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Scoring Rules, Ballot Truncation, and the Truncation Paradox

Eric Kamwa*

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Abstract

A voting rule that permits some voters to favor a candidate by providing only the initial segment of their sincere rankings is said to be vulnerable to the truncation paradox. In this paper, we consider four models for counting truncated ballots, optimistic, pessimistic (the most common), averaged, and round-down. Under the impartial anonymous culture assumption, the choice of model generally has a real impact on truncation-paradox vulnerability, but there are exceptions. When the election is decided by a one-shot scoring rule, the optimistic model is invulnerable to the truncation paradox, but all other models are vulnerable. We identify new voting rules immune to the truncation paradox, such as the *Modified Borda Count*. To obtain a more complete picture of the impact of processing model, we assess the likelihood of the truncation paradox in three-candidate elections with large electorates, focusing not only on one-shot scoring rules but also scoring rules with one-by-one or below-average elimination. Our assessment confirms that the processing model for truncated ballots may really matter.

Keywords: Truncation, Rankings, Scoring model, Probability, Paradox, Impartial and Anonymous Culture.
JEL Classification Number: D71, D72

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1 Introduction

Since the pioneering work of Arrow (1963), Gibbard (1973) and Satterthwaite (1975), it has been known that there is no "best" voting rule. Unless it is dictatorial, any voting rule is manipulable, i.e. a voter or a group of voters may have a vested interest in changing their sincere preference for an insincere one in order to get a more favorable outcome. The choice of a voting rule should therefore not be made in ignorance of the strategic behaviors to which the rule may be vulnerable. One way to discriminate between voting rules is to look at their vulnerability, or the frequencies with which they are likely to lead to voting paradoxes.[1]

Alongside Gibbard-Satterthwaite-style strategic behavior, Brams (1982) and Fishburn and Brams (1984) described another strategic behavior in which voters try to manipulate "while maintaining their sincerity": *sincere truncation*. Many democratic organizations and societies have recourse to voting systems in which individuals vote by submitting a ranking of the options (candidates) offered up for assessment. Since producing a complete ranking on a large number of candidates is not always obvious or even possible, it is often permitted for individuals to produce partial (incomplete or truncated) rankings, even on a small number of candidates. When a voter provides a truncated ranking, one cannot be sure of the real reasons for this: either he is unable to produce a complete ranking, or he finds the unranked candidates undesirable, etc. According to Brams (1982) and Fishburn and Brams (1984), the possibility of producing truncated rankings could be used for strategic purposes: a voter or a group of voters can favor a preferred outcome by providing only a part of their sincere rankings on the competing candidates, rather than listing their (entire) sincere preference rankings on all the competing candidates. This is known as the *sincere truncation paradox* or simply, *the truncation paradox*. The work of Brams (1982), Felsenthal (2012), Fishburn and Brams (1983, 1984) and Nurmi (1999), among others, shows that almost all well-known voting rules are vulnerable to the truncation paradox; among these rules are the scoring rules, the iterative scoring rules and Condorcet consistent rules.[2] The few exceptions are the *Plurality rule*, *Plurality runoff* and *Approval voting*.

In this paper, we want to emphasize that the manipulability of voting rules by sincere truncation may depend on how incomplete (sincere) preferences are treated. If this turns out to be the case, states or organizations that, in their collective decision-making process, resort to voting rules vulnerable to the paradox of truncation should frankly question the "best" way to deal with incomplete preferences. Throughout the paper, we will assume that voters are aware, prior to voting, of how the truncated ballots will be processed. This assumption is consistent with what is done in real life.

Before going further, let us note that when the truncation paradox was introduced by Brams (1982) and Fishburn and Brams (1984), explicit allusion was made to the fact that when a voter truncates, only the candidate(s) mentioned on the ballot will receive points

[1] For an overview of the vast literature devoted to the evaluation of the probabilities of voting paradoxes, the reader may refer to the recent books by Gehrlein and Lepelley (2011, 2017) and Diss and Merlin (2021).

[2] A Condorcet consistent rule is a voting system that always elects the Condorcet winner when he exists. A Condorcet winner is a candidate who defeats each of the other candidates in pairwise comparisons.

from that voter while the others will receive nothing. This way of managing incomplete preferences is known in the literature as the *pessimistic model* (see Baumeister et al., 2012).[3] The pessimistic model is used for political elections in Slovenia and in Kiribati where the voting rule is the Borda rule.[4] Although the pessimistic model seems to be the most prevalent both in the literature and in practice, other ways of dealing with truncated preferences exist, including the *optimistic model* (see Baumeister et al., 2012, Saari, 2008), the *averaged model* (see Dummett, 1997) and the *round-down model* (see Narodytska and Walsh, 2014).[5] Under the optimistic model, if out of m candidates in the running, a voter only ranks k of them, each of the $m-k$ unranked candidates will be awarded the points that would have been associated with the $k+1$th position of the voter's sincere ranking. The optimistic model is used for the election of the leader of the Irish Green Party. Under the averaged model, each non-ranked candidate is awarded a number of points equal to the average of the total points that all non-ranked candidates would have received if they had been ranked. Under the round-down model, if a voter ranks k candidates out of m, a candidate ranked jth ($j \leq k$) will receive from this voter a number of points associated with position $m-(k-j)-1$, while unranked candidates will receive a number of points associated with position m. Notice that an instance of the round-down model is the Modified Borda Count (MBC). This rule is inspired by the Borda rule and was introduced by Emerson (2007, 2013). MBC allows voters to express strict preferences that may be either complete or incomplete. With MBC, when m candidates are in contention, if a voter ranks exactly k candidates ($1 \leq k \leq m$), the candidate ranked qth will receive $k-q+1$ points from that voter; the winner will be the candidate with the highest total number of points. It is obvious that if $k=m$, we have the classical Borda rule.

According to Baumeister et al. (2012), the drawback of the pessimistic model is that it gives incentives for voters to rank only a single candidate, so that the impact of the vote on the score of this candidate, relative to the scores of other candidates, is greatest; on the other hand, the optimistic model rewards the voters who rank more candidates: the more candidates one ranks, the more points (in relative terms) these candidates receive.

When considering the pessimistic model, the vulnerability of most voting rules is clearly established, as noted above. What do we have with regards to the optimistic, the round-down and the averaged models? As far as we know, not much. To offer a contribution in this sense, we focus our attention on the family of one-shot scoring rules and on that of iterative scoring rules. We will show that: all these voting rules are vulnerable to the truncation paradox under the averaged model; if the optimistic model is used to deal with truncated preferences, all the one-round scoring rules are immune to the truncation paradox while this is not the case for the iterative scoring rules. We will also show that the *Modified*

[3]The pessimistic model is called the "*round up model*" in Narodytska and Walsh (2014).

[4]With m competing candidates, the Borda rule is the scoring rule which gives $m-k$ points to a candidate each time he is ranked kth by a voter; the winner is the candidate with the greatest total number of points. In the Slovenian national elections, this rule is used for the reserved legislative seats for Hungarian and Italian ethnic minorities.

[5]For an overview of schemes for dealing with incomplete preferences in collective decision-making, the reader may refer among others to Baumeister et al. (2012), Kruger and Terzopoulou (2020), Menon and Larson (2017), Narodytska and Walsh (2014) and Terzopoulou and Endriss (2021, 2019).

Borda Count, popularized by Emerson (2007, 2013), is among the voting rules immune to the truncation paradox.

Only a few papers have tried to assess the likelihood of the truncation paradox; they all consider only the pessimistic model. Using simulations based on the spatial model for three-candidate voting situations with an electorate size varying from ten to a million, Plassmann and Tideman (2014) evaluated the likelihood of the *strong* truncation paradox[6] by focusing, among other things, on certain scoring rules and iterative scoring rules. They found that the likelihood of the strong truncation paradox tends to decrease as the number of voters increases. For their part, Kilgour et al. (2019) assessed the significance of ballot truncation in ranked-choice elections with four, five, and six candidates using intensive simulations on real data under both spatial and random models of voter preferences. In a more recent paper, Kamwa and Moyouwou (2021) characterized for three-candidate elections and large electorates, all the voting situations where the truncation paradox can occur for the whole family of one-shot scoring rules and scoring runoff rules; for these families of rules, they computed the likelihood of the truncation paradox under the impartial and anonymous culture assumption (defined later). The present contribution stands out in that our characterization results are not limited to the pessimistic model as in Kamwa and Moyouwou (2021), Kilgour et al. (2019), Plassmann and Tideman (2014); we include the other models and our results are valid regardless of the number of candidates. Moreover, our probabilistic results for three-candidate elections provide, under each of the models, exact formulas for the limiting probabilities of the truncation paradox.

Using the tools of computational social choice,[7] Baumeister et al. (2012), Menon and Larson (2017) and Narodytska and Walsh (2014) tried to evaluate the feasibility and the complexity of manipulation by truncation for certain voting systems included in the family of scoring rules and scoring runoff rules. They all tend to conclude that the computational complexity of computing a strategic vote may vary depending on how truncated ballots are processed. Their framework of analysis is quite different from ours since we are not interested in the complexity or the computational aspects of truncation.

This paper demonstrates that how truncated ballots are counted may make the truncation paradox more or less likely. Moreover, we think that using the pessimistic model for dealing with sincere truncated preferences leaves ample possibilities for strategic behavior. Therefore, it may come as no surprise that for a given voting rule, the probability of the truncation paradox is higher with the pessimistic model than with the optimistic, the averaged or the round-down models. But in what proportion? Of what order of magnitude are these differences in probability? In attempting to provide answers to these questions, we consider three-candidate elections with large electorates, and for each of the models for dealing with incomplete rankings we provide a characterization of the voting situations where the truncation paradox can occur; we consider the whole family of scoring rules and the

[6]The strong truncation paradox occurs if one voter reports only part of his ranking, then a candidate will win whom the voter ranks higher than the candidate who would win if the voter reported his complete ranking of the candidates.

[7]Computational social choice is at the intersection of social choice theory, computer science, and the problems of multi-agent systems. It analyzes preference aggregation problems from a computational perspective.

whole family of two types of iterative scoring rules. Then, we compute the likelihood of the truncation paradox under the impartial and anonymous culture.

The rest of the paper is organized as follows: Section 2 is devoted to basic definitions. In Section 3, we derive some general results on the behavior of scoring rules and iterative scoring rules with regard to the sincere truncation under the pessimistic, the round-down and the averaged models. Section 4 is devoted to specific results for three-candidate elections: given an infinite number of voters having strict rankings, we characterize all the voting situations where the truncation paradox can occur; then we provide our computation results on the likelihood of the truncation paradox for all the scoring rules and iterative scoring rules under each of the models. Section 5 concludes. The proofs of the propositions and the probability representations are provided in the online supplementary material.

2 Notation and definitions

2.1 Preferences

Let N be a set of n voters ($n \geq 2$) and $A = \{x, y, \ldots\}$ a set of $m \geq 3$ candidates. Individual preferences are linear orders, and these are complete, asymmetric, and transitive binary relations on A. We assume that voters sincerely know their strict rankings on the candidates in A. With m candidates, there are exactly $m!$ linear orders $P_1, P_2, \ldots, P_{m!}$ on A. For $x, y, z \in A$, we simply write $\begin{matrix} x \\ y \\ z \end{matrix}$ or xyz to denote the linear order on A according to which x is strictly preferred to y, y is strictly preferred to z, and by transitivity, x is strictly preferred to z. A *voting situation* is an $m!$-tuple $\pi = (n_1, n_2, ..., n_t, ..., n_{m!})$ that indicates the total number n_t of voters casting the complete linear order of type t such that $\sum_{t=1}^{m!} n_t = n$.

For the particular case of three-candidate elections, Table 1 describes a voting situation on $A = \{x, y, z\}$.

Table 1: Strict rankings on $A = \{x, y, z\}$

	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6
Nb. of voters →	n_1	n_2	n_3	n_4	n_5	n_6
rankings	x	x	y	y	z	z
	y	z	x	z	x	y
	z	y	z	x	y	x

Given $x, y \in A$ and a voting situation π, we denote by $n_{xy}(\pi)$ (or simply n_{xy}) the total number of voters who strictly prefer x to y. If $n_{xy} > n_{yx}$, we say that candidate x majority

dominates candidate y; or equivalently x beats y in a pairwise majority voting. In such a case, we will simply write $x\mathbf{M}y$.

2.2 Voting rules

In this paper we focus on the whole family of one-shot scoring rules and on that of scoring rules with eliminations. Let us define all these rules.

2.2.1 One-shot scoring rules

Scoring rules are voting systems that give points to candidates according to the position they have in voters' rankings. For a given scoring rule, the total number of points received by a candidate defines his score for this rule. The winner is the candidate with the highest score. In general, with m candidates and complete strict rankings, a scoring vector is an m-tuple $w = (w_1, w_2, \ldots, w_{m-1}, w_m)$ of real numbers such that $w_1 \geq w_2 \geq \ldots \geq w_{m-1} \geq w_m$ with $w_1 > w_m$. In the sequel, we will assume that $w_m = 0$. Given π and w, each candidate receives w_k each time she is ranked k^{th} $(k = 1, 2, \ldots, m)$ by a voter. The score of a candidate $x \in A$ is the sum $S(\pi, w, x) = \sum_{t=1}^{m!} n_t w_{r(t,x)}$ where $r(t, x)$ is the rank of candidate x according to voters of type t.

