

NONEXISTENCE OF DEC SPIN FILL-INS

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ABSTRACT. In this note, we show that a closed spin Riemannian manifold does not admit a spin fill-in satisfying the dominant energy condition (DEC) if a certain generalized mean curvature function is point-wise large.

Given a closed n -dimensional Riemannian manifold (Σ, γ) , it is a very interesting question to know whether there exists an $(n + 1)$ -dimensional compact Riemannian manifold (Ω, g) with nonnegative scalar curvature (NNSC) whose boundary is isometric to (Σ, γ) . If so, the Riemannian manifold is called a fill-in of (Σ, γ) and the set of all such fill-ins, referred to as NNSC fill-ins of (Σ, γ) , is denoted by $\mathcal{F}(\Sigma, \gamma)$. The problem of the existence of such fill-ins has recently been solved by Shi, Wang and Wei [SWW22] where it is shown that if Σ is the boundary of an $(n + 1)$ -dimensional compact manifold Ω then, any metric γ on Σ can be extended to a Riemannian metric g on Ω with positive scalar curvature.

One can also try to find a fill-in whose mean curvature is prescribed by a smooth function H on Σ . This problem is tightly related to the Bartnik quasi-local mass [Bar89] and a triplet (Σ, γ, H) is then usually called a Bartnik data.

In [Mia21], Miao proved that if Σ is the boundary of some compact $(n+1)$ -dimensional manifold Ω , then given any Riemannian metric γ on Σ , there exists a constant H_0 , depending on γ and Ω , such that, if $\min_{\Sigma} H \geq H_0$, there does not exist NNSC fill-ins of (Σ, γ, H) . The proof makes use of the work of Shi, Wang and Wei [SWW22] and of Schoen and Yau's results on closed manifolds [SY79, SY19]. Such a result is also obtained for fill-ins with a negative scalar curvature lower bound.

This fact was previously demonstrated by Gromov [Gro18] for spin manifolds. More precisely, he showed that if (Ω, g) is a NNSC spin fill-in of (Σ, γ, H) , then

$$\min_{\Sigma} H \leq \frac{n}{\text{Rad}(\Sigma, \gamma)}$$

where $\text{Rad}(\Sigma, \gamma)$ is a constant only depending on (Σ, γ) and known as the hyperspherical radius of (Σ, γ) . The main remark here, which is the key point of our argument, is that the nonexistence of NNSC spin fill-ins can be obtained by taking another route. Indeed, it can be deduced from an eigenvalue estimate on the first eigenvalue $\lambda_1(\Sigma, \gamma)$ of the Dirac operator of (Σ, γ) proved by Hijazi, Montiel and Zhang [HMZ01] and which states that

$$\min_{\Sigma} H \leq 2|\lambda_1(\Sigma, \gamma)|. \tag{1}$$

Since $\lambda_1(\Sigma, \gamma)$ depends only on Σ , γ and the involved spin structure, the nonexistence of NNSC spin fill-ins is a direct consequence of (1). This inequality is sharp since round balls in the Euclidean space satisfy the equality case of (1). Note that a similar result can

be deduced for spin fill-ins with a negative scalar curvature lower bound from [HMR03] as in [Gro18].

In this note, we use the aforementioned observation to generalize this result in the context of spin fill-ins satisfying the dominant energy condition (DEC). Following [Bar97], a 5-uple $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$ is called a spacetime Bartnik data set if (Σ, γ) is an oriented, closed Riemannian manifold, H and \mathfrak{h} are smooth functions on Σ and α is a smooth 1-form on Σ . In this situation, a triplet (Ω, g, k) is a fill-in of such a spacetime Bartnik data if

- (1) (Ω, g, k) is a compact initial data set, that is (Ω, g) is an $(n + 1)$ -dimensional compact Riemannian manifold with boundary and k is a smooth symmetric $(0, 2)$ -tensor field on Ω ,
- (2) there exists an isometry $f : (\Sigma, \gamma) \rightarrow (\partial\Omega, g|_{\partial\Omega})$ such that
 - (a) $f^*H_g = H$, where H_g is the mean curvature of $\partial\Omega$ in (Ω, g) with respect to the outward unit normal $\tilde{\nu}$,
 - (b) $f^*(k(\tilde{\nu}, \cdot)^T) = \alpha$,
 - (c) $f^*(\text{Tr}_{g|_{\partial\Omega}} k) = \mathfrak{h}$.

