \|g\|_\infty^2 \|i\partial h \wedge \bar{\partial} h\|_{p, \alpha}. \end{aligned}$$

The assertion follows from Definition 3.8. \square

3.5. The semi-norm $\|\cdot\|_{\langle p, \alpha, \gamma \rangle}$. The following semi-norm defines the final space of functions that we will use in our study of the transfer operator. We use here some ideas from the theory of interpolation between Banach spaces, see also [Tri95].

Definition 3.11. For all real numbers $d^{-1} \leq \alpha < 1$, $\gamma > 0$ and $p > 0$, we define for a continuous function $g: \mathbb{P}^k \rightarrow \mathbb{R}$

$$(3.3) \quad \|g\|_{\langle p, \alpha, \gamma \rangle} := \inf \left\{ c \geq 0: \forall 0 < \epsilon \leq 1 \exists g_\epsilon^{(1)}, g_\epsilon^{(2)}: \right. \\ \left. g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \|g_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} \leq c(1/\epsilon)^{1/\gamma}, \|g_\epsilon^{(2)}\|_\infty \leq c\epsilon \right\}.$$

When such a number c does not exist, we set $\|g\|_{\langle p, \alpha, \gamma \rangle} := \infty$.

Remark 3.12. Lemma 2.4 applied for $s = 1$ implies that $\|\cdot\|_{\langle p, \alpha, \gamma \rangle} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}$ because $\|\cdot\|_{\langle p, \alpha \rangle} \lesssim \|\cdot\|_{\mathcal{C}^1}$, see Lemma 3.9.

The following two lemmas are the counterparts of Lemmas 3.9 and 3.10 for the semi-norm $\|\cdot\|_{\langle p, \alpha, \gamma \rangle}$. Recall that q_1 is defined in Lemma 3.9.

Lemma 3.13. *For all positive numbers p, α, γ, q satisfying $p > 3/2$, $d^{-1} \leq \alpha < d^{-5/(2p+2)}$ and $q < q_2 := \frac{\gamma}{\gamma+1} q_1$, there is a positive constant $c = c(p, \alpha, \gamma, q)$ such that*

$$\|g\|_{\log^q} \leq c \|g\|_{\langle p, \alpha, \gamma \rangle} \quad \text{and} \quad \|g\|_{\langle p, \alpha, \gamma \rangle} \leq \|g\|_{\langle p, \alpha \rangle}$$

for every continuous function $g: \mathbb{P}^k \rightarrow \mathbb{R}$. Moreover, if $\chi: I \rightarrow \mathbb{R}$ is a Lipschitz function with Lipschitz constant κ on an interval $I \subset \mathbb{R}$ containing the image of g , then we have

$$\|\chi(g)\|_{\langle p, \alpha, \gamma \rangle} \leq \kappa \|g\|_{\langle p, \alpha, \gamma \rangle}.$$

Proof. Let us prove the first inequality. We can assume that $\|g\|_{\langle p, \alpha \rangle, \gamma} \leq 1$. Lemma 3.9 implies that $g_\epsilon^{(1)}$ has $\|\cdot\|_{\log^{q'}}$ semi-norm bounded by a constant times $(1/\epsilon)^{1/\gamma}$ when $q' < q_1$. Therefore, we have for $r > 0$

$$m(g, r) \leq m(g_\epsilon^{(1)}, r) + m(g_\epsilon^{(2)}, r) \lesssim \frac{(1/\epsilon)^{1/\gamma}}{(\log^* r)^{q'}} + \epsilon.$$

Choosing $\epsilon = (\log^* r)^{-K}$ with $K = q'/(1 + 1/\gamma)$ gives

$$m(g, r) \lesssim (\log^* r)^{-q'+K/\gamma} + (\log^* r)^{-K} = 2(\log^* r)^{-q'/(1+1/\gamma)}.$$

The first assertion of the lemma follows by choosing q' close enough to q_1 .

The second inequality follows from the definition of the semi-norm $\|\cdot\|_{\langle p, \alpha \rangle, \gamma}$, by taking $g_\epsilon^{(2)} = 0$ in the decomposition $g = g_\epsilon^{(1)} + g_\epsilon^{(2)}$ for every ϵ .

We prove now the last assertion. Since we can approximate χ uniformly by smooth functions χ_n with $|\chi'_n| \leq 1$, we can assume for simplicity that χ is smooth. Define $h := \chi(g)$ and recall that we are assuming that $\|g\|_{\langle p, \alpha \rangle, \gamma} \leq 1$. For every $0 < \epsilon \leq 1$, we have the decomposition

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)} \quad \text{with} \quad \|g_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} \leq (1/\epsilon)^{1/\gamma} \quad \text{and} \quad \|g_\epsilon^{(2)}\|_\infty \leq \epsilon.$$

Write

$$h = h_\epsilon^{(1)} + h_\epsilon^{(2)} \quad \text{with} \quad h_\epsilon^{(1)} := \chi(g_\epsilon^{(1)}) \quad \text{and} \quad h_\epsilon^{(2)} := h - h_\epsilon^{(1)}.$$

We have

$$\|h_\epsilon^{(2)}\|_\infty = \|\chi(g) - \chi(g_\epsilon^{(1)})\|_\infty \lesssim \kappa \|g - g_\epsilon^{(1)}\|_\infty = \kappa \|g_\epsilon^{(2)}\|_\infty \leq \kappa \epsilon$$

and also

$$\|h_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} = \|i\partial\chi(g_\epsilon^{(1)}) \wedge \bar{\partial}\chi(g_\epsilon^{(1)})\|_{p, \alpha}^{1/2} \leq \kappa \|i\partial g_\epsilon^{(1)} \wedge \bar{\partial} g_\epsilon^{(1)}\|_{p, \alpha}^{1/2} = \kappa \|g_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} \leq \kappa (1/\epsilon)^{1/\gamma}.$$

It follows that $\|h\|_{\langle p, \alpha \rangle, \gamma} \leq \kappa$. This completes the proof of the lemma. \square

Lemma 3.14. *For all positive numbers p, α, γ such that $d^{-1} \leq \alpha < 1$ we have*

$$\|gh\|_{\langle p, \alpha \rangle, \gamma} \leq 3(\|g\|_{\langle p, \alpha \rangle, \gamma} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p, \alpha \rangle, \gamma})$$

for every continuous functions $g, h: \mathbb{P}^k \rightarrow \mathbb{R}$.

Proof. We can assume that $\|g\|_{\langle p, \alpha \rangle, \gamma} = \|h\|_{\langle p, \alpha \rangle, \gamma} = 1$. For every $0 < \epsilon \leq 1$, we need to find a decomposition $gh = L_\epsilon^{(1)} + L_\epsilon^{(2)}$ with $L_\epsilon^{(1)}, L_\epsilon^{(2)}$ such that

$$\|L_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} \leq 3(\|g\|_\infty + \|h\|_\infty)(1/\epsilon)^{1/\gamma} \quad \text{and} \quad \|L_\epsilon^{(2)}\|_\infty \leq 3(\|g\|_\infty + \|h\|_\infty)\epsilon.$$

Case 1. Assume that $\|g\|_\infty \leq 3\epsilon$ and $\|h\|_\infty \leq 3\epsilon$. Choose $L_\epsilon^{(1)} = 0$ and $L_\epsilon^{(2)} = gh$. Clearly, these functions satisfy the desired estimates.

Case 2. Assume now that $\|g\|_\infty + \|h\|_\infty \geq 3\epsilon$. By the definition of the semi-norm $\|\cdot\|_{\langle p, \alpha \rangle, \gamma}$, we have the decompositions $g = g_\epsilon^{(1)} + g_\epsilon^{(2)}$ and $h = h_\epsilon^{(1)} + h_\epsilon^{(2)}$ with

$$\|g_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} \leq (1/\epsilon)^{1/\gamma}, \quad \|g_\epsilon^{(2)}\|_\infty \leq \epsilon, \quad \|h_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} \leq (1/\epsilon)^{1/\gamma}, \quad \|h_\epsilon^{(2)}\|_\infty \leq \epsilon.$$

Observe that $\|g_\epsilon^{(1)}\|_\infty \leq \|g\|_\infty + \epsilon$ and $\|h_\epsilon^{(1)}\|_\infty \leq \|h\|_\infty + \epsilon$, which imply that

$$\|g_\epsilon^{(1)}\|_\infty + \|h_\epsilon^{(1)}\|_\infty \leq 2(\|g\|_\infty + \|h\|_\infty).$$

Set

$$L_\epsilon^{(1)} := g_\epsilon^{(1)}h_\epsilon^{(1)} \quad \text{and} \quad L_\epsilon^{(2)} := g_\epsilon^{(1)}h_\epsilon^{(2)} + g_\epsilon^{(2)}h_\epsilon^{(1)} + g_\epsilon^{(2)}h_\epsilon^{(2)}.$$

The desired estimate for $\|L_\epsilon^{(1)}\|_{\langle p, \alpha \rangle}$ follows from Lemma 3.10 and the one for $\|L_\epsilon^{(2)}\|_\infty$ is obtained by a direct computation. This ends the proof of the lemma. \square

4. SPECTRAL GAP FOR THE TRANSFER OPERATOR

In this section we prove our main Theorem 1.2. Theorem 1.1 and (1.2) give the scaling ratio λ , the density function ρ as an eigenfunction for the operator $\mathcal{L} = \mathcal{L}_\phi$, and the probability measures m_ϕ and μ_ϕ , all under the hypothesis that $\|\phi\|_{\log^q} < \infty$ for some $q > 2$. The semi-norms $\|\cdot\|_{\langle p, \alpha \rangle}$ and $\|\cdot\|_{\langle p, \alpha \rangle, \gamma}$ were introduced in Sections 3.4 and 3.5, respectively.

4.1. Some preliminary results. For positive real numbers q, M , and Ω with $q > 2$ and $\Omega < \log d$, consider the following set of weights

$$\mathcal{P}(q, M, \Omega) := \{\phi: \mathbb{P}^k \rightarrow \mathbb{R} : \|\phi\|_{\log^q} \leq M, \Omega(\phi) \leq \Omega\}$$

and the uniform topology induced by the sup norm. Observe that this family is equicontinuous. The following two lemmas were obtained in [BD20], see Lemmas 4.6 and 4.7 therein. Since we will use them several times in this section, we restate them here. We use the index ϕ or parameter ϕ for objects which depend on ϕ , e.g., we write $\lambda_\phi, \mathcal{L}_\phi, \rho_\phi$ instead of $\lambda, \mathcal{L}, \rho$.

Lemma 4.1. *Let q, M , and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. The maps $\phi \mapsto \lambda_\phi$, $\phi \mapsto m_\phi$, $\phi \mapsto \mu_\phi$, and $\phi \mapsto \rho_\phi$ are continuous on $\phi \in \mathcal{P}(q, M, \Omega)$ with respect to the standard topology on \mathbb{R} , the weak topology on measures, and the uniform topology on functions. In particular, ρ_ϕ is bounded from above and below by positive constants which are independent of $\phi \in \mathcal{P}(q, M, \Omega)$. Moreover, $\|\lambda_\phi^{-n} \mathcal{L}_\phi^n\|_\infty$ is bounded by a constant which is independent of n and of $\phi \in \mathcal{P}(q, M, \Omega)$.*

Lemma 4.2. *Let q, M , and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. Let \mathcal{F} be a uniformly bounded and equicontinuous family of real-valued functions on \mathbb{P}^k . Then the family*

$$\{\lambda_\phi^{-n} \mathcal{L}_\phi^n(g) : n \geq 0, \phi \in \mathcal{P}(p, M, \Omega), g \in \mathcal{F}\}$$

is equicontinuous. Moreover, $\|\lambda_\phi^{-n} \mathcal{L}_\phi^n(g) - \langle m_\phi, g \rangle\|_\infty$ tends to 0 uniformly on $\phi \in \mathcal{P}(p, M, \Omega)$ and $g \in \mathcal{F}$ when n goes to infinity.

4.2. Main result and first step of the proof. The following is the main result of this section. We will use it in order to prove Theorem 1.2 with a suitable norm and another value of β .

Theorem 4.3. *Let $f, \phi, \lambda, m_\phi, \rho$ be as in Theorem 1.1 and \mathcal{L} the Perron-Frobenius operator associated to ϕ . Let $p, \alpha, \gamma, A, \Omega$ be positive constants and q_2 as in Lemma 3.13 such that $p > 3/2$, $d^{-1} \leq \alpha < d^{-5/(2p+2)}$, $\Omega < \log(d\alpha)$, and $q_2 > 2$. Assume that $\|\phi\|_{\langle p, \alpha \rangle, \gamma} \leq A$ and $\Omega(\phi) \leq \Omega$. Then we have*

$$\|\lambda^{-n} \mathcal{L}^n\|_{\langle p, \alpha \rangle, \gamma} \leq c, \quad \|\rho\|_{\langle p, \alpha \rangle, \gamma} \leq c, \quad \text{and} \quad \|1/\rho\|_{\langle p, \alpha \rangle, \gamma} \leq c$$

for some positive constant $c = c(p, \alpha, \gamma, A, \Omega)$ independent of ϕ and n . Moreover, for every constant $0 < \beta < 1$ there is a positive integer $N = N(p, \alpha, \gamma, A, \Omega, \beta)$ independent of ϕ such that

$$(4.1) \quad \|\lambda^{-N} \mathcal{L}^N g\|_{\langle p, \alpha \rangle, \gamma} \leq \beta \|g\|_{\langle p, \alpha \rangle, \gamma}$$

for every function $g: \mathbb{P}^k \rightarrow \mathbb{R}$ with $\langle m_\phi, g \rangle = 0$. Furthermore, we can take $\beta = \delta^{-N}$ for every given constant $0 < \delta < (d\alpha)^{\gamma/(2\gamma+2)}$, provided that A is small enough.

Notice that Lemma 3.13 and the assumption $q_2 > 2$ imply that $\|\phi\|_{\log^q} < \infty$ for some $q > 2$. Hence, the scaling ratio λ , the density function ρ , and the measures m_ϕ and μ_ϕ are well defined by Theorem 1.1. Notice also that $q_2 > 2$ implies that the condition $\alpha < d^{-5/(2p+2)}$ is automatically satisfied.

The proof of Theorem 4.3 will be reduced to a comparison between suitable currents and their norms. Theorem 4.3 will then follow from some interpolation techniques (see Section 4.4). A crucial estimate that we will need here is the following.

Proposition 4.4. *Let f be as in Theorem 1.1. Take $0 < \alpha < 1$ and $p > 0$. Given n functions $\phi^{(j)}: \mathbb{P}^k \rightarrow \mathbb{R}$ for $j = 1, \dots, n$, set*

$$\Phi_m := \alpha^{-m} d^{(k-1)m} e^{\sum_{j=1}^m \max(\phi^{(j)})} \quad \text{and} \quad \mathcal{L}_{m,n} := \mathcal{L}_{\phi^{(m)}} \circ \dots \circ \mathcal{L}_{\phi^{(1)}}.$$

Then there exists a positive constant $c = c(p, \alpha)$, independent of $\phi^{(j)}$, such that

$$\|\mathcal{L}_{1,n}g\|_{\langle p, \alpha \rangle} \leq c \|\mathcal{L}_{1,n}\mathbb{1}\|_{\infty}^{1/2} \Phi_n^{1/2} \|g\|_{\langle p, \alpha \rangle} + c \sum_{m=1}^n \|\phi^{(m)}\|_{\langle p, \alpha \rangle} m^{3/2} \Phi_m^{1/2} \|\mathcal{L}_{1,m}\mathbb{1}\|_{\infty}^{1/2} \|\mathcal{L}_{m+1,n}g\|_{\infty}$$

for every function $g: \mathbb{P}^k \rightarrow \mathbb{R}$.