In three-candidate elections, one can characterize the whole family of scoring rules by using the normalized scoring vector $w_\lambda = (1, \lambda, 0)$ with $0 \leq \lambda \leq 1$. For $\lambda = 0$, we obtain the *Plurality rule*, also called the First-Past-The-Post. For $\lambda = 1$, we have the *Antiplurality rule* and for $\lambda = \frac{1}{2}$, we have the *Borda rule*. For $\lambda = \frac{1}{4}$, we get the *Dowdall rule*, a variant of the Borda rule which is in use in the island nation of Nauru. From now on, we will denote by $S(\pi, \lambda, x)$ or simply $S(\pi, x)$, the score of candidate x when the scoring vector is $w_\lambda = (1, \lambda, 0)$ and the voting situation is π; without loss of generality, F_λ will be used to refer to a one-shot scoring rule associated with λ in a three-candidate election.

2.2.2 Scoring rules with eliminations

Scoring rules with eliminations proceed via steps in which one or more candidates are eliminated at each step. We consider two main families of such rules:

- ***Scoring runoff rules with one-by-one eliminations***: at each step, the alternative with the lowest score is eliminated. This family includes voting rules such as the *Plurality elimination rule* (also *called Instant Runoff Voting*), *Negative plurality elimination rule* and the *Borda elimination rule* also known as the Baldwin rule (Baldwin, 1926).
- ***Scoring runoff rules with eliminations according to the average***: at each step, any candidate who obtains (strictly) less than the average of the scores is eliminated.

There are at least two and at most $m-1$ rounds in the elimination process; thus, rules are needed for the second and subsequent rounds, when the number of not-yet-eliminated candidates is more than 1 but fewer than m. Sequences of scoring vectors can be used to characterize scoring runoff rules. Let us denote these sequences by $W = (w^2, w^3, \ldots, w^{p-1}, w^p)$ where w^p is the scoring vector used when there are p $(2 \leq p \leq m)$ candidates in contention; so, $w^m = w$ while $w^2 = (1, 0)$ represents a majority duel.

For the specific case of three-candidate elections, runoff systems with one-by-one eliminations involve two rounds of voting such that in the first round, the candidate with the smallest score is eliminated; and in the second round, a majority contest determines who is the winner. Given $w_\lambda = (1, \lambda, 0)$, if the runoff system is associated with $\lambda = 0$, we get the *Plurality elimination rule*; with $\lambda = 1$ we have the *Negative plurality elimination rule*; and for $\lambda = \frac{1}{2}$ we have the *Borda elimination rule*. In the sequel, we will use to F'_λ to denote a scoring runoff rule with one-by-one eliminations associated with λ in a three-candidate election.

In three-candidate elections, the following scenarios are viable when eliminations proceed according to the average: (a) the ballot could stop in the first round if a single candidate has obtained more than the average of the scores; or (b) two candidates score more than the average and a majority contest in the second round determines the winner. If the iterative scoring rule in this class is associated with $\lambda = 1$, this defines the *Kim-Roush voting rule* (Kim and Roush, 1996), while we get the *Nanson rule* (Nanson, 1883) for $\lambda = \frac{1}{2}$. We will use $\tilde{F}'_\lambda$ to refer to a scoring runoff rule with eliminations according to the average associated with λ in a three-candidate election.

3 Dealing with truncated preferences and the truncation paradox

As stated in Section 1, the literature suggests several ways to proceed when voters submit truncated rankings. These methods include among others the pessimistic model, the optimistic model, the averaged model and the round-down model. Now let us introduce each of these models in a formal way.

Consider N the set of voters, $A = \{x, y, \ldots\}$ the set of $m \geq 3$ candidates and $w = (w_1, w_2, \ldots, w_{m-1}, w_m)$ the scoring vector. Recall that, given w, a candidate receives w_j points each times he is ranked jth by a voter. Assume a voter who only ranks k candidates out of the m in the running; this truncated ballot will be processed in the following manner depending on the model considered:

Truncation Rule	Candidate ranked jth	All unranked candidates
Pessimistic	w_j	w_m
Optimistic	w_j	w_{k+1}
Averaged	w_j	$\frac{1}{m-k}\sum_{h=k+1}^{m} w_h$
Round-Down	$w_{m-k+j-1}$	w_m

In this paper we assume that voters are truly engaging in strategic voting behavior by not revealing their true preferences. It is important to draw the attention that in some voting situations, voters would not achieve the same outcome by sincerely rating their preferred candidate higher and strategically rating a competitor to their preferred candidate lower. More, withholding their complete ranking can be the best strategy to achieve their objective of manipulation. To illustrate this, let us consider the voting situation in Example 1.

Example 1. *Consider the following voting situation with 3 candidates $\{x, y, z\}$ and 14 voters. Let the voting rule be the Borda rule and incomplete preferences be treated according to the pessimistic model.*

$2 : xyz$	$2 : xzy$	$4 : yxz$	$1 : yzx$	$2 : zxy$	$3 : zyx$

With this profile where voters have given their full rankings, candidate y is the winner since he gets 7.5 points while x and z respectively get 7 and 6.5 points. Assume that the voters with the ranking xyz try to manipulate. To do so, they can change their preference to xzy (lowering y in their preferences and increasing z): as a result, z (their worst candidate) becomes the winner; this strategy is therefore not rationally feasible. Any other action (including even abstention) that would remove x from the top would also be ineffective. However, if these voters simply choose to truncate, i.e. they just indicate $x - -$, then candidate x becomes the winner. Truncation is therefore the best strategy for these voters to manipulate.[8]

Based on Example 1, there may therefore be voting situations in which the only way for voters to manipulate effectively is through truncation. Thus, voters are really engaging in strategic voting behavior by withholding their complete ranking. Assume in Example 1 that we add 3 voters with xyz and that the election involves a runoff. Candidate x wins the first round election with 10 points while y receives 9 points and z gets 6.5 points. If the voters with the ranking xyz switch y and z, candidate y drops out after the first round while candidates x and z contest in the runoff election. In such a scenario where voters are interested in who their preferred candidate faces in a runoff election, truncation may not be the best strategy.

As stated before, when the pessimistic model is used, almost all the one-shot scoring rules (except the Plurality rule) and all the iterative scoring rules (except the Plurality elimination rule) are vulnerable to the truncation paradox. Our goal in this section is to

[8]The same conclusion can be reached with similar examples (even with more candidates) for each of our voting rules and models of truncated preferences. We can provide a Maple spreadsheet to obtain such voting situations given a voting rule, a number of voters and a model of preference truncation.

provide a comprehensive overview of the vulnerability of scoring rules and iterative rules to the truncation paradox. Propositions 1 to 3 give us a picture of what is going on underneath the other models. The proofs of these propositions are provided in the online supplementary material.

Proposition 1. *If there are at least three candidates and two voters, then under optimistic truncation all one-shot scoring rules are immune to the truncation paradox, whereas all iterative scoring rules except Plurality elimination are vulnerable.*

Proposition 2. *If there are at least three candidates and two voters, then under averaged truncation all one-shot scoring rules (except the Plurality rule) and all iterative scoring rules(except the Plurality elimination) are vulnerable to the truncation paradox.*

Proposition 3. *Consider a voting situation with at least three candidates and two voters. Given the scoring vector $w = (w_1, w_2, \ldots, w_{m-1}, w_m)$, assume that some voters who rank the winner at position l $(2 \leq l \leq m-1)$ in their sincere ranking truncate and now report only k candidates $(1 \leq k \leq l-1)$ in order to favor a more preferred candidate ranked jth $(1 \leq j \leq k)$. When the round-down model is used, it is always possible to find a profile such that this manipulation is successful under all the iterative scoring rules; under one-shot scoring rules, the manipulation is possible only if $w_j - w_{m-k+j-1} < w_l$.*

Since the Plurality rule and the Plurality elimination rule only take into account the first places, it is obvious that they are not manipulable under the round-down model. In Section 1, we noticed that the Modified Borda Count (MBC) is an instance of the round-down model. Proposition 4 comes as a corollary of Proposition 3 as it tells us that MBC can be added to the short list of voting rules not manipulable by sincere truncation.

Proposition 4. *For all voting situation with at least three candidates and two voters, the Modified Borda Count is immune to the sincere truncation paradox.*

It comes from Propositions 1 to 4 that depending on voting rules, the choice of how to deal with incomplete preferences can be crucial. Thus, this choice should not be neglected. To support our argument, let us now extend the analysis to the calculation of the frequencies of appearance of the truncation paradox. To do so, we will focus, in Section 4, on the specific case of elections with three candidates.

4 Specific results for three-candidate elections

In our setting, voters sincerely provide complete strict rankings on the competing candidates. With three candidates, when a voter of a given type truncates, he just states his top-ranked candidate and erases the others who are assumed to be less preferred. Then, the normalized vector $w_\lambda = (1, \lambda, 0)$ has to be modified for truncated rankings according to the four models. Assume that a voter with the ranking xyz truncates and submits $x - -$.

- with the pessimistic model, candidate x will receive 1 point in the new voting situation while y and z both receive zero points. Thus, for three-candidate elections, the scoring vector applied to truncated rankings is $w'_\lambda = (1, 0, 0)$.
- with the optimistic model, candidate x will still receive 1 point while y and z both will receive λ points. Here, the scoring vector applied to truncated rankings is $w'_\lambda = (1, \lambda, \lambda)$.
- with the averaged model, candidate x will still receive 1 point while y and z will both receive $\frac{\lambda}{2}$ points. So, the scoring vector applied to truncated rankings is $w'_\lambda = (1, \frac{\lambda}{2}, \frac{\lambda}{2})$.
- with the round-down model, candidate x will receive λ points while y and z will both receive no points. So, the scoring vector applied to truncated rankings is $w'_\lambda = (\lambda, 0, 0)$.

Remark 1. *With three candidates, it can easily been seen that if $\lambda = 1$, the vector w'_1 is the same under both the pessimistic and the round-down models; so, for $\lambda = 1$, these two models are equivalent. Also, the pessimistic, the averaged and the optimistic models are equivalent when $\lambda = 0$. If $\lambda = \frac{1}{2}$, under the optimistic model, we get $w'_{\frac{1}{2}} = (1, \frac{1}{2}, \frac{1}{2})$ which is equivalent to $(\frac{1}{2}, 0, 0)$: thus, for $\lambda = \frac{1}{2}$, the optimistic and the round-down models are equivalent. The averaged and the round-down models are equivalent when $\lambda = \frac{2}{3}$.*

Notice that in three-candidate elections, when some voters truncate, *(i)* under the pessimistic model: only the scores of candidates ranked second by some of these voters are affected and diminish; *(ii)* under the optimistic model: only the scores of candidates ranked last by some of these voters are affected and increase; *(iii)* under the averaged model: the scores of candidates ranked second by some of these voters diminish while those of candidates ranked last by some of these voters increase; *(iv)* under the round-down model: only the last ranked candidate is not affected while the two other candidates record a decrease in their scores.

In our framework, we assume that truncation is only possible at the first round of runoff systems. Given π, we will denote by $\pi([xyz])$ the voting situation obtained from π when all voters of type 1 with the ranking xyz truncate their preferences.

4.1 Characterization results

Throughout this section, we will stipulate our results by considering a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{x, y, z\}$ and the scoring vector $w_\lambda = (1, \lambda, 0)$. The proof of each of the coming propositions is given in the online supplementary material.

For one-shot scoring rules, Proposition 5 identifies all the voting situations in which the truncation paradox is possible under each of the models.

Proposition 5. *Given the one-shot scoring rule associated with $0 < \lambda \leq 1$ and a pair $\{x, y\}$ of candidates with $A \setminus \{x, y\} = \{z\}$,*

i) if x is the winner at π, the truncation paradox can occur at π in favor of y under the pessimistic model or the averaged model if and only if y is the winner at $\pi([yxz, yzx])$.[9]

ii) all the one-shot scoring rules are immune to sincere truncation when the optimistic model is used.

iii) under the round-down model, all the one-shot scoring rules such that $\lambda \in [0\ \frac{1}{2}]$ are immune to the truncation paradox while they are not for $\lambda \in]\frac{1}{2}\ 1]$.

Given a scoring runoff rule with one-by-one eliminations associated with the scoring vector $w_\lambda = (1, \lambda, 0)$, Propositions 6.A to 6.D characterize the voting situations where the truncation paradox can occur. In these propositions, it is assumed on $A = \{x, y, z\}$ at π that z is eliminated in the first round and x wins against y in the second round.

Proposition 6.A. *Under the pessimistic model and the averaged model, the truncation paradox can occur at π in favor of y if and only if at $\pi([yxz])$, x is eliminated in the first round and y wins the majority duel against z.*

Proposition 6.B. *Under the pessimistic model and the averaged model, the truncation paradox can occur at π in favor of z if and only if he wins the majority duel against y and x is the first-round loser at $\pi([zxy])$; or, if z wins the majority duel against x and y is the first-round loser at $\pi([zyx])$.*

Proposition 6.C. *Under the optimistic model, the truncation paradox can occur at π only in favor of y if and only if y wins the majority duel against z and x is the first-round loser at $\pi([yxz])$. Nonetheless, the truncation paradox cannot occur at π in favor of z.*

Proposition 6.D. *Under the round-down model, the truncation paradox can occur at π in favor of y if and only if y wins the majority duel against z and x is the first-round loser at $\pi([yxz])$. It can occur at π in favor of z if and only if $\lambda > \frac{1}{2}$ and x is the first-round loser at $\pi([zxy])$ or y is the first-round loser at $\pi([zyx])$.*

Propositions 7.A to 7.F provide the characterization for scoring runoff rules with eliminations below the average.