Here $\text{Tr}_{g|_{\partial\Omega}}$ denotes the trace operator on $\partial\Omega$ and ω^T is the tangent part to Σ of a 1-form ω defined along Σ . In the following, we will omit the isometry f in the identification between Σ and $\partial\Omega$. Then, a fill-in (Ω, g, k) of $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$ satisfies the dominant energy condition, or is a DEC fill-in, if

$$\mu \geq |J|_g$$

where μ and J are respectively the mass density and the current density defined by

$$\mu = \frac{1}{2}(R_g + (\text{Tr}_g k)^2 - |k|_g^2)$$

and

$$J = \text{div}_g(k - (\text{Tr}_g k)g).$$

Here R_g and Tr_g denote respectively the scalar curvature and the trace operator of (Ω, g) . In this situation, a natural generalization of the mean curvature is given by the function

$$\mathcal{H}_g := H_g - \sqrt{|k(\tilde{\nu}, \cdot)^T|_g^2 + (\text{Tr}_{g|_{\partial\Omega}} k)^2}$$

which corresponds, for a spacetime Bartnik data, to

$$\mathcal{H} := f^*\mathcal{H}_g = H - \sqrt{|\alpha|_\gamma^2 + \mathfrak{h}^2}. \quad (2)$$

When Σ is endowed with a spin structure, we will say that (Ω, g, k) is a DEC spin fill-in of the spacetime Bartnik data $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$ if (Ω, g, k) is a DEC fill-in and if Ω is a spin manifold which induces the given spin structure on Σ . We then have the following result.

Theorem 1. *Given any Riemannian metric γ on an n -dimensional spin manifold Σ , there exists a constant \mathcal{H}_0 , depending only on Σ , γ and the spin structure of Σ , such that, if $\min_\Sigma \mathcal{H} \geq \mathcal{H}_0$, there do not exist DEC spin fill-ins of $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$.*

This question was tackled by Tsang in [Tsa22] where several partial results are proved. We remark that if $k = 0$ then $\alpha = 0$ and $\mathfrak{h} = 0$, and the DEC gives the nonnegativity of the scalar curvature so that a DEC fill-in of (Σ, γ, H) corresponds to a NNSC fill-in. In a same way, if $k = cg$, $c \neq 0$, then $\alpha = 0$ and $\mathfrak{h} = nc$, and the DEC condition implies that the scalar curvature of (Ω, g) is bounded from below by a negative constant, namely $R_g \geq -n(n+1)c^2$. These remarks imply that our result covers both of these cases. As mentioned above, the proof is a direct consequence of an eigenvalue estimate for the first eigenvalue of the Dirac operator of a spacetime Bartnik spin data which admits a DEC spin fill-in. This lower bound is stated as follow.

Theorem 2. *If a spacetime Bartnik spin data $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$ admits a DEC spin fill-in (Ω, g, k) with $\mathcal{H} > 0$, then the first eigenvalue $\lambda_1(\Sigma, \gamma)$ of the Dirac operator of (Σ, γ) satisfies*

$$\min_{\Sigma} \mathcal{H} \leq 2|\lambda_1(\Sigma, \gamma)|.$$

One can consider $\mathcal{F}_{spin}(\Sigma, \gamma)$, the set of the DEC spin fill-ins of the Riemannian spin manifold (Σ, γ) without specifying the data H , α and \mathfrak{h} . Then Theorem 2 implies that if $\mathcal{F}_{spin}(\Sigma, \gamma) \neq \emptyset$, it holds that

$$\sup_{(\Omega, g, k) \in \mathcal{F}_{spin}(\Sigma, \gamma)} \min_{\Sigma} \mathcal{H} \leq 2|\lambda_1(\Sigma, \gamma)| < \infty.$$