Proof. By Definition 3.8 of the semi-norm $\|\cdot\|_{\langle p, \alpha \rangle}$, we need to bound the current $i\partial\mathcal{L}_{1,n}g \wedge \bar{\partial}\mathcal{L}_{1,n}g$.

Consider the product space $(\mathbb{P}^k)^{n+1}$ and denote by (x_0, \dots, x_n) its elements. Define the manifold $\Gamma_n \subset (\mathbb{P}^k)^{n+1}$ by

$$\Gamma_n := \{(x, f(x), \dots, f^n(x)) : x \in \mathbb{P}^k\},$$

which can also be seen as the graph of the map (f, f^2, \dots, f^n) in the product space $(\mathbb{P}^k)^{n+1}$. Denote by π_n the restriction to Γ_n of the projection of $(\mathbb{P}^k)^{n+1}$ to its last component.

We have, using a direct computation,

$$\partial(e^{\phi^{(n)}(x_0) + \dots + \phi^{(1)}(x_{n-1})} g(x_0)) = \Theta_1 + \Theta_2$$

with

$$\Theta_1 := \mathfrak{h}'' \partial g(x_0), \quad \Theta_2 := \mathfrak{h}'' g(x_0) \sum_{m=0}^{n-1} \partial \phi^{(n-m)}(x_m), \quad \text{and} \quad \mathfrak{h}'' := e^{\phi^{(n)}(x_0) + \dots + \phi^{(1)}(x_{n-1})}.$$

Using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} i\partial\mathcal{L}_{1,n}g \wedge \bar{\partial}\mathcal{L}_{1,n}g &= i(\pi_n)_*(\Theta_1 + \Theta_2) \wedge (\pi_n)_*(\bar{\Theta}_1 + \bar{\Theta}_2) \\ &\leq 2i(\pi_n)_*(\Theta_1) \wedge (\pi_n)_*(\bar{\Theta}_1) + 2i(\pi_n)_*(\Theta_2) \wedge (\pi_n)_*(\bar{\Theta}_2). \end{aligned}$$

We need to bound the norm $\|\cdot\|_{p, \alpha}$ of the two terms in the last sum by the square of the RHS of the inequality in the proposition.

For the first term, using again Cauchy-Schwarz's inequality, the definition of Φ_m as in the statement, and Lemma 3.7, we get (notice that $(\pi_n)_*(\mathfrak{h}'') = \mathcal{L}_{1,n}\mathbb{1}$)

$$\begin{aligned} \|i(\pi_n)_*(\Theta_1) \wedge (\pi_n)_*(\bar{\Theta}_1)\|_{p, \alpha} &\leq \|(\pi_n)_*(\mathfrak{h}'')(\pi_n)_*(\mathfrak{h}'' i\partial g(x_0) \wedge \bar{\partial}g(x_0))\|_{p, \alpha} \\ &\leq \|\mathcal{L}_{1,n}\mathbb{1}\|_{\infty} e^{\sum_{j=1}^n \max \phi^{(j)}} \|(f^n)_*(i\partial g \wedge \bar{\partial}g)\|_{p, \alpha} \\ &\leq \|\mathcal{L}_{1,n}\mathbb{1}\|_{\infty} \Phi_n \|i\partial g \wedge \bar{\partial}g\|_{p, \alpha}. \end{aligned}$$

This gives the desired estimate for the first term.

For the second term, observe that $i(\pi_n)_*(\Theta_2) \wedge (\pi_n)_*(\bar{\Theta}_2)$ is equal to

$$\begin{aligned} &\sum_{0 \leq m, m' < n} i(\pi_n)_*(\mathfrak{h}'' g(x_0) \partial \phi^{(n-m)}(x_m)) \wedge (\pi_n)_*(\mathfrak{h}'' g(x_0) \bar{\partial} \phi^{(n-m')}(x_{m'})) \\ &\leq 2 \sum_{0 \leq m' \leq m < n} |i(\pi_n)_*(\mathfrak{h}'' g(x_0) \partial \phi^{(n-m)}(x_m)) \wedge (\pi_n)_*(\mathfrak{h}'' g(x_0) \bar{\partial} \phi^{(n-m')}(x_{m'}))|. \end{aligned}$$

Using Cauchy-Schwarz's inequality, we can bound the current in the absolute value signs by

$$\begin{aligned} &(m - m' + 1)^{-2} i(\pi_n)_*(\mathfrak{h}'' g(x_0) \partial \phi^{(n-m)}(x_m)) \wedge (\pi_n)_*(\mathfrak{h}'' g(x_0) \bar{\partial} \phi^{(n-m')}(x_{m'})) \\ &\quad + (m - m' + 1)^2 i(\pi_n)_*(\mathfrak{h}'' g(x_0) \partial \phi^{(n-m')}(x_{m'})) \wedge (\pi_n)_*(\mathfrak{h}'' g(x_0) \bar{\partial} \phi^{(n-m)}(x_m)) \end{aligned}$$

and deduce that $i(\pi_n)_*(\Theta_2) \wedge (\pi_n)_*(\bar{\Theta}_2)$ is bounded by a constant times

$$\sum_{0 \leq m < n} (n - m)^3 i(\pi_n)_*(\mathfrak{h}'' g(x_0) \partial \phi^{(n-m)}(x_m)) \wedge (\pi_n)_*(\mathfrak{h}'' g(x_0) \bar{\partial} \phi^{(n-m)}(x_m)).$$

Therefore, in order to get the proposition, setting $\eta := (\pi_n)_*(\mathfrak{h}''g(x_0)\partial\phi^{(n-m)}(x_m))$ we only need to show that

$$\|i\eta \wedge \bar{\eta}\|_{p,\alpha} \leq \|\phi^{(n-m)}\|_{(p,\alpha)}^2 \Phi_{n-m} \|\mathcal{L}_{1,n-m}\mathbb{1}\|_\infty \|\mathcal{L}_{n-m+1,n}g\|_\infty^2.$$

Consider the map $\pi' : \Gamma_n \rightarrow (\mathbb{P}^k)^{n-m+1}$ defined by $\pi'(x) := x' := (x_m, \dots, x_n)$. Denote by Γ' the image of Γ_n by π' . Consider also the map $\pi'' : \Gamma' \rightarrow \mathbb{P}^k$ defined by $\pi''(x') := x_n$. Both $\pi' : \Gamma_n \rightarrow \Gamma'$ and $\pi'' : \Gamma' \rightarrow \mathbb{P}^k$ are ramified coverings, respectively of degrees d^{km} and $d^{k(n-m)}$, and we have $\pi_n = \pi'' \circ \pi'$.

With the notation as above, we see that

$$\eta = \pi''_*(\mathcal{L}_{n-m+1,n}g(x_m)\mathfrak{h}_m\partial\phi^{(n-m)}(x_m)) \quad \text{with} \quad \mathfrak{h}_m := e^{\phi^{(n-m)}(x_m) + \dots + \phi^{(1)}(x_{n-1})}.$$

It follows from Cauchy-Schwarz's inequality that

$$\begin{aligned} i\eta \wedge \bar{\eta} &\leq \|\mathcal{L}_{n-m+1,n}g\|_\infty^2 \pi''_*(\mathfrak{h}_m)\pi''_*(\mathfrak{h}_m i\partial\phi^{(n-m)}(x_m) \wedge \bar{\partial}\phi^{(n-m)}(x_m)) \\ &\leq \|\mathcal{L}_{n-m+1,n}g\|_\infty^2 \|\pi''_*(\mathfrak{h}_m)\|_\infty \|\mathfrak{h}_m\|_\infty \|(f^{n-m})_*(i\partial\phi^{(n-m)} \wedge \bar{\partial}\phi^{(n-m)})\|. \end{aligned}$$

Thus, by Lemma 3.7 and the definition of π'' , we get

$$\|i\eta \wedge \bar{\eta}\|_{p,\alpha} \leq \|\mathcal{L}_{n-m+1,n}g\|_\infty^2 \|\mathcal{L}_{1,n-m}\mathbb{1}\|_\infty \Phi_{n-m} \|i\partial\phi^{(n-m)} \wedge \bar{\partial}\phi^{(n-m)}\|_{p,\alpha}.$$

This ends the proof of the proposition. \square

4.3. Proof of Theorem 4.3. We will need the following elementary lemma.

Lemma 4.5. *Let $\phi, \phi^{(j)}, \psi : \mathbb{P}^k \rightarrow \mathbb{R}$ be such that $\sum_{j=1}^\infty \|\phi - \phi^{(j)}\|_\infty \leq a$ for some positive constant a . Define $\vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\vartheta(t) := t^{-1}(e^t - 1)$ and $\vartheta(0) = 1$, which is a smooth increasing function. Then we have*

- (i) $\|\mathcal{L}_\phi - \mathcal{L}_\psi\|_\infty \leq \vartheta(\|\phi - \psi\|_\infty) \|\mathcal{L}_\phi\|_\infty \|\phi - \psi\|_\infty$;
- (ii) for every $n \geq 1$, $\|\mathcal{L}^n - \mathcal{L}_{\phi^{(1)}} \circ \dots \circ \mathcal{L}_{\phi^{(n)}}\|_\infty \leq \vartheta(a)a \|\mathcal{L}^n\|_\infty$.

Proof. Observe that for $\phi, \psi : \mathbb{P}^k \rightarrow \mathbb{R}$ we have, using the definition of \mathcal{L}_ϕ and \mathcal{L}_ψ ,

$$\|\mathcal{L}_\phi(g) - \mathcal{L}_\psi(g)\|_\infty \leq \|(1 - e^{\psi-\phi})\|g\|_\infty\|_\infty \|\mathcal{L}_\phi(\mathbb{1})\|_\infty = \|1 - e^{\psi-\phi}\|_\infty \|\mathcal{L}_\phi\|_\infty \|g\|_\infty.$$

The first item in the lemma follows. For the second item, notice that

$$\left\| (\phi + \phi \circ f + \dots + \phi \circ f^{n-1}) - (\phi^{(n)} + \phi^{(n-1)} \circ f + \dots + \phi^{(1)} \circ f^{n-1}) \right\|_\infty \leq \sum_{j=1}^n \|\phi - \phi^{(j)}\|_\infty.$$

Therefore, using this estimate and the expansions of $\mathcal{L}^n(g)$ and $\mathcal{L}_{\phi^{(1)}} \circ \dots \circ \mathcal{L}_{\phi^{(n)}}(g)$, we obtain the result in the same way as the first item. \square

We continue the proof of Theorem 4.3 We first prove the following result.

Proposition 4.6. *Under the hypotheses of Theorem 4.3, there exists a positive integer $N_0 = N_0(p, \alpha, \gamma, A, \Omega, \beta)$ independent of ϕ and g and such that (4.1) holds for all $N \geq N_0$.*

By subtracting from ϕ a constant, we can assume that ϕ belongs to the family of weights

$$\mathcal{Q}_0 := \{\phi : \mathbb{P}^k \rightarrow \mathbb{R} : \min \phi = 0, \|\phi\|_{(p,\alpha),\gamma} \leq A, \Omega(\phi) \leq \Omega\}.$$

Observe that we can apply Lemmas 4.1 and 4.2 because, by Lemma 3.13 and the assumptions on α and p , the family \mathcal{Q}_0 is contained in $\mathcal{P}_0(q, M, \Omega)$ for suitable $q > 2$ and M . Observe also that $\|\phi\|_\infty = \Omega(\phi) \leq \Omega$ and $\|\phi\|_\infty \lesssim \|\phi\|_{(p,\alpha),\gamma} \leq A$.

Consider two constants $K \geq 1$ and $K' \geq 1$ whose values depend on β and will be specialised later. We will not fix A but we assume $A \leq A_0$ for some fixed constant $A_0 > 0$. For a large part of this section we can take $A = A_0$, but at the end of the proof of Theorem 4.3 we will consider $A \rightarrow 0$. This is the reason why we will keep the constant A in the estimates below. Note that the constants hidden in the signs \lesssim below are independent of the parameters A, β, K, K', n and also of the constant $0 < \epsilon \leq 1$ and the integer j that we consider now.

Since $\|\phi\|_{\langle p, \alpha \rangle, \gamma} \leq A$, for every $j \geq 1$ there are functions $\phi^{(j)}$ and $\psi^{(j)}$ such that

$$\phi = \phi^{(j)} + \psi^{(j)}, \quad \|\phi^{(j)}\|_{\langle p, \alpha \rangle} \leq A(Kj^2)^{1/\gamma}(1/\epsilon)^{1/\gamma}, \quad \text{and} \quad \|\psi^{(j)}\|_\infty \leq AK^{-1}j^{-2}\epsilon.$$

Observe that $\|\phi^{(j)}\|_\infty$ is bounded by a constant since $\|\phi\|_\infty$ is bounded by a constant.

We can assume for simplicity that $\|g\|_{\langle p, \alpha \rangle, \gamma} \leq 1$, which implies that $\Omega(g)$ is bounded by a constant. Since $\langle m_\phi, g \rangle = 0$ by hypothesis, we deduce that $\|g\|_\infty$ is bounded by a constant. By the definition of the semi-norm $\|\cdot\|_{\langle p, \alpha \rangle, \gamma}$, we can find two functions $g_\epsilon^{(1)}$ and $g_\epsilon^{(2)}$ satisfying

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \quad \|g_\epsilon^{(1)}\|_{\langle p, \alpha \rangle} \leq K^{1/\gamma}(1/\epsilon)^{1/\gamma}, \quad \|g_\epsilon^{(2)}\|_\infty \leq 2K^{t-1}\epsilon, \quad \langle m_\phi, g_\epsilon^{(1)} \rangle = \langle m_\phi, g_\epsilon^{(2)} \rangle = 0.$$

Notice that without the condition $\langle m_\phi, g_\epsilon^{(2)} \rangle = 0$ we would not need the coefficient 2 in the above estimate of $\|g_\epsilon^{(2)}\|_\infty$. We obtain this condition by adding to $g_\epsilon^{(2)}$ a suitable constant and subtracting the same constant from $g_\epsilon^{(1)}$. The condition $\langle m_\phi, g_\epsilon^{(1)} \rangle = 0$ is deduced from the hypothesis $\langle m_\phi, g \rangle = 0$ when we have $\langle m_\phi, g_\epsilon^{(2)} \rangle = 0$. Since $\|g\|_\infty$ is bounded by a constant, $\|g_\epsilon^{(1)}\|_\infty$ is also bounded by a constant.