Proposition 7.A. *Assume that x wins in the first round (*i.e.*, only x's score is above the average), the truncation paradox cannot occur under both the optimistic and the averaged models.*

Proposition 7.B. *Assume that x wins in the first round (*i.e.*, only x's score is above the average), under the pessimistic model, the truncation paradox can occur in favor of one of the losers, let us say y: if at $\pi([yxz])$, only y now scores above the average, or if y wins the second round against x or z.*

Proposition 7.C. *Assume that x wins in the first round (*i.e.*, only x's score is above the average), under the round-down model, the truncation paradox may occur in favor of one of the losers, let us say y: if only if $\lambda > \frac{2}{3}$ and that at $\pi([yxz])$, y is now the only one to score above the average or he wins the second round against x or z.*

[9] This result is in line with the one obtained by Kamwa and Moyouwou (2021) for the pessimistic model.

Proposition 7.D. *Assume that x wins in the second round against y (*i.e.*, only z's score is below the average). Under both the optimistic and the averaged models, the truncation paradox is never possible in favor of z; it is possible at $\pi([yxz])$ in favor of y if after truncation, he is the only one to score above the average or if he wins the second round against z.*

Proposition 7.E. *Assume that x wins in the second round against y (*i.e.*, only z's score is below the average). Under the pessimistic model, the truncation paradox can occur in favor of y if and only if at $\pi([yxz])$, only y scores above the average or if he wins the second round against z; it may occur in favor of z if and only if at $\pi([zxy])$, z wins the second round against y.*

Proposition 7.F. *Assume that x wins in the second round against y (*i.e.*, only z's score is below the average). Under the round-down model, the truncation paradox may occur in favor of y if and only if $\lambda > \frac{1}{3}$ and that at $\pi([yxz])$, only y scores above the average; or, if he wins the second round against z; it may occur in favor of z if and only if $\lambda > \frac{1}{3}$ and that he wins the second round against x at $\pi([zxy])$ or against y at $\pi([zyx])$.*

With these characterization results, we can now turn to the evaluation of the probability of occurrence of the truncation paradox according to the processing model.

4.2 Computation results

The impartial and anonymous culture (IAC) assumption introduced by Kuga and Hiroaki (1974) and Gehrlein and Fishburn (1976) is one of the most widely used assumptions in social choice theory literature when computing the likelihood of voting events. Under IAC, the likelihood of a given event is calculated with respect to the ratio between the number of voting situations in which the event is likely over the total number of possible voting situations. It is known that the total number of possible voting situations in three-candidate elections is given by the following five-degree polynomial in n: $C^n_{n+3!-1} = \frac{(n+5)!}{n!5!}$. The number of voting situations associated with a given event can be reduced to the solutions of a finite system of linear constraints with rational coefficients. As recently pointed out, the appropriate mathematical tools to find these solutions are the Ehrhart polynomials (Gehrlein and Lepelley, 2011, 2017, Lepelley et al., 2008, Pritchard and Wilson, 2007). This technique has been widely used in numerous studies analyzing the probability of electoral events in three-candidate elections under the IAC assumption. As we only deal with large electorates, we follow a procedure that was developed in Cervone et al. (2005). This technique which is based on the computation of polytope volumes has recently been used in many research papers; see among others Diss and Gehrlein (2015, 2012), Diss et al. (2020, 2018, 2012), El Ouafdi et al. (2020), Gehrlein et al. (2015), Kamwa (2019), Kamwa and Moyouwou (2021), Kamwa and Valognes (2017), Moyouwou and Tchantcho (2015). In the online supplementary material we report Propositions 8 to 10 which give the probability representations that we obtain from this technique. Thus, in this section, we focus only on general comments about our probabilities.

In what follows, we will denote by $P_\theta\left(V(\lambda)\right)$ the limiting probability (with an infinite number of voters) of the truncation paradox given a voting rule $V(\lambda)$ $(0 \leq \lambda \leq 1)$ when θ is the model used for dealing with truncated ballots.

4.2.1 The case of one-shot scoring rules

From Proposition 5, we derive a system of linear inequalities which we solve by using the same technique as in Cervone et al. (2005). From the solutions of this system of inequalities, we reach Proposition 8 (see the online supplementary material) which provides the representation forms $P_{\text{Pe}}\left(F_\lambda\right)$ for the pessimistic model, $P_{\text{Av}}\left(F_\lambda\right)$ for the averaged model, and $P_{\text{Rd}}\left(F_\lambda\right)$ for the round-down model. For space constraints, we choose to skip the computation details for Proposition 8, but these are available upon request. Figure 1 gives a complete overview of the behavior of our probabilities given λ. In Table 2, we report some numerical evaluations.

Figure 1: Vulnerability of one-shot scoring rules to the truncation paradox

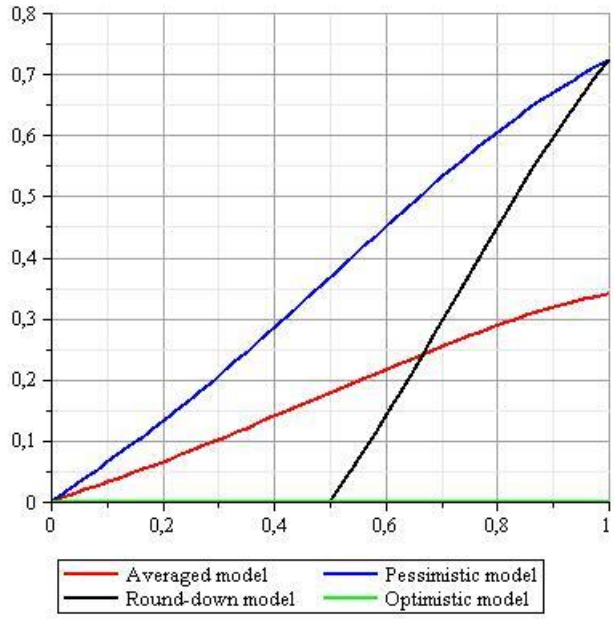

Table 2: Likelihood of the truncation paradox for one-shot scoring rules

	λ										
Models	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	0	0.0632	0.1315	0.2052	0.2839	0.3664	0.4497	0.5302	0.6042	0.6689	0.7222
Averaged	0	0.0315	0.0652	0.1011	0.1390	0.1778	0.2163	0.2537	0.2883	0.3180	0.3403
Round-down	0	0	0	0	0	0	0.1395	0.2937	0.4495	0.5947	0.7222
Optimistic	0	0	0	0	0	0	0	0	0	0	0

We notice that as the number of voters tends to infinity, the limiting probability of the truncation paradox for a one-shot scoring rule F_λ increases as λ increases from 0 (the Plurality rule) to 1 (the Antiplurality rule). It grows from 0 to 72.22% under both the pessimistic and the round-down model and from 0 to 34.03% under the averaged model. We also notice that for any $\lambda \in]0\ 1]$, while the paradox does not occur under the optimistic scoring model, the pessimistic model is the most likely to lead to the paradox. For $\lambda \in]0\ \frac{2}{3}[$, the round-down model is less likely to lead to the paradox than the averaged model and we have the opposite for $\lambda \in]\frac{2}{3}\ 1[$. For $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$ and $\lambda = 1$, our probabilities are well in accord with Remark 1.

The lesson that can be drawn therefore is that, for one-shot scoring rules, any manipulation by truncation is futile if we resort to the optimistic model; while the widespread pessimistic model can open up significant possibilities for electoral manipulation. Thus, adopting the optimistic model would be an effective way to discourage strategic truncation behavior in an electoral system where voters have to rank candidates in order to elect a winner.

4.2.2 The case of iterative scoring rules with one-by-one eliminations

The conditions in Propositions 6.A to 6.D completely describe all the scenarios that support possible occurrences of the truncation paradox. From these conditions, we draw some sets of linear constraints in order to characterize all possible occurrences of the truncation paradox under F'_λ given θ. By solving the systems with the same technique as in Cervone et al. (2005), we obtain the representation forms provided in Proposition 9 (see the online supplementary material).[10]

In Table 3, we report some numerical evaluations of $P_{\mathrm{Pe}}(F'_\lambda)$, $P_{\mathrm{Op}}(F'_\lambda)$, $P_{\mathrm{Av}}(F'_\lambda)$, and $P_{\mathrm{Rd}}(F'_\lambda)$, while Figure 2 gives an overview of their behavior.

[10]For space constraints, we omit the calculation details of Proposition 9, but they are available upon request.

Figure 2: Vulnerability of scoring runoff rules (with one-by-one eliminations) to the truncation paradox

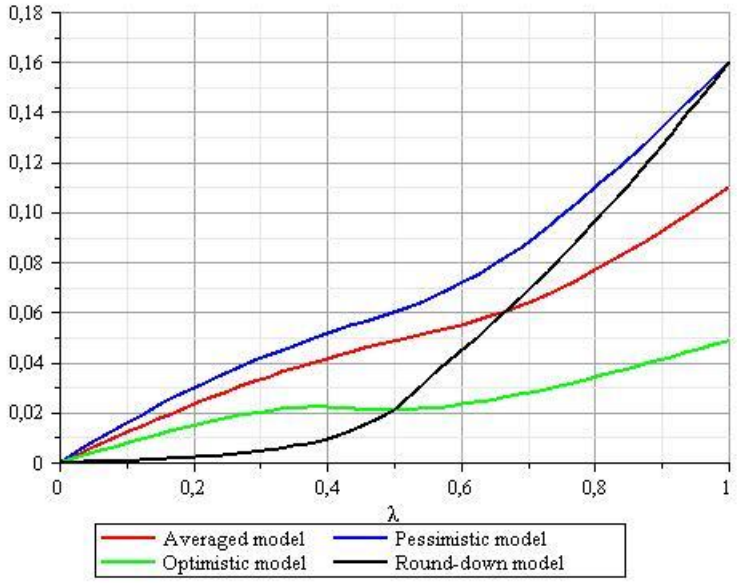

Table 3: Likelihood of the truncation paradox for scoring runoff rules with one-by-one eliminations

	λ										
Models	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	0	0.0157	0.0296	0.0414	0.0513	0.0683	0.0715	0.0882	0.1094	0.1337	0.1597
Averaged	0	0.01185	0.0229	0.0329	0.0415	0.0485	0.0548	0.0637	0.0759	0.0924	0.1100
Round-down	0	0.00072	0.0019	0.0044	0.0095	0.0208	0.0445	0.0686	0.0959	0.1265	0.1597
Optimistic	0	0.0076	0.0147	0.0202	0.0219	0.0208	0.0232	0.0279	0.0341	0.0411	0.0486

It comes out that probabilities tend to increase with λ and they are highest under each model when $\lambda = 1$ (Antiplurality runoff). The configuration here is less clear-cut than in the case of one-shot scoring rules. For $\lambda \in]0\ 1]$, the pessimistic model is the most likely to lead to the truncation paradox; for $\lambda \in]0\ \frac{1}{2}[$, the round-down model is least likely to lead to the paradox than the optimistic model which does better than the averaged model; for $\lambda \in]\frac{1}{2}\ 1]$, the optimistic model is the less likely to lead to the truncation paradox; for $\lambda \in]\frac{1}{2}\ \frac{2}{3}]$, the round-down model does better than the averaged model and we get the opposite for $\lambda \in]\frac{2}{3}\ 1]$. Hence, our probabilities are well in accord with Remark 1 when $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$ or $\lambda = 1$.

Our results tell us that the model under which we operate does indeed have a significant impact on the probability of the truncation paradox for scoring runoff rules with one-by-one eliminations. The choice of a model to minimize strategic truncation depends closely on λ, the voting rule.

4.2.3 The case of iterative scoring rules with eliminations below the average

Using the sets of linear constraints derived from the conditions in Propositions 7.A to 7.F, we were able to compute the likelihood of the truncation paradox for each of the four models as displayed in Proposition 10 (see the online supplementary material).[11] Figure 3 provides a complete overview of the evolution of these probabilities for $\lambda \in [0\ 1]$. In Table 4 we report some numerical values for $P_{\text{Pe}}\left(\tilde{F}'_\lambda\right)$, $P_{\text{Op}}\left(\tilde{F}'_\lambda\right)$, $P_{\text{Rd}}\left(\tilde{F}'_\lambda\right)$, and $P_{\text{Av}}\left(\tilde{F}'_\lambda\right)$.