The proof of Theorem 2 relies on spin geometry and we refer especially to [BHM⁺15, Fri98, Gin09] for more details on this subject. Let us briefly recall what we need here. Since (Ω, g) is a Riemannian spin manifold, there exists a smooth Hermitian vector bundle over Ω , the spinor bundle, denoted by $\mathbf{S}\Omega$, whose sections are called spinor fields. The Hermitian scalar product is denoted by $\langle \cdot, \cdot \rangle$. Moreover, the tangent bundle $T\Omega$ acts on $\mathbf{S}\Omega$ by Clifford multiplication $X \otimes \psi \mapsto c(X)\psi$ for any tangent vector fields X and any spinor fields ψ . On the other hand, the Riemannian Levi-Civita connection ∇ lifts to the so-called spin Levi-Civita connection, also denoted by ∇ , and defines a metric covariant derivative on $\mathbf{S}\Omega$ that preserves the Clifford multiplication. A quadruplet $(\mathbf{S}\Omega, c, \langle \cdot, \cdot \rangle, \nabla)$ which satisfies the previous assumptions is usually referred to as a Dirac bundle. The Dirac operator is then the first order elliptic differential operator acting on $\mathbf{S}\Omega$ defined by $D := c \circ \nabla$. The spin structure on Ω induces, via a choice of an unit normal field to $\partial\Omega \simeq \Sigma$, a spin structure on Σ . This allows to define the *extrinsic* spinor bundle $\mathcal{S} := \mathbf{S}\Omega|_{\Sigma}$ over Σ on which there exists a Clifford multiplication ℓ and a metric covariant derivative ∇ . The quadruplet $(\mathcal{S}, \ell, \langle \cdot, \cdot \rangle, \nabla)$ is thus endowed with a Dirac bundle structure. Similarly, the extrinsic Dirac operator is defined by taking the Clifford trace of the covariant derivative ∇ that is $\mathcal{D} := \ell \circ \nabla$. It is by now well-known that this operator can be expressed using the Dirac operator D_{Σ} of (Σ, γ) endowed with the induced spin structure. What is important to us here is that the first nonnegative eigenvalue of the extrinsic Dirac operator \mathcal{D} corresponds to $|\lambda_1(\Sigma, \gamma)|$, the absolute value of the first eigenvalue of D_{Σ} and so it only depends on (Σ, γ) and the spin structure on Σ .

Proof of Theorem 2: Let (Ω, g, k) be a DEC spin fill-in of the spacetime Bartnik spin data $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$ and consider the modified spin covariant derivatives defined by

$$\nabla_X^{\pm} \psi := \nabla_X \psi \pm \frac{i}{2} c(k(X)) \psi \tag{3}$$

for $X \in \Gamma(T\Omega)$ and $\psi \in \Gamma(\mathbf{S}\Omega)$. The associated Dirac-type operators given by $D^\pm := c \circ \nabla^\pm$ are easily seen to satisfy

$$D^\pm \psi = D\psi \mp \frac{i}{2}(\text{Tr}_g k)\psi. \quad (4)$$

These are first order elliptic differential operators whose formal adjoints, with respect to the L^2 scalar product on $\mathbf{S}\Omega$, are D^\mp as deduced from the following integration by parts formulae

$$\int_\Omega \langle D^\pm \psi, \varphi \rangle d\mu = \int_\Omega \langle \psi, D^\mp \varphi \rangle d\mu - \int_\Sigma \langle c(\nu)\psi, \varphi \rangle d\sigma \quad (5)$$

for all smooth spinor fields ψ, φ on Ω . Here $d\mu$ (resp. $d\sigma$) denotes the Riemannian volume (resp. area) element of (Ω, g) (resp. (Σ, γ)) and ν is the inner unit normal to Σ in (Ω, g) . In a same way, a straightforward computation implies that