Define as above $\mathcal{L}_{m,n} := \mathcal{L}_{\phi^{(m)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}}$ and write

$$(4.2) \quad \lambda^{-n} \mathcal{L}^n g = \lambda^{-n} \mathcal{L}_{1,n} g_\epsilon^{(1)} + \lambda^{-n} (\mathcal{L}^n g_\epsilon^{(1)} - \mathcal{L}_{1,n} g_\epsilon^{(1)}) + \lambda^{-n} \mathcal{L}^n g_\epsilon^{(2)} =: G_{n,\epsilon}^{(a)} + G_{n,\epsilon}^{(b)} + G_{n,\epsilon}^{(c)}.$$

Lemma 4.7. *When K and K' are large enough, we have for every $n \geq 1$*

$$\|G_{n,\epsilon}^{(b)}\|_\infty \leq \frac{1}{2}\beta\epsilon \quad \text{and} \quad \|G_{n,\epsilon}^{(c)}\|_\infty \leq \frac{1}{2}\beta\epsilon.$$

Proof. The above estimate on $\|\psi^{(j)}\|_\infty$ implies that $\sum \|\psi^{(j)}\|_\infty \lesssim AK^{-1}\epsilon$. Therefore, using Lemma 4.5 and the fact that the sequence $\lambda^{-n} \mathcal{L}^n \mathbb{1}$ is bounded uniformly on n and ϕ (see Lemma 4.1), we get

$$\|G_{n,\epsilon}^{(b)}\|_\infty \lesssim AK^{-1}\epsilon \|g_\epsilon^{(1)}\|_\infty \lesssim AK^{-1}\epsilon \leq A_0 K^{-1}\epsilon$$

because $\|g_\epsilon^{(1)}\|_\infty$ is bounded by a constant. So we get the first estimate in the lemma when K is large enough (depending on A_0 and β).

For the second estimate, using again that the sequence $\lambda^{-n} \mathcal{L}^n \mathbb{1}$ is uniformly bounded, we obtain

$$\|G_{n,\epsilon}^{(c)}\|_\infty \lesssim \|g_\epsilon^{(2)}\|_\infty \leq K'^{-1}\epsilon.$$

The result follows provided that K' is large enough. \square

Lemma 4.8. *When $K \geq 1$ and K' are fixed, there is a constant $0 < \epsilon_0 \leq 1$ independent of ϕ and g such that, for all $0 < \epsilon \leq \epsilon_0$ and all n large enough, also independent of ϕ and g ,*

$$\|G_{n,\epsilon}^{(a)}\|_{\langle p, \alpha \rangle} \leq \beta(1/\epsilon)^{1/\gamma}.$$

Proof. Fix an $\epsilon_0 > 0$ small enough. We will apply Proposition 4.4. First, using the estimates $\sum \|\psi^{(j)}\|_\infty \lesssim AK^{-1}\epsilon \leq A_0$ and $\lambda \geq d^k e^{\min \phi}$, we obtain

$$\Phi_m \lesssim \alpha^{-m} d^{(k-1)m} e^{m \max \phi} \lesssim \alpha^{-m} d^{-m} \lambda^m e^{m \Omega(\phi)} \leq \alpha^{-m} d^{-m} \lambda^m e^{m \Omega}.$$

By Lemmas 4.5 and 4.1, we have

$$\|\mathcal{L}_{1,m} \mathbb{1}\|_\infty \lesssim \|\mathcal{L}^m \mathbb{1}\|_\infty \lesssim \lambda^m \quad \text{and} \quad \|\mathcal{L}_{m+1,n} \mathbb{1}\|_\infty \lesssim \|\mathcal{L}^{n-m} \mathbb{1}\|_\infty \lesssim \lambda^{n-m}$$

and also, again by Lemma 4.5,

$$\begin{aligned} \|\mathcal{L}_{m+1,n} g_\epsilon^{(1)}\|_\infty &\leq \|\mathcal{L}^{n-m} g_\epsilon^{(1)}\|_\infty + \|\mathcal{L}_{m+1,n} g_\epsilon^{(1)} - \mathcal{L}^{n-m} g_\epsilon^{(1)}\|_\infty \\ &\lesssim \|\mathcal{L}^{n-m} g_\epsilon^{(1)}\|_\infty + \|\mathcal{L}^{n-m} \mathbb{1}\|_\infty \|g_\epsilon^{(1)}\|_\infty AK^{-1}\epsilon \\ &\lesssim \|\mathcal{L}^{n-m} g\|_\infty + \|\mathcal{L}^{n-m} g_\epsilon^{(2)}\|_\infty + \lambda^{n-m} AK^{-1}\epsilon \\ &\lesssim \|\mathcal{L}^{n-m} g\|_\infty + \lambda^{n-m} K'^{-1}\epsilon + \lambda^{n-m} AK^{-1}\epsilon. \end{aligned}$$

This, Proposition 4.4, and the estimates in the definitions of $g_\epsilon^{(1)}$ and $\phi^{(j)}$ allow us to bound $\|G_{n,\epsilon}^{(a)}\|_{\langle p,\alpha \rangle}$ by a constant times

$$(4.3) \quad \left[K^{1/\gamma} \left(\frac{e^\Omega}{d\alpha} \right)^{n/2} + \sum_{m=1}^n AK^{1/\gamma} m^{2/\gamma+3/2} \left(\frac{e^\Omega}{d\alpha} \right)^{m/2} \left(\|\lambda^{-n+m} \mathcal{L}^{n-m} g\|_\infty + (K^{l-1} + AK^{-1}) \epsilon_0 \right) \right] \epsilon^{1/\gamma}.$$

Recall that g belongs to a uniformly bounded and equicontinuous family of functions, see Lemma 3.13. It follows from Lemma 4.2 that $\|\lambda^{-n+m} \mathcal{L}^{n-m} g\|_\infty$ tends to 0, uniformly on ϕ and g , when $n-m$ tends to infinity. This and the fact that $e^\Omega < d\alpha$ imply that, when n tends to infinity, the sum between brackets in (4.3) converges to

$$\sum_{m=1}^{\infty} AK^{1/\gamma} m^{2/\gamma+3/2} \left(\frac{e^\Omega}{d\alpha} \right)^{m/2} (K^{l-1} + AK^{-1}) \epsilon_0.$$

This sum is smaller than β because ϵ_0 is chosen small enough. Therefore, we get the estimate in the lemma for n large enough, independently of ϕ and g , because the last convergence is uniform on ϕ and g . \square

Proof of Proposition 4.6. Take N large enough, independent of ϕ . It suffices to show that we can write

$$\lambda^{-N} \mathcal{L}^N g = G_{N,\epsilon}^{(1)} + G_{N,\epsilon}^{(2)} \quad \text{with} \quad \|G_{N,\epsilon}^{(1)}\|_{\langle p,\alpha \rangle} \leq \beta(1/\epsilon)^{1/\gamma} \quad \text{and} \quad \|G_{N,\epsilon}^{(2)}\|_\infty \leq \beta\epsilon.$$

We apply Lemmas 4.7 and 4.8 to $n := N$. When $\epsilon \leq \epsilon_0$, it is enough to choose $G_{N,\epsilon}^{(1)} := G_{N,\epsilon}^{(a)}$ and $G_{N,\epsilon}^{(2)} := G_{N,\epsilon}^{(b)} + G_{N,\epsilon}^{(c)}$. Assume now that $\epsilon_0 \leq \epsilon \leq 1$ and choose $G_{N,\epsilon}^{(1)} := 0$ and $G_{N,\epsilon}^{(2)} := \lambda^{-N} \mathcal{L}^N g$. With N large enough, we have $\|G_{N,\epsilon}^{(2)}\|_\infty \leq \beta\epsilon_0 \leq \beta\epsilon$ because $\|\lambda^{-N} \mathcal{L}^N g\|_\infty$ tends to 0 uniformly on ϕ and g when n goes to infinity, see Lemma 4.2. Thus, we have the desired decomposition of $\lambda^{-N} \mathcal{L}^N g$ and hence the property (4.1) for all N large enough. \square

Proposition 4.9. *Under the hypotheses of Theorem 4.3, there is a positive constant $c = c(p, \alpha, \gamma, A, \Omega)$ independent of ϕ and n such that*

$$\|\lambda^{-n} \mathcal{L}^n\|_{\langle p,\alpha \rangle, \gamma} \leq c, \quad \|\rho\|_{\langle p,\alpha \rangle, \gamma} \leq c, \quad \text{and} \quad \|1/\rho\|_{\langle p,\alpha \rangle, \gamma} \leq c.$$

Proof. We prove the first inequality. We will use the above computations for $K = K^l = \epsilon_0 = 1$. Consider any function $g: \mathbb{P}^k \rightarrow \mathbb{R}$ such that $\|g\|_{\langle p,\alpha \rangle, \gamma} \leq 1$. We do not assume that $\langle m_\phi, g \rangle = 0$. As before, for any $0 < \epsilon \leq 1$ we can write

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)} \quad \text{with} \quad \|g_\epsilon^{(1)}\|_{\langle p,\alpha \rangle} \leq (1/\epsilon)^{1/\gamma} \quad \text{and} \quad \|g_\epsilon^{(2)}\|_\infty \leq \epsilon.$$

We also consider as above the decomposition

$$\lambda^{-n} \mathcal{L}^n g = G_{n,\epsilon}^{(1)} + G_{n,\epsilon}^{(2)} \quad \text{with} \quad G_{n,\epsilon}^{(1)} := G_{n,\epsilon}^{(a)} \quad \text{and} \quad G_{n,\epsilon}^{(2)} := G_{n,\epsilon}^{(b)} + G_{n,\epsilon}^{(c)},$$

see (4.2). The computations in Lemmas 4.7 and 4.8 give $\|G_{n,\epsilon}^{(2)}\|_\infty \lesssim \epsilon$ and $\|G_{n,\epsilon}^{(1)}\|_{\langle p,\alpha \rangle} \lesssim \epsilon^{1/\gamma}$. We use here the fact that $\|\lambda^{-n+m} \mathcal{L}^{n-m} g\|_\infty$ is bounded by a constant, see Lemma 4.2. Therefore, $\|\lambda^{-n} \mathcal{L}^n g\|_{\langle p,\alpha \rangle, \gamma}$ is bounded by a constant. Thus, the first inequality in the proposition holds.

Consider now the second inequality. Observe that

$$\rho = \lim_{n \rightarrow \infty} \lambda^{-n} \mathcal{L}^n \mathbf{1} = \mathbf{1} + \sum_{n=0}^{\infty} \lambda^{-n} \mathcal{L}^n g \quad \text{with} \quad g := \lambda^{-1} \mathcal{L} \mathbf{1} - \mathbf{1}.$$

The $\lambda^{-1} \mathcal{L}^*$ -invariance of m_ϕ implies that $\langle m_\phi, g \rangle = 0$. Therefore, Proposition 4.6 and the first inequality in the present proposition imply that $\|\lambda^{-n} \mathcal{L}^n g\|_{\langle p,\alpha \rangle, \gamma} \lesssim \beta^{n/N}$. We deduce that $\|\rho\|_{\langle p,\alpha \rangle, \gamma}$ is bounded by a constant.

For the last inequality in the lemma, observe that ρ is bounded from above and below by positive constants which are independent of ϕ , see Lemma 4.1. The result is then a consequence of Lemma 3.13 applied to the function $\chi(t) := 1/t$. The proof is complete. \square

End of the proof of Theorem 4.3. By Propositions 4.6 and 4.9, it only remains to prove the last assertion in this theorem. We continue to use the computations in Lemmas 4.7 and 4.8 and take $K = 1$, $K' = \delta'^N$, and $\epsilon_0 = 1$ for some δ' and d' such that $\delta < \delta' = d'^{\gamma/(2\gamma+2)}$ and $d' < d\alpha$. As above, we consider the decomposition

$$\lambda^{-N}\mathcal{L}^N g = G_{N,\epsilon}^{(1)} + G_{N,\epsilon}^{(2)} \quad \text{with} \quad G_{N,\epsilon}^{(1)} := G_{N,\epsilon}^{(a)} \quad \text{and} \quad G_{N,\epsilon}^{(2)} := G_{N,\epsilon}^{(b)} + G_{N,\epsilon}^{(c)}.$$

Take $A \rightarrow 0$, which also implies that $\Omega(\phi) \rightarrow 0$. So we can fix an Ω as small as needed and assume that $d' < e^{-\Omega}d\alpha$. Then, the estimates in Lemmas 4.7 and 4.8 give

$$\|G_{N,\epsilon}^{(1)}\|_{\langle p,\alpha \rangle} \leq c(K'^{1/\gamma}d'^{-N/2} + O(A))\epsilon^{1/\gamma} \quad \text{and} \quad \|G_{N,\epsilon}^{(2)}\|_{\infty} \leq c(K'^{-1} + O(A))\epsilon,$$

where c is a positive constant. With N large enough and A small enough, since $\delta < \delta'$ and $K' = \delta'^N$, we get

$$\|G_{N,\epsilon}^{(2)}\|_{\infty} \leq \delta^{-N}\epsilon \quad \text{and} \quad \|G_{N,\epsilon}^{(1)}\|_{\langle p,\alpha \rangle} \leq \delta^{-N}\epsilon^{1/\gamma}.$$

In other words, we can take $\beta = \delta^{-N}$. This completes the proof of the theorem. \square

4.4. Proof of Theorem 1.2. The statement is a consequence of Theorem 4.3, namely, of the estimate (4.1). Note that the constant β in Theorem 1.2 is not the one in (4.1). Given ϕ as in the statement, we first choose α sufficiently close to 1 so that $\Omega(\phi) < \log(\alpha d)$. Then, we choose p large enough so that $q < q_2$, where q and γ are as in the statement and q_2 is defined in Lemma 3.13 (this also implies that $\alpha < d^{-5/(2p+2)}$ since $q > 2$). Recall that the semi-norm $\|\cdot\|_{\langle p,\alpha,\gamma \rangle}$ is almost a norm. Define

$$\|\cdot\|_{\diamond_1} := \|\cdot\|_{\infty} + \|\cdot\|_{\langle p,\alpha,\gamma \rangle}.$$

This is now a norm, which is independent of ϕ . By Lemmas 3.13 and 2.4, we have

$$\|\cdot\|_{\infty} + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}.$$

By Lemma 4.1, the quantities $\|\lambda^{-n}\mathcal{L}^n\|_{\infty}$, $\|\rho\|_{\infty}$, and $\|1/\rho\|_{\infty}$ are bounded by a constant when $\|\phi\|_{\diamond_1} \leq A$. By Theorem 4.3, $\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha,\gamma \rangle}$, $\|\rho\|_{\langle p,\alpha,\gamma \rangle}$, and $\|1/\rho\|_{\langle p,\alpha,\gamma \rangle}$ are also bounded by a constant. We deduce that $\|\lambda^{-n}\mathcal{L}^n\|_{\diamond_1}$, $\|\rho\|_{\diamond_1}$, and $\|1/\rho\|_{\diamond_1}$ satisfy the same property.

Let N and β_0 be as in Theorem 4.3 (we write β_0 instead of β to distinguish it from the constant that we use now for Theorem 1.2). Fix a constant β such that $\beta_0^{1/N} < \beta < 1$ and consider the following norms

$$\|g\|_{\diamond} := |c_g| + \|g'\|_{\langle p,\alpha,\gamma \rangle} \quad \text{and} \quad \|g\|_{\diamond_2} := |c_g| + \sum_{n=0}^{\infty} \beta^{-n} \|\lambda^{-n}\mathcal{L}^n g'\|_{\langle p,\alpha,\gamma \rangle}$$

for every function $g: \mathbb{P}^k \rightarrow \mathbb{R}$, where $c_g := \langle m_\phi, g \rangle$ and $g' := g - c_g\rho$.