Figure 3: Vulnerability of scoring runoff rules (with eliminations below the average) to the truncation paradox

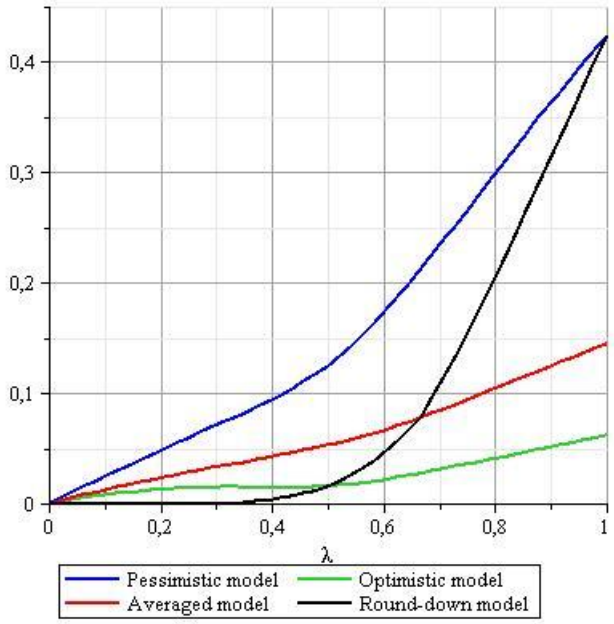

Table 4: Likelihood of the truncation paradox for scoring runoff rules with eliminations below the average

	λ										
Models	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Pessimistic	0	0.0243	0.0481	0.0709	0.0945	0.1247	0.1726	0.2336	0.2986	0.3627	0.4236
Averaged	0	0.0123	0.0234	0.0331	0.0427	0.0532	0.0664	0.0841	0.1041	0.1248	0.1453
Round-down	0	0	0	0	0.0035	0.0156	0.0459	0.1075	0.2053	0.3131	0.4236
Optimistic	0	0.0076	0.0126	0.0148	0.0145	0.0156	0.0218	0.0308	0.0409	0.0513	0.0614

The configuration here is quite similar to the one we had with the scoring runoff rules with one-by-one eliminations. The probabilities tend to increase with λ and they are highest

[11] Computation details are available upon request.

under each model when $\lambda = 1$ (Kim-Roush rule). For $\lambda \in]0\ 1]$, the pessimistic model appears to be the most likely to lead to the truncation paradox; for $\lambda \in]0\ \frac{1}{2}[$, the round-down model is less likely to lead to the paradox than the optimistic model which does better than the averaged model; for $\lambda \in]\frac{1}{2}\ 1]$, the optimistic model is the less likely to lead to the truncation paradox; for $\lambda \in]\frac{1}{2}\ \frac{2}{3}]$, the round-down model does better than the averaged model and we get the opposite for $\lambda \in]\frac{2}{3}\ 1]$. Our probabilities are well in accord with Remark 1 when $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$ and $\lambda = 1$.

It has been established that the average-elimination rules behave rather well in the face of voting paradoxes compared to scoring rules with one-to-one eliminations and one-shot scoring rules (Favardin and Lepelley, 2006, Kamwa, 2019, Kim and Roush, 1996, Lepelley and Valognes, 2003). A comparison of the figures in Tables 3 and 4 shows that apart from the optimistic model, the scoring runoff rules with eliminations below the average tend to exhibit the truncation paradox more than scoring rules with one-to-one eliminations. However, they do better than one-shot rules. It comes out that, to our knowledge, the truncation paradox is one of the rare voting events where the runoff rules with eliminations according to the average perform less well than scoring runoff rules with one-by-one eliminations.

In Table 5, we report, in summary, the probabilities for some well-known voting rules included in our framework.

Table 5: Summary results for well-known voting rules

	Voting rules								
Models	Plurality	Antiplurality	Borda	Dowdall	Plurality runoff	Antiplurality runoff	Baldwin	Nanson	Kim-Roush
Pessimistic	0	0.7222	0.3664	0.1677	0	0.1597	0.0683	0.1247	0.4236
Averaged	0	0.3403	0.1778	0.0829	0	0.1100	0.0485	0.0532	0.1453
Round-down	0	0.7222	0	0	0	0.1597	0.0208	0.0156	0.4236
Optimistic	0	0	0	0	0	0.0486	0.0208	0.0156	0.0614

5 Concluding remarks

The first objective of this paper was to draw the attention of policy-makers to the fact that the choice of how to deal with truncated preferences in elections should not be overlooked; we have shown that this choice can encourage strategic behaviors as well as discourage them. The other objective of this paper was to contribute to the small pool of works on the evaluation of the probability of occurrence of the truncation paradox. First, we have characterized, for any number of candidates and for one-shot scoring rules and scoring runoff rules, all the voting situations under each of the models for dealing with incomplete preferences: the pessimistic, the optimistic, the averaged and the round-down models. It comes out from our analysis that the vulnerability of a voting rule to the truncation paradox depends fundamentally on how one deals with truncated ballots. From our general results, we also derived that the Modified Borda Count is also a voting rule immune to the truncation paradox.

To corroborate our result that the vulnerability of voting rules to the truncation paradox would depend on the way truncated preferences are treated, we paid particular attention to three-candidate elections. First, we have characterized all the voting situations under which the truncation paradox is likely to occur. Then, we computed the limiting probability of the truncation paradox for each of the models. From our characterization and computation results, it comes out that the occurrence of the truncation paradox is highly dependent on the model that is applied to truncated ballot: the pessimistic model is the model most likely to give way to the truncation paradox compared to the other models. So, the widespread pessimistic model is less to be recommended. Since it leaves the door open for a considerable degree of manipulation, organizations and countries like Slovenia and Kiribati that use the pessimistic model should abandon this model in favor of one that leaves little room for manipulation.

The conclusions that we obtained for the case of elections with three candidates are only a first step in a work that we plan to extend to situations with more than three candidates. Even if this proves to be a greater challenge from a computational point of view, one way to do it would be to follow the same methodology as Kilgour et al. (2019) or Plassmann and Tideman (2014), based on numerical simulations from real or merely simulated data.

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6 Appendices

6.1 Proofs of Propositions 1 to 4

6.1.1 Proof of Proposition 1

Consider a voting situation π where $n \geq 2$ voters have sincere strict rankings on $m \geq 3$ candidates of a set A. Assume for a given scoring rule that candidate x is the winner; this means that for all $y \in A \setminus \{x\}$, we have $S(\pi, w, x) > S(\pi, w, y)$. Let consider a group of $\bar{n}$ voters who rank candidate x at position $p > 1$ and who want to favor by sincere truncation a more preferred candidate, say y, ranked at one of the positions $k < p$. If they truncate after position l ($k \leq l < p$) and that the optimistic model is in effect, this implies that the score of all the candidates they rank from the top to position $l+1$ are not affected (including that of the candidate they wish to favor) while the new score of candidate x is equal to $S(\pi, w, x) + (-w_p + w_{l+1})\bar{n}$. Since by definition $w_{l+1} \geq w_p$, candidate x records an increase of his score and he is still get more than y. Thus, when the optimistic model is used to deal with truncated preferences, no one-shot scoring rule is manipulable by sincere truncation.

Now, let us assume the following profile on a the set $A = \{x, y, z\}$ with $6r - 3$ voters ($r \in \mathbb{N}$ and $r > 4$).

$r+1$	$r-2$	$r-1$	r	r	$r-1$
x	x	y	y	z	z
y	z	x	z	x	y
z	y	z	x	y	x

Consider iterative scoring rules with one-by-one eliminations ($\forall \lambda \in]0\ 1]$). One can check that z is eliminated at the first round and x wins the majority duel against y. Now, assume that the $r-1$ voters with the ranking yxz truncate. If the optimistic model is used to deal with the truncated ballot, we note that x is now rushed out at the first round and y wins the majority duel against z. So, it always possible to build a voting situation where the truncation paradox occurs for a given iterative scoring rule with one-by-one eliminations under the optimistic model. Instead, if we consider all the iterative rules with eliminations on the average ($\forall \lambda \in]0\ 1]$), x and y are qualified for the second round since they score above the average and x wins this round. When the $r-1$ voters with the ranking yxz truncate, if the optimistic model is assumed, x is now the only one scoring below the average and y defeats z in the last round. Thus, all iterative rules with eliminations according to the average are vulnerable to the truncation paradox when the optimistic model is assumed.

6.1.2 Proof of Proposition 2

Let us assume the following profile on a the set $A = \{x, y, z\}$ with $6r - 3$ voters ($r \in \mathbb{N}$ and $r > 2$).

$r+1$	$r-2$	r	$r-1$	r	$r-1$
x	x	y	y	z	z
y	z	x	z	x	y
z	y	z	x	y	x

With this profile, one can easily check that when the r voters with the ranking yxz truncate, the truncation paradox occurs for all the one-shot scoring rules ($\forall \lambda \in]0\ 1]$) and for all the iterative rules with eliminations according to the average both under the pessimistic model and the averaged model. The profile provided in the proof of Proposition 1 can be used to show that all the runoff rules with one-by-one eliminations are vulnerable to the truncation paradox when the averaged model is used to deal with truncated ballots.

6.1.3 Proof of Proposition 3

Given a voting situation π and a scoring vector $w = (w_1, w_2, \ldots, w_{m-1}, w_m)$, assume a one-shot scoring rule such that candidate x is the winner and that n_t voters who rank the x at position l ($2 \leq l \leq m-1$) in their sincere ranking truncate and now report only k candidates ($1 \leq k \leq l-1$) in order to favor a more preferred candidate ranked in their sincere ranking. After truncation, if the round-down model is assumed, the new scores of the candidates in the truncated voting situation π' are as follows: for a candidate y sincerely ranked jth ($1 \leq j \leq k$) by these voters, we get $S(\pi', w, y) = S(\pi, w, y) - (w_j - w_{m-k+j-1})n_t$ and $S(\pi', w, z) = S(\pi, w, z) - w_j n_t$ for all candidate z ranked pth ($k+1 \leq p \leq m$). The truncation paradox occurs if a candidate between the top and the position $l-1$ of the sincere ranking wins.

Given the new scores, if a candidate y ranked jth such that $k+1 \leq j \leq l-1$) now scores better than x, this means that

$$\begin{aligned} S(\pi', w, x) < S(\pi', w, y) &\Rightarrow S(\pi, w, x) - w_l n_t < S(\pi, w, y) - w_j n_t \\ &\Rightarrow -w_l n_t + w_j n_t < S(\pi, w, y) - S(\pi, w, x) \\ &\Rightarrow -w_l n_t + w_j n_t < 0 \quad \text{since } S(\pi, w, y) < S(\pi, w, x) \\ &\Rightarrow w_j < w_l \end{aligned}$$

This last inequality contradicts that y is better ranked than x; so, after truncation, a candidate y ranked jth with $k+1 \leq j \leq l-1$ cannot be favored. Let us now assume that a candidate y ranked jth ($j \leq k$) scores better than x; this means that

$$\begin{aligned} S(\pi', w, x) < S(\pi', w, y) &\Rightarrow S(\pi, w, x) - w_l n_t < S(\pi, w, y) - (w_j - w_{m-k+j-1})n_t \\ &\Rightarrow -w_l n_t + (w_j - w_{m-k+j-1})n_t < S(\pi, w, y) - S(\pi, w, x) \\ &\Rightarrow -w_l n_t + (w_j - w_{m-k+j-1})n_t < 0 \quad \text{since } S(\pi, w, y) < S(\pi, w, x) \\ &\Rightarrow w_j - w_{m-k+j-1} < w_l \end{aligned}$$

Thus, following this last inequality, a candidate may be favoured under a one-shot scoring rule if $w_j - w_{m-k+j-1} < w_l$. Assuming $w_j - w_{m-k+j-1} > w_l$, one can easily reach a contradiction with $S(\pi', w, x) < S(\pi', w, y)$. So, the manipulation is possible only if $w_j - w_{m-k+j-1} < w_l$.

In order to show that the truncation paradox can affect scoring runoff rules with eliminations, let us just exhibit a profile showing that this can be the case.

Consider the following profile on a the set $A = \{x, y, z\}$ with $4r - 5$ voters ($r \in \mathbb{N}$ and $r > 4$).

$r-1$	$r-3$	r	$r-1$
x	y	y	z
y	x	z	x
z	z	x	y

Assume that the voting rule is a scoring runoff rule with one-by-one eliminations with the normalized scoring vector $(1, \lambda, 0)$. For all $0 < \lambda \leq 1$, the winner in this profile is candidate x who defeats candidate y in the second round since the scores at the first round are as follow: $S(\pi, x) = r - 1 + \lambda(2r - 4)$, $S(\pi, y) = 2r - 3 + \lambda(r - 1)$ and $S(\pi, z) = r - 1 + \lambda r$. Assume that all the voters with the ranking yxz truncate and that the round-down model is assumed; the new scores are now $r - 1 + \lambda(r - 1)$ for x, $r + \lambda(2r - 4)$ for y while that of z remains unchanged. It follows that $\forall \lambda \in]0\ 1]$, x now gets the lowest score and is rushed out and y wins the majority contest against z. So, it is always possible to find a profile where the truncation paradox is possible when the round-down model is assumed.

So, when the round-down model is assumed, it is always possible to find a profile such that this manipulation is successful under the scoring runoff rules with one-by-one eliminations or with eliminations according to the average.

6.1.4 Proof of Proposition 4

Under MBC as under the Borda rule, the scoring vector $w = (w_1, w_2, \ldots, w_{m-1}, w_m)$ is such that for two candidates ranked pth and $p + 1$th, we get $w_p - w_{p+1} = 1$ $(1 \leq p \leq m)$ and $w_p = w_1 - p$ with $w_1 = m$. Assume that the truncation paradox is possible; so, it follows from Proposition 3 that

$$\begin{aligned} w_j - w_{m-k+j-1} < w_l \quad &\Rightarrow \quad w_1 - j - (w_1 - (m - k + j - 1)) < w_1 - l \\ &\Rightarrow \quad w_1 - j - w_1 + m - k + j - 1 < w_1 - l \\ &\Rightarrow \quad m - j - m + m - k + j - 1 < m - l \\ &\Rightarrow \quad -k - 1 < -l \\ &\Rightarrow \quad k + 1 > l \end{aligned}$$

This last inequality contradicts that $1 \leq k \leq l - 1$. The proof is completed.