$$(\nabla^\pm)^* \nabla^\pm \psi = - \sum_{j=1}^{n+1} \nabla_{e_j}^\mp \nabla_{e_j}^\pm \psi \quad (6)$$

where $(\nabla^\pm)^*$ denote the formal adjoints of the modified connection ∇^\pm and $\{e_1, \dots, e_{n+1}\}$ is a local g -orthonormal frame of $T\Omega$. In particular, the Stokes formula leads to

$$\int_\Omega \langle (\nabla^\pm)^* \nabla^\pm \psi, \psi \rangle d\mu = \int_\Omega |\nabla^\pm \psi|^2 d\mu + \int_\Sigma \langle \nabla_\nu^\pm \psi, \psi \rangle d\sigma. \quad (7)$$

Then, it follows from the fact that $(D^\pm)^* = D^\mp$, from (6) and the classical Schrödinger-Lichnerowicz formula

$$D^2 \psi = \nabla^* \nabla \psi + \frac{R_g}{4} \psi$$

that

$$(D^\pm)^* D^\pm \psi = (\nabla^\pm)^* \nabla^\pm \psi + \frac{1}{2}(\mu\psi \pm ic(J)\psi) \quad (8)$$

for all $\psi \in \Gamma(\mathbf{S}\Omega)$. This is the $(n+1)$ -dimensional Riemannian counterpart of the formula obtained by Witten [Wit81] in his proof of the positive energy theorem. Now observe that since the point-wise symmetric endomorphism $J^\pm := \pm ic(J)$ satisfies $(J^\pm)^2 \psi = |J|_g^2 \psi$ it holds that

$$\langle J^\pm \psi, \psi \rangle \geq -|J|_g |\psi|^2$$

so that the DEC and (8) imply the point-wise inequalities

$$\langle (D^\pm)^* D^\pm \psi, \psi \rangle \geq \langle (\nabla^\pm)^* \nabla^\pm \psi, \psi \rangle$$

for all $\psi \in \Gamma(\mathbf{S}\Omega)$. Now integrating by part these estimates on Ω using (5) and (7) leads to the following important integral inequalities

$$\int_\Omega \left(|\nabla^\pm \psi|^2 - |D^\pm \psi|^2 \right) d\mu \leq - \int_\Sigma \langle \nabla_\nu^\pm \psi + c(\nu)D^\pm \psi, \psi \rangle d\sigma. \quad (9)$$

From the very definitions (3) and (4) of the modified covariant derivatives and the associated Dirac operators, we compute that

$$-\nabla_\nu^\pm \psi - c(\nu)D^\pm \psi = \not{D}\psi - \frac{1}{2}(H\psi \mp ic(\mathcal{V})\psi) \quad (10)$$

where

$$\mathcal{V} := \alpha^\sharp + \mathfrak{h}\nu \in \Gamma(T\Omega|_\Sigma)$$

since (Ω, g, k) is a fill-in of the data $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$. Here $\sharp : T^*\Omega \rightarrow T\Omega$ denotes the classical musical isomorphism between the cotangent bundle and the tangent bundle. Observe that the endomorphisms $\mathcal{V}^\pm := \pm ic(\mathcal{V})$ of \mathcal{S} is point-wise symmetric with respect to the Hermitian structure and satisfies $(\mathcal{V}^\pm)^2\psi = |\mathcal{V}|^2\psi$ in such a way that

$$\langle \mathcal{V}^\pm\psi, \psi \rangle \geq -|\mathcal{V}|_g|\psi|^2 = -\sqrt{\mathfrak{h}^2 + |\alpha|_\gamma^2}|\psi|^2 \quad (11)$$

for all $\psi \in \Gamma(\mathcal{S})$. Combining (9), (10) and (11) yields the following integral inequalities

$$\int_\Omega \left(|\nabla^\pm\psi|^2 - |D^\pm\psi|^2 \right) d\mu \leq \int_\Sigma \left\langle \mathcal{D}\psi - \frac{1}{2}\mathcal{H}\psi, \psi \right\rangle d\sigma. \quad (12)$$

which hold for all $\psi \in \Gamma(\mathbf{S}\Omega)$ and where \mathcal{H} is the generalized mean curvature function defined in (2).