Lemma 4.10. *We have $\|gh\|_{\diamond_1} \leq 3\|g\|_{\diamond_1}\|h\|_{\diamond_1}$ for all functions $g, h: \mathbb{P}^k \rightarrow \mathbb{R}$. Moreover, both of the norms $\|\cdot\|_{\diamond}$ and $\|\cdot\|_{\diamond_2}$ are equivalent to $\|\cdot\|_{\diamond_1}$.*

Proof. The first assertion is a direct consequence of Lemma 3.14. We prove now the second assertion. Since m_ϕ is a probability measure and $\|\rho\|_{\langle p,\alpha,\gamma \rangle}$ is bounded, we have

$$\|g\|_{\diamond} = |c_g| + \|g - c_g\rho\|_{\langle p,\alpha,\gamma \rangle} \leq |c_g| + \|g\|_{\langle p,\alpha,\gamma \rangle} + |c_g|\|\rho\|_{\langle p,\alpha,\gamma \rangle} \lesssim \|g\|_{\infty} + \|g\|_{\langle p,\alpha,\gamma \rangle} = \|g\|_{\diamond_1}.$$

Conversely, assume that $\|g\|_{\diamond} \leq 1$, then $|c_g| \leq 1$ and $\|g'\|_{\langle p,\alpha,\gamma \rangle} \leq 1$. It follows that $\|g\|_{\langle p,\alpha,\gamma \rangle}$ is bounded by a constant because it is bounded by $\|g'\|_{\langle p,\alpha,\gamma \rangle} + |c_g|\|\rho\|_{\langle p,\alpha,\gamma \rangle}$. By Lemma 3.13, $\Omega(g)$ is also bounded by a constant. This and the inequality $|\langle m_\phi, g \rangle| = |c_g| \leq 1$ imply that $\|g\|_{\infty}$ is bounded by a constant. We deduce that $\|\cdot\|_{\diamond}$ is equivalent to $\|\cdot\|_{\diamond_1}$.

Observe that $\|\cdot\|_{\diamond} \leq \|\cdot\|_{\diamond_2}$. To complete the proof, it is enough to show that $\|g\|_{\diamond_2} \lesssim \|g\|_{\diamond}$ for every function g . Recall that ρ is invariant by $\lambda^{-1}\mathcal{L}$ and $\langle m_\phi, \rho \rangle = 1$. Therefore, we have $\langle m_\phi, g' \rangle = 0$. Theorem 4.3 and Proposition 4.9 imply that $\|\lambda^{-n}\mathcal{L}^n g'\|_{\langle p,\alpha,\gamma \rangle} \lesssim \beta_0^{n/N} \|g'\|_{\langle p,\alpha,\gamma \rangle}$. Hence

$$\|g\|_{\diamond_2} \lesssim |c_g| + \|g'\|_{\langle p,\alpha,\gamma \rangle} \sum_{n=0}^{\infty} \beta^{-n} \beta_0^{n/N} \lesssim |c_g| + \|g'\|_{\langle p,\alpha,\gamma \rangle} = \|g\|_{\diamond}.$$

The last infinite sum is finite because $\beta > \beta_0^{1/N}$. This ends the proof of the lemma. \square

Consider now a function g with $c_g = \langle m_\phi, g \rangle = 0$, which implies $g = g'$. We also have $\langle m_\phi, \lambda^{-1} \mathcal{L}g \rangle = 0$ because m_ϕ is invariant by $\lambda^{-1} \mathcal{L}^*$. From the definition of $\|\cdot\|_{\diamond_2}$, we get

$$\|\lambda^{-1} \mathcal{L}g\|_{\diamond_2} = \beta(\|g\|_{\diamond_2} - \|g\|_{\diamond}) \leq \beta \|g\|_{\diamond_2}.$$

This is the desired contraction. Finally, the last assertion in Theorem 1.2 is a direct consequence of the last assertion in Theorem 4.3 by taking α close enough to 1. The proof of Theorem 1.2 is now complete.

4.5. Spectral gap in the limit case. The semi-norm $\|\cdot\|_{\langle p, \alpha \rangle}$ can be seen as the limit of the semi-norm $\|\cdot\|_{\langle p, \alpha \rangle, \gamma}$ as γ goes to infinity. In order to complete our study, we will prove here a spectral gap with respect to this limit norm. The following is an analogue of Theorem 4.3.

Theorem 4.11. *Let $f, \phi, \lambda, m_\phi, \rho$ be as in Theorem 1.1 and \mathcal{L} the Perron-Frobenius operator associated to ϕ . Let p, α, A, Ω be positive constants and q_1 as in Lemma 3.9 such that $p > 3/2$, $d^{-1} \leq \alpha < d^{-5/(2p+2)}$, $\Omega < \log(d\alpha)$, and $q_1 > 2$. Assume that $\|\phi\|_{\langle p, \alpha \rangle} \leq A$ and $\Omega(\phi) \leq \Omega$. Then we have*

$$\|\lambda^{-n} \mathcal{L}^n\|_{\langle p, \alpha \rangle} \leq c, \quad \|\rho\|_{\langle p, \alpha \rangle} \leq c, \quad \text{and} \quad \|1/\rho\|_{\langle p, \alpha \rangle} \leq c$$

for some positive constant $c = c(p, \alpha, A, \Omega)$ independent of ϕ and n . Moreover, for every constant $0 < \beta < 1$ there is a positive integer $N = N(p, \alpha, A, \Omega, \beta)$ independent of ϕ such that

$$\|\lambda^{-N} \mathcal{L}^N g\|_{\langle p, \alpha \rangle} \leq \beta \|g\|_{\langle p, \alpha \rangle}$$

for every function $g: \mathbb{P}^k \rightarrow \mathbb{R}$ with $\langle m_\phi, g \rangle = 0$. Furthermore, we can take $\beta = \delta^{-N}$ for every given constant $0 < \delta < (d\alpha)^{1/2}$, provided that A is small enough.

Notice that Lemma 3.9 and the assumption $q_1 > 2$ imply that $\|\phi\|_{\log^q}$ is finite for some $q > 2$. Hence, the scaling ratio λ , the density function ρ , and the measures m_ϕ and μ_ϕ are well defined by Theorem 1.1. Notice also that $q_1 > 2$ implies that the condition $\alpha < d^{-5/(2p+2)}$ is automatically satisfied.

Proof. The proof follows the same lines as the one of Theorem 4.3. It is however simpler because the definition of the semi-norm $\|\cdot\|_{\langle p, \alpha \rangle}$ is simpler than the one of $\|\cdot\|_{\langle p, \alpha \rangle, \gamma}$. In particular, we do not need any decomposition of $\lambda^{-n} \mathcal{L}^n g$. Applying directly Proposition 4.4 with $\phi^{(j)} := \phi$ for all $j \geq 1$ and recalling that $\|\mathbb{1}_n\|_\infty \lesssim \lambda^n$ we obtain

$$\|\lambda^{-n} \mathcal{L}^n g\|_{\langle p, \alpha \rangle} \lesssim \|g\|_{\langle p, \alpha \rangle} \left(\frac{e^\Omega}{d\alpha}\right)^{n/2} + \|\phi\|_{\langle p, \alpha \rangle} \sum_{m=1}^n m^{3/2} \left(\frac{e^\Omega}{d\alpha}\right)^{m/2} \|\lambda^{-n+m} \mathcal{L}^{n-m} g\|_\infty.$$

With this estimate, the rest of the proof is the same as that of Theorem 4.3. \square

As in the last section, we obtain the following counterpart of Theorem 1.2 as a consequence of the last result.

Theorem 4.12. *Let $f, p, \alpha, A, \Omega, \phi, \rho, \lambda, m_\phi$ and \mathcal{L} be as in Theorem 4.11. Then there is an explicit norm $\|\cdot\|_{\diamond_0}$, depending on f, p, α, ϕ and equivalent to $\|\cdot\|_\infty + \|\cdot\|_{\langle p, \alpha \rangle}$, such that when $\|\phi\|_{\langle p, \alpha \rangle} \leq A$ and $\Omega(\phi) \leq \Omega$ we have*

$$\|\lambda^{-n} \mathcal{L}^n\|_{\langle p, \alpha \rangle} \leq c, \quad \|\rho\|_{\langle p, \alpha \rangle} \leq c, \quad \|1/\rho\|_{\langle p, \alpha \rangle} \leq c, \quad \text{and} \quad \|\lambda^{-1} \mathcal{L}g\|_{\diamond_0} \leq \beta \|g\|_{\diamond_0}$$

for every $g: \mathbb{P}^k \rightarrow \mathbb{R}$ with $\langle m_\phi, g \rangle = 0$, and for some positive constants $c = c(f, p, \alpha, A, \Omega)$ and $\beta = \beta(f, p, \alpha, A, \Omega)$ with $\beta < 1$, both independent of ϕ, n , and g . Furthermore, given any constant $0 < \delta < (d\alpha)^{1/2}$, when A is small enough, the norm $\|\cdot\|_{\diamond_0}$ can be chosen so that we can take $\beta = 1/\delta$.

Note that Lipschitz functions have finite $\|\cdot\|_{\langle p, \alpha \rangle}$ semi-norm (this follows from Lemma 3.9, since Lipschitz functions can be uniformly approximated by \mathcal{C}^1 ones whose norm is dominated by the Lipschitz constant, see also the proof of Lemma 3.13). So the last theorem can be applied to Lipschitz functions. For such functions we can take any p large enough and α close to 1. The rate of contraction is then almost equal to $d^{-1/2}$ when A is small enough (i.e., when ϕ is close to a constant function). This rate is likely optimal as it corresponds to known results obtained in the setting of zero weight, see [DS10a].

5. STATISTICAL PROPERTIES OF EQUILIBRIUM STATES

In this section we prove Theorem 1.3. The definitions of the statistical properties that we consider are all recalled in Appendix A, as well as criteria ensuring their validity in an abstract setting. We work under the hypotheses of Theorems 1.2 and 1.3 and with the equivalent norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ as in Theorem 1.2, see Section 4.4. For simplicity, we define the operator L as $L(g) := (\lambda\rho)^{-1}\mathcal{L}(\rho g)$.

5.1. Exponential equidistribution of preimages of points. The following result gives a quantitative version of the equidistribution of preimages in Theorem 1.1. Because of Lemma 2.4 and of the definition of the norm $\|\cdot\|_{\diamond_1}$, it applies in particular to Hölder continuous test functions, see Remark 3.12.

Theorem 5.1. *Under the hypotheses of Theorem 1.2, for every $x \in \mathbb{P}^k$, as n tends to infinity the points in $f^{-n}(x)$, with suitable weights, are equidistributed exponentially fast with respect to the conformal measure m_ϕ . More precisely, we have*

$$\left| \left\langle \lambda^{-n} \sum_{f^n(a)=x} e^{\phi(a)+\dots+\phi(f^{n-1}(a))} \delta_a - \rho(x)m_\phi, g \right\rangle \right| \leq c\beta^n \|g\|_{\diamond_1},$$

for all $g: \mathbb{P}^k \rightarrow \mathbb{R}$, where $0 < \beta < 1$ is the constant in Theorem 1.2 and c is a positive constant independent of x and g .

Proof. For simplicity, assume that $\|g\|_{\diamond_1} = 1$. Define $g' := g - c_g\rho$ with $c_g := \langle m_\phi, g \rangle$. By Lemma 4.10, both $|c_g|$ and $\|g'\|_{\diamond_1}$ are bounded by a constant. Since the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ are equivalent, we deduce from Theorem 1.2 that $\|\lambda^{-n}\mathcal{L}^n(g')\|_{\diamond_1} \lesssim \beta^n$. Since

$$\left\langle \lambda^{-n} \sum_{f^n(a)=x} e^{\phi(a)+\dots+\phi(f^{n-1}(a))} \delta_a, g \right\rangle = \lambda^{-n}(\mathcal{L}^n g)(x),$$

the LHS of the inequality in the theorem is bounded by

$$\|\lambda^{-n}\mathcal{L}^n(g) - c_g\rho\|_\infty = \|\lambda^{-n}\mathcal{L}^n(g - c_g\rho)\|_\infty = \|\lambda^{-n}\mathcal{L}^n(g')\|_\infty \lesssim \beta^n.$$

This implies the result. □

We also have the following useful result.

Proposition 5.2. *There exists a positive constant c such that, for every $g: \mathbb{P}^k \rightarrow \mathbb{R}$ with $\|g\|_{\diamond_1} \leq 1$, setting $c_{\rho g} := \langle m_\phi, \rho g \rangle = \langle \mu_\phi, g \rangle$ we have*

$$\left\langle \mu_\phi, e^{\beta^{-n}|L^n(g) - c_{\rho g}|} \right\rangle \leq c \quad \text{and} \quad \|L^n g - c_{\rho g}\|_{\diamond_1} \leq c\beta^n.$$

Proof. Clearly, the first inequality follows from the second one by increasing the constant c if necessary. For the second inequality, define $\tilde{g} = \rho g$ and $\tilde{g}' := \tilde{g} - c_{\tilde{g}}\rho$. We have $\langle m_\phi, \tilde{g}' \rangle = 0$. By Lemma 4.10 and Theorem 1.2, we have that $\|\tilde{g}'\|_{\diamond_1}$ is bounded by a constant because it is at most equal to $3\|\rho\|_{\diamond_1}\|g\|_{\diamond_1}$. We deduce that $c_{\tilde{g}}$ and hence $\|\tilde{g}'\|_{\diamond_1}$ are both bounded by a constant. Using again Lemma 4.10 and Theorem 1.2, we also obtain

$$\|L^n g - c_{\rho g}\|_{\diamond_1} = \|\lambda^{-n}\rho^{-1}\mathcal{L}^n\tilde{g} - c_{\tilde{g}}\|_{\diamond_1} = \|\rho^{-1}\lambda^{-n}\mathcal{L}^n\tilde{g}'\|_{\diamond_1} \leq 3\|\rho^{-1}\|_{\diamond_1}\|\lambda^{-n}\mathcal{L}^n\tilde{g}'\|_{\diamond_1} \lesssim \beta^n.$$

The result follows. □

5.2. Exponential mixing of the equilibrium state. The speed of mixing in Theorem 1.1 is not controlled for $g_0, \dots, g_r, g, l \in L^2(\mu_\phi)$. We establish here some uniform exponential bound for the speed of mixing of the system $(\mathbb{P}^k, f, \mu_\phi)$ for more regular observables. In the case of Hölder continuous weight ϕ and observable g , this was established in [Hay99] for $k = 1$ (see also [DPU96] for a uniform sub-exponential speed) and in [SUZ14] for $k > 1$, see also [DS10a] for the case when ϕ is constant.

Theorem 5.3. *Under the hypotheses of Theorem 1.2, for every integer $r \geq 0$, there is a positive constant $c = c(r)$ such that*

$$\left| \langle \mu_\phi, g_0 (g_1 \circ f^{n_1}) \cdots (g_r \circ f^{n_r}) \rangle - \prod_{j=0}^r \langle \mu_\phi, g_j \rangle \right| \leq c\beta^n \left(\prod_{j=0}^{r-1} \|g_j\|_{\diamond_1} \right) \|g_r\|_{L^1(\mu_\phi)}$$

for $0 = n_0 \leq n_1 \leq \dots \leq n_r$ and $n := \min_{0 \leq j < r} (n_{j+1} - n_j)$. Here, the constant $0 < \beta < 1$ is the one from Theorem 1.2.

Proof. We prove the theorem by induction on r . Since when $r = 0$ the assertion is trivial, we can assume that the theorem holds for $r - 1$ and prove it for r .