6.2 Proofs of Propositions 5 to 7.F

6.2.1 Proof of Proposition 5

The proof for the pessimistic model can been drawn from Kamwa and Moyouwou (2021). The proof for the average model follows the same scheme as that of the pessimistic model; so, we skip it.[12] The proof of (iii) directly comes from Prop. 1. When the round-down model is assumed, we derive from the condition in Proposition 3 (with $j = 1$, $k = 1$ and $l = 2$) that the truncation paradox occurs only if $1 - \lambda < \lambda$ which implies that $\lambda > \frac{1}{2}$.

6.2.2 Proof of Proposition 6.A to 6.D

We choose to skip the proof for the averaged model as it follows the same scheme as that of the pessimistic model that one can find in appendix C of Kamwa and Moyouwou (2021). So, we only need to focus on the proofs for the optimistic and the round-down models. Consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on A and a runoff rule associated with $0 < \lambda \leq 1$, where x is the winner, y is the challenger, and z is the first-round loser.

- *Let us assume the optimistic model.*

By truncating their true preferences, voters who strictly prefer y to x (those with the ranking yxz) increase the score of z while those of x and y remain unchanged in such a way that x is ruled out in the first round at $\pi([yxz])$ and y wins the second round against z. Hence, the truncation paradox occurs in favor of y. This is not possible if voters with the ranking yzx wanted to favor y since in the first round of $\pi([yzx])$, the score of x increases while those of y and z remain unchanged and x still defeats y in the second round.

Assume that voters who strictly prefer z to x want to engage in manipulation. If those with the ranking zxy truncate, this increases the score of y while those of x and z are unchanged: z is ruled out in the first round at $\pi([zxy])$. If those with the ranking zyx truncate, it does not affect the scores of y and z while that of x increases: z is ruled out in the first round at $\pi([zyx])$. Thus, there is no way to favor candidate z.

- *Let us now assume the round-down model.*

In order to favor y, if voters the ranking yxz truncate, the scores of x and y decrease while that of z remains unchanged. If the number of strategic voters is sufficient in such a way that x is ruled out in the first round at $\pi([yxz])$ then y wins the second round against z. Hence, the truncation paradox occurs in favor of y. This is not possible when voters with the ranking yzx truncate since the score of x will remain unchanged.

[12] Nonetheless, this is still available upon request.

If after truncation, z wins this means that he wins the majority duel either against x or against y. He wins against x if when voters with zyx truncates his score which is still lower than that of x, is now greater than that y; then, in the first round at $\pi([zyx])$, we get $S(\pi,z)-(1-\lambda)n_6 > S(\pi,y)-\lambda)n_6$ which leads to $S(\pi,z)-S(\pi,y) > (1-\lambda)n_6-\lambda)n_6$; since $S(\pi,y) > S(\pi,z)$ at π, it follows that $(1-\lambda)n_6-\lambda)n_6 < 0$ which implies that $\lambda > \frac{1}{2}$. In a similar way, z wins against y if when voters with zxy truncates his score which is still lower than that of y, is now greater than that x: $S(\pi,z)-(1-\lambda)n_5 > S(\pi,x)-\lambda)n_5$ which also leads to $\lambda > \frac{1}{2}$. So, it is only possible to favor z if $\lambda > \frac{1}{2}$.

6.2.3 Proof of Proposition 7.A to 7.F

Here, we only focus on the proof of the first part of the proposition; that of the second part follows an almost similar approach. Let us consider a voting situation $\pi = (n_1, n_2, n_3, n_4, n_5, n_6)$ on $A = \{x, y, z\}$ and a scoring rule $\tilde{F}'_\lambda$ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$.

Assume that x wins at the first round. By definition, we get: $S(\pi,\lambda,x) \geq T(\pi)$, $S(\pi,\lambda,y) < T(\pi)$ and $S(\pi,\lambda,z) < T(\pi)$ with $T(\pi) = \frac{S(\pi,\lambda,x)+S(\pi,\lambda,y)+S(\pi,\lambda,z)}{3}$.

Assume that voters with the ranking yxz truncate in order to favor y; if the the optimistic model is assumed, it follows that the scores of x and y are still unchanged while that of z increases by λn_3. The average of the scores at $\pi([yxz])$ is now equal to $T(\pi)+\frac{\lambda}{3}n_3$. Since $S(\pi,\lambda,y) < T(\pi)$ it follows that $S(\pi,\lambda,y) < T(\pi)+\frac{\lambda}{3}n_3$: candidate y is certainly rushed out at the first run. Also in order to favor candidate y, if voters with yzx truncates, it is easy to see that this is unfruitful at $\pi([yzx])$. We reach the same conclusion if one to favor candidate z at $\pi([zxy])$ or $\pi([zyx])$. So, the truncation paradox is not liable under the optimistic model.

If the averaged model is assumed when voters with the ranking yxz (resp. yzx) truncate, it follows at $\pi([yxz])$ (resp. at $\pi([yzx])$) that the score of y is still unchanged while that of z increases (resp. decreases) by $\frac{\lambda}{2}n_3$ (resp. by $\frac{\lambda}{2}n_4$) and that of x decreases (resp. increases) by $\frac{\lambda}{2}n_3$ (resp. by $\frac{\lambda}{2}n_4$). The average of the scores at $\pi([yxz])$ (resp. at $\pi([yzx])$) remains the same as at π: in fact, x remains the winner. We get the same conclusion if one tries to favor z at $\pi([zxy])$ or at $\pi([zxy])$. So, the truncation paradox is not liable under the averaged model.

Let us assume the pessimistic model and that voters with the ranking yxz truncate. At $\pi([yxz])$, it follows that one the score of x decreases by λn_3 and the new average of the scores is T' such that $T' = T(\pi) - \frac{\lambda n_3}{3}$. If the set of voters who truncate is enough, candidate y wins at $\pi([yxz])$ if he scores above T' : either he is the only one in this case or he wins the majority duel against one of his competitors meeting this condition. At $\pi([zxy])$, we get the same conclusion in favor of z.

Under the round-down model, if the voters with the ranking yxz truncate in order to favor y, it follows at $\pi([yxz])$ that the new score of x is $S(\pi,\lambda,x)-\lambda n_3$, that of y is $S(\pi,\lambda,y)-(1-\lambda)n_3$ while the score of z remains unchanged. So, the new average of the scores is $T' = T(\pi) - \frac{n_3}{3}$. The truncation favors b if he is the only one to score above

T' or he beats x or z in the second round. In all the case, it requires b to score above T' at $\pi([yxz])$; this means that $S(\pi,\lambda,a)-(1-\lambda)n_3 > T(\pi)-\frac{n_3}{3}$ which is equivalent to $S(\pi,\lambda,y)-T(\pi) > (1-\lambda)n_3-\frac{n_3}{3}$. Since at π we get $T(\pi) > S(\pi,\lambda,y)$, this implies that $0 > (1-\lambda)n_3-\frac{n_3}{3}$; we then derive that $\lambda > \frac{2}{3}$. At $\pi([zxy])$, we get the same conclusion in favor of z.

6.3 Probability representations

Proposition 8. *For three-candidate elections and the IAC assumption, the likelihood of the truncation paradox for a given scoring rule F_λ ($\lambda \in [0\ 1]$) under respectively the optimistic, the average, and the pessimistic models is given by:*

$$P_{Op}(F_\lambda) = \ 0, \quad \textit{if} \ \ 0 \le \lambda \le 1.$$

$$P_{Av}(F_\lambda) = \begin{cases} \dfrac{\left(\begin{array}{c} 84\,\lambda^{12}-265\,\lambda^{11}-591\,\lambda^{10}-11351\,\lambda^9+126949\,\lambda^8 \\ -517521\,\lambda^7+1185665\,\lambda^6-1720474\,\lambda^5 \\ +1636446\,\lambda^4-999044\,\lambda^3+356664\,\lambda^2-56592\,\lambda \end{array}\right)}{9(4-3\lambda)(-6+7\lambda)(-7\lambda+5\lambda^2+6)(-3+2\lambda)(-6+\lambda)(\lambda-1)^2(\lambda-2)(\lambda-4)}, & \textit{if} \ \ 0<\lambda\le\frac{1}{2}; \\ \dfrac{\left(\begin{array}{c} 24\,\lambda^{12}+106\,\lambda^{11}-2836\,\lambda^{10}+16430\,\lambda^9-56040\,\lambda^8 \\ +136282\,\lambda^7-245600\,\lambda^6+322203\,\lambda^5-297754\,\lambda^4 \\ +184484\,\lambda^3-69752\,\lambda^2+13632\,\lambda-1152 \end{array}\right)}{18\,\lambda^2(\lambda-1)(6-\lambda)(-4+3\,\lambda)(\lambda-2)(-7\,\lambda+5\,\lambda^2+6)(-3+2\,\lambda)(\lambda-4)}, & \textit{if} \ \ \frac{1}{2}\le\lambda\le 2-\sqrt{2}; \\ \dfrac{\left(\begin{array}{c} 12\,\lambda^{10}-298\,\lambda^9+3214\,\lambda^8-15754\,\lambda^7-3196\,\lambda+336 \\ 40654\,\lambda^6-62044\,\lambda^5+59581\,\lambda^4-36013\,\lambda^3+13312\,\lambda^2 \end{array}\right)}{18\,\lambda^2(\lambda-5)(4-3\,\lambda)(-7\,\lambda+5\,\lambda^2+6)(-3+\lambda)(\lambda-2)}, & \textit{if} \ \ 2-\sqrt{2}\le\lambda\le 1. \end{cases}$$

$$P_{Pe}(F_\lambda) = \begin{cases} \dfrac{\left(\begin{array}{c} 4153\,\lambda^4-1234\,\lambda^5-200\,\lambda^6+260\,\lambda^7-50\,\lambda^8 \\ -10\,\lambda^9-1179\,\lambda-5995\,\lambda^3+4243\,\lambda^2+4\,\lambda^{10} \end{array}\right)}{18\,(3+\lambda)(-3+2\,\lambda)(3-2\,\lambda+\lambda^2)(-1+\lambda)^2(-2+\lambda)^2}, & \textit{if} \ \ 0<\lambda\le\frac{1}{2}; \\ \dfrac{\left(\begin{array}{c} -1733\,\lambda^4+806\,\lambda^5-62\,\lambda^6-106\,\lambda^7+34\,\lambda^8 \\ +2\,\lambda^9+1598\,\lambda^3-12+128\,\lambda-551\,\lambda^2 \end{array}\right)}{18\,\lambda^2(-2+\lambda)^2(3+\lambda)(3-2\,\lambda+\lambda^2)}, & \textit{if} \ \ \frac{1}{2}<\lambda\le 1. \end{cases}$$

$$P_{Rd}(F_\lambda) = \begin{cases} 0 & \text{if } 0 < \lambda \leq \frac{1}{2}; \\ \dfrac{\left(\begin{array}{c} (-1+2\,\lambda)\,(90\,\lambda^{15} - 3006\,\lambda^{14} + 7754\,\lambda^{13} + 4282\,\lambda^{12} - 33018\,\lambda^{11} \\ +32929\,\lambda^{10} + 3710\,\lambda^{9} - 17631\,\lambda^{8} - 6216\,\lambda^{7} + 25986\,\lambda^{6} \\ -20610\,\lambda^{5} + 5972\,\lambda^{4} + 586\,\lambda^{3} - 780\,\lambda^{2} + 184\,\lambda - 16) \end{array} \right)}{9\,(\lambda-1)^2(-2+5\,\lambda)(\lambda+1)^2(2-3\,\lambda+4\,\lambda^2)\lambda^2(-4+\lambda^2)(9\,\lambda^2-1)}, & \text{if } \frac{1}{2} < \lambda \leq \frac{-1+\sqrt{5}}{2}; \\ \dfrac{\left(\begin{array}{c} 360\,\lambda^{16} - 10944\,\lambda^{15} + 434\,\lambda^{14} + 100375\,\lambda^{13} \\ -124879\,\lambda^{12} - 71483\,\lambda^{11} + 112454\,\lambda^{10} + 52918\,\lambda^{9} \\ -143388\,\lambda^{8} + 61488\,\lambda^{7} + 35388\,\lambda^{6} - 40675\,\lambda^{5} \\ +10489\,\lambda^{4} + 1745\,\lambda^{3} - 1334\,\lambda^{2} + 212\,\lambda - 8 \end{array} \right)}{\left(\begin{array}{c} 9\,(\lambda+1)^2\left(3\,\lambda-1+\lambda^2\right)\left(2-3\,\lambda+4\,\lambda^2\right) \\ (-2+5\,\lambda)\,(-2+\lambda)\,\lambda^2\,(1+2\,\lambda)\left(9\,\lambda^2-1\right) \end{array} \right)} & \text{if } \frac{-1+\sqrt{5}}{2} < \lambda \leq 1. \end{cases}$$

