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# Chapter 3

## The Theory of Mathematical Working Spaces in Brief

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**Abstract.** This chapter contains an overview of the main points of the theory of MWS in relation to frequently asked questions. The aim is to form the basis of an online glossary of the theory, to be completed according to the contributions of researchers.

### Main purpose of the theory:

Describing, understanding and (trans)forming mathematical work in a school context.

### 3.1 Mathematical work

In the theory of MWS, *mathematical work* must be understood as an on-going intellectual human process of production, the orientation and finality of which are defined and supported by mathematics and more generally by mathematical culture.

Mathematical work is intended to perform a task, solve a problem or overcome an obstacle in relation to mathematics. It requires the sustained mobilization and development of material or intellectual resources derived from mathematical culture (1.3.1).

A reasonable goal of mathematics education is that, at the end of their studies, students should have a good idea and practice of mathematical work (7.2.2).

#### 3.1.1 Purpose, processes and results

Mathematical work involves consideration of three aspects related to its performance and development (1.3.1):

*The purpose of the work.* This attribute helps us to distinguish the notion of ‘work’ from a simple activity. This is done by assigning a goal to an action,

clarifying the stakes associated with it, and inscribing it in the long term by demonstrating its general relevance.

*The processes of the work.* These are related to the procedures and constraints of implementing given tasks. Appropriation of the goals of the task and compliance with the rules of functioning are important aspects for observing and characterizing the work.

*The results of the work.* Results should be valid and coherent within the mathematical domain in question.

### **3.2 Epistemological and cognitive aspects of mathematical work**

Through the lens of MWS, the development by an individual, whether generic or not, of appropriate mathematical work is a gradual and progressively constructed process of bridging epistemological and cognitive planes.

#### **3.2.1 The epistemological plane and its components (1.4.2.1)**

The epistemological plane, organized according to purely mathematical criteria, has three interacting components: a set of concrete and tangible objects (representamen); a set of artefacts such as drawing tools or software; and a theoretical system of reference (the referential) based on definitions, properties and theorems.

##### *Signs or representamen*

The sign, or representamen, is “something which stands to somebody for something in some respect or capacity” (Peirce). Signs or representamen summarize the component using concrete and tangible symbols. Depending on the mathematical field at stake, the signs used can be geometrical images, algebraic symbols, graphics, or even tokens, models or photos in the case of problems involving modeling.

##### *Artifacts*

An artifact includes everything that has undergone a transformation, however small, of human origin. Its meaning is not limited to material objects and can include a strong symbolic dimension. In the theory of MWS, mathematical artifacts will generally be associated with material objects to avoid confusion with other components of the epistemological plane. These objects will be intimately linked to rules and techniques of construction or calculation (Euclidean division, "classical" constructions with ruler and compass, etc.). These techniques are based on algorithms, the validation and theoretical status of which are no longer problematic or are no longer questioned by their user.

##### *Theoretical frame of reference or referential*

The referential includes the set of properties, definitions, theorems and axioms and refers to the theoretical part of the mathematical work. This set is not only

a collection of properties: it supports the deductive discourse of proof specific to mathematics, and this requires that it should be organized in a coherent way, well-adapted to the tasks the students are requested to solve and model. The referential is not absolute and determined in a unique way once and for all; its constitution and organization depend, in particular, on the mathematical paradigms and work targeted by the educational institutions or organizations.

### **3.2.2 The cognitive plane and cognitive processes (1.4.2.2)**

The cognitive plane relates to the processes and procedures used by individuals in task-solving activities. It is structured around three cognitive processes: visualization, construction and proving.

#### *Visualization*

The process of *visualization* can be considered as the cognitive process associated with identifying and developing a set of treatments and transformations of the representamen or signs –which, in this respect, can very well be perceived as acoustic images in strictly oral contexts. It is thus possible to study the role of signs in different mathematical domains, such as mathematical analysis, which give rise to a specific work of visualization based on spreadsheet data or numerical writings. This visualization must be distinguished from the simple vision or perception of signs.