The fact that the statement holds for $r - 1$ and the invariance of μ_ϕ imply the result for r when g_0 is constant. So, by subtracting from g_0 the constant $c_{\rho g_0} := \langle m_\phi, \rho g_0 \rangle = \langle \mu_\phi, g_0 \rangle$, we can assume that $\langle \mu_\phi, g_0 \rangle = 0$. Setting $g' := g_1 (g_2 \circ f^{n_2 - n_1}) \cdots (g_r \circ f^{n_r - n_1})$ and recalling that $L(\cdot) = (\lambda\rho)^{-1}\mathcal{L}(\rho\cdot)$, computing as at the end of the proof of Theorem 1.1 we get

$$\begin{aligned} |\langle \mu_\phi, g_0 (g' \circ f^{n_1}) \rangle| &= |\langle \mu_\phi, L^{n_1}(g_0 (g' \circ f^{n_1})) \rangle| = |\langle \mu_\phi, L^{n_1}(g_0) g' \rangle| \\ &\leq \|L^{n_1}(g_0)\|_\infty \|g'\|_{L^1(\mu_\phi)} \\ &\leq \|L^{n_1}(g_0)\|_\infty \|g_1\|_\infty \cdots \|g_{r-1}\|_\infty \|g_r \circ f^{n_r - n_1}\|_{L^1(\mu_\phi)} \\ &= \|L^{n_1}(g_0)\|_\infty \|g_1\|_\infty \cdots \|g_{r-1}\|_\infty \|g_r\|_{L^1(\mu_\phi)}. \end{aligned}$$

The assertion follows from Proposition 5.2 applied to g_0 instead of g . \square

5.3. Central Limit Theorem (CLT). We prove here the CLT for observables g such that $\|g\|_{\diamond_1} < \infty$ and which are not coboundaries, see Appendix A for the definitions. In the case of Hölder continuous weight ϕ and observable g , this was established in [DPU96] for $k = 1$ and in [SUZ14] for $k > 1$, see also [DS10a] for the case when ϕ is constant. We start with the following version that does not need the introduction of perturbed operators. More refined versions of this theorem will be given later in Sections 5.5 and 5.6.

Theorem 5.4. *Under the hypotheses of Theorem 1.2, consider a function $g: \mathbb{P}^k \rightarrow \mathbb{R}$ such that $\|g\|_{\diamond_1} < \infty$ and $\langle \mu_\phi, g \rangle = 0$. Assume that g is not a coboundary. Then g satisfies the CLT with variance $\sigma > 0$ given by*

$$(5.1) \quad \sigma^2 := \langle \mu_\phi, g^2 \rangle + 2 \sum_{n \geq 1} \langle \mu_\phi, g \cdot (g \circ f^n) \rangle.$$

Moreover, the Berry-Esseen theorem holds for the observable g .

We refer to Definition A.1 for the statement of the CLT and to Theorem A.3 for the Berry-Esseen theorem. Recall that g is a coboundary if there exists $h \in L^2(\mu_\phi)$ such that $g = h \circ f - h$, and this is the case if and only if $\sigma = 0$. It is not difficult to check that a coboundary cannot satisfy the CLT as in Definition A.1.

Proof. Let \mathcal{B} denote the Borel σ -algebra on \mathbb{P}^k . By Gordin Theorem A.2, we only need to show that

$$\sum_{n \geq 1} \left\| \mathbb{E} (g | f^{-n} \mathcal{B}) \right\|_{L^2(\mu_\phi)}^2 < \infty.$$

We will transform the above series into a series of the norms $\|L^n g\|_{L^2(\mu_\phi)}$, that we will bound by means of Proposition 5.2. Recall that $\langle \mu_\phi, (L^n g)h \rangle = \langle \mu_\phi, g(h \circ f^n) \rangle$ for all $g, h: \mathbb{P}^k \rightarrow \mathbb{R}$. So

$$\langle \mu_\phi, g(h \circ f^n) \rangle = \langle \mu_\phi, (L^n g)h \rangle = \langle \mu_\phi, ((L^n g) \circ f^n)(h \circ f^n) \rangle,$$

where in the last passage we used the f_* -invariance of μ_ϕ . This precisely says that $\mathbb{E}(g, f^{-n}\mathcal{B}) = L^n(g) \circ f^n$. Note that the norm of f^* on $L^2(\mu_\phi)$ is 1. So we have

$$\sum_{n \geq 1} \|\mathbb{E}(g|f^{-n}\mathcal{B})\|_{L^2(\mu_\phi)}^2 \leq \sum_{n \geq 1} \|L^n(g)\|_{L^2(\mu_\phi)}^2 \leq \sum_{n \geq 1} \|L^n(g)\|_\infty^2.$$

The last sum converges because of Proposition 5.2 and the hypothesis $\langle \mu_\phi, g \rangle = 0$. For the Berry-Esseen theorem, see Theorem A.3 and Section 5.4 below. This completes the proof. \square

We have the following characterization of coboundaries with bounded $\|\cdot\|_{\diamond_1}$ norm that will be used in Section 5.5.

Proposition 5.5. *Let g be a coboundary and assume that $\|g\|_{\diamond_1} < \infty$. Then there exists a function \tilde{h} such that $\|\tilde{h}\|_{\diamond_1} < \infty$ and $g = \tilde{h} \circ f - \tilde{h}$ on the small Julia set of f .*

Proof. By the definition of coboundary, there exists a function $h \in L^2(\mu_\phi)$ such that $g = h \circ f - h$ in $L^2(\mu_\phi)$. This identity implies that $\langle \mu_\phi, g \rangle = 0$. By adding a constant to h we can assume that $\langle \mu_\phi, h \rangle = 0$. Since

$$L(h \circ f) = \lambda^{-1} \rho^{-1} \mathcal{L}(\rho(h \circ f)) = \lambda^{-1} \rho^{-1} \mathcal{L}(\rho)h = h$$

a direct computation gives

$$Lg + \dots + L^n g = h - L^n h.$$

We claim that $L^n h \rightarrow 0$ in $L^2(\mu_\phi)$ when n tends to infinity. Indeed, fix any $\epsilon > 0$ and choose a continuous function h' with $\|h - h'\|_{L^2(\mu_\phi)} < \epsilon$ and $\langle \mu_\phi, h' \rangle = 0$. By [BD20, Lemma 4.3] there exists a positive constant c such that $\|L^n\|_{L^2(\mu_\phi)} \leq c$ for all $n \geq 0$. Moreover, $\|L^n h'\|_\infty = \|L^n h' - \langle \mu_\phi, h' \rangle\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ by Theorem 1.1 and the definition of L . Hence,

$$\|L^n h\|_{L^2(\mu_\phi)} \leq \|L^n(h - h')\|_{L^2(\mu_\phi)} + \|L^n h'\|_{L^2(\mu_\phi)} \lesssim \epsilon$$

as $n \rightarrow \infty$. By taking $\epsilon \rightarrow 0$, this proves that $L^n h \rightarrow 0$ in $L^2(\mu_\phi)$. Proposition 5.2 and the identity $\langle \mu_\phi, g \rangle = 0$ imply that the LHS of the last identity converges to a function \tilde{h} with finite $\|\cdot\|_{\diamond_1}$ norm. We conclude that $\tilde{h} = h$ μ_ϕ -almost everywhere. Therefore, we have $g = \tilde{h} \circ f - \tilde{h}$ μ_ϕ -almost everywhere, and thus everywhere on $\text{supp}(\mu_\phi)$ as both sides are continuous functions. The proposition follows because the support of μ_ϕ is equal to the small Julia set, see Theorem 1.1. \square

5.4. Properties of perturbed Perron-Frobenius operators. The next statistical properties will be proved by means of spectral methods, and more precisely by the introduction of suitable (complex) perturbations of the operator $\mathcal{L} = \mathcal{L}_\phi$. This method was originally developed by Nagaev [Nag57] in the context of Markov chains. More details are given in Appendix A. We introduce here some notations and prove some preliminary results. We mainly follow the approach as presented in [Bro96; DNS07; Rou83].

Definition 5.6. Given functions $\phi, g: \mathbb{P}^k \rightarrow \mathbb{R}$, $h: \mathbb{P}^k \rightarrow \mathbb{C}$, and a parameter $\theta \in \mathbb{C}$ we set

$$\mathcal{L}_{\phi+\theta g} h := \mathcal{L}_{\phi+\theta g} \Re h + i \mathcal{L}_{\phi+\theta g} \Im h,$$

where the operator in the RHS is the natural extension of (1.1) in the case with complex weight.

Since from now on we fix ϕ and g , we will just denote the above operator by $\mathcal{L}_{[\theta]}$ when no possible confusion arises. In particular, we have $\mathcal{L}_{[0]} = \mathcal{L}_\phi$. By means of Definition 5.6, we extend the operator \mathcal{L} to complex weights and complex test functions. We naturally extend the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ to these function spaces by setting

$$\|h\|_{\diamond_1} := \|\Re h\|_{\diamond_1} + \|\Im h\|_{\diamond_1} \quad \text{and} \quad \|h\|_{\diamond_2} := \|\Re h\|_{\diamond_2} + \|\Im h\|_{\diamond_2}.$$

We will be in particular interested in the case where θ is small or pure imaginary. We may develop our study given in the previous sections for a complex weight directly, but we prefer to use a shortcut which only requires to treat the real case. The next lemma collects the main properties of the family of operators $\mathcal{L}_{[\theta]}$ that we need.

Lemma 5.7. *Assume that $\|g\|_{\diamond_1}$ is finite. Then the following assertions hold for both of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$.*

- (i) $\mathcal{L}_{[0]} = \mathcal{L}_\phi$;
- (ii) For every $\theta \in \mathbb{C}$, $\mathcal{L}_{[\theta]}$ is a bounded operator;
- (iii) The map $\theta \mapsto \mathcal{L}_{[\theta]}$ is analytic in θ ;
- (iv) For every $n \in \mathbb{N}$, $\theta \in \mathbb{C}$, and $h: \mathbb{P}^k \rightarrow \mathbb{C}$, we have

$$(5.2) \quad \mathcal{L}_{[\theta]}^n h = \mathcal{L}_{[0]}^n (e^{\theta S_n(g)} h),$$

where

$$(5.3) \quad S_0(g) := 0 \quad \text{and} \quad S_n(g) := \sum_{j=0}^{n-1} g \circ f^j \quad \text{for } n \geq 1.$$

Proof. The first item is true by definition. For the second and the third items, it is enough to prove them for the norm $\|\cdot\|_{\diamond_1}$ since the norm $\|\cdot\|_{\diamond_2}$ is equivalent, see Lemma 4.10. By Lemma 4.10 we have, for all $h: \mathbb{P}^k \rightarrow \mathbb{C}$ and $n \in \mathbb{N}$,

$$\|\mathcal{L}_{[0]}(g^n h)\|_{\diamond_1} \leq \|\mathcal{L}_{[0]}\|_{\diamond_1} \|g^n h\|_{\diamond_1} \leq 3^n \|\mathcal{L}_{[0]}\|_{\diamond_1} \|g\|_{\diamond_1}^n \|h\|_{\diamond_1}.$$

So, the series $\sum_{n \geq 0} (\theta^n/n!) \mathcal{L}_{[0]}(g^n h)$ converges normally in norm $\|\cdot\|_{\diamond_1}$. The limit is equal to $\mathcal{L}_{[\theta]}(h)$ and we have

$$\|\mathcal{L}_{[\theta]}(h)\|_{\diamond_1} \leq e^{3|\theta|\|g\|_{\diamond_1}} \|\mathcal{L}_{[0]}\|_{\diamond_1} \|h\|_{\diamond_1}.$$

This proves the second and the third items.

The last item follows by a direct induction. The case $n = 0$ is trivial and we have

$$\begin{aligned} \mathcal{L}_{[0]}^n (e^{\theta S_n(g)} h) &= \mathcal{L}_{[0]} \left(\mathcal{L}_{[0]}^{n-1} (e^{\theta \cdot g \circ f^{n-1}} e^{\theta S_{n-1}(g)} h) \right) \\ &= \mathcal{L}_{[0]} \left(e^{\theta g} \mathcal{L}_{[0]}^{n-1} (e^{\theta S_{n-1}(g)} h) \right) = \mathcal{L}_{[\theta]} \mathcal{L}_{[0]}^{n-1} (e^{\theta S_{n-1}(g)} h) = \mathcal{L}_{[\theta]} \mathcal{L}_{[\theta]}^{n-1} h. \end{aligned}$$

The proof is complete. \square

Recall that the operator $\lambda^{-1} \mathcal{L}_{[0]}$ has ρ as its unique (up to a multiplicative constant) eigenfunction of eigenvalue 1. It is a contraction with respect to the norm $\|\cdot\|_{\diamond_2}$ (which is equivalent to $\|\cdot\|_{\diamond_1}$) on the space of functions whose integrals with respect to m_ϕ are zero, see Theorem 1.2. The following is then a consequence of the Rellich perturbation method described in [DS58, Ch. VII], see also [Bro96, Proposition 5.2] and [Kat13; Le 82]. Note that the last assertion of Theorem 1.3 is a direct consequence of the analyticity of α given by the fourth item.

Proposition 5.8. *Assume that $\|g\|_{\diamond_1}$ is finite and let $0 < \beta < 1$ be the constant in Theorem 1.2. Then, for all $\beta < \beta' < 1$, the following holds for θ sufficiently small and all $n \in \mathbb{N}$: there exists a decomposition*

$$\lambda^{-n} \mathcal{L}_{[\theta]}^n = \alpha(\theta)^n \Phi_\theta + \Psi_\theta^n$$

as operators on $\{h : \|h\|_{\diamond_1} < \infty\}$ such that

- (i) $\alpha(\theta)$ is the (only) largest eigenvalue of $\mathcal{L}_{[\theta]}$, $\alpha(0) = 1$ and $|\alpha(\theta)| > \beta'$;
- (ii) Φ_θ is the projection on the (one dimensional) eigenspace associated to $\alpha(\theta)$ and we have $\Phi_0(h) = \langle m_\phi, h \rangle \rho$;
- (iii) Ψ_θ is a bounded operator on $\{h : \|h\|_{\diamond_1} < \infty\}$ whose spectral radius is $< \beta'$ and

$$\Psi_\theta \circ \Phi_\theta = \Phi_\theta \circ \Psi_\theta = 0;$$

- (iv) the maps $\theta \mapsto \Psi_\theta, \theta \mapsto \Phi_\theta$, and $\theta \mapsto \alpha(\theta)$ are analytic.

The last property that we will need is the second order expansion of $\alpha(\theta)$ for θ near 0. It is a consequence of the above results.

Lemma 5.9. *Assume that $\|g\|_{\diamond_1}$ is finite and $\langle \mu_\phi, g \rangle = 0$. Let $\sigma > 0$ be as in (5.1) and $\alpha(\theta)$ be given by Proposition 5.8. Then we have*

$$\alpha(\theta) = e^{\frac{\sigma^2 \theta^2}{2} + o(\theta^2)} = 1 + \frac{\theta^2 \sigma^2}{2} + o(\theta^2).$$

Proof. We know from Lemma 5.7 and Proposition 5.8 that $\alpha(0) = 1$ and that $\theta \mapsto \alpha(\theta)$ is analytic near 0. So, we only need to prove that $\alpha'(0) = 0$ and $\alpha''(0) = \sigma^2$.

