Mean-Field Limit Derivation of a Monokinetic Spray Model with Gyroscopic Effects.

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ABSTRACT. In this paper we derive a two dimensional spray model with gyroscopic effects as the mean-field limit of a system modeling the interaction between an incompressible fluid and a finite number of solid particles. This spray model has been studied by Moussa and Sueur (Asymptotic Anal., 2013), in particular the mean-field limit was established in the case of $W^{1,\infty}$ interactions.
First we prove the local in time existence and uniqueness of strong solutions of a monokinetic version of the model with a fixed point method. Then we adapt the proof of Duerinckx and Serfaty (Duke Math. J., 2020) to establish the mean-field limit to the spray model in the monokinetic regime in the case of Coulomb interactions.

1. INTRODUCTION

The purpose of this paper is to establish the mean-field limit derivation of a system of partial differential equations introduced by Moussa and Sueur in [40] to describe a two dimensional spray modeled by an incompressible fluid and a dispersed phase of solid particles with the following interactions: The fluid particles move through the velocity field V generated by the fluid and the solid particles whereas the solid particles are submitted to a gyroscopic effect related to their velocities and to V. We define

$$g(x) := -\frac{1}{2\pi}\ln|x| \tag{1.1}$$

as the opposite of the Green kernel on the plane. Let $\omega(t,x)$ be the vorticity of the fluid and $f(t,x,\xi)$ be the density of solid particles, then this system can be written

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega V) = 0 \\ \partial_t f + \xi \cdot \nabla_x f + \operatorname{div}_\xi \left((\xi - V)^\perp f \right) = 0 \\ V = -\nabla^\perp g * (\omega + \rho) \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^2} f(t,x,\xi)\,\mathrm{d}\xi \end{cases} \tag{1.2}$$

where $u^\perp := (-u_2, u_1)$ and ρ is the space density of solid particles.

Replacing $\nabla^\perp g$ with a $W^{1,\infty}$ kernel, Moussa and Sueur derived these equations as the mean-field limit of a model describing the dynamics of a finite number of particles moving in an incompressible fluid (see [40, Corollary 1]). Namely, for N solid particles immersed in a fluid of vorticity $\omega_N(t,x)$ with initial condition ω_0, if the number of particles becomes large and if at

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time zero their empirical measure $f_N(0)$ is close to a regular density f_0, then for any time t $(f_N(t), \omega_N(t))$ is also close to the solution $(f(t), \omega(t))$ of (1.2) starting from (f_0, ω_0). The system modeling the interaction between a fluid of vorticity ω_N and N solid particles with positions $q_1, ..., q_N$ and velocities $p_1, ..., p_N$ is the following:

$$\begin{cases} \partial_t \omega_N + \operatorname{div}(\omega_N V_N) = 0 \\ \dot{q}_i = p_i \\ \dot{p}_i = p_i^\perp - \nabla g * \omega_N(q_i) - \dfrac{1}{N} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla g(q_i(t) - q_j(t)) \\ V_N = -\nabla^\perp g * (\omega_N + \rho_N) \\ \rho_N = \dfrac{1}{N} \displaystyle\sum_{k=1}^{N} \delta_{q_i} \end{cases} \tag{1.3}$$

where $\dfrac{1}{N}$ represents both the mass of a solid particle and the circulation of velocity around it. This model was established by Glass, Lacave and Sueur in [21] by looking at a rigid body in a fluid and assuming that its size is going to zero. Its well-posedness was studied in [33] by Lacave and Miot. Remark that we can formally obtain this system from System (1.2) if we take

$$f = \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i} \otimes \delta_{p_i}.$$

By Theorem 1.2 of [33], we know that there exists a unique global weak solution of System (1.3) on $\mathbb{R}_+ \times \mathbb{R}^2$.

In this paper we adapt the proof of Duerinckx and Serfaty in [50] to extend the mean-field convergence result of [40] for the true Coulombian interaction, that is we prove the convergence of (1.3) to (1.2) in the monokinetic regime, or more precisely to the following system:

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega V) = 0 \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \partial_t v + (v \cdot \nabla) v = (v - V)^\perp \\ V = -\nabla^\perp g * (\omega + \rho). \end{cases} \tag{1.4}$$

It can be obtained by taking formally $f(t, x, \xi) = \rho(t, x) \otimes \delta_{\xi = v(t,x)}$ in System (1.2). A rigorous derivation of System (1.4) from (1.2) was proved in [40] replacing $\nabla^\perp g$ with a $W^{1,\infty}$ kernel.

Before establishing the mean-field limit, we will justify the local in time existence and uniqueness of strong solutions of System (1.4). The local well-posedness of Euler-Poisson system (that is the system we get if we take $\omega = 0$ and add a pressure term in the equation on v) was studied in [39] in the case $d = 3$ using the usual estimates on hyperbolic systems that were proved in [32]. In Section 2 we extend this result to System (1.4). We will not study the existence of weak solutions of our system, for more details on this subject one can refer to the bibliography of the appendix of [50].

Mean-field limits for regular kernels were first established by compactness arguments in [6, 42] or by optimal transport theory and Wasserstein distances by Dobrushin in [17]. The latest method is the one used in [40] to prove the mean-field convergence of (1.3) to (1.2). In the Coulomb case, the kernel is no longer regular and their proof no longer holds. However, there are other works which prove mean-field limits for some systems with Coulombian or Riesz interactions. Let us consider N particles $x_1, ..., x_N$ satisfying the differential equations:

$$\dot{x}_i = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} K(x_i - x_j). \tag{1.5}$$

Then the mean-field limit to a density μ satisfying

$$\partial_t \mu + \mathrm{div}((K * \mu)\mu) = 0 \tag{1.6}$$

has been rigorously justified in different cases:

Schochet proved in [48] the mean-field limit of the point vortex system (that is $K = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$ in dimension two) to a measure-valued solution of the Euler equations up to a subsequence, using arguments previously developed in [47] and [15] to prove existence of such solutions. This result was extended later to include convergence to vortex sheets in [37].

For sub-coulombic interactions, that is $|K(x)|, |x||\nabla K(x)| \leqslant C|x|^{-\alpha}$ with $0 < \alpha < d-1$, the mean-field limit was proved by Hauray in [24] assuming $\mathrm{div}(K) = 0$ and using a Dobruschin-type approach. It was also used by Carillo, Choi and Hauray to deal with the mean-field limit of some aggregation models in [11, 12].