Proposition 9. *Consider the scoring runoff rule F'_λ associated with the scoring vector $w_\lambda = (1, \lambda, 0)$ with $0 < \lambda \leq 1$. As the total number n of voters tends to infinity, the limiting probability $P.(F'_\lambda)$ of observing a voting situation in which the truncation paradox may occur is given by :*

$$P_{Pe}(F'_\lambda) = \begin{cases} -\dfrac{\left(\begin{array}{c} 996\,096\lambda^{20} - 25\,010\,368\lambda^{19} + 286\,101\,152\lambda^{18} - 2000\,804\,220\lambda^{17} \\ +9664\,972\,152\lambda^{16} - 34\,453\,144\,125\lambda^{15} + 94\,322\,255\,778\lambda^{14} \\ -203\,353\,434\,975\lambda^{13} + 350\,716\,379\,871\lambda^{12} - 488\,312\,722\,095\lambda^{11} \\ +551\,142\,449\,552\lambda^{10} - 504\,159\,008\,281\lambda^{9} + 372\,136\,194\,567\lambda^{8} \\ -219\,653\,377\,992\lambda^{7} + 102\,140\,474\,607\lambda^{6} - 36\,558\,733\,185\lambda^{5} \\ +9711\,109\,602\lambda^{4} - 1801\,641\,852\lambda^{3} + 208\,222\,083\lambda^{2} - 11\,278\,359\lambda \end{array} \right)}{\left(\begin{array}{c} 96\,(\lambda-1)^2\,(\lambda-2)^2\,(2\lambda-3)^2\,(4\lambda-3)^2\,(5\lambda-3)^2\left(-2\lambda+\lambda^2+3\right) \\ \left(-5\lambda+\lambda^2+3\right)^2\left(-4\lambda+2\lambda^2+3\right)\left(-7\lambda+3\lambda^2+3\right) \end{array} \right)} & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\ \dfrac{\left(\begin{array}{c} 132\lambda + 9346\lambda^2 - 55\,961\lambda^3 + 161\,587\lambda^4 - 283\,660\lambda^5 \\ +330\,502\lambda^6 - 265\,921\lambda^7 + 149\,437\lambda^8 - 57\,766\lambda^9 \\ +14\,560\lambda^{10} - 2112\lambda^{11} + 128\lambda^{12} - 180 \end{array} \right)}{288\lambda^3(\lambda-2)^2(3-2\lambda)(-2\lambda+\lambda^2+3)(-4\lambda+2\lambda^2+3)} & \text{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

$$P_{Rd}(F'_\lambda) = \begin{cases} \dfrac{\left(\begin{array}{c} 2266\,\lambda^7 - 3843\,\lambda^6 + 4142\,\lambda^5 - 2947\,\lambda^4 - 764\,\lambda^8 \\ +1368\,\lambda^3 - 386\,\lambda^2 + 108\,\lambda^9 + 52\,\lambda \end{array} \right)}{\left(\begin{array}{c} 576\,\left(2-\lambda\,(\lambda-1)\right)^3\left(3\,\lambda^2+2-4\,\lambda\right)\left(5\,\lambda^2-5\,\lambda+2\right)(-2+3\,\lambda) \end{array} \right)} & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\ \dfrac{\left(\begin{array}{c} 576\,\lambda^{12} - 4176\,\lambda^{11} + 11624\,\lambda^{10} - 14085\,\lambda^9 + 1243\,\lambda^8 + 18200\,\lambda^7 \\ -24797\,\lambda^6 + 16760\,\lambda^5 - 6554\,\lambda^4 + 1472\,\lambda^3 - 188\,\lambda^2 + 19\,\lambda - 2 \end{array} \right)}{144\,(-8\,\lambda^2+3\,\lambda^3-2+8\,\lambda)(3\,\lambda-1)\lambda^3(2-\lambda)(\lambda+1)} & \text{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

$$P_{Op}\left(F'_{\lambda}\right)=\begin{cases}\dfrac{\left(\begin{array}{c}-10611\lambda+550072\lambda^{6}+668616\lambda^{4}-785172\lambda^{5}\\-213643\lambda^{7}+36244\lambda^{8}+92682\lambda^{2}-337434\lambda^{3}\\-3170\lambda^{9}+3296\lambda^{10}-896\lambda^{11}\end{array}\right)}{864\,(2\,\lambda^{2}-4\,\lambda+3)(\lambda-1)^{3}(-3+4\,\lambda)(-3+7\,\lambda)(-3+\lambda)(-2+\lambda)} & if\ \ 0\leq\lambda\leq\frac{1}{3}\\ \dfrac{\left(\begin{array}{c}54+32322\,\lambda^{4}-17544\,\lambda^{3}-35958\,\lambda^{5}+25142\,\lambda^{6}-11246\,\lambda^{7}\\+3188\,\lambda^{8}-460\,\lambda^{9}-80\,\lambda^{10}+32\,\lambda^{11}+5391\,\lambda^{2}-837\,\lambda\end{array}\right)}{864\,(\lambda-1)^{3}(2\,\lambda^{2}-4\,\lambda+3)(3-\lambda)(-2+\lambda)\lambda^{2}} & if\ \ \frac{1}{3}\leq\lambda\leq\frac{1}{2}\\ \dfrac{\left(\begin{array}{c}-1173\,\lambda-1024\,\lambda^{5}-11745\,\lambda^{3}-3120\,\lambda^{8}+224\,\lambda^{9}\\+126+5048\,\lambda^{2}+12236\,\lambda^{7}-15400\,\lambda^{6}+13484\,\lambda^{4}\end{array}\right)}{1728\,(2-\lambda)(\lambda+1)\lambda^{3}(-3+7\,\lambda)(-3+\lambda)} & if\ \ \frac{1}{2}\leq\lambda\leq1\end{cases}$$

$$P_{Av}(F'_\lambda) = \begin{cases} \dfrac{\begin{pmatrix} 33216569100\,\lambda^{28} - 578325461985\,\lambda^{27} \\ +9304334549790\,\lambda^{26} - 108047886220115\,\lambda^{25} \\ +764944151235819\,\lambda^{24} - 3274064370679568\,\lambda^{23} \\ +7376416701981204\,\lambda^{22} + 3478879519154483\,\lambda^{21} \\ -99656142723761351\,\lambda^{20} + 449206760805885263\,\lambda^{19} \\ -1299296723118306018\,\lambda^{18} + 2820724443926997914\,\lambda^{17} \\ -4863903300584438368\,\lambda^{16} + 6849485494037368872\,\lambda^{15} \\ -8000357606231143984\,\lambda^{14} + 7819248718632596480\,\lambda^{13} \\ -6423381226780596592\,\lambda^{12} + 4440315222181111920\,\lambda^{11} \\ -2578541106422978592\,\lambda^{10} + 1252158152437843488\,\lambda^{9} \\ -504536511000085632\,\lambda^{8} + 166715248881851136\,\lambda^{7} \\ -44409266486558208\,\lambda^{6} + 9300309071357952\,\lambda^{5} \\ -1474068154705920\,\lambda^{4} + 166182281994240\,\lambda^{3} \\ -11872009617408\,\lambda^{2} + 403914055680\,\lambda \end{pmatrix}}{\begin{pmatrix} 864\,\left(13\,\lambda^2 - 18\,\lambda + 6\right)(-3+5\,\lambda)\left(\lambda^2+3\,\lambda-2\right) \\ (-2+3\,\lambda)^2\left(-10\,\lambda+5\,\lambda^2+6\right)(3\,\lambda-4) \\ (-3+4\,\lambda)\left(9\,\lambda^2-16\,\lambda+6\right)\left(6-11\,\lambda+\lambda^2\right) \\ (-6+11\,\lambda)^2(\lambda-1)^3(-2+\lambda) \\ \left(\lambda^2-4\,\lambda+2\right)\left(\lambda^2-2\,\lambda+2\right)(5\,\lambda-6)^2 \end{pmatrix}} & \text{if } \ 0 \le \lambda \le \frac{1}{2} \\ \dfrac{\begin{pmatrix} 960\,\lambda^{12} - 15683\,\lambda^{11} + 97885\,\lambda^{10} - 336844\,\lambda^{9} \\ +734471\,\lambda^{8} - 1075974\,\lambda^{7} + 1076390\,\lambda^{6} - 721992\,\lambda^{5} \\ +302272\,\lambda^{4} - 61008\,\lambda^{3} - 4912\,\lambda^{2} + 5216\,\lambda - 768 \end{pmatrix}}{864\,(-10\,\lambda+5\,\lambda^2+6)(3\,\lambda-4)\lambda^3(\lambda^2-2\,\lambda+2)(2-\lambda)(\lambda-1)} & \text{if } \ \frac{1}{2} \le \lambda \le \frac{2}{3} \\ \dfrac{\begin{pmatrix} 153435744000\,\lambda^{22} - 1080861013100\,\lambda^{21} \\ -5207821958625\,\lambda^{20} + 70974670832542\,\lambda^{19} \\ -269023597095227\,\lambda^{18} + 355035781965385\,\lambda^{17} \\ +793163784390930\,\lambda^{16} - 4915171010844029\,\lambda^{15} \\ +12479613659707154\,\lambda^{14} - 20862380675468342\,\lambda^{13} \\ +25549374150267504\,\lambda^{12} - 23940244324060448\,\lambda^{11} \\ +17531522545688352\,\lambda^{10} - 10137854810435680\,\lambda^{9} \\ +4647403823436288\,\lambda^{8} - 1688760012632192\,\lambda^{7} \\ +485435977494528\,\lambda^{6} - 110225975092224\,\lambda^{5} \\ +19800884477952\,\lambda^{4} - 2817139378176\,\lambda^{3} \\ +310563569664\,\lambda^{2} - 23999348736\,\lambda \\ +955514880 \end{pmatrix}}{\begin{pmatrix} 1728\,(\lambda+6)^2(5\,\lambda-4)(-3+5\,\lambda)(5\,\lambda-1)(9\,\lambda-5) \\ (7\,\lambda-4)\left(37\,\lambda^2-42\,\lambda+12\right)(17\,\lambda-10)(-6+11\,\lambda)^2 \\ (-1+2\,\lambda)\,\lambda^3(1-\lambda)(-2+\lambda)\left(\lambda^2-2\,\lambda+2\right) \end{pmatrix}} & \text{if } \ \frac{2}{3} \le \lambda \le -1+\sqrt{3} \\ \dfrac{\begin{pmatrix} 15343574400\,\lambda^{21} - 11373633920\,\lambda^{20} - 1050657135517\,\lambda^{19} \\ +2490627167603\,\lambda^{18} + 12639372321284\,\lambda^{17} - 47275784649312\,\lambda^{16} \\ +11356839708139\,\lambda^{15} + 190308912327531\,\lambda^{14} - 426342123649578\,\lambda^{13} \\ +415729617310034\,\lambda^{12} - 110189736092600\,\lambda^{11} - 225675759314224\,\lambda^{10} \\ +351934483916640\,\lambda^{9} - 278570652123472\,\lambda^{8} + 149018782047360\,\lambda^{7} \\ -58376289741312\,\lambda^{6} + 17394914835072\,\lambda^{5} - 4029648632064\,\lambda^{4} \\ +727429800960\,\lambda^{3} - 98372413440\,\lambda^{2} + 8843986944\,\lambda - 388177920 \end{pmatrix}}{\begin{pmatrix} 1728\,(\lambda+6)^2\left(37\,\lambda^2-42\,\lambda+12\right)(3-5\,\lambda) \\ (17\,\lambda-10)\left(\lambda^2+3\,\lambda-2\right)(\lambda+1)(-6+11\,\lambda)^2 \\ \left(\lambda^2-4\,\lambda+2\right)\lambda^3(7\,\lambda-4)(9\,\lambda-5)(5\,\lambda-1) \end{pmatrix}} & \text{if } \ -1+\sqrt{3} \le \lambda \le 1 \end{cases}$$