We start with the first equality. First of all notice that, for all $n \in \mathbb{N}$ and $h: \mathbb{P}^k \rightarrow \mathbb{C}$, the invariance of m_ϕ implies

$$(5.4) \quad \langle \mu_\phi, h \rangle = \langle m_\phi, \rho h \rangle = \left\langle m_\phi, \lambda^{-n} \mathcal{L}_{[0]}^n(\rho h) \right\rangle.$$

Applying these identities to $e^{S_n(g)/n}$ instead of h (with $S_n(g)$ as in (5.3)) and using (5.2), we get

$$\langle \mu_\phi, e^{S_n(g)/n} \rangle = \langle m_\phi, \lambda^{-n} \mathcal{L}_{[1/n]}^n(\rho) \rangle.$$

Then, the decomposition given in Proposition 5.8 (applied with $1/n$ instead of θ) gives

$$(5.5) \quad \langle \mu_\phi, e^{S_n(g)/n} \rangle = \alpha(1/n)^n \langle m_\phi, \Phi_{1/n}(\rho) \rangle + \langle m_\phi, \Psi_{1/n}^n(\rho) \rangle.$$

By Proposition 5.8, the second term in the RHS goes to zero exponentially fast as $n \rightarrow \infty$. Using that $\alpha(\theta) = 1 + \alpha'(0)\theta + o(\theta)$ and $\Phi_\theta(\rho) = \rho + o(1)$ as $\theta \rightarrow 0$ and the identity $\langle m_\phi, \rho \rangle = 1$, we get

$$\lim_{n \rightarrow \infty} \langle \mu_\phi, e^{S_n(g)/n} \rangle = e^{\alpha'(0)}.$$

Recall that μ_ϕ is mixing and hence ergodic. Moreover, g is continuous and $\langle \mu_\phi, g \rangle = 0$. Birkhoff's ergodic theorem then implies that $S_n(g)/n \rightarrow 0$ μ_ϕ -almost surely as n goes to infinity. We conclude that $\alpha'(0) = 0$ as desired.

Let us now prove that $\alpha''(0) = \sigma^2$. The f -invariance of μ_ϕ and a direct computation give

$$\sigma^2 = \lim_{n \rightarrow \infty} \left\langle \mu_\phi, \left(\frac{S_n g}{\sqrt{n}} \right)^2 \right\rangle \quad \text{and} \quad \left\langle \mu_\phi, \left(\frac{S_n g}{\sqrt{n}} \right)^2 \right\rangle = \frac{\partial^2}{\partial \theta^2} \left\langle \mu_\phi, e^{(\theta/\sqrt{n}) S_n g} \right\rangle \Big|_{\theta=0}.$$

We use again (5.2) and Proposition 5.8 (applied with θ/\sqrt{n} instead of θ) to get

$$\left\langle \mu_\phi, e^{(\theta/\sqrt{n}) S_n g} \right\rangle = \alpha(\theta/\sqrt{n})^n \left\langle m_\phi, \Phi_{\theta/\sqrt{n}}(\rho) \right\rangle + \left\langle m_\phi, \Psi_{\theta/\sqrt{n}}^n(\rho) \right\rangle.$$

A direct computation, together with the identities $\alpha(0) = 1$, $\alpha'(0) = 0$, $\langle m_\phi, \rho \rangle = 1$ and the properties in Proposition 5.8, gives

$$\frac{\partial^2}{\partial \theta^2} \left\langle \mu_\phi, e^{(\theta/\sqrt{n}) S_n g} \right\rangle \Big|_{\theta=0} = \alpha''(0) + o(1).$$

It is now clear that $\alpha''(0) = \sigma^2$. This completes the proof of the lemma. \square

5.5. Local Central Limit Theorem (LCLT). We establish here an improvement of the CLT, see Definition A.4, for observables satisfying a further cocycle condition. Our result is new for $k = 1$, ϕ non-constant, and for $k > 1$, even when $\phi = 0$; for $k = 1$ and $\phi = 0$, see [DNS07]. We need the following definition.

Definition 5.10. Let $g: \mathbb{P}^k \rightarrow \mathbb{R}$ be a measurable function. We say that g is a *multiplicative cocycle* if there exist $t > 0$, $s \in \mathbb{R}$, and a measurable function $\xi: \mathbb{P}^k \rightarrow \mathbb{C}$, not equal to zero μ_ϕ -almost everywhere, such that $e^{itg(z)}\xi(z) = e^{is}\xi(f(z))$. We say that g is a (\mathcal{C}^0, ϕ) -*multiplicative cocycle* (resp. $(\|\cdot\|_{\diamond_1}, \phi)$ -*multiplicative cocycle*) if there exist $t > 0$, $s \in \mathbb{R}$, and $\xi: \mathbb{P}^k \rightarrow \mathbb{C}$, not identically zero on the small Julia set of f , which is continuous (resp. with finite $\|\cdot\|_{\diamond_1}$ norm), such that $e^{itg(z)}\xi(z) = e^{is}\xi(f(z))$ on the small Julia set of f .

Recall that the supports of μ_ϕ and m_ϕ are both equal to the small Julia set of f , see Theorem 1.1.

Theorem 5.11. *Under the hypotheses of Theorem 1.2, let $g: \mathbb{P}^k \rightarrow \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite, $\langle \mu_\phi, g \rangle = 0$, and g is not a $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycle. Then g satisfies the LCLT with variance σ given by (5.1).*

Remark 5.12. The LCLT is a refined version of the CLT. Notice that it requires a stronger assumption on the observable g .

We first need some properties of multiplicative cocycles. Note that the following lemma still holds if we only assume that ϕ and g have bounded $\|\cdot\|_{\log^q}$ norms for some $q > 2$.

Lemma 5.13. *There is a positive constant c independent of g, t , and n such that $\|\lambda^{-n} \mathcal{L}_{[it]}^n\|_\infty \leq c$. Let K be a compact subset of $\mathbb{R} \setminus \{0\}$ (e.g., a singleton in $\mathbb{R} \setminus \{0\}$). Let \mathcal{F} be a uniformly bounded and equicontinuous family of functions on \mathbb{P}^k . Then the family*

$$\mathcal{F}_{\mathbb{N}}^K := \{\lambda^{-n} \mathcal{L}_{[it]}^n h : t \in K, h \in \mathcal{F}, n \in \mathbb{N}\}$$

is also uniformly bounded and equicontinuous. Furthermore, if $K \subset (0, \infty)$ and g is not a (\mathcal{C}^0, ϕ) -multiplicative cocycle, then $\|\lambda^{-n} \mathcal{L}_{[it]}^n h\|_\infty$ tends to 0 when n goes to infinity, uniformly in $t \in K$ and $h \in \mathcal{F}$.

Proof. Define $\phi_t := \phi + itg$. Observe that $|e^{\phi_t}| = e^\phi$. It follows that $\|\mathcal{L}_{[it]}^n\|_\infty \leq \|\mathcal{L}^n\|_\infty \leq c\lambda^n$ for some positive constant c , according to (1.2). Using this, we can follow the proof of [BD20, Lemma 3.9], with ϕ_t instead of ϕ , and obtain that $\mathcal{F}_{\mathbb{N}}^K$ is uniformly bounded and equicontinuous. It remains to prove the last assertion in the lemma.

Let \mathcal{F}_∞^K denote the family of the limit functions of all sequences $\lambda^{-n_j} \mathcal{L}_{[it_j]}^{n_j}(h_j)$ with $t_j \in K$, $h_j \in \mathcal{F}$, and n_j going to infinity. By Arzelà-Ascoli theorem, this is a uniformly bounded and equicontinuous family of functions which is compact for the uniform topology. Define

$$M := \max \{|l(a)/\rho(a)| : l \in \mathcal{F}_\infty^K, a \text{ in the small Julia set}\}.$$

Claim 1. If $M = 0$, then \mathcal{F}_∞^K only contains the zero function.

Proof of Claim 1. Assume that $M = 0$. Consider a function l in \mathcal{F}_∞^K . We show that $l = 0$. Consider functions $l_{-n} \in \mathcal{F}_\infty^K$ such that $l = \lambda^{-n} \mathcal{L}_{[it]}^n l_{-n}$ for some $t \in K$ (take t as a limit of t_j above). Since $M = 0$, the function l_{-n} vanishes on the support of m_ϕ (which is the small Julia set) and we have $c_{|l_{-n}|} = \langle m_\phi, |l_{-n}| \rangle = 0$. Moreover, we also have

$$|l| = |\lambda^{-n} \mathcal{L}_{[it]}^n l_{-n}| \leq \lambda^{-n} \mathcal{L}^n |l_{-n}|.$$

By Lemma 4.2 applied to the family $\{|h| : h \in \mathcal{F}_\infty^K\}$, the last function converges to 0 when n tends to infinity. The claim follows. \square

Claim 2. There is $l \in \mathcal{F}_\infty^K$ such that $|l/\rho| = M$ on the small Julia set.

Proof of Claim 2. Choose a function l in \mathcal{F}_∞^K such that $\max |l/\rho| = M$. Consider l_{-n} and t as in the proof of Claim 1. Let a in the small Julia set be such that $|l(a)/\rho(a)| = M$. Then we have

$$\begin{aligned} |l(a)| &= |\lambda^{-n} \mathcal{L}_{[it]}^n l_{-n}(a)| = \left| \lambda^{-n} \sum_{b \in f^{-n}(a)} e^{\phi_t(b) + \dots + \phi_t(f^{n-1}(b))} l_{-n}(b) \right| \\ &\leq \lambda^{-n} \sum_{b \in f^{-n}(a)} e^{\phi(b) + \dots + \phi(f^{n-1}(b))} M \rho(b) = M \rho(a). \end{aligned}$$

So the last inequality is actually an equality. In particular, we have $|l_{-n}/\rho| = M$ on $f^{-n}(a)$. Recall that when n goes to infinity, $f^{-n}(a)$ tends to the small Julia set. Therefore, if l_∞ is a limit of l_{-n} , we have $|l_\infty/\rho| = M$ on the small Julia set. Replacing l by l_∞ gives the claim. \square

Claim 3. There are $t \in K$ and $l \in \mathcal{F}_\infty^K$ such that $|\lambda^{-N} \mathcal{L}_{[it]}^N l| = M\rho$ on the small Julia set for every $N \geq 0$.

Proof of Claim 3. Choose l satisfying Claim 2 and t as in its proof. Define l_{-n} and $l_{-\infty}$ as in the proof of that claim. Fix an $N \geq 0$. For every $n \geq N$, on the small Julia set we have

$$\begin{aligned} M\rho &= |l| = |\lambda^{-n} \mathcal{L}_{[it]}^n l_{-n}| = |\lambda^{-n+N} \mathcal{L}_{[it]}^{n-N} (\lambda^{-N} \mathcal{L}_{[it]}^N l_{-n})| \\ &\leq \lambda^{-n+N} \mathcal{L}^{n-N} |\lambda^{-N} \mathcal{L}_{[it]}^N l_{-n}| \leq \lambda^{-n+N} \mathcal{L}^{n-N} \lambda^{-N} \mathcal{L}^N (M\rho) = M\rho. \end{aligned}$$

So the two last inequalities are in fact equalities and we deduce that $|\lambda^{-N} \mathcal{L}_{[it]}^N l_{-n}| = M\rho$. It follows that $|\lambda^{-N} \mathcal{L}_{[it]}^N l_{-\infty}| = M\rho$ for every $N \geq 0$. Replacing l by $l_{-\infty}$ gives the claim. \square

Assume now that g is not a (\mathcal{C}^0, ϕ) -multiplicative cocycle. By Claim 1, we only need to show that $M = 0$. Assume by contradiction that $M \neq 0$. Consider t and l as in Claim 3. Define $\xi(a) := l(a)/\rho(a)$ for $a \in \mathbb{P}^k$ and $\vartheta(a) := e^{itg(a)}\xi(a)/\xi(f(a))$ for a in the small Julia set. These functions are continuous and we have $|\xi(a)| = M$ and $|\vartheta(a)| = 1$ on the small Julia set. We have for a in the small Julia set

$$\begin{aligned} M\rho(a) &= |\lambda^{-n} \mathcal{L}_{[it]}^n l(a)| = \left| \lambda^{-n} \sum_{b \in f^{-n}(a)} e^{\phi_t(b) + \dots + \phi_t(f^{n-1}(b))} l(b) \right| \\ &\leq \lambda^{-n} \sum_{b \in f^{-n}(a)} e^{\phi(b) + \dots + \phi(f^{n-1}(b))} M\rho(b) = M\rho(a). \end{aligned}$$

So, the last inequality is an equality. Using the function ξ , we can rewrite this equality as (we remove the factors λ^{-n} and also the factors $|\xi(f^n(b))|$ and M as they are both equal to $|\xi(a)|$ and independent of $b \in f^{-n}(a)$)

$$\left| \sum_{b \in f^{-n}(a)} \vartheta(b) \dots \vartheta(f^{n-1}(b)) e^{\phi(b) + \dots + \phi(f^{n-1}(b))} \rho(b) \right| = \sum_{b \in f^{-n}(a)} e^{\phi(b) + \dots + \phi(f^{n-1}(b))} \rho(b).$$

As $|\vartheta| = 1$, we deduce that if b and b' are two points in $f^{-n}(a)$ then

$$\vartheta(b) \dots \vartheta(f^{n-1}(b)) = \vartheta(b') \dots \vartheta(f^{n-1}(b')).$$

This and a similar equality for $f(b), f(b'), n-1$ instead of b, b', n imply that $\vartheta(b) = \vartheta(b')$. We conclude that ϑ is constant on $f^{-n}(a)$ for every n . As $f^{-n}(a)$ tends to the small Julia set when n going to infinity, it follows that ϑ is constant. From the definition of ϑ , and since ξ is continuous, we obtain that g is a (\mathcal{C}^0, ϕ) -multiplicative cocycle for a suitable real number s such that $\vartheta = e^{is}$. This is a contradiction. So we have $M = 0$, as desired. \square

Recall that ϕ and g have bounded $\|\cdot\|_{\diamond_1}$ norms.

Lemma 5.14. *Let K be a compact subset of \mathbb{R} . There is a positive constant c such that $\|\lambda^{-n} \mathcal{L}_{[it]}^n\|_{\diamond_1} \leq c$ for every $n \geq 0$ and $t \in K$. If $K \subset \mathbb{R} \setminus \{0\}$ and g is not a $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycle, then there are constants $c > 0$ and $0 < r < 1$ such that $\|\lambda^{-n} \mathcal{L}_{[it]}^n\|_{\diamond_1} \leq cr^n$ for every $t \in K$ and $n \geq 0$.*

Proof. Consider the functional ball $\mathcal{F} := \{h : \|h\|_{\diamond_1} \leq 1\}$. By Arzelà-Ascoli theorem this ball is compact for the uniform topology. Define \mathcal{F}_∞^K as in the proof of Lemma 5.13. Using that lemma, we can follow the proof of Proposition 4.9 and obtain that $\|\lambda^{-n} \mathcal{L}_{[it]}^n\|_{\langle p, \alpha, \gamma \rangle}$ is bounded uniformly on n and $t \in K$. It follows that a similar property holds for the norm $\|\cdot\|_{\diamond_1}$. This gives the first assertion in the lemma. We also obtain that the family \mathcal{F}_∞^K is bounded in the $\|\cdot\|_{\diamond_1}$ norm and, using again Arzelà-Ascoli theorem, we obtain that it is compact in the uniform topology.