In [18] Duerinckx gave another proof of the mean-field limit of several Riesz interaction gradient flows using a "modulated energy" that was introduced by Serfaty in [49]. Together they also established the mean-field limit of Ginzburg-Landau vortices with pinning and forcing effects in [19].

In [50], Serfaty proved the mean-field convergence of such systems where K was a kernel given by Coulomb, logarithmic or Riesz interaction, that is $K = \nabla g$ for $g(x) = |x|^{-s}$ with $\max(d-2, 0) \leqslant s < d$ for $d \geqslant 1$ or $g(x) = -\ln|x|$ for $d = 1$ or 2. For this purpose $K * \mu$ is supposed to be Lipschitz.

Rosenzweig proved in [46] the mean-field convergence of the point vortex system without assuming Lipschitz regularity of the limit velocity field, using the same energy as in [50] with refined estimates. Remark that it ensures that the point vortex system converges to any Yudovich solutions of the Euler equations (see [51]). This result was extended later for higher dimensional systems ($d \geqslant 3$) in [44] by the same author.

Numerous mean-field limit results were proved for interacting particles with noise with regular or singular interaction kernels in [3, 5, 7, 8, 13, 20, 36, 43].

For systems of order two satisfying Newton's second law:

$$\ddot{x}_i = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} K(x_i - x_j) \tag{1.7}$$

the mean-field convergence to Vlasov-like equations remains open in the Coulombian case but was established for some singular kernels:

In [26, 25], Hauray and Jabin treated the case of some sub-coulombian interactions, or more precisely they considered a kernel $K = \nabla g$ where $|\nabla g(x)| \leqslant C|x|^{-s}$ and $|\nabla g(x)| \leqslant C|x|^{-s-1}$ where $0 < s < 1$. For this purpose they used the same kind of arguments Hauray used in [24].

In [31, 30], Jabin and Wang treated the case of bounded and $W^{-1,\infty}$ gradients.

In [4, 34, 35, 27] the same kind of results is proved with some cutoff of the interaction kernel.

In the appendix of [50], Duerinckx and Serfaty treated the case of particles with Coulombian interactions converging to the Vlasov equations in the monokinetic regime, that is the pressureless Euler-Poisson equations. This was used later by Carillo and Choi in [10] to prove the mean-field limit of some swarming models with alignment interactions.

In [23], Han-Kwan and Iacobelli proved the mean-field limit of particles satisfying Newton's second law to the Euler equations in a quasineutral regime or in a gyrokinetic limit. This result was improved later by Rosenzweig in [45] who treated the case of quasineutral regime for a larger choice of scaling between the number of particles and the coupling constant.

For a general introduction to the subject of mean-field limits one can have a look at the reviews [22, 29].

1.1. **Main results.**

If ν is a probability measure on $\mathbb{R}^2$, we will denote

$$\nu^{\otimes 2} := \nu \otimes \nu.$$

Recall that g is the opposite of the Green kernel on the plane:

$$g(x) := -\frac{1}{2\pi}\ln|x|.$$

Δ will denote the diagonal of $(\mathbb{R}^2)^2$:

$$\Delta := \{(x,x)\ ;\ x \in \mathbb{R}^2\}.$$

The main result in this paper is Theorem 1.9 which proves the mean-field limit of solutions of System (1.3) to solutions of (1.4) with some regularity assumptions. We will use the following definition of weak solutions:

Definition 1.1. *We say that (ρ, ω, v) is a weak solution of (1.4) if*

(1) $\rho, \omega \in \mathcal{C}([0,T], L^1 \cap L^\infty(\mathbb{R}^2, \mathbb{R}))$ with compact supports.

(2) For all $t \in [0,T]$, $\int_{\mathbb{R}^2} \rho(t) = \int_{\mathbb{R}^2} \omega(t) = 1$.

(3) $v \in W^{1,\infty}([0,T] \times \mathbb{R}^2, \mathbb{R}^2)$

(4) The equation on the velocity is satisfied almost everywhere and the continuity equations are satisfied in the sense of distributions, that

is for every $\varphi \in W^{1,\infty}([0,T],\mathcal{C}^1_c(\mathbb{R}^2))$ *and for every* $t \in [0,T]$, *we have:*

$$\begin{aligned}\int_{\mathbb{R}^2}(\rho(t)\varphi(t)-\rho_0\varphi(0)) &= \int_0^t\int_{\mathbb{R}^2}\rho(s,x)(\partial_s\varphi+\nabla\varphi\cdot v)(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ \int_{\mathbb{R}^2}(\omega(t)\varphi(t)-\omega_0\varphi(0)) &= \int_0^t\int_{\mathbb{R}^2}\omega(s,x)(\partial_s\varphi+\nabla\varphi\cdot V)(s,x)\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{1.8}$$

Remark that by conservation of mass it is enough to ask

$$\int_{\mathbb{R}^2}\rho_0 = \int_{\mathbb{R}^2}\omega_0 = 1$$

to get Assumption (2).

In Section 2 we will prove existence and uniqueness of solutions of (1.4) in a space strictly included in $\mathcal{C}([0,T],L^1\cap L^\infty)^2\times W^{1,\infty}$ (see Theorem 2.1). For the microscopic system (1.3), we will use the following definition of weak solutions, introduced in [33]:

Definition 1.2. (ω_N,Q_N,P_N) *is a weak solution of* (1.3) *on* $[0,T]$ *if*

(1) $\omega_N \in L^\infty([0,T],L^1\cap L^\infty)\cap\mathcal{C}([0,T],L^\infty-w^*)$ *with compact support.*

(2) For all $t\in[0,T]$, $\displaystyle\int_{\mathbb{R}^2}\omega_N(t)=1$.

(3) $q_1,...,q_N \in \mathcal{C}^2([0,T],\mathbb{R}^2)$

(4) The partial differential equation on ω_N *is satisfied in the sense of distributions (which means that it also verifies* (1.8)*) and the ordinary differential equations are satisfied in the classical sense.*

Remark that by conservation of mass it is enough to ask

$$\int_{\mathbb{R}^2}\omega_N(0) = 1$$

to get Assumption (2).