#### *Construction*

The process of *construction* is associated with actions triggered by the use of artifacts, actions which lead to tangible results such as figures, graphs or the results of a calculation based on algorithms or produced by a numerical machine. These results are concrete products of some mathematical entities which are generally not isolated and form configurations to be observed and explored (8.5.1).

#### *Proving*

The process of *proof* considered here relies on deductive and logical discourse and must be based on, or lead to, assertions with a clear theoretical status. In contrast, simple empirical validation is more likely to relate to construction or visualization, as discussed above. In addition to deductively organized arguments, these assertions may well be definitions, hypotheses, conjectures, or statements of counterexamples.

### **3.3 Mathematical Working Spaces – MWS**

A Mathematical Working Space refers to an abstract structure organized in such a way as to account for mathematical work and enable individuals to conduct operations with problems in a specific domain (geometry, probability, etc.).

The theory of MWS aims to highlight the mathematical work proposed by educational institutions and organizations involved in the teaching and learning of mathematics.

### 3.3.1 The MWS diagram

The MWS diagram gives a general overview of the different relationships and interactions existing between the components of the epistemological plane and cognitive plane of the Mathematical Working Space. The diagram is closely associated with the development of the theory, of which it is in a way the emblem, but should not be confused with the theory as a whole.

Its use as a methodological tool requires selection of the most appropriate modes of representation to reflect the evolution of the work (3.8.3).

### 3.3.2 The three mathematical work geneses in the MWS

Development by an individual, whether generic or not, of appropriate mathematical work is a gradual process which is progressively developed, as a process of bridging the epistemological and cognitive planes. It evolves in accordance with different specific yet intertwined generative developments: semi-otic, instrumental and discursive geneses.

Each of these geneses can be visualized through one of the dimensions of the MWS diagram.

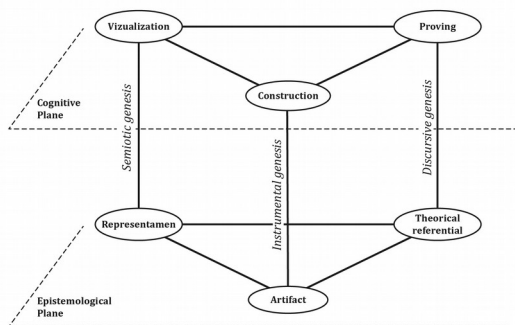


Fig. 3.1 The Mathematical Working Space diagram

### 3.3.3 Top-down and bottom-up perspectives on geneses

Each of these geneses can be described according to two perspectives: one stance focuses on epistemological components (top-down perspective) and a second oriented towards cognitive processes (bottom-up perspective). The terms bottom-up and top-down are only relative to the position of the epistemological and cognitive planes in the MWS diagram.

### **3.3.4 The Semiotic genesis**

The *Semiotic genesis* relates signs and representamen with visualization. It explains the dialectical relationship between syntactic and semantic perspectives on mathematical objects, represented and organized by semiotic systems. The semiotic genesis confers to the representamen their status of operational mathematical objects. In this way, it establishes the links between the function and structure of the expressed signs. The bottom-up perspective associated with this genesis can be considered as a process of decoding and interpreting signs from a given representamen through visualization. The top-down perspective, directed from visualization towards a representamen, can be evoked when a sign is produced or specified.

### **3.3.5 The Instrumental genesis**

The *instrumental genesis* establishes the link between the artifacts and the construction processes contributing to the completion of mathematical work. The bottom-up perspective describes the actions by which the user appropriates the various techniques associated with the artifact. The bottom-down perspective relates to the adequate choice of the artifact required to perform the intended actions. In the theory of MWS, the two perspectives, epistemological and cognitive, are essential because if using the artifacts modifies the users' way of doing and thinking, in return, the mathematics they develop will also be transformed.

### **3.3.6 The Discursive genesis of proof**

The *discursive genesis* of proof is the process by which the properties and results organized in the referential are actuated in order to be available for mathematical reasoning and discursive validations. By this we mean validations going beyond graphic, empirical or instrumented verification, but which could nevertheless be triggered by these. In this genesis, the bottom-up perspective is related to a deductive discourse of proof supported by properties structured into the referential. Conversely, the identification of properties and definitions to be included in the referential, possibly suggested by instrumental, computational or visual processes, relate to the top-down perspective.