Consider now the second assertion and assume that g is not a $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycle. We first show that \mathcal{F}_∞^K reduces to $\{0\}$. Assume by contradiction that this is not true. Consider t and l as in Claim 3 in the proof of Lemma 5.13. Recall that both $\|l\|_{\diamond_1}$ and $\|1/\rho\|_{\diamond_1}$ are finite,

see Theorem 1.2. By Lemma 3.14, the function $\xi := l/\rho$ satisfies the same property and we conclude, as at the end of the proof of Lemma 5.13, that g is a $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycle. This contradicts the hypothesis.

So \mathcal{F}_∞^K is reduced to $\{0\}$. By definition of \mathcal{F}_∞^K , we obtain that $\lambda^{-n}\mathcal{L}_{[it]}^n h$ converges to 0 uniformly on $h \in \mathcal{F}$ and $t \in K$. Using this property, we can follow the proof of Proposition 4.6 (take $\beta = 1/4$) to obtain that $\|\lambda^{-N}\mathcal{L}_{[it]}^N h\|_{(p,\alpha),\gamma} \leq 1/4$ for N large enough and for all $h \in \mathcal{F}$ and $t \in K$. When N is large enough, we also have $\|\lambda^{-N}\mathcal{L}_{[it]}^N h\|_\infty \leq 1/4$. Therefore, we have $\|\lambda^{-N}\mathcal{L}_{[it]}^N\|_{\diamond_1} \leq 1/2$ which implies the desired property with $r = 2^{-1/N}$. \square

We have the following characterizations of multiplicative cocycles.

Proposition 5.15. *Let $g: \mathbb{P}^k \rightarrow \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite. Then the following properties are equivalent:*

- (ia) g is a multiplicative cocycle;
- (ib) g is a (\mathcal{C}^0, ϕ) -multiplicative cocycle;
- (ic) g is a $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycle;
- (iia) there exists a number $t > 0$ such that the spectral radius with respect to the norm $\|\cdot\|_{\diamond_1}$ of $\lambda^{-1}\mathcal{L}_{[it]}$ is ≥ 1 .
- (iib) there exists a number $t > 0$ such that the spectral radius with respect to the norm $\|\cdot\|_{\diamond_1}$ of $\lambda^{-1}\mathcal{L}_{[it]}$ is equal to 1.

Moreover, every coboundary with finite $\|\cdot\|_{\diamond_1}$ norm is a $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycle.

Proof. The last assertion is a direct consequence of Proposition 5.5. Indeed, if g is such a coboundary, we have $g = \tilde{h} \circ f - \tilde{h}$ on the small Julia set for some \tilde{h} with $\|\tilde{h}\|_{\diamond_1} < \infty$. So g is a $(\|\cdot\|_{\diamond_1}, \phi)$ -multiplicative cocycle as in Definition 5.10 for $t := 1$, $s := 0$ and $\xi := e^{i\tilde{h}}$. We prove now the equivalence of the above five properties. It is clear that (ic) \Rightarrow (ib) \Rightarrow (ia) and (iib) \Rightarrow (iia). Lemma 5.14 implies that (iia) \Rightarrow (ic). The following implications complete the proof.

(iia) \Rightarrow (iib). By Lemma 5.14, $\|\lambda^{-n}\mathcal{L}_{[it]}^n\|_{\diamond_1}$ is bounded uniformly on n . Thus, the spectral radius of $\lambda^{-1}\mathcal{L}_{[it]}$ with respect to the norm $\|\cdot\|_{\diamond_1}$ is at most equal to 1. We easily deduce that (iia) \Rightarrow (iib).

(ia) \Rightarrow (iia). Assume by contradiction that (ia) is true and (iia) is not true. Consider t, s , and ξ as in Definition 5.10. If $\xi(z) \neq 0$ we divide it by $|\xi(z)|$. This allows us to assume that ξ is bounded. Define $h := \xi\rho$. Using the cocycle property of g and the $\lambda^{-1}\mathcal{L}$ -invariance of ρ , we have

$$\lambda^{-1}\mathcal{L}_{[it]}h = \lambda^{-1}\mathcal{L}(e^{itg}\xi\rho) = e^{is}\lambda^{-1}\mathcal{L}((\xi \circ f)\rho) = e^{is}\xi\lambda^{-1}\mathcal{L}(\rho) = e^{is}h.$$

It follows that $\lambda^{-n}\mathcal{L}_{[it]}^n h = e^{ins}h$ for $n \geq 1$. Fix an arbitrary positive constant ϵ and choose a smooth function \tilde{h} such that $\|h - \tilde{h}\|_{L^1(m_\phi)} \leq \epsilon$. Since (iia) is not true, $\lambda^{-n}\mathcal{L}_{[it]}^n \tilde{h}$ converges uniformly to 0. Moreover, we have

$$|h| = |\lambda^{-n}\mathcal{L}_{[it]}^n h| \leq |\lambda^{-n}\mathcal{L}_{[it]}^n \tilde{h}| + |\lambda^{-n}\mathcal{L}_{[it]}^n (h - \tilde{h})| \leq |\lambda^{-n}\mathcal{L}_{[it]}^n \tilde{h}| + \lambda^{-n}\mathcal{L}^n \|h - \tilde{h}\|.$$

This and the $\lambda^{-1}\mathcal{L}^*$ -invariance of m_ϕ imply that

$$\|h\|_{L^1(m_\phi)} \leq \|\lambda^{-n}\mathcal{L}_{[it]}^n \tilde{h}\|_{L^1(m_\phi)} + \|h - \tilde{h}\|_{L^1(m_\phi)} \leq \|\lambda^{-n}\mathcal{L}_{[it]}^n \tilde{h}\|_{L^1(m_\phi)} + \epsilon.$$

By taking n going to infinity, we obtain that $\|h\|_{L^1(m_\phi)} \leq \epsilon$, which implies that $h = 0$ and hence $\xi = 0$, both m_ϕ -almost everywhere. This contradicts the requirement on ξ in Definition 5.10 and ends the proof of the proposition. \square

Proof of Theorem 5.11. We follow here the proof of [DNS07, Th. C], which is based on [Bre92, Theorem 10.17]. Let ψ be any real-valued function in $L^1(\mathbb{R})$ whose Fourier transform

$$\hat{\psi}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t) e^{-itx} dt$$

is a continuous function with support contained in some interval $[-\delta, \delta]$. Define

$$(5.6) \quad A_n(x) := \sigma\sqrt{2\pi n} \langle \mu_\phi, \psi(x + S_n(g)) \rangle - e^{-x^2/(2\sigma^2 n)} \int_{-\infty}^{\infty} \psi(t) dt.$$

By [Bre92, Th. 10.7], it suffices to show that $A_n(x)$ converges to 0 as n goes to infinity, uniformly on x .

Claim. We have

$$(5.7) \quad A_n(x) = \int_{-\delta\sigma\sqrt{n}}^{\delta\sigma\sqrt{n}} \hat{\psi}\left(\frac{t}{\sigma\sqrt{n}}\right) e^{itx/(\sigma\sqrt{n})} \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it/(\sigma\sqrt{n})]}^n \rho \rangle dt - \hat{\psi}(0) \int_{-\infty}^{\infty} e^{itx/(\sigma\sqrt{n})} e^{-t^2/2} dt.$$

Proof of Claim. The second term in the RHS of (5.6) is equal to the second term in the RHS of (5.7) because

$$e^{-x^2/(2\sigma^2 n)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx/(\sigma\sqrt{n})} e^{-t^2/2} dt \quad \text{and} \quad \hat{\psi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t) dt.$$

It remains to compare the first terms. Using the identities

$$\langle m_\phi, e^{itS_n(g)} h \rangle = \langle m_\phi, \lambda^{-n} \mathcal{L}^n(e^{itS_n(g)} h) \rangle = \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it]}^n h \rangle \quad \text{and} \quad \psi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} \hat{\psi}(t) e^{itz} dt$$

and the fact that $\mu_\phi = \rho m_\phi$, we obtain

$$\begin{aligned} \sigma\sqrt{2\pi n} \langle \mu_\phi, \psi(x + S_n(g)) \rangle &= \sigma\sqrt{n} \int_{-\delta}^{\delta} \hat{\psi}(t) e^{itx} \langle \mu_\phi, e^{itS_n(g)} \rangle dt \\ &= \sigma\sqrt{n} \int_{-\delta}^{\delta} \hat{\psi}(t) e^{itx} \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it]}^n \rho \rangle dt. \end{aligned}$$

The last expression is equal to the first term in the RHS of (5.7) by the change of variable $t \mapsto t/(\sigma\sqrt{n})$. This ends the proof of the claim. \square

Notice that for any constant $\delta_0 > 0$ we have the following partial estimate for the second term in the RHS of (5.7):

$$(5.8) \quad \lim_{n \rightarrow \infty} \left| \int_{|t| > \delta_0 \sigma\sqrt{n}} e^{itx/(\sigma\sqrt{n})} e^{-t^2/2} dt \right| \leq \lim_{n \rightarrow \infty} \int_{|t| > \delta_0 \sigma\sqrt{n}} e^{-t^2/2} dt = 0.$$

Moreover, it follows from Proposition 5.8, Lemma 5.9, and the identity $\langle m_\phi, \rho \rangle = 1$, that

$$(5.9) \quad \lim_{n \rightarrow \infty} \hat{\psi}\left(\frac{t}{\sigma\sqrt{n}}\right) \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it/(\sigma\sqrt{n})]}^n \rho \rangle - \hat{\psi}(0) e^{-t^2/2} = 0 \quad \text{for all } t \in \mathbb{R},$$

and, for $\delta_0 < \delta$ sufficiently small,

$$(5.10) \quad \left| \hat{\psi}\left(\frac{t}{\sigma\sqrt{n}}\right) \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it/(\sigma\sqrt{n})]}^n \rho \rangle - \hat{\psi}(0) e^{-t^2/2} \right| \lesssim e^{-t^2/4} \|\hat{\psi}\|_\infty \quad \text{when } |t| \leq \delta_0 \sigma\sqrt{n}.$$

Since the RHS in (5.10) defines an integrable function of $t \in \mathbb{R}$, (5.9) and (5.10) imply that

$$\lim_{n \rightarrow \infty} \int_{-\delta_0 \sigma\sqrt{n}}^{\delta_0 \sigma\sqrt{n}} \hat{\psi}\left(\frac{t}{\sigma\sqrt{n}}\right) e^{itx/(\sigma\sqrt{n})} \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it/(\sigma\sqrt{n})]}^n \rho \rangle dt - \hat{\psi}(0) \int_{-\delta_0 \sigma\sqrt{n}}^{\delta_0 \sigma\sqrt{n}} e^{itx/(\sigma\sqrt{n})} e^{-t^2/2} dt = 0.$$

Using this, (5.8), and the above claim, it is enough to prove that (compare with the first term in the RHS of (5.7))

$$A'_n(x) := \int_{\delta_0 \leq |t/(\sigma\sqrt{n})| < \delta} \hat{\psi}\left(\frac{t}{\sigma\sqrt{n}}\right) e^{itx/(\sigma\sqrt{n})} \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it/(\sigma\sqrt{n})]}^n \rho \rangle dt$$

converges to 0. Here is where we use the assumption that g is not a multiplicative cocycle. By Lemma 5.14 and Proposition 5.15, we can find two constants $c > 0$ and $0 < r < 1$ such that $\|\lambda^{-n} \mathcal{L}_{[it/(\sigma\sqrt{n})]}^n\|_{\diamond_1} \leq cr^n$ for $|t/(\sigma\sqrt{n})| \in [\delta_0, \delta]$. This shows that the term $\langle \cdot, \cdot \rangle$ in the definition

of $A'_n(x)$ goes to zero exponentially fast, and we deduce that $A'_n(x) = O(\sqrt{nr}^n)$ as n goes to infinity. This completes the proof of the theorem. \square

5.6. Almost Sure Invariant Principle (ASIP) and consequences. We can now prove the ASIP for observables which are not coboundaries, see Definition A.5. The ASIP was proved by Dupont [Dup10] in the case where $\phi = 0$ for observables which are Hölder continuous, or admit analytic singularities, by using [PS75], see also Przytycki-Urbański-Zdunik [PUZ89] for $k = 1$ and $\phi = 0$.

Theorem 5.16. *Under the hypotheses of Theorem 1.2, let $g: \mathbb{P}^k \rightarrow \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite and $\langle \mu_\phi, g \rangle = 0$. Assume that g is not a coboundary. Then g satisfies the ASIP with error rate $o(n^q)$ for all $q > 1/4$.*

Proof. We will prove that the assumptions of Theorem A.6 are satisfied by the system $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$ and the operators $\mathcal{T}_t := \lambda^{-1} \mathcal{L}_{[it]}$ for $t \in \mathbb{R}$. These operators act on the functional space $\mathcal{H} := \{h : \|h\|_{\diamond_1} < \infty\}$ which is endowed with the norm $\|\cdot\|_{\diamond_1}$. We consider here the probability measures $\nu := \mu_\phi$, $\nu^* := m_\phi$, the function $\xi := \rho$, and the constant p (of Theorem A.6) large enough. We only need to check the strong coding condition in Theorem A.6. Indeed, the spectral description is a consequence of Theorem 1.2, the weak regularity follows from Lemma 5.7 or Lemma 5.14, and the last condition is satisfied since g is continuous.

Let us prove that the strong coding condition holds for all $t_0, \dots, t_{n-1} \in \mathbb{R}$. As in the proof of (5.2), we obtain

$$\begin{aligned} \mathcal{L}_{[0]}^n(e^{i \sum_{l=0}^{n-1} t_l g \circ f^l} \rho) &= \mathcal{L}_{[0]} \left(\mathcal{L}_{[0]}^{n-1}(e^{it_{n-1} g \circ f^{n-1}} e^{i \sum_{l=0}^{n-2} t_l g \circ f^l} \rho) \right) \\ &= \mathcal{L}_{[0]} \left(e^{it_{n-1} g} \mathcal{L}_{[0]}^{n-1}(e^{i \sum_{l=0}^{n-2} t_l g \circ f^l} \rho) \right) \\ &= \mathcal{L}_{[it_{n-1}]} \mathcal{L}_{[0]}^{n-1}(e^{i \sum_{l=0}^{n-2} t_l g \circ f^l} \rho). \end{aligned}$$

Hence, by induction, we get

$$\mathcal{L}_{[0]}^n(e^{i \sum_{l=0}^{n-1} t_l g \circ f^l} \rho) = \mathcal{L}_{[it_{n-1}]} \circ \dots \circ \mathcal{L}_{[it_0]} \rho.$$

The identity (5.4) and the last one imply

$$\langle \mu_\phi, e^{i \sum_{l=0}^{n-1} t_l g \circ f^l} \rangle = \langle m_\phi, \lambda^{-n} \mathcal{L}_{[0]}^n(e^{i \sum_{l=0}^{n-1} t_l g \circ f^l} \rho) \rangle = \langle m_\phi, \lambda^{-n} \mathcal{L}_{[it_{n-1}]} \circ \dots \circ \mathcal{L}_{[it_0]} \rho \rangle.$$

This is the desired strong coding condition. Applying Theorem A.6 gives the result. \square

We also have the following direct consequence of the ASIP and Theorem A.3, see Definitions A.7, A.8, and Theorem A.9. The LIL was established in [SUZ14] in the case where both the weight ϕ and the observable g are Hölder continuous.