Remark 1.3. By Theorems 1.4 and 1.5 of [33] we know that for $\omega_N(0)\in L^\infty(\mathbb{R}^2)$ compactly supported and $q_1(0),...,q_N(0)$ distinct outside of the support of $\omega_N(0)$ there exists a unique weak solution of (1.3) on $[0,T]$ for any $T>0$ and no collision between the solid particles occurs in finite time. It follows by [33, Corollary A.2] that for all $1\leqslant p\leqslant\infty$, $\|\omega_N(t)\|_{L^p}=\|\omega_N(0)\|_{L^p}$.

Remark 1.4. One could replace the compact support assumptions by some logarithmic decrease of the solutions ω and ρ at infinity as done in [18] and [46] but for the sake of simplicity we will only consider solutions with compact support.

In order to show that the limit of a sequence (ω_N,Q_N,P_N) of solutions of (1.3) converges to a solution (ω,ρ,v) of (1.4), we will control a modulated energy similar to the one defined in [50]. Let $X_N=(x_1,...,x_N)\in(\mathbb{R}^2)^N$ be such that $x_i\neq x_j$ if $i\neq j$ and let μ be a $L^1\cap L^\infty$ probability density with compact support, then the following quantity is well defined:

$$\mathcal{F}(X_N,\mu) := \int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\left(\mu-\sum_{i=1}^N\delta_{x_i}\right)(\mathrm{d}x)\left(\mu-\sum_{i=1}^N\delta_{x_i}\right)(\mathrm{d}y). \tag{1.9}$$

This is the "modulated energy" used in [50, 46] to prove the mean-field limit of (1.5) to (1.6). As we will see later this quantity controls the distance between μ and the empirical distribution on X_N in a weak sense. More precisely we have the following proposition proved in [50] (number 3.6 in the article):

Proposition 1.5 (proved in [50])**.** *For any $0 < \theta \leqslant 1$, there exists $\lambda > 0$ and $C > 0$ such that for ξ smooth and $\mu \in L^\infty$ probability density with compact support,*

$$\left|\int_{\mathbb{R}^2} \xi \left(\mu - \frac{1}{N}\sum_{i=1}^N \delta_{x_i}\right)\right| \leqslant C\Bigg(|\xi|_{\mathcal{C}^{0,\theta}} N^{-\lambda} + \|\nabla\xi\|_{L^2}\left(\mathcal{F}(X_N,\mu) + (1+\|\mu\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\right)^{\frac{1}{2}}\Bigg)$$

where

$$|\xi|_{\mathcal{C}^{0,\theta}} := \sup_{x\neq y} \frac{|\xi(x)-\xi(y)|}{|x-y|^\theta}.$$

Remark 1.6. In [50, Proposition 3.6] the coercivity inequality is stated with the Hölder norm $\|\xi\|_{\mathcal{C}^{0,\theta}}$ but by inequality [50, Inequality (3.27)] we can replace this Hölder norm by the semi-norm $|\xi|_{\mathcal{C}^{0,\theta}}$.

We will also need the following functional inequality, proved by Serfaty in [50] (number 1.1 in the article).

Proposition 1.7. *There exists $\lambda, C > 0$ such that for any probability density $\mu \in L^\infty$ with compact support, $\psi \in W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2)$ and $X_N \in (\mathbb{R}^2)^N$, we have*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (\psi(x)-\psi(y))\cdot\nabla g(x-y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\right)(x)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\right)(y) \leqslant C\,\|\psi\|_{W^{1,\infty}}\,(\mathcal{F}(X_N,\mu)+(1+\|\mu\|_{L^\infty})N^{-\lambda}).$$

This proposition is one of the main result of [50] as it is used to perform a Grönwall estimate on the modulated energy from which the mean-field result is deduced.

Now let ρ, ω, ω_N be $(L^1\cap L^\infty)(\mathbb{R}^2,\mathbb{R})$ probability densities with compact supports, $v \in W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2)$, $Q_N, P_N \in (\mathbb{R}^2)^N$ be such that $q_i \neq q_j$ if $i \neq j$. We define:

$$\begin{aligned}\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N) :=& \frac{1}{N}\sum_{i=1}^N |v(q_i)-p_i|^2 \\ &+ \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho+\omega-\rho_N-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \|\omega-\omega_N\|_{L^2}^2 + BN^{-\gamma}\end{aligned}$$

where

$$\rho_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{q_i}$$

and γ and B are constants ensuring that $\mathcal{H}$ is nonnegative, as explained in the following result:

Proposition 1.8. *For any $0<\gamma<1$, there exists a constant B depending only on γ, $\|\omega\|_{L^1\cap L^\infty}$, $\|\rho\|_{L^1\cap L^\infty}$ and $\sup_N \|\omega_N\|_{L^1\cap L^\infty}$ such that:*

$$\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho+\omega-\rho_N-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) + BN^{-\gamma} \geqslant 0. \tag{1.10}$$

and

$$\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N)\geqslant 0.$$

Remark that if we remove $BN^{-\gamma}$ and if we set $\omega_N=\omega=0$ our quantity $\mathcal{H}$ is the functionnal used by Duerinckx in the appendix of [50] to prove the mean-field limit of particles satisfying (1.7) to the Euler-Poisson equations.

Our main result is the following theorem:

Theorem 1.9. *Let (ρ,ω,v) be a weak solution of System (1.4) in the sense of Definition 1.1 and (ω_N,Q_N,P_N) be a weak solution of System (1.3) in the sense of Definition 1.2. Then we define*

$$\mathcal{H}_N(t) := \mathcal{H}(\omega(t),\rho(t),v(t),\omega_N(t),Q_N(t),P_N(t)). \tag{1.11}$$

Suppose that $\nabla\omega\in L^\infty$, $\nabla v\in\mathcal{C}^0([0,T]\times\mathbb{R}^2,\mathbb{R}^2)$ and that

$$\sup_{N\in\mathbb{N}} \left\|\omega_N^0\right\|_{L^1\cap L^\infty} < +\infty \tag{1.12}$$

$$q_1(0),...,q_N(0)\notin \mathrm{supp}(\omega_N^0) \tag{1.13}$$

$$\forall i\neq j, q_i(0)\neq q_j(0). \tag{1.14}$$

Then there exist positive constants C and β depending only on T,ρ,ω and $\|\omega_N\|_{L^\infty}$ such that for all $t\in[0,T]$,

$$\mathcal{H}_N(t)\leqslant C(\mathcal{H}_N(0)+N^{-\beta}). \tag{1.15}$$

Remark 1.10. By Sobolev embeddings the solutions of the spray system (1.4) given by Theorem 2.1 are also solutions in the sense of Definition 1.1 that satisfy the hypothesis of Theorem 1.9 and thus Theorem 2.1 gives the existence of sufficiently regular solutions of System (1.4) that can be approached as mean-field limits of solutions of System (1.3) (even if Theorem 1.9 does not recquire solutions to be as regular as the solutions obtained in Theorem 2.1).