Proposition 10. *Consider the scoring runoff rule associated with the scoring vector* $w_\lambda = (1, \lambda, 0)$ *with* $0 < \lambda \leq 1$ *and where eliminations proceed according to the average. As the total number* n *of voters tends to infinity, the limiting probability of observing a voting situation in which the truncation paradox may occur is given by :*

$$P_{Av}\left(\tilde{F}'_\lambda\right) = \begin{cases} \dfrac{\left(\begin{array}{c}(2880\,\lambda^{12} - 16512\,\lambda^{11} + 35217\,\lambda^{10} - 164910\,\lambda^9 \\ +980311\,\lambda^8 - 3247752\,\lambda^7 + 6419912\,\lambda^6 - 8127852\,\lambda^5 \\ +6766612\,\lambda^4 - 3684952\,\lambda^3 + 1263440\,\lambda^2 - 247520\,\lambda + 21120)\lambda\end{array}\right)}{5184(-3\,\lambda+2)(\lambda^2+2-2\,\lambda)(\lambda^2-4\,\lambda+2)(\lambda-2)(-1+\lambda)^4(5\,\lambda-2)} & \text{if } 0 \leq \lambda \leq \frac{2}{7} \\ \dfrac{\left(\begin{array}{c}192\,\lambda^{11} - 512\,\lambda^{10} - 4165\,\lambda^9 \\ +24304\,\lambda^8 - 59575\,\lambda^7 + 85782\,\lambda^6 - 79784\,\lambda^5 \\ +50732\,\lambda^4 - 22860\,\lambda^3 + 7040\,\lambda^2 - 1216\,\lambda + 64\end{array}\right)}{5184\lambda(\lambda-1)^4(-\lambda^2+4\lambda-2)(\lambda^2+2-2\lambda)} & \text{if } \frac{2}{7} \leq \lambda \leq \frac{1}{2} \\ \dfrac{\left(\begin{array}{c}640\,\lambda^{12} - 3616\,\lambda^{11} + 3664\,\lambda^{10} + 12871\,\lambda^9 \\ -44588\,\lambda^8 + 62770\,\lambda^7 - 52659\,\lambda^6 + 36661\,\lambda^5 \\ -27394\,\lambda^4 + 17106\,\lambda^3 - 6780\,\lambda^2 + 1456\,\lambda - 128\end{array}\right)}{5184(-5\,\lambda+2)(\lambda^2+2-2\,\lambda)(-1+\lambda)^2\lambda^4} & \text{if } \frac{1}{2} \leq \lambda \leq \frac{2}{3} \\ \dfrac{\left(\begin{array}{c}384\,\lambda^{13} - 4064\,\lambda^{12} + 13552\,\lambda^{11} - 9903\,\lambda^{10} \\ -36474\,\lambda^9 + 91171\,\lambda^8 - 55009\,\lambda^7 - 75181\,\lambda^6 \\ +160347\,\lambda^5 - 129347\,\lambda^4 + 56934\,\lambda^3 - 14114\,\lambda^2 \\ +1780\,\lambda - 80\end{array}\right)}{5184\lambda^3(\lambda-1)^2(1-3\lambda)(\lambda^2-2\lambda+2)(\lambda^2-5\lambda+2)} & \text{if } \frac{2}{3} \leq \lambda \leq -1+\sqrt{3} \\ \dfrac{\left(\begin{array}{c}384\,\lambda^{12} - 4064\,\lambda^{11} + 11632\,\lambda^{10} + 5044\,\lambda^9 \\ -56144\,\lambda^8 + 35238\,\lambda^7 + 47947\,\lambda^6 - 74683\,\lambda^5 \\ +42124\,\lambda^4 - 12446\,\lambda^3 + 2244\,\lambda^2 - 320\,\lambda + 32\end{array}\right)}{5184(\lambda^2-4\,\lambda+2)(-3\,\lambda+1)(-5\,\lambda+\lambda^2+2)\lambda^4} & \text{if } -1+\sqrt{3} \leq \lambda \leq 1 \end{cases}$$

$$P_{Op}\left(\tilde{F}'_{\lambda}\right)=\begin{cases}\dfrac{\begin{pmatrix}(2007040\,\lambda^{16}-31567968\,\lambda^{15}+254487636\,\lambda^{14}\\-1673042594\,\lambda^{13}+7891593436\,\lambda^{12}-25242771985\,\lambda^{11}\\+56041630779\,\lambda^{10}-89218569797\,\lambda^{9}+104324126150\,\lambda^{8}\\-90845074086\,\lambda^{7}+59183341011\,\lambda^{6}-28718089245\,\lambda^{5}\\+10225905993\,\lambda^{4}-2593961334\,\lambda^{3}+443687625\,\lambda^{2}\\-45867951\,\lambda+2165130)\lambda\end{pmatrix}}{\begin{pmatrix}432\left(-7\,\lambda+3+\lambda^{2}\right)\left(7\,\lambda-3\right)^{2}\left(8\,\lambda-3\right)\left(-1+\lambda\right)^{4}\\\left(-3+\lambda\right)\left(3+2\,\lambda^{2}-4\,\lambda\right)\left(2\,\lambda-3\right)\left(4\,\lambda-3\right)^{2}\left(-3+5\,\lambda\right)\end{pmatrix}} & if\ \ 0\le\lambda\le\frac{1}{4}\\[2ex]\dfrac{\begin{pmatrix}25088\,\lambda^{15}+209394\,\lambda^{14}-6426216\,\lambda^{13}+40984961\,\lambda^{12}\\-135428896\,\lambda^{11}+282711391\,\lambda^{10}-405851583\,\lambda^{9}+418674743\,\lambda^{8}\\-317925857\,\lambda^{7}+179789199\,\lambda^{6}-75771612\,\lambda^{5}+23495913\,\lambda^{4}\\-5184162\,\lambda^{3}+759618\,\lambda^{2}-64152\,\lambda+2187\end{pmatrix}}{\begin{pmatrix}432\lambda\,\left(-7\,\lambda+3+\lambda^{2}\right)\left(7\,\lambda-3\right)^{2}\left(-1+\lambda\right)^{4}\\\left(3+2\,\lambda^{2}-4\,\lambda\right)\left(-3+\lambda\right)\left(4\,\lambda-3\right)^{2}\end{pmatrix}} & if\ \ \frac{1}{4}\le\lambda\le\frac{1}{3}\\[2ex]\dfrac{\begin{pmatrix}32\,\lambda^{12}-552\,\lambda^{11}+3636\,\lambda^{10}-13918\,\lambda^{9}\\+33134\,\lambda^{8}-51899\,\lambda^{7}+55638\,\lambda^{6}-42130\,\lambda^{5}\\+23152\,\lambda^{4}-9234\,\lambda^{3}+2520\,\lambda^{2}-405\,\lambda+27\end{pmatrix}}{432(-7\,\lambda+3+\lambda^{2})(-3+\lambda)(3+2\,\lambda^{2}-4\,\lambda)\lambda^{2}(-1+\lambda)^{4}} & if\ \ \frac{1}{3}\le\lambda\le-1+\sqrt{2}\\[2ex]\dfrac{\begin{pmatrix}64\,\lambda^{12}-606\,\lambda^{11}+2412\,\lambda^{10}-5345\,\lambda^{9}\\+7132\,\lambda^{8}-5470\,\lambda^{7}+1662\,\lambda^{6}+664\,\lambda^{5}\\-538\,\lambda^{4}-138\,\lambda^{3}+234\,\lambda^{2}-81\,\lambda+9\end{pmatrix}}{432(1+2\,\lambda)(-3+\lambda)(3+2\,\lambda^{2}-4\,\lambda)\lambda^{3}(-1+\lambda)^{4}} & if\ \ -1+\sqrt{2}\le\lambda\le\frac{1}{2}\\[2ex]\dfrac{\begin{pmatrix}301056\,\lambda^{16}-3833088\,\lambda^{15}+13750656\,\lambda^{14}-6734912\,\lambda^{13}\\-32548680\,\lambda^{12}+4100776\,\lambda^{11}+77490304\,\lambda^{10}-39137456\,\lambda^{9}\\-54511130\,\lambda^{8}+47314071\,\lambda^{7}+7194988\,\lambda^{6}-20050104\,\lambda^{5}\\+6805017\,\lambda^{4}+521721\,\lambda^{3}-819072\,\lambda^{2}+184680\,\lambda-13851\end{pmatrix}}{\begin{pmatrix}31104\left(-7\,\lambda+3+\lambda^{2}\right)\left(7\,\lambda-3\right)^{2}\left(8\,\lambda-3\right)\\\left(1+2\,\lambda\right)\left(3-\lambda\right)\lambda^{4}\left(-1+\lambda^{2}\right)\end{pmatrix}} & if\ \ \frac{1}{2}\le\lambda\le\frac{3}{5}\\[2ex]\dfrac{\begin{pmatrix}1505280\,\lambda^{15}-9230592\,\lambda^{14}+7289472\,\lambda^{13}+36742208\,\lambda^{12}\\-45946536\,\lambda^{11}-19131136\,\lambda^{10}+37859624\,\lambda^{9}-4031512\,\lambda^{8}\\+1437614\,\lambda^{7}-15432807\,\lambda^{6}+10976459\,\lambda^{5}-956646\,\lambda^{4}\\-1752966\,\lambda^{3}+804843\,\lambda^{2}-142155\,\lambda+9234\end{pmatrix}}{\begin{pmatrix}31104\,(1+2\,\lambda)\,(5\,\lambda-2)\,(8\,\lambda-3)\\\left(7\,\lambda-3\right)^{2}\left(3-\lambda\right)\lambda^{4}\left(-1+\lambda^{2}\right)\end{pmatrix}} & if\ \ \frac{3}{5}\le\lambda\le\frac{3}{4}\\[2ex]\dfrac{\begin{pmatrix}1505280\,\lambda^{16}-13558272\,\lambda^{15}+31075584\,\lambda^{14}+23913152\,\lambda^{13}\\-107966000\,\lambda^{12}-66112144\,\lambda^{11}+241896312\,\lambda^{10}-86654760\,\lambda^{9}\\-115938132\,\lambda^{8}+116580298\,\lambda^{7}-34211530\,\lambda^{6}-7077125\,\lambda^{5}\\+10409724\,\lambda^{4}-4974300\,\lambda^{3}+1430676\,\lambda^{2}-236925\,\lambda+17010\end{pmatrix}}{\begin{pmatrix}31104\,(4\,\lambda-1)\,(5\,\lambda-2)\,(1+2\,\lambda)\,(7\,\lambda-3)^{2}\\\left(-8\,\lambda+3+2\,\lambda^{2}\right)\left(3-\lambda\right)\left(1+\lambda\right)\lambda^{4}\end{pmatrix}} & if\ \ \frac{3}{4}\le\lambda\le1\end{cases}$$

$$P_{Rd}\left(\tilde{F}'_{\lambda}\right) = \begin{cases} 0 & \text{if } \ 0 \le \lambda \le \frac{1}{3} \\ \dfrac{\begin{pmatrix} (1728\,\lambda^{11} - 11808\,\lambda^{10} + 37239\,\lambda^{9} - 71792\,\lambda^{8} \\ +94302\,\lambda^{7} - 88938\,\lambda^{6} + 61735\,\lambda^{5} - 31733\,\lambda^{4} \\ +11936\,\lambda^{3} - 3165\,\lambda^{2} + 541\,\lambda - 46)\,(3\,\lambda - 1) \end{pmatrix}}{-432\,(6\,\lambda^{2}-5\,\lambda+2)(2-4\,\lambda+3\,\lambda^{2})(3\,\lambda-2)^{2}(\lambda-1)^{4}} & \text{if } \ \frac{1}{3} \le \lambda \le \frac{1}{2} \\ \dfrac{\begin{pmatrix} 606528\,\lambda^{15} - 2068416\,\lambda^{14} + 1369872\,\lambda^{13} + 2481408\,\lambda^{12} - 4290615\,\lambda^{11} \\ +1741803\,\lambda^{10} + 1309122\,\lambda^{9} - 2209356\,\lambda^{8} + 1802404\,\lambda^{7} - 1199730\,\lambda^{6} \\ +653374\,\lambda^{5} - 254468\,\lambda^{4} + 63391\,\lambda^{3} - 9125\,\lambda^{2} + 660\,\lambda - 20 \end{pmatrix}}{-3888\,\lambda^{3}(6\,\lambda^{2}-5\,\lambda+2)(3\,\lambda^{2}+\lambda-1)(3\,\lambda-2)(3\,\lambda-1)(\lambda^{2}-1)} & \text{if } \ \frac{1}{2} \le \lambda \le \frac{5}{9} \\ \dfrac{\begin{pmatrix} 1819584\,\lambda^{15} - 7822656\,\lambda^{14} + 11040624\,\lambda^{13} - 2001024\,\lambda^{12} - 10285173\,\lambda^{11} \\ +8806665\,\lambda^{10} + 4695795\,\lambda^{9} - 14405958\,\lambda^{8} + 13332827\,\lambda^{7} - 7148232\,\lambda^{6} \\ +2404127\,\lambda^{5} - 479830\,\lambda^{4} + 40934\,\lambda^{3} + 2855\,\lambda^{2} - 750\,\lambda + 20 \end{pmatrix}}{-3888\,\lambda^{3}(9\,\lambda^{2}-8\,\lambda+2)(6\,\lambda^{2}-5\,\lambda+2)(3\,\lambda-1)^{2}(\lambda^{2}-1)} & \text{if } \ \frac{5}{9} \le \lambda \le \frac{2}{3} \\ \dfrac{\begin{pmatrix} 245643840\,\lambda^{21} - 649539000\,\lambda^{20} - 402399252\,\lambda^{19} + 2752929990\,\lambda^{18} \\ -1899488232\,\lambda^{17} - 3692770641\,\lambda^{16} + 6966396981\,\lambda^{15} - 2123626086\,\lambda^{14} \\ -5490784557\,\lambda^{13} + 7663459545\,\lambda^{12} - 4380347997\,\lambda^{11} + 783081786\,\lambda^{10} \\ +538407255\,\lambda^{9} - 420896863\,\lambda^{8} + 120550128\,\lambda^{7} - 5104996\,\lambda^{6} \\ -8231734\,\lambda^{5} + 3381167\,\lambda^{4} - 750364\,\lambda^{3} + 102906\,\lambda^{2} - 7012\,\lambda + 48 \end{pmatrix}}{\begin{pmatrix} -11664\,\left(3\,\lambda^{2} - 4\,\lambda + 3\right)\left(-4\,\lambda + 6\,\lambda^{2} + 1\right)\left(3\,\lambda - 2 + 3\,\lambda^{2}\right) \\ (5\,\lambda - 2)\,\lambda^{4}\,(3\,\lambda + 1)^{2}\,(3\,\lambda - 1)^{2}\left(\lambda^{2} - 1\right) \end{pmatrix}} & \text{if } \ \frac{2}{3} \le \lambda \le \frac{5}{6} \\ \dfrac{\begin{pmatrix} 1228219200\,\lambda^{21} - 3436651800\,\lambda^{20} + 59915052\,\lambda^{19} + 8331354630\,\lambda^{18} \\ -7425083700\,\lambda^{17} - 6659800641\,\lambda^{16} + 13401783315\,\lambda^{15} + 113558922\,\lambda^{14} \\ -17064597591\,\lambda^{13} + 18914133177\,\lambda^{12} - 9318096090\,\lambda^{11} + 979704618\,\lambda^{10} \\ +1559208630\,\lambda^{9} - 1040317061\,\lambda^{8} + 293239879\,\lambda^{7} - 13617760\,\lambda^{6} \\ -21825567\,\lambda^{5} + 9839845\,\lambda^{4} - 2330476\,\lambda^{3} + 322646\,\lambda^{2} - 21132\,\lambda + 144 \end{pmatrix}}{\begin{pmatrix} -11664\,(\lambda + 1)\left(3\,\lambda^{2} - 4\,\lambda + 3\right)(-3 + 5\,\lambda)\left(-4\,\lambda + 6\,\lambda^{2} + 1\right) \\ \left(3\,\lambda - 2 + 3\,\lambda^{2}\right)(5\,\lambda - 2)\,\lambda^{4}\,(3\,\lambda + 1)^{2}\,(3\,\lambda - 1)^{2} \end{pmatrix}} & \text{if } \ \frac{5}{6} \le \lambda \le 1 \end{cases}$$