### **3.3.7 Interactions between geneses and vertical planes**

The vertical planes (named Sem-Ins, Ins-Dis and Sem-Dis) make interactions between the different geneses visible: in particular, they enable expression of the main coordination games involved in the mathematical work and the geneses that are temporarily subordinated to such work.

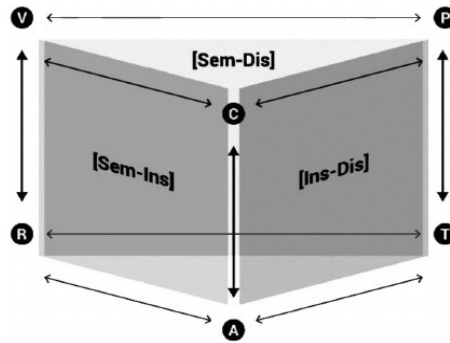


Fig. 3.2 The three vertical planes in the MWS diagram

A deep understanding of the different genes and the three planes of interaction between these genes ([Sem-Ins], [Ins-Dis] and [Sem-Dis]) is gained from looking at methodological investigation, but also at more specific and detailed studies of the proofs and validations used to solve tasks.

From a methodological perspective, recognition of active genes and interactions between genes is supported by the identification of tools (in the epistemological plane) and instruments (in the cognitive plane) used to perform the task (3.7).

### 3.4 Diversity of mathematical work associated with various MWS

#### 3.4.1 The reference mathematical work in connection with the reference MWS

The reference mathematical work corresponds to the work normally expected by educational institutions and organizations in charge of teaching mathematics. The reference MWS is naturally defined according to mathematical criteria. But cultural, economic, social and political criteria also contribute towards shaping it, to such an extent that describing it requires thorough studies: of the intended curriculum, of treatises written by mathematicians, of essays and articles by mathematics educators. This aspect in turn supposes that epistemological vigilance is required. It means that the operating rules involved in this MWS enable knowledge to be organized into a well-defined and coherent mathematical field. The set of rules, procedures and questioning which help to define the reference work refers to the notion of paradigm (3.5). Pragmatically in educational contexts, the constitution of a MWS does not rely on a single paradigm, but rather on the interplay between different paradigms.

### **3.4.2 The personal mathematical work and personal MWS**

The personal work and personal MWS will depend on the actors and subjects implicated in the work --a pupil or student, a teacher or teacher trainer. This MWS reflects the reality of the work of individuals who perform a task with their own knowledge and cognitive capacities. The personal work is very fluid and its evolution depends on the tasks the subjects are led to perform.

### **3.4.3 The idone or suitable MWS**

This MWS refers to an intermediate state of transmission and mediation of knowledge where tension exists between the teacher's expectations and the re-definition of the task and problems in order to enable students' personal work to progress. The MWS in question is designated as suitable or idone MWS precisely to emphasize these transformations by which it is characterized in order to best adjust to the educational project and to the reality of the learners.

It can be seen as a connecting structure from reference MWS to personal MWS mediated by designer-actors (teacher or researcher) to help receptor-actors (students) build their personal mathematical work.

It is necessary to consider two states of this MWS: a potential state, which corresponds to what the teacher foresees, and an actual or effective state, which comprises the work produced in the classroom.

These three MWS are not independent but are rather interrelated (4.2). Each requires a specific methodology which must be understood (Part II).

### **3.4.4 Mathematical domain and specific MWS**

A mathematical domain should be organized into a consistent structure within mathematics. The best proof that a given segment of mathematics should be called a domain is that it is regarded as such by the community of mathematicians.

In this way, a specific MWS is associated with each domain. Thus, we may speak of a  $MWS_{Geometry}$ ,  $MWS_{Algebra}$ ,  $MWS_{Analysis}$ ,  $MWS_{Probability}$ , etc.