We will also prove a coerciveness result about this energy.

Proposition 1.11. *Let $Q_N,P_N\in(\mathbb{R}^2)^N$ and let $\omega,\omega_N,\rho\in L^1\cap L^\infty(\mathbb{R}^2,\mathbb{R})$ be probability densities with compact supports and $v\in W^{2,\infty}(\mathbb{R}^2,\mathbb{R}^2)$. Assume that*

$$\sup_{N\in\mathbb{N}}\|\omega_N\|_{L^\infty}<+\infty. \tag{1.16}$$

Then there exist positive constants C and β such that

$$\left\| \frac{1}{N}\sum_{i=1}^{N} \delta_{(q_i,p_i)} - \rho \otimes \delta_{\xi=v(x)} \right\|_{H^{-5}} \leqslant C(1+\|\nabla v\|^2_{W^{1,\infty}}) \times (\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N)^{\frac{1}{2}} + (1+\|\rho\|_{L^\infty})N^{-\beta})^{\frac{1}{2}}. \tag{1.17}$$

In particular, if we assume that

$$\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N) \underset{N\to\infty}{\longrightarrow} 0$$

then for any $a < -1$,

$$\rho_N \underset{N\to+\infty}{\longrightarrow} \rho \ \ in\ H^a$$
$$\omega_N - \omega \underset{N\to+\infty}{\longrightarrow} 0 \ \ in\ L^2 \cap \dot{H}^{-1}$$
$$\frac{1}{N}\sum_{i=1}^{N} \delta_{(q_i,p_i)} \underset{N\to+\infty}{\longrightarrow} \rho \otimes \delta_{\xi=v(x)} \ \ in\ H^{-5}.$$

Remark 1.12. The H^{-5} norm is not optimal, but it is sufficient to justify that $\mathcal{H}_N$ controls the convergence to a monokinetic distribution in a weak sense.

As a consequence we get that if a sequence of solutions (ω_N, Q_N, P_N) of (1.3) satisfying the hypothesis of Theorem 1.9 are such that

$$\mathcal{H}_N(0) \underset{N\to+\infty}{\longrightarrow} 0$$

then for any $t \in [0,T]$ we have

$$\mathcal{H}_N(t) \underset{N\to+\infty}{\longrightarrow} 0$$

and it follows by Proposition 1.11 that for any $t \in [0,T]$ and $a < -1$,

$$\rho_N(t) \underset{N\to+\infty}{\longrightarrow} \rho(t) \ \text{ in } H^a$$
$$\omega_N(t) - \omega(t) \underset{N\to+\infty}{\longrightarrow} 0 \ \text{ in } L^2 \cap \dot{H}^{-1}$$
$$\frac{1}{N}\sum_{i=1}^{N} \delta_{(q_i(t),p_i(t))} \underset{N\to+\infty}{\longrightarrow} \rho(t) \otimes \delta_{\xi=v(t,x)} \ \text{ in } H^{-5}.$$

Since $\frac{1}{N}\sum_{i=1}^{N} \delta_{(q_i(t),p_i(t))}$ is bounded in the dual of continuous bounded functions, we can extract a subsequence which will converge in the weak-$*$ topology of signed measures $\mathcal{M}(\mathbb{R}^2 \times \mathbb{R}^2)$. Since it necessarily converges to $\rho(t) \otimes \delta_{\xi=v(t,x)}$, by weak-$*$ compactness we can deduce that for all $t \in [0,T]$,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{(q_i(t),p_i(t))} \overset{*}{\rightharpoonup} \rho(t) \otimes \delta_{\xi=v(t,x)} \ \text{ in } \mathcal{M}(\mathbb{R}^2)$$

and thus we have the mean-field convergence of (1.3) to (1.4). One can look at [46, Corollary 1.2] for a more detailed proof of such a compactness argument.

Remark 1.13. If we suppose that $\omega_{N,0}-\omega_0$ converges to 0 in $\dot{H}^{-1}$ and that $\rho_{N,0}$ converges to ρ_0 in the weak-$*$ topology of signed measures, then the convergence of

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho_0+\omega_0-\rho_{N,0}-\omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)$$

to zero can be seen as a well-preparedness condition on the initial data, as stated in the following proposition:

Proposition 1.14. *Let us suppose that $\omega_{N,0},\omega_0,\rho_0\in L^2(\mathbb{R}^2,\mathbb{R})$ are probability densities with compact support and that $(q_1^0,...,q_N^0)$ are such that $q_i^0\neq q_j^0$ if $i\neq j$. Then if we suppose*

$$\sup_{N\in\mathbb{N}}\left\|\omega_N^0\right\|_{L^1\cap L^\infty}<+\infty \tag{1.18}$$

and

$$\omega_{N,0}-\omega_0 \underset{N\to+\infty}{\longrightarrow} 0 \ \text{ in } \dot{H}^{-1}$$
$$\rho_{N,0} \underset{N\longrightarrow+\infty}{\overset{*}{\rightharpoonup}} \rho_0 \ \text{ in } \mathcal{M}(\mathbb{R}^2)$$
$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g(q_i^0-q_j^0) \underset{N\to+\infty}{\longrightarrow} \iint g(x-y)\rho_0(x)\rho_0(y)\,\mathrm{d}x\,\mathrm{d}y$$

we have

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho_0+\omega_0-\rho_{N,0}-\omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \underset{N\to+\infty}{\longrightarrow} 0.$$

The latter statement strongly relies on the results proved in [18]. One could have more details about these well-preparedness assumptions by reading the introduction of [18].

The remainder of this paper is organized as follows. In Section 2 we establish local well-posedness of strong solutions of (1.4). Then in Section 3 we provide the proof of Proposition 1.8, Theorem 1.9, Proposition 1.11 and Proposition 1.14. Sections 2 and 3 are independent of each other.