Now we are going to show that one can extend any spinor fields on Σ in a suitable way. For this, we recall that the map $\chi := ic(\nu)$ is a boundary chirality operator (in the sense of [BB12, Example 7.26]) and so it is an orthogonal involution of \mathcal{S} which induces an orthogonal splitting $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ into the eigenbundles of χ for the eigenvalues ± 1 . The associated projection maps $P^\pm : \mathcal{S} \rightarrow \mathcal{S}^\pm$ define elliptic local boundary conditions for the Dirac-type operators D^\pm (see [BB12, Corollary 7.23] for example). This implies that the operators

$$D_\pm^+ : \{ \psi \in H^1(\mathbf{S}\Omega) / P^\pm\psi|_\Sigma = 0 \} \longrightarrow L^2(\mathbf{S}\Omega)$$

are of Fredholm type and that if $\Phi \in \Gamma(\mathbf{S}\Omega)$ and $\Psi \in \Gamma(\mathcal{S})$ are smooth spinor fields, any solutions $\psi \in \Gamma(\mathbf{S}\Omega)$ of the boundary problem

$$\begin{cases} D^+\psi = \Phi & \text{on } \Omega \\ P^\pm\psi|_\Sigma = P^\pm\Psi & \text{on } \Sigma \end{cases} \quad (13)$$

are smooth. The same holds for the operators D_\pm^- . It turns out that these operators are isomorphisms. To prove this fact, we notice that it is enough to show that D_\pm^+ and D_\pm^- are one-to-one since it follows from the integration by parts formulae (5) that the adjoint of D_\pm^+ is D_\mp^- . So take ψ_\pm , non trivial, in the kernel of D_\pm^+ , that is $\psi_\pm \in \Gamma(\mathbf{S}\Omega)$ satisfies (13) with $\Phi = 0$ and $\Psi = 0$. In particular, ψ_\pm is smooth on Ω . On the other hand, from the self-adjointness of the Dirac operator \mathcal{D} and the fact that $\mathcal{D}\chi = -\chi\mathcal{D}$, we get that

$$\int_\Sigma \langle \mathcal{D}\psi, \psi \rangle d\sigma = 2 \int_\Sigma \operatorname{Re} \langle \mathcal{D}(P^\pm\psi), P^\mp\psi \rangle d\sigma \quad (14)$$

for all $\psi \in \Gamma(\mathbf{S}\Omega)$. Using this formula for $\psi = \psi_\pm$, we deduce from (12) that

$$0 \leq \int_\Omega |\nabla^\pm\psi_\pm|^2 d\mu \leq -\frac{1}{2} \min_\Sigma \mathcal{H} \int_\Sigma |\psi_\pm|^2 d\sigma.$$

Since we assumed that \mathcal{H} is positive on Σ , we conclude that ψ_\pm is zero on Σ and $\nabla^\pm\psi_\pm = 0$. This leads to a contradiction since this last property implies that ψ_\pm is nowhere vanishing on Ω . The same holds for D_\pm^- .

Now take $\Psi_1 \in \Gamma(\mathcal{S})$ be an eigenspinor for the operator \mathcal{D} associated with the eigenvalue $|\lambda_1(\Sigma, \gamma)|$ and so, from the previous discussion, there exists a unique smooth solution $\varphi \in \Gamma(\mathbf{S}\Omega)$ satisfying

$$\begin{cases} D^+\varphi = 0 & \text{on } \Omega \\ P^+\varphi|_\Sigma = P^+\Psi_1 & \text{on } \Sigma. \end{cases}$$