Corollary 5.17. *Under the hypotheses of Theorem 1.2, let g be such that $\|g\|_{\diamond_1}$ is finite and $\langle \mu_\phi, g \rangle = 0$. Assume that g is not a coboundary. Then g satisfies the ASCLT, the LIL, and the CLT with error rate $O(n^{-1/2})$.*

5.7. Large Deviation Principle (LDP). We conclude the statistical study of $(\mathbb{P}^k, f, \mu_\phi)$ with the following property, see Definition A.10. This is new in this generality for all $k \geq 1$, even for $\phi = 0$ (see [CR11] for the case when $k = 1$ and some kind of weak hyperbolicity is assumed). The LDP in particular implies the Large Deviation Theorem, which is proved in [DNS10] in the case $\phi = 0$, see also [PS96; DS10a].

Theorem 5.18. *Under the hypotheses of Theorem 1.2, let $g: \mathbb{P}^k \rightarrow \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite and $\langle \mu_\phi, g \rangle = 0$. Assume that g is not a coboundary. Then g satisfies the LDP.*

Proof. We apply Theorem A.11 for the random variables $\mathcal{S}_n := S_n(g)$ (see (5.3)) with respect to the probability μ_ϕ , and for $\Lambda(\theta) := \log \alpha(\theta)$, where $\alpha(\theta)$ is defined in Proposition 5.8. So, $\Lambda(\theta)$ is analytic and well defined for θ small enough. Since g is not a coboundary, we have $\sigma > 0$ and

the convexity of Λ follows from the expansion of $\alpha(\theta)$ given in Lemma 5.9. Finally, as in (5.5), using θ instead of $1/n$, we get

$$\langle \mu_\phi, e^{\theta S_n(g)} \rangle = \alpha(\theta)^n \langle m_\phi, \Phi_\theta(\rho) \rangle + \langle m_\phi, \Psi_\theta^n(\rho) \rangle$$

which implies (note that $\langle m_\phi, \Phi_\theta(\rho) \rangle$ does not vanish for θ small as it is equal to 1 when $\theta = 0$)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \langle \mu_\phi, e^{\theta S_n(g)} \rangle = \log \alpha(\theta) = \Lambda(\theta).$$

Thus, Condition (A.5) in Theorem A.11 is satisfied. Applying that theorem gives the result. \square

APPENDIX A. ABSTRACT STATISTICAL THEOREMS

We provide here the precise definitions for the statistical properties studied in Section 5, as well as the criteria, used in the previous section, ensuring their validity in an abstract setting. We will consider in what follows a dynamical system $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$, where \mathcal{F} is a given σ -algebra, and ν a probability measure which is invariant with respect to T . Given any observable (real-valued measurable function) $g \in L^1(\nu)$ and $n \geq 1$ we denote by $S_n(g)$ the Birkhoff sum $S_n(g) := \sum_{j=0}^{n-1} g \circ T^j$. We are interested in comparing the behaviour of the sequence $S_n(g)$ with that of a sum of independent identically distributed random variables Z_n of mean $\langle \nu, g \rangle$. Notice that the invariance of ν precisely implies that the functions $g \circ T^j$, considered as random variables, are identically distributed. The goal is to prove that, under suitable assumptions, the sequence of *weakly dependent* random variables $g \circ T^j$ and their sums $S_n(g)$ enjoy many of the statistical properties of the sequences Z_n and $\sum_{j=0}^{n-1} Z_j$. For simplicity, we only consider observables g of zero mean, i.e., such that $\langle \nu, g \rangle = 0$. It is easy to extend the discussion to the case of non-zero mean.

In what follows, we will denote by $\mathbb{E}(h|\mathcal{G})$ the conditional expectation of an \mathcal{F} -measurable function h with respect to a σ -algebra $\mathcal{G} \subseteq \mathcal{F}$. The σ -algebra $T^{-n}\mathcal{F}$ is generated by the sets of the form $T^{-n}(B)$ with B in \mathcal{F} . We will say that h is a *coboundary* if there exists $\tilde{h} \in L^2(\nu)$ such that $h = \tilde{h} \circ T - \tilde{h}$. A coboundary belongs to $L^2(\nu)$ and has zero mean.

Definition A.1 (CLT - Central Limit Theorem). Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g \rangle = 0$. We say that g satisfies the Central Limit Theorem (CLT) with variance $\sigma > 0$ if for any interval $I \subset \mathbb{R}$ we have

$$(A.1) \quad \lim_{n \rightarrow \infty} \nu \left\{ \frac{1}{\sqrt{n}} S_n(g) \in I \right\} = \frac{1}{\sqrt{2\pi\sigma}} \int_I e^{-\frac{t^2}{2\sigma^2}} dt.$$

We give two criteria to ensure that an observable g satisfies the CLT. The first one, due to Gordin (see also [Liv96]), is more classical and easier to use.

Theorem A.2 (Gordin [Gor69]). Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued function in $L^2(\nu)$ which is not a coboundary. Assume that

$$(A.2) \quad \sum_{n \geq 1} \|\mathbb{E}(g|T^{-n}\mathcal{F})\|_{L^2(\nu)}^2 < \infty.$$

Then g satisfies the CLT with variance $\sigma > 0$ given by

$$(A.3) \quad \sigma^2 := \langle \nu, g^2 \rangle + 2 \sum_{n \geq 1} \langle \nu, g \cdot (g \circ T^n) \rangle.$$

Note that under the hypothesis (A.2), the observable g is not a coboundary if and only if the constant σ defined in (A.3) is non-zero. Note also that this hypothesis implies that g has zero mean because each term in the infinite sum in (A.2) is larger than or equal to $|\langle \nu, g \rangle|^2$ thanks to Cauchy-Schwarz's inequality and the invariance of ν .

The second criterion is based on the theory of perturbed operators, which also allows to study the speed of convergence in (A.1), i.e., to get a version of Berry-Esseen theorem. Recall that the

spectrum of a linear operator \mathcal{T} , from a complex Banach space to itself, is the closure of the set of $z \in \mathbb{C}$ such that the operator $zI - \mathcal{T}$ is not invertible. The essential spectrum is obtained from the spectrum by removing its isolated points corresponding to eigenvalues of finite multiplicity. The spectral radius $r(\mathcal{T})$ and the essential spectral radius $r_{\text{ess}}(\mathcal{T})$ are the radii of the smallest disks centered at the origin which contain the spectrum and the essential spectrum, respectively. The following criterion, based on ideas on Nagaev and Guivarc'h [Nag57; GH88], is a translation in our setting of [Gou15, Th. 3.7].

Theorem A.3 (Nagaev-Guivarc'h-Gouëzel). *Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function. Assume that there exist a complex Banach space \mathcal{H} , a family of operators \mathcal{T}_t acting on \mathcal{H} , for $t \in [-\delta, \delta]$ with some constant $\delta > 0$, and elements $\xi \in \mathcal{H}$ and $\nu^* \in \mathcal{H}^*$ such that*

- (i) *coding: for all $n \in \mathbb{N}$ and all $t \in [-\delta, \delta]$, we have $\langle \nu, e^{itS_n(g)} \rangle = \langle \nu^*, \mathcal{T}_t^n \xi \rangle$;*
- (ii) *spectral description: we have $r_{\text{ess}}(\mathcal{T}_0) < 1$, and \mathcal{T}_0 has a unique eigenvalue of modulus ≥ 1 ; moreover, this eigenvalue has multiplicity 1 and is located at 1;*
- (iii) *regularity: the family $t \mapsto \mathcal{T}_t$ is of class C^3 .*

Assume also that the largest eigenvalue $\alpha(t)$ of \mathcal{T}_t for t near 0 admits the order 2 expansion $\alpha(t) = e^{iAt - \sigma^2 t^2/2 + o(t^2)}$ for some $A, \sigma \in \mathbb{R}$ with $\sigma > 0$. Then $g - A$ satisfies the CLT with variance σ and the Berry-Esseen theorem holds, i.e., the speed of convergence in (A.1) is of order $O(n^{-1/2})$.

The following is an improvement of the CLT, see also Remark 5.12.

Definition A.4 (LCLT - Local Central Limit Theorem). *Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system, and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g \rangle = 0$. We say that g satisfies the Local Central Limit Theorem (LCLT) with variance $\sigma > 0$ if for every bounded interval $I \subset \mathbb{R}$ the convergence*

$$\lim_{n \rightarrow \infty} \left| \sigma \sqrt{n} \nu \{x + S_n(g) \in I\} - \frac{1}{\sqrt{2\pi}} e^{-x^2/(2\sigma^2 n)} |I| \right| = 0$$

holds uniformly in $x \in \mathbb{R}$. Here $|I|$ denotes the length of I .

Partial sums of independent and identically distributed random variables can be compared with Brownian motions of suitable variance, and one can get precise estimates for the fluctuations around the mean value. In the context of dynamical systems, we have the following analogous definition.

Definition A.5 (ASIP - Almost Sure Invariant Principle). *Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g \rangle = 0$. We say that g satisfies the Almost Sure Invariance Principle (ASIP) of variance $\sigma > 0$ and error rate n^γ if there exist, on some probability space \mathcal{X} , a sequence of random variables $(\mathcal{S}_n)_{n \geq 0}$ and a Brownian motion \mathcal{W} of variance σ such that*

- (i) $\mathcal{S}_n = \mathcal{W}(n) + o(n^\gamma)$ almost everywhere on \mathcal{X} ;
- (ii) $\mathcal{S}_n(g)$ and \mathcal{S}_n have the same distribution for any $n \geq 0$.

General criteria that allow one to establish the ASIP in various contexts are given in [PS75]. We state here a criterion by Gouëzel, see [Gou10, Th. 1.2] and [Gou15, Th. 5.2]. Notice that Gouëzel's result actually holds in the more general case of random variables with values in \mathbb{R}^d .

Theorem A.6 (Gouëzel). *Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function. Assume that there exist a complex Banach space \mathcal{H} , a family of operators \mathcal{T}_t acting on \mathcal{H} , for $t \in [-\delta, \delta]$ with some constant $\delta > 0$, and elements $\xi \in \mathcal{H}$ and $\nu^* \in \mathcal{H}^*$ such that*

- (i) *strong coding: for all $n \in \mathbb{N}$ and all $t_0, \dots, t_{n-1} \in [-\delta, \delta]$, we have*

$$\langle \nu, e^{i \sum_{j=0}^{n-1} t_j g \circ T^j} \rangle = \langle \nu^*, \mathcal{T}_{t_{n-1}} \circ \dots \circ \mathcal{T}_{t_0} \xi \rangle;$$

- (ii) *spectral description*: we have $r_{\text{ess}}(\mathcal{T}_0) < 1$, and \mathcal{T}_0 has a unique eigenvalue of modulus ≥ 1 ; moreover, this eigenvalue has multiplicity 1 and is located at 1;
- (iii) *weak regularity*: either $t \mapsto \mathcal{T}_t$ is continuous, or $\|\mathcal{T}_t^n\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq c$ for some constant $c > 0$ and for all $t \in [-\delta, \delta]$ and $n \geq 0$;
- (iv) *there exist* $p > 2$ and $c > 0$ such that $\|g \circ T^n\|_{L^p(\nu)} \leq c$ for all $n \geq 0$.

Assume also that the largest eigenvalue $\alpha(t)$ of \mathcal{T}_t for t near 0 admits the order 2 expansion $\alpha(t) = e^{iAt - \sigma^2 t^2/2 + o(t^2)}$ for some $A, \sigma \in \mathbb{R}$ with $\sigma > 0$. Then $g - A$ satisfies the ASIP of variance σ and error rate $o(n^q)$ for all q such that

$$q > \frac{p}{4p-4} = \frac{1}{4} + \frac{1}{4p-4}.$$

The following properties are general consequences of the ASIP, see for instance [PS75; LP89; CG07].

Definition A.7 (LIL - Law of iterated logarithms). Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function such that $\langle \nu, g \rangle = 0$. We say that g satisfies the Law of Iterated Logarithms (LIL) with variance $\sigma > 0$ if

$$\limsup_{n \rightarrow \infty} \frac{S_n(g)}{\sigma \sqrt{2n \log \log n}} = 1 \quad \nu\text{-almost everywhere.}$$

Definition A.8 (ASCLT - Almost sure Central Limit Theorem). Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function such that $\langle \nu, g \rangle = 0$. We say that g satisfies the Almost Sure Central Limit Theorem (ASCLT) with variance $\sigma > 0$ if, for ν -almost every point $x \in X$,

$$\frac{1}{\log n} \sum_{j=1}^n \frac{1}{j} \delta_{j^{-1/2} S_j(g)(x)} \rightarrow \mathcal{N}(0, \sigma).$$

In particular, ν -almost surely

$$\frac{1}{\log n} \sum_{j=1}^n \frac{1}{j} \mathbb{1}_{\{j^{-1/2} S_j(g) \leq t_0\}} \rightarrow \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t_0} e^{-\frac{t^2}{2\sigma^2}} dt$$

for all $t_0 \in \mathbb{R}$.

Theorem A.9. Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function with $\langle \nu, g \rangle = 0$. Assume that g satisfies the ASIP with variance $\sigma > 0$ and error rate n^γ for some $\gamma < 1/2$. Then g satisfies the LIL and the ASCLT, both with the same variance σ .

The last property that we recall is the Large Deviation Principle. It gives very precise estimates on the measure of the set where the partial sums are far from the mean value. It implies in particular the Large Deviation Theorem, which only requires an upper bound for the measure in (A.4) below.

Definition A.10 (LDP - Large Deviation Principle). Let $T: (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$ be a dynamical system and ν an invariant probability measure. Let g be a real-valued integrable function such that $\langle \nu, g \rangle = 0$. We say that g satisfies the Large Deviation Principle (LDP) if there exists a non-negative, strictly convex function c which is defined on a neighbourhood of $0 \in \mathbb{R}$, vanishes only at 0, and such that, for all $\epsilon > 0$ sufficiently small,

$$(A.4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \nu \left\{ x \in X : \frac{S_n(g)(x)}{n} > \epsilon \right\} = -c(\epsilon).$$

The following result, established in [BL85], is a local version of Gärtner-Ellis Theorem [Gär77; Ell84; DZ98] which can be used to prove the LDP, see [HH01, Lem. XIII.2].

Theorem A.11 (Bougerol-Lacroix). *For all $n \geq 1$ denote by \mathbb{P}_n a probability distribution, by \mathbb{E}_n the corresponding expectation operator and by \mathcal{S}_n a real-valued random variable. Assume that on some interval $[-\theta_0, \theta_0]$, with $\theta_0 > 0$, we have*

$$(A.5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_n(e^{\theta \mathcal{S}_n}) = \Lambda(\theta),$$

where Λ is a \mathcal{C}^1 , strictly convex function satisfying $\Lambda'(0) = 0$. Then, for all $0 < \epsilon < \Lambda(\theta_0)/\theta_0$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_n(\mathcal{S}_n > n\epsilon) = -c(\epsilon) < 0,$$

where $c(\epsilon) := \sup\{\theta\epsilon - \Lambda(\theta) : |\theta| \leq \theta_0\}$ is the Legendre transform of Λ , which is non-negative, strictly convex, and vanishes only at 0.

Notice that the symmetric statement for the measure of the set where $n^{-1}\mathcal{S}_n(g) < -\epsilon$ in (A.4) can be obtained by applying the result above to the sequence of random variables $-\mathcal{S}_n$.

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