$$P_{Pe}\left(\tilde{F}'_{\lambda}\right)=\begin{cases}\dfrac{\left(\begin{array}{c}182784\,\lambda^{22}-7413888\,\lambda^{21}+99868080\,\lambda^{20}-703368340\,\lambda^{19}\\+3058365062\,\lambda^{18}-8892589638\,\lambda^{17}+18279260681\,\lambda^{16}\\-30012537373\,\lambda^{15}+55534171593\,\lambda^{14}-143651776680\,\lambda^{13}\\+374612620377\,\lambda^{12}-781360184994\,\lambda^{11}+1250819876673\,\lambda^{10}\\-1545441575841\,\lambda^{9}+1484488185678\,\lambda^{8}-1108937998494\,\lambda^{7}\\+638860202676\,\lambda^{6}-278530036749\,\lambda^{5}+88854888168\,\lambda^{4}\\-19553977935\,\lambda^{3}+2650355316\,\lambda^{2}-166518180\,\lambda\end{array}\right)}{\left(\begin{array}{c}3888\,\left(-9+\lambda^{2}\right)\left(-3+2\,\lambda\right)^{2}\left(3-2\,\lambda+\lambda^{2}\right)\left(\lambda-1\right)^{4}\left(-5\,\lambda+3\,\lambda^{2}+3\right)\\\left(2\,\lambda^{2}-3\,\lambda+3\right)\left(2\,\lambda^{2}-6\,\lambda+3\right)\left(3+\lambda^{2}-5\,\lambda\right)\left(4\,\lambda-3\right)\left(-3+7\,\lambda\right)\end{array}\right)} & \textit{if}\ \ 0\leq\lambda\leq\frac{1}{3}\\[2ex]\dfrac{\left(\begin{array}{c}531441-9034497\,\lambda-13742370009\,\lambda^{9}+8895112200\,\lambda^{8}\\-5952314889\,\lambda^{7}+3386845791\,\lambda^{6}-1632035628\,\lambda^{5}\\+694868949\,\lambda^{4}-246313062\,\lambda^{3}+62001450\,\lambda^{2}\\-51440504634\,\lambda^{11}+26349425253\,\lambda^{10}+81371993442\,\lambda^{12}\\-96382478184\,\lambda^{13}+10247568\,\lambda^{23}-93984692\,\lambda^{22}\\+530222012\,\lambda^{21}-1972391674\,\lambda^{20}+4914002198\,\lambda^{19}\\-7592752426\,\lambda^{18}+3746081246\,\lambda^{17}+14820695230\,\lambda^{16}\\-48580336425\,\lambda^{15}+82863073740\,\lambda^{14}-595152\,\lambda^{24}+13056\,\lambda^{25}\end{array}\right)}{\left(\begin{array}{c}3888\,\left(-9+\lambda^{2}\right)\left(-3+2\,\lambda\right)^{2}\left(3-2\,\lambda+\lambda^{2}\right)\left(\lambda-1\right)^{4}\lambda^{2}\left(-7\,\lambda+3+\lambda^{2}\right)\\\left(2\,\lambda^{2}-3\,\lambda+3\right)\left(2\,\lambda^{2}-6\,\lambda+3\right)\left(3+\lambda^{2}-5\,\lambda\right)\left(1+2\,\lambda\right)\left(-5\,\lambda+3\,\lambda^{2}+3\right)\end{array}\right)} & \textit{if}\ \ \frac{1}{3}\leq\lambda\leq1-\frac{\sqrt{10}}{5}\\[2ex]\dfrac{\left(\begin{array}{c}-1594323+38263752\,\lambda-588606151086\,\lambda^{9}+3181533876\,\lambda^{8}\\+146651512095\,\lambda^{7}-108880376742\,\lambda^{6}+47802094434\,\lambda^{5}\\-14368806513\,\lambda^{4}+3017758194\,\lambda^{3}-430998651\,\lambda^{2}-3336632295732\,\lambda^{11}\\+1758731414013\,\lambda^{10}-1837680\,\lambda^{26}+39168\,\lambda^{27}+4787900328618\,\lambda^{12}\\-5495430514569\,\lambda^{13}+1849248554\,\lambda^{23}-6918294320\,\lambda^{22}\\+15209372665\,\lambda^{21}-5929711070\,\lambda^{20}-97174000058\,\lambda^{19}\\+448443199831\,\lambda^{18}-1227439450120\,\lambda^{17}+2482849588620\,\lambda^{16}\\-3968650335033\,\lambda^{15}+5155072480809\,\lambda^{14}-315369348\,\lambda^{24}+32898312\,\lambda^{25}\end{array}\right)}{\left(\begin{array}{c}3888\,\left(-3+2\,\lambda\right)^{2}\left(\lambda-1\right)^{4}\left(-4+3\,\lambda\right)\lambda^{3}\left(-5\,\lambda+3\,\lambda^{2}+3\right)\\\left(2\,\lambda^{2}-3\,\lambda+3\right)\left(-7\,\lambda+3+\lambda^{2}\right)\left(1+2\,\lambda\right)\left(3+\lambda^{2}-5\,\lambda\right)\\\left(2\,\lambda^{2}-6\,\lambda+3\right)\left(3-2\,\lambda+\lambda^{2}\right)\left(-9+\lambda^{2}\right)\end{array}\right)} & \textit{if}\ \ 1-\frac{\sqrt{10}}{5}\leq\lambda\leq-1+\sqrt{2}\\[2ex]\dfrac{\left(\begin{array}{c}-4369626\,\lambda-96556789584\,\lambda^{9}+67892860719\,\lambda^{8}-39049896447\,\lambda^{7}\\+18037355157\,\lambda^{6}-6531899778\,\lambda^{5}+1798835931\,\lambda^{4}-362167200\,\lambda^{3}\\+50388480\,\lambda^{2}-111685140678\,\lambda^{11}+113755653873\,\lambda^{10}\\+91199583777\,\lambda^{12}-61188027975\,\lambda^{13}+177147+19584\,\lambda^{23}-738624\,\lambda^{22}\\+9092208\,\lambda^{21}-57219708\,\lambda^{20}+209625442\,\lambda^{19}-410817890\,\lambda^{18}\\-58130301\,\lambda^{17}+3379320706\,\lambda^{16}-13234824797\,\lambda^{15}+32807109456\,\lambda^{14}\end{array}\right)}{\left(\begin{array}{c}3888\,\left(-3+2\,\lambda\right)^{2}\left(3+\lambda\right)\left(\lambda-1\right)^{4}\left(-4+3\,\lambda\right)\lambda^{3}\left(-5\,\lambda+3\,\lambda^{2}+3\right)\\\left(3+\lambda^{2}-5\,\lambda\right)\left(2\,\lambda^{2}-6\,\lambda+3\right)\left(2\,\lambda^{2}-3\,\lambda+3\right)\left(3-2\,\lambda+\lambda^{2}\right)\end{array}\right)} & \textit{if}\ \ -1+\sqrt{2}\leq\lambda\leq\frac{1}{2}\\[2ex]\dfrac{\left(\begin{array}{c}1741824\,\lambda^{18}-10907136\,\lambda^{17}-758016\,\lambda^{16}+168682080\,\lambda^{15}\\-542778056\,\lambda^{14}+381266732\,\lambda^{13}+2078569192\,\lambda^{12}-7655867648\,\lambda^{11}\\+13684744844\,\lambda^{10}-15725654684\,\lambda^{9}+12680394759\,\lambda^{8}-7818174426\,\lambda^{7}\\+4168442997\,\lambda^{6}-1906236342\,\lambda^{5}+503338293\,\lambda^{4}+66250062\,\lambda^{3}\\-98990181\,\lambda^{2}+28811538\,\lambda-2886840\end{array}\right)}{\left(\begin{array}{c}46656\,\left(3+\lambda\right)\left(2\,\lambda^{2}-3\,\lambda+3\right)\left(3-2\,\lambda+\lambda^{2}\right)\left(\lambda-1\right)\left(-7\,\lambda+3\right)\\\left(1+3\,\lambda\right)\left(-4+3\,\lambda\right)\left(-3+2\,\lambda\right)^{2}\lambda^{4}\end{array}\right)} & \textit{if}\ \ \frac{1}{2}\leq\lambda\leq\frac{3}{5}\end{cases}$$

$$
= \begin{cases}
\dfrac{\begin{pmatrix} 1244160\,\lambda^{20} - 16464384\,\lambda^{19} + 57270528\,\lambda^{18} + 83844384\,\lambda^{17} \\ -1061276248\,\lambda^{16} + 2693408772\,\lambda^{15} - 664982416\,\lambda^{14} - 12981572404\,\lambda^{13} \\ +38640837008\,\lambda^{12} - 58851878656\,\lambda^{11} + 52663055577\,\lambda^{10} - 23828524786\,\lambda^{9} \\ -886000740\,\lambda^{8} + 5706458679\,\lambda^{7} - 293767497\,\lambda^{6} - 2264168214\,\lambda^{5} \\ +1151368020\,\lambda^{4} - 78824583\,\lambda^{3} - 98240040\,\lambda^{2} + 31195368\,\lambda - 2939328 \end{pmatrix}}{\begin{pmatrix} 46656\,(3+\lambda)\left(2\,\lambda^{2}-3\,\lambda+3\right)\left(3-2\,\lambda+\lambda^{2}\right)\left(-7\,\lambda+3+\lambda^{2}\right) \\ (5\,\lambda-2)\,(1+3\,\lambda)\,(-\lambda+1)\,(-4+3\,\lambda)\,(-3+2\,\lambda)^{2}\,\lambda^{4} \end{pmatrix}} & \textit{if} \quad \frac{3}{5} \le \lambda \le \frac{5}{7} \\
\dfrac{\begin{pmatrix} 414720\,\lambda^{19} - 4935168\,\lambda^{18} + 12509952\,\lambda^{17} + 50678584\,\lambda^{16} \\ -389881692\,\lambda^{15} + 1122604750\,\lambda^{14} - 1889543531\,\lambda^{13} + 2231491822\,\lambda^{12} \\ -2814863390\,\lambda^{11} + 4920482481\,\lambda^{10} - 7665836555\,\lambda^{9} + 7933448364\,\lambda^{8} \\ -4689397074\,\lambda^{7} + 925995555\,\lambda^{6} + 597329316\,\lambda^{5} - 421617150\,\lambda^{4} \\ +71923140\,\lambda^{3} + 15707034\,\lambda^{2} - 7247718\,\lambda + 734832 \end{pmatrix}}{\begin{pmatrix} 46656\,(3+\lambda)\left(2\,\lambda^{2}-3\,\lambda+3\right)\left(3-2\,\lambda+\lambda^{2}\right)(\lambda-1) \\ \lambda^{4}\,(-3+2\,\lambda)^{2}\left(-7\,\lambda+3+\lambda^{2}\right)(-5\,\lambda+2)\,(1+3\,\lambda) \end{pmatrix}} & \textit{if} \quad \frac{5}{7} \le \lambda \le \frac{3}{4} \\
\dfrac{\begin{pmatrix} 414720\,\lambda^{20} - 7008768\,\lambda^{19} + 36356352\,\lambda^{18} + 118840\,\lambda^{17} - 701475188\,\lambda^{16} \\ +3173845802\,\lambda^{15} - 7336790313\,\lambda^{14} + 9998128561\,\lambda^{13} - 7604183170\,\lambda^{12} + 2406538637\,\lambda^{11} \\ -113742558\,\lambda^{10} + 74001411\,\lambda^{9} + 3583353636\,\lambda^{8} - 9901165347\,\lambda^{7} + 11050921233\,\lambda^{6} \\ -6093026712\,\lambda^{5} + 1307401722\,\lambda^{4} + 295874856\,\lambda^{3} - 242061534\,\lambda^{2} + 54062640\,\lambda - 4330260 \end{pmatrix}}{\begin{pmatrix} 46656\,\left(3-2\,\lambda+\lambda^{2}\right)(3+\lambda)\,\lambda^{4}\left(2\,\lambda^{2}-3\,\lambda+3\right)(-3+2\,\lambda)^{2} \\ \left(-6\,\lambda+3+\lambda^{2}\right)\left(-7\,\lambda+3+\lambda^{2}\right)(-5\,\lambda+2)\,(1+3\,\lambda) \end{pmatrix}} & \textit{if} \quad \frac{3}{4} \le \lambda 1
\end{cases}
$$