It is also worth reflecting upon the way new transversal themes — such as Algorithmics or Modeling — may be integrated into MWS, and what kind of reorganization of the framework is necessary to foster this integration.

## **3.5 Paradigms in the theory of MWS**

A paradigm stands for the combination of beliefs, convictions, techniques, methods and values shared by a community. This community can be scientific, but in the theory of MWS it will be educational organizations engaged in mathematics education.

In the theory, different paradigms can exist and coexist. Identifying these paradigms and their interactions help us to understand and orient mathematical work.



Paradigms ensure the conformity of work processes and the accuracy of the results produced. Mathematical work is said to be compliant when the processes and procedures used are valid and consistent with the expectations present in the preferred paradigm(s) of work.

The search for paradigms associated with the MWS specific to a mathematical domain is an important focus for the theory of MWS.

### **3.6 Mathematical Tasks**

#### **3.6.1 Mathematical tasks and the learner's work**

In the theory of MWS, tasks must be interpreted in a broad and varied sense. They can thus be associated with traditional problems formulated with clear assumptions and known to be solvable in a predictable amount of time by students. They can also be given with relatively vague outlines as support for open and rich problematization or modeling situations. In this way, a range of tasks, from simple to rich tasks, exists. Performance of the tasks depends on their level of mathematical sophistication and on the degree of initiative the students are free to use.

Tasks are the essential units which trigger the subject's activity. More precisely, mathematical actions with well-determined goals are seen as the visible manifestation of a subject's cognitive activity (or mathematical thinking) when actively performing tasks. From analysis of the mathematical actions identified being performed by subjects, it becomes a matter of identifying their mathematical work with the idea of adapting or transforming it.

#### **3.6.2 Task environment and mathematical work**

A task is determined by its goals and the conditions necessary for its performance which can be individual or collective.

In mathematics education, tasks are conceived and implemented in various environments, which can relate to different didactical and theoretical traditions such as task design, lesson studies or didactical engineering.

The tasks within their environment and the mathematical actions performed form a set of observables which enable the mathematical work to be investigated. Studying these observables through the lens of the MWS enables generic forms of work in different mathematical domains to be characterized. In the long term, this characterization of forms serves as a solid basis for forming or transforming mathematical work to conform to the expectations of the given paradigm.

### 3.7 Tools and instruments of mathematical work

Solving a mathematical problem implies the use of different mathematical objects which can be considered as tools or instruments with which to perform the task.

In the theory, and whenever it is useful, the term *tool* is reserved for objects of the epistemological plane that have a specific potential use in the context of problem solving. Tools have a productive purpose for performing the task.

Use of the word *instrument* is prioritized as soon as a subject (pupil, student or teacher) has developed their use of a tool to effectively solve the proposed task. It supposes implementation of techniques by the active subject and thus concerns the cognitive aspects of the work.

In relation to the different components of the epistemological plane and the different geneses of mathematical work, it is possible to consider, a priori, three types of tools and instruments:

*Semiotic tools* are non-material tools – like diagrams, symbols or figures – that may be *used as instruments* for operating on semiotic representations of mathematical objects.

*Technological tools* are artifacts – like drawing tools or routinized techniques – which are “to hand”, based on algorithms or calculators with implemented calculation or drawing algorithms *used as instruments* for constructing mathematical objects such as figures, graphs, numbers, etc.

*Theoretical tools* are elements of the referential non-material tools like theorems which may be *used as instruments* for mathematical reasoning based on the logical and mathematical properties of mathematical objects.

### 3.8 The MWS diagram: a methodological tool to describe, characterize and transform mathematical work

#### 3.8.1 Describing the circulation of work

The individuals involved in performing a task produce a series of actions oriented by the goals to be reached and triggered by knowledge. The set of dynamics thus created enables us to speak of a circulation of work.

The MWS diagram makes it possible to describe and visualize the evolution and dynamics of the various interactions between geneses during the performance of a task or set of tasks by individuals (Chapter 2.). This dynamic is called circulation of work.