2. Local Well-Posedness

In this section, if μ is a continuous function defined on $\mathbb{R}^2$ with compact support, we will denote

$$R[\mu]:=\sup\{|x|\ ;\ x\in\mathbb{R}^2,\mu(x)\neq 0\}$$

and

$$R_T[\mu]:=\sup_{0\leqslant t\leqslant T} R[\mu(t)]$$

if μ depends on time. If B is a Banach space and $1\leqslant p\leqslant\infty$, we will denote

$$L^p_T B:=L^p([0,T],B).$$

We will use the same convention for the Hölder spaces $\mathcal{C}^k_T B$ and the Sobolev spaces $W^{k,p}_T B$. Let us also recall that g is the opposite of the Green kernel on the plane defined in (1.1).

C will refer to a constant independent of time and of any other parameter that can change value from one line to another. We will denote $C(A, B)$ for a constant depending only on some quantities A and B.

We want to show that System (1.4) has a unique regular solution on $[0, T]$ for T small enough. In [39], Makino builds such a solution for the following compressible Euler-Poisson system in three dimensions:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = F * \rho \\ \partial_t \rho + \operatorname{div}(\rho u) = 0 \end{cases}$$

where p is a function of ρ and $F := \nabla G$ where G is the Green function on $\mathbb{R}^3$. There are three main differences with our system (1.4):

(1) We have no pressure term, but we have a gyroscopic effect.
(2) We have a continuity equation on ω that we also need to solve.
(3) On the plane $\mathbb{R}^2$, the function $V = -\nabla^\perp g * (\rho + \omega)$ is not in L^2 except if we assume that $\displaystyle\int (\rho + \omega) = 0$.

In order to deal with the third point, we will assume that $v_0 = u_0 + \overline{V}$ where $u_0 \in L^2$ and $\overline{V}$ is a function of x that we will specify later. If we try to find a solution of (1.4) when $v = u + \overline{V}$, we find that (u, ρ, ω) evolves according to the following equations:

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega V) = 0 \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \partial_t u + ((u + \overline{V}) \cdot \nabla)u + (u \cdot \nabla)\overline{V} = u^\perp + f \\ V = -\nabla^\perp g * (\omega + \rho) \\ f = (\overline{V} - V)^\perp - (\overline{V} \cdot \nabla)\overline{V} \\ v = u + \overline{V}. \end{cases} \tag{2.1}$$

Thus if we choose $\overline{V}$ such that $f \in L^2$, we will find an equation that we expect to have a solution in L^2. We can achieve this goal choosing the following value of $\overline{V}$:

$$\overline{V} := -\left(\int_{\mathbb{R}^2} \omega_0 + \rho_0\right) \nabla^\perp g * \chi$$

where χ is some compactly supported function such that $\displaystyle\int_{\mathbb{R}^2} \chi = 1$. We make such a choice because $\displaystyle\int_{\mathbb{R}^2} \rho$ and $\displaystyle\int_{\mathbb{R}^2} \omega$ are conserved and we will justify later that for μ compactly supported,

$$-\nabla^\perp g * \mu(x) = \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} \mu\right) \frac{x^\perp}{|x|^2} + \underset{|x| \to \infty}{\mathcal{O}}(|x|^{-2}).$$

Since we assumed that ρ and ω have compact support, we are not concerned by the fact that V is not L^2 on the whole plane. Remark also that the space

$$-\left(\int_{\mathbb{R}^2} \omega_0 + \rho_0\right) \nabla^\perp g * \chi + L^2$$

does not depend on the choice of χ. Now we are able to write the main theorem of this section:

Theorem 2.1. *Let s be an integer such that $s \geqslant 3$ and $u_0 \in H^{s+1}(\mathbb{R}^2, \mathbb{R}^2)$, $\rho_0, \omega_0 \in H^s(\mathbb{R}^2, \mathbb{R})$ such that ω_0 and ρ_0 have compact support, then if T is small enough (with respect to some quantity depending only on $\|\omega_0\|_{H^s}$, $\|\rho_0\|_{H^s}$, $\|u_0\|_{H^{s+1}}$, $R[\omega_0]$ and $R[\rho_0]$), there exists a unique (u, ω, ρ) with $(\rho, \omega) \in (C_T H^s \cap C_T^1 H^{s-1})^2$ and $u \in C_T H^{s+1} \cap C_T^1 H^s$ solution of (2.1).*

The proof of Theorem 2.1 proceeds as follows:

(1) We fix $T > 0$ and define

$$R_0 := R[\rho_0 + \omega_0]$$
$$M_0 := \max(\|\rho_0\|_{H^s}, \|\omega_0\|_{H^s}, \|u_0\|_{H^{s+1}})$$

$$\begin{aligned} X_T := \Big\{ & (\omega, \rho) \in L_T^\infty H^s \cap C_T H^{s-1} \Big| \omega(0) = \omega_0, \rho(0) = \rho_0, \\ & \|\rho\|_{L_T^\infty H^s} \leqslant 2M_0, \|\omega\|_{L_T^\infty H^s} \leqslant 2M_0, R_T[\rho + \omega] \leqslant 2R_0, \\ & \forall t \in [0,T], \int (\rho(t) + \omega(t)) = \int (\rho_0 + \omega_0), \\ & \forall t, t' \in [0,T], \left\|\rho(t) - \rho(t')\right\|_{H^{s-1}} \leqslant L|t - t'|, \\ & \left\|\omega(t) - \omega(t')\right\|_{H^{s-1}} \leqslant L|t - t'|, \Big\} \end{aligned} \tag{2.2}$$

where $L > 0$ is a quantity depending only on R_0 and M_0. Remark that X_T is a subspace of $(C_T H^s \cap C_T^1 H^{s-1})^2$. Then we fix $(\omega, \rho) \in X_T$ and we define

$$V := -\nabla^\perp g * (\rho + \omega) \tag{2.3}$$

$$\overline{V} := -\left(\int_{\mathbb{R}^2} \omega_0 + \rho_0\right) \nabla^\perp g * \chi \tag{2.4}$$

$$f := (\overline{V} - V)^\perp - \overline{V} \cdot \nabla \overline{V}. \tag{2.5}$$

Note that we will prove in Subsection 2.1 that $f \in \mathcal{C}_T H^s \cap L_T^\infty H^{s+1}$.

(2) In Subsection 2.2 we solve in $C_{T_1} H^{s+1} \cap C_{T_1}^1 H^s$ the equation

$$\partial_t u + ((u + \overline{V}) \cdot \nabla) u + (u \cdot \nabla) \overline{V} = u^\perp + f$$

for initial condition u_0 and $T_1 \leqslant T$ small enough depending only on M_0 and L.