Taking $\psi = \varphi$ in (12) and using the fact that

$$2\operatorname{Re}\langle P^-\Psi_1, P^-\varphi \rangle \leq |P^-\Psi_1|^2 + |P^-\varphi|^2,$$

finally lead to

$$0 \leq \left(|\lambda_1(\Sigma, \gamma)| - \frac{1}{2} \min_\Sigma \mathcal{H} \right) \int_\Sigma |\varphi|^2 d\sigma$$

which implies the estimate of Theorem 2. To get this last inequality, we implicitly used the fact that

$$\begin{aligned} |\lambda_1(\Sigma, \gamma)| \int_\Sigma |P^-\Psi_1|^2 d\sigma &= \int_\Sigma \langle \mathcal{D}(P^+\varphi), P^-\Psi_1 \rangle d\sigma \\ &= \int_\Sigma \langle P^+\varphi, \mathcal{D}(P^-\Psi_1) \rangle d\sigma \\ &= \int_\Sigma \langle P^+\varphi, P^+(\mathcal{D}\Psi_1) \rangle d\sigma \\ &= |\lambda_1(\Sigma, \gamma)| \int_\Sigma |P^+\varphi|^2 d\sigma \end{aligned}$$

which follows from the identity (14) and the self-adjointness of \mathcal{D} . \square

We conclude this note by noticing that this method is probably easily generalized to other situations (like the Einstein-Maxwell equations with nonnegative cosmological constant).

REFERENCES

- [Bar89] Robert Bartnik. New definition of quasilocal mass. *Phys. Rev. Lett.*, 62(20):2346–2348, 1989.
- [Bar97] R. Bartnik. Energy in general relativity. In *Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991)*, pages 5–27. Int. Press, Cambridge, MA, 1997.
- [BB12] C. Bär and W. Ballmann. Boundary value problems for elliptic differential operators of first order. In *Surveys in differential geometry. Vol. XVII*, volume 17 of *Surv. Differ. Geom.*, pages 1–78. Int. Press, Boston, MA, 2012.
- [BHM⁺15] J.-P. Bourguignon, O. Hijazi, J.-L. Milhorat, A. Moroianu, and S. Moroianu. *A spinorial approach to Riemannian and conformal geometry*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2015.
- [Fri98] T. Friedrich. On the spinor representation of surfaces in Euclidean 3-space. *J. Geom. Phys.*, 28(1-2):143–157, 1998.
- [Gin09] N. Ginoux. *The Dirac spectrum*, volume 1976 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
- [Gro18] M. Gromov. Scalar curvature of manifolds with boundaries: natural questions and artificial constructions. *arXiv:1811.04311*, 2018.
- [HMR03] O. Hijazi, S. Montiel, and A. Roldán. Dirac operators on hypersurfaces of manifolds with negative scalar curvature. *Ann. Global Anal. Geom.*, 23(3):247–264, 2003.
- [HMZ01] O. Hijazi, S. Montiel, and X. Zhang. Dirac operator on embedded hypersurfaces. *Math. Res. Lett.*, 8(1-2):195–208, 2001.

- [Mia21] P. Miao. Nonexistence of NNSC fill-ins with large mean curvature. *Proc. Amer. Math. Soc.*, 149(6):2705–2709, 2021.
- [SWW22] Y. Shi, W. Wang, and G. Wei. Total mean curvature of the boundary and nonnegative scalar curvature fill-ins. *J. Reine Angew. Math.*, 784:215–250, 2022.
- [SY79] R. M. Schoen and S.-T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.
- [SY19] R. M. Schoen and S.-T. Yau. Positive scalar curvature and minimal hypersurface singularities. In *Surveys in differential geometry 2019. Differential geometry, Calabi-Yau theory, and general relativity*, volume 24 of *Surv. Differ. Geom.*, pages 441–480. Int. Press, Somerville, MA, 2019.
- [Tsa22] T.-Y. Tsang. On a spacetime positive mass theorem with corners. *arXiv:2109.11070*, 2022.
- [Wit81] E. Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80(3):381–402, 1981.

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