#### 3.8.2 Characterizing mathematical work

When the circulation of work engages all the geneses and the different planes of the diagram, the mathematical work will be considered complete.

The diagram can be used to demonstrate the completeness of the mathematical work. It also helps to identify the reasons for a blockage which may be due

to confinements in a vertical plane or in a genesis. It provides an insight into the rebounds of this work to break through blockages and confinements.

### **3.8.3 (Trans)forming the circulation of work**

By focusing on the various interactions which help or hinder mathematical work, the diagram can serve as a support for thinking about and rethinking the circulation of work (by tuning tasks and their implementation). In the long term, this can contribute towards forming or transforming the mathematical work.

## **3.9 Further aspects on specificity and originality of the theory**

### **3.9.1 Epistemological anchoring**

The theory of MWS is a didactic theory strongly supported by mathematical content. It achieves this by closely combining epistemological and cognitive aspects around the unifying notion of mathematical work.

### **3.9.2 A triadic approach**

The synthetic and triadic vision of mathematical work gives it coherence and facilitates its analysis. In this conception of mathematical work, it is the arrangement and relationships between the different epistemological components and cognitive processes which ensure the coherent and complete development of mathematical work. The elaboration of the latter is seen as harmonious interactions of three types of genesis: semiotic, instrumental and discursive.

### **3.9.3 Processes and results**

Work is based on a set of organized and finalized actions, which emphasizes the duality between, on the one hand, the processes of work and, on the other hand, the results of work.

### **3.9.5 Importance of the idea of ‘idonéité’**

The ‘suitability’ hallmark of a MWS is gradually constructed and adapted along the paths and vagaries of the teaching route. This explains our fondness for the French term ‘idoine’, the translation of which to ‘suitable’ in English may not fully convey the meaning suggested by Gonseth (1945–1952, our translation):

A metaphorical way of presenting suitability in the theory of MWS is to think about it as resulting from recursive feedback loops, converging towards a stable state when one exists. Instead of trying to reduce the uncertainty related to any new experience, Gonseth proposes a recursive adaptation to it.

### **3.9.6 The link with learning theories**

Accordingly, one of the main strengths of studies based on MWS is their investigation of the interactions between the cognitive and epistemological dimen-

sions. The underlying learning model is not, as in many theories which emerged in the 1970s, the socioconstructivist model.

As far as the MWS are concerned, there is no prescribed learning model. Studies can look at other learning principles such as Socrates' maieutics or behaviorism, or even approaches which emphasize the imitation of models presented to students.

### **3.9.7 Link to institutional choices**

The theory of MWS does not give priority to a determined institutional model with regards to the development of an epistemological basis, if only for the purpose of expecting — and this is no small matter — an approach which is globally coherent, in particular with respect to validation within a given paradigm or network of paradigms, meant to be consistent and rational. The notion of a paradigm is essential here and, in a certain way, mathematical work could be seen as a quest for harmonious articulation between coherent paradigms.

### **3.9.8 Points not addressed by the theory**

- The impact on the development of the mental apparatus of a subject of information and retroactions induced by the environment.
- The nature of the black box in precise reference to the psychological constitution of the geneses.
- The existence and possible place in the brain of all kinds of schemas which can intervene in the mathematical work.

These are issues which the theory has neither the ambition nor the means to explore. The theory relies on the visible effects of a subject's actions through explicit observables in order to understand cognitive processes. In this sense, it can be seen as a materialist-type scientific approach.

### **3.9.9 Dialogue with other theories**

The theory of MWS is not a holistic theory and has close links with theories which share its materialist approach on people's way of thinking when they are engaging in mathematics.

Moreover, the whole activity of teaching and learning mathematics is not limited to mathematical work. Many complementary and indispensable elements are external to this work yet participate in its undertaking in practice: affects and emotions, collaborative and collective work, language exchanges, cultural and social context, and so on.

These represent many points and perspectives which are not directly related to the core of the theory, but which can contribute to fruitful exchanges with specialists in these areas. These exchanges and interactions are part of the theoretical and methodological innovation which has characterized MWS research to date.