(3) In Subsection 2.3 we define $v = u + \overline{V}$ and solve in $(C_{T_1} H^s \cap C_{T_1}^1 H^{s-1})^2$ the system

$$\begin{cases} \partial_t \widetilde{\omega} + \operatorname{div}(\widetilde{\omega} V) = 0 \\ \partial_t \widetilde{\rho} + \operatorname{div}(\widetilde{\rho} v) = 0. \end{cases}$$

(4) In Subsection 2.4 we apply a fixed-point theorem by showing that the map defined on X_{T_2} by $(\omega, \rho) \longmapsto (\widetilde{\omega}, \widetilde{\rho})$ is a contraction for the $\mathcal{C}_T L^2$ norm if $T_2 \leqslant T_1$ is small enough, using the estimates proved for the previous equations.

Remark 2.2. X_T is strictly included in $(C_{T_1}H^s \cap C^1_{T_1}H^{s-1})^2$ but since we prove in step (3) that the image of the application Φ sending (ω, ρ) to $(\widetilde{\omega}, \widetilde{\rho})$ is contained in $(C_{T_1}H^s \cap C^1_{T_1}H^{s-1})^2$ we have the expected regularity for the solutions of our system.

Remark 2.3. Uniqueness is established in the space X_T which is bigger than $(C_TH^s \cap C^1_TH^{s-1})^2$. It ensures uniqueness for the whole system: If (ρ_1, ω_1, u_1) and (ρ_2, ω_2, u_2) are two solutions of (1.4), then $(\rho_1, \omega_1) = (\rho_2, \omega_2)$ by uniqueness of the fixed point and $u_1 = u_2$ follows by uniqueness of solutions of Equation (2.6). Remark also that using energy estimates one could prove uniqueness in a space of smaller regularity.

Before doing these different steps, we give some results about the Biot-Savart kernel $-\nabla^\perp g$ that we will need later. In this section we will use the following definition of uniformly local Sobolev spaces:

Definition 2.4. *We define $H^s_{\mathrm{ul}}(\mathbb{R}^2)$ as the space of locally H^s functions verifying*

$$\|u\|_{H^s_{\mathrm{ul}}(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} \|u\|_{H^s(B(x,1))} < +\infty.$$

For a more complete introduction to these spaces we refer to Section 2.2 of [32].

2.1. **Properties of the Biot-Savart kernel on the plane.** In this subsection we prove Proposition 2.5, which contains several results about the Biot-Savart kernel $-\nabla^\perp g$.

Proposition 2.5. *Let $s \geqslant 3$ and let μ be a H^s function on $\mathbb{R}^2$ with compact support. Denote*

$$V := -\nabla^\perp g * \mu.$$

Then we have the following inequalities:

(1) $V \in H^{s+1}_{\mathrm{ul}}$ *and* $\|V\|_{H^{s+1}_{\mathrm{ul}}} \leqslant C(1 + R[\mu]) \|\mu\|_{H^s}$.
(2) $\|\nabla V\|_{H^s} \leqslant C \|\mu\|_{H^s}$.
(3) $V \in L^\infty$ *and we have the three following bounds:*

$$\|V\|_{L^\infty} \leqslant CR[\mu] \|\mu\|_{L^\infty}$$
$$\|V\|_{L^\infty} \leqslant CR[\mu]^{\frac{1}{3}} \|\mu\|_{H^1}$$
$$\|V\|_{L^\infty} \leqslant C \|\mu\|_{L^1}^{\frac{1}{2}} \|\mu\|_{L^\infty}^{\frac{1}{2}}.$$

(4) $\|(V \cdot \nabla)V\|_{H^s} \leqslant C(1 + R[\mu]) \|\mu\|^2_{H^s}$.
(5) $V(x) = \dfrac{1}{2\pi}\left(\displaystyle\int_{\mathbb{R}^2} \mu\right) \dfrac{x^\perp}{|x|^2} + \underset{|x|\to\infty}{\mathcal{O}}(|x|^{-2})$.
(6) If μ has mean zero, then $V \in L^2$ and

$$\|V\|_{L^2} \leqslant C(R[\mu] + R[\mu]^3)^{\frac{1}{2}} \|\mu\|_{L^2}.$$

Estimates (1) to (4) are consequences of the two following propositions. The first one is the usual potential estimate of a velocity field given by the Biot-Savart law:

Proposition 2.6 (Potential estimates in L^p)**.** *If $2 < p < \infty$ and $\omega \in L^1 \cap L^p$, then*

$$\|\nabla g * \omega\|_{L^\infty} \leqslant C_p \|\omega\|_{L^1}^{\frac{p-2}{2p-2}} \|\omega\|_{L^p}^{\frac{p}{2p-2}}$$
$$\|\nabla g * \omega\|_{L^\infty} \leqslant C \|\omega\|_{L^1}^{\frac{1}{2}} \|\omega\|_{L^\infty}^{\frac{1}{2}} .$$

For the proof of this proposition see for example [28, Lemma 1]. The second is the Calderón-Zygmund inequality:

Proposition 2.7 (Calderón-Zygmund inequality)**.** *If $1 < p < +\infty$,*

$$\left\|\nabla^2 g * \omega\right\|_{L^p} \leqslant C_p \|\omega\|_{L^p} .$$

For the proof of this inequality we refer to [2, Proposition 7.5].

Claims (5) and (6) giving the behavior of V at infinity are classical results in fluid dynamics (see for example [38, Proposition 3.3]) that we will prove to have the specific L^2 bound we need on V.

The first consequence of Proposition 2.5 is the following:

Corollary 2.8. *Let $\rho_0, \omega_0 \in H^s$ with compact support, χ be a smooth function with compact support and $(\rho, \omega) \in X_T$ where X_T is the space defined by (2.2). Let us consider the functions $\overline{V}$ and f defined by (2.4) and (2.5), then we have $\overline{V} \in H^{s+2}_{\mathrm{ul}}$, $f \in L^\infty_T H^{s+1} \cap \mathcal{C}_T H^s$ and*

$$\left\|\overline{V}\right\|_{H^{s+2}_{\mathrm{ul}}} \leqslant C(R_0, M_0)$$
$$\|f\|_{L^\infty_T H^{s+1}} \leqslant C(R_0, M_0).$$

Proof of Proposition 2.5. Let us begin by the second inequality. We have:

$$\|\nabla V\|_{H^s} = \left\|\nabla^2 g * \mu\right\|_{H^s} \leqslant C \sum_{|\alpha| \leqslant s} \left\|\nabla^2 g * \partial^\alpha \mu\right\|_{L^2} \leqslant C \|\mu\|_{H^s}$$

by Proposition 2.7.

Let us now prove the third Claim. By Proposition 2.6, we have

$$\begin{aligned}
\|V\|_{L^\infty} &\leqslant C \|\mu\|_{L^1}^{\frac{1}{2}} \|\mu\|_{L^\infty}^{\frac{1}{2}} \\
&\leqslant C \|\mu\|_{L^\infty}^{\frac{1}{2}} \left(\int_{B(0,R[\mu])} |\mu| \right)^{\frac{1}{2}} \\
&\leqslant C \|\mu\|_{L^\infty} \left(\int_{B(0,R[\mu])} 1 \right)^{\frac{1}{2}} \\
&\leqslant C R[\mu] \|\mu\|_{L^\infty} .
\end{aligned}$$

For the second inequality of (3), we use Proposition 2.6 again to get

$$\|V\|_{L^\infty} \leqslant C \|\mu\|_{L^1}^{\frac{1}{3}} \|\mu\|_{L^4}^{\frac{2}{3}} .$$

Moreover, by Cauchy-Schwartz inequality,

$$\|\mu\|_{L^1} \leqslant C \left\|\mathbf{1}_{B(0,R[\mu])}\right\|_{L^2} \|\mu\|_{L^2} \leqslant C R[\mu] \|\mu\|_{H^1} .$$

and therefore by the embedding of H^1 into L^4 (see for example [9, Corollary 9.11]) we have

$$\|V\|_{L^\infty} \leqslant C R[\mu]^{\frac{1}{3}} \|\mu\|_{H^1} .$$

The third inequality of (3) is the second inequality of Proposition 2.6.

The first inequality follows from the two Claims we just proved: Since all derivatives of V of order k for $1 \leqslant k \leqslant s+1$ belong to L^2 and since $\|V\|_{L^2_{\mathrm{ul}}} \leqslant C\|V\|_{L^\infty}$, we get

$$\begin{aligned}\|V\|_{H^{s+1}_{\mathrm{ul}}} &\leqslant C(\|\mu\|_{H^s} + R[\mu]\,\|\mu\|_{L^\infty}) \\ &\leqslant C(1+R[\mu])\,\|\mu\|_{H^s}\end{aligned}$$

because $H^s \hookrightarrow L^\infty$.

Now let us prove the fourth point. Let α be a multi-index such that $|\alpha| \leqslant s$, then $\partial^\alpha((V\cdot\nabla)V)$ is a combination of $(\partial^{\alpha_1}V\cdot\nabla)\partial^{\alpha_2}V$ where $\alpha_1+\alpha_2=\alpha$. If $\alpha_1 = 0$,

$$\|(\partial^{\alpha_1}V\cdot\nabla)\partial^{\alpha_2}V\|_{L^2} \leqslant \|V\|_{L^\infty}\,\|\nabla V\|_{H^s}\,.$$

If $1 \leqslant |\alpha_1| \leqslant s-1$, then

$$\begin{aligned}\|(\partial^{\alpha_1}V\cdot\nabla)\partial^{\alpha_2}V\|_{L^2} &\leqslant \|\partial^{\alpha_1}V\|_{L^\infty}\,\|\nabla V\|_{H^s} \\ &\leqslant \|\nabla V\|^2_{H^s}\,.\end{aligned}$$

Finally if $|\alpha_1| = s$,

$$\begin{aligned}\|(\partial^{\alpha_1}V\cdot\nabla)\partial^{\alpha_2}V\|_{L^2} &\leqslant \|\partial^{\alpha_1}V\|_{L^2}\,\|\nabla V\|_{L^\infty} \\ &\leqslant \|\nabla V\|^2_{H^s}\,.\end{aligned}$$

We conclude using (2) and (3).

We now prove the fifth claim by a standard argument. Let us set

$$W(x+iy) = V_1(x,y) - iV_2(x,y).$$

Then we have

$$\begin{aligned}(\partial_x + i\partial_y)W &= (\partial_x V_1 + \partial_y V_2) + i(\partial_y V_1 - \partial_x V_2) \\ &= \operatorname{div}(V) - i\operatorname{curl}(V) \\ &= 0 - i\mu.\end{aligned}$$

Thus W is holomorphic on $\mathbb{C}\backslash B(0,R)$ with $R = R[\mu]$ (since it is a solution of Cauchy-Riemann equations) and we can write it as the sum of a Laurent serie:

$$W(z) = \sum_{k=-\infty}^{+\infty} a_k z^{-k}.$$

Remark that since we have $W(z) \underset{z\to\infty}{\longrightarrow} 0$, $a_k = 0$ for k nonpositive. Now we compute a_1 by a contour integral in the counter clockwise sense:

$$\begin{aligned}a_1 &= \frac{1}{2i\pi}\int_{\partial B(0,R)} W(z)\,\mathrm{d}z \\ &= \frac{1}{2i\pi}\int_0^{2\pi} W(Re^{i\theta})Rie^{i\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} (V_1 - iV_2)(R\cos(\theta), R\sin(\theta))(R\cos(\theta) + iR\sin(\theta))\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{\partial B(0,R)} (V\cdot n + iV^\perp\cdot n)\,\mathrm{d}\sigma\end{aligned}$$

n beeing the outer normal vector to $B(0,R)$ (or equivalently the inner normal vector to $B(0,R)^c$) and σ its unit measure. Thus by Stokes theorem,

$$\begin{aligned} a_1 &= \frac{1}{2\pi}\int_{B(0,R)} \operatorname{div}(V) + \frac{i}{2\pi}\int_{B(0,R)} \operatorname{div}(V^\perp) \\ &= -\frac{i}{2\pi}\int_{\mathbb{R}^2} \operatorname{curl}(V) \\ &= -\frac{i}{2\pi}\int_{\mathbb{R}^2} \mu. \end{aligned}$$

Finally, we get

$$\begin{aligned} V_1 + iV_2 &= \frac{i}{2\pi}\Big(\int_{\mathbb{R}^2}\mu\Big)\frac{1}{\overline{z}} + O(|z|^{-2}) \\ &= \frac{1}{2\pi}\Big(\int_{\mathbb{R}^2}\mu\Big) i\frac{z}{|z|^2} + O(|z|^{-2}) \end{aligned}$$

which gives us the fifth claim.

Now let us assume that $\displaystyle\int_{\mathbb{R}^2}\mu = 0$ and bound the L^2 norm of V. Let $\psi = g * \mu$, then by the fifth point of the inequality, W is holomorphic in $B(0,R)^c$ and has a holomorphic primitive. Thus we get $\psi(x) = D + O(|x|^{-1})$ and for $A > 0$ big enough,

$$\begin{aligned} \left|\int_{\partial B(0,A)} V\psi\right| &\leqslant C\,\|V\|_{L^\infty(\partial B(0,A))}\,\|\psi\|_{L^\infty(\partial B(0,A))}\,2\pi A \\ &\leqslant \frac{CD}{A} \underset{A\to+\infty}{\longrightarrow} 0. \end{aligned}$$

This fact allows us to compute the following integral by parts:

$$\begin{aligned} \int_{\mathbb{R}^2} |V|^2 &= \int_{\mathbb{R}^2} \nabla\psi\cdot\nabla\psi \\ &= -\int_{\mathbb{R}^2} \psi\Delta\psi \\ &= \int_{\mathbb{R}^2} \psi\mu \\ &\leqslant \|\psi\|_{L^\infty(\operatorname{supp}(\mu))}\,\|\mu\|_{L^1} \\ &\leqslant CR[\mu]\,\|\mu\|_{L^2}\,\|\psi\|_{L^\infty(\operatorname{supp}(\mu))}. \end{aligned}$$

Now if $x \in B(0,R[\mu])$, we have

$$\begin{aligned} |\psi(x)| &\leqslant C\Big(-\int_{B(x,1)\cap B(0,R[\mu])} \ln(|x-y|)|\mu(y)|\,\mathrm{d}y \\ &\quad + \int_{B(x,1)^c\cap B(0,R[\mu])} (|x|+|y|)|\mu(y)|\,\mathrm{d}y\Big) \\ &\leqslant C\Big(-\int_{B(x,1)} \ln(|x-y|)|\mu(y)|\,\mathrm{d}y \\ &\quad + \int_{B(0,R[\mu])} (2R[\mu])|\mu(y)|\,\mathrm{d}y\Big) \end{aligned}$$

$$\leqslant C(1+R[\mu]^2)\,\|\mu\|_{L^2}\,.$$

Thus

$$\int |V|^2 \leqslant CR[\mu](1+R[\mu]^2)\,\|\mu\|_{L^2}^2$$

which is the sixth claim of our proposition. □

Now we prove the uniform bounds we need on f and $\overline{V}$:

Proof of Corollary 2.8. First remark that

$$\begin{aligned}\|\overline{V}\|_{H^{s+2}_{\mathrm{ul}}} &= \left|\int_{\mathbb{R}^2}\rho_0+\omega_0\right|\,\|\nabla g*\chi\|_{H^{s+2}_{\mathrm{ul}}}\\ &\leqslant C\left|\int\rho_0+\omega_0\right|\\ &\leqslant C\,\|\rho_0+\omega_0\|_{L^1}\\ &\leqslant CR[\rho_0+\omega_0]\,\|\rho_0+\omega_0\|_{L^2}\\ &\leqslant 2CR_0M_0\end{aligned}$$

by Claims (1) and (3) of Proposition 2.5. Moreover, if we denote

$$h=\rho+\omega-\left(\int_{\mathbb{R}^2}\rho_0+\omega_0\right)\chi$$

we have

$$\begin{aligned}\|f\|_{L^\infty_T H^{s+1}} &\leqslant \left\|(V-\overline{V})^\perp-(\overline{V}\cdot\nabla)\overline{V}\right\|_{L^\infty_T H^{s+1}}\\ &\leqslant \left\|(V-\overline{V})^\perp\right\|_{L^\infty_T L^2}+\left\|\nabla(V-\overline{V})^\perp\right\|_{L^\infty_T H^s}\\ &\quad+\left\|(\overline{V}\cdot\nabla)\overline{V}\right\|_{L^\infty_T H^{s+1}}\\ &\leqslant \|\nabla g*h\|_{L^\infty_T L^2}+\left\|\nabla^2 g*h\right\|_{L^\infty_T H^s}+\left\|(\overline{V}\cdot\nabla)\overline{V}\right\|_{L^\infty_T H^{s+1}}\\ &\leqslant C(R_T[h]+R_T[h]^3)^{\frac12}\,\|h\|_{L^\infty_T L^2}+C\,\|h\|_{L^\infty_T H^s}\\ &\quad+C(1+R_T[\chi])\left|\int\rho_0+\omega_0\right|^2\|\chi\|^2_{L^\infty_T H^{s+1}}\\ &\leqslant C(R_0,M_0)\end{aligned}$$

where we used Claims (2), (4) and (6) of Proposition 2.5.

Now let us justifiy that $f\in\mathcal{C}_T H^s$. If $t_1,t_2\in[0,T]$, we have

$$\begin{aligned}\|f(t_1)-f(t_2)\|_{H^s} &= \left\|\nabla^\perp g*(\rho(t_1)+\omega(t_1)-\rho(t_2)-\omega(t_2))\right\|_{H^s}\\ &\leqslant \left\|\nabla^\perp g*(\rho(t_1)+\omega(t_1)-\rho(t_2)-\omega(t_2))\right\|_{L^2}\\ &\quad+\left\|\nabla^2 g*(\rho(t_1)+\omega(t_1)-\rho(t_2)-\omega(t_2))\right\|_{H^{s-1}}\\ &\leqslant C(R_T[\rho+\omega]+R_T[\rho+\omega]^3)^{\frac12}\\ &\quad\times\|\rho(t_1)+\omega(t_1)-\rho(t_2)-\omega(t_2)\|_{L^2}\\ &\quad+C\,\|\rho(t_1)+\omega(t_1)-\rho(t_2)-\omega(t_2)\|_{H^{s-1}}\end{aligned}$$

where we used points (6) and (2) of Proposition 2.5 and therefore $f \in \mathcal{C}_T H^s$ follows from $\rho, \omega \in \mathcal{C}_T H^{s-1}$. □

2.2. **Pressureless Euler equations.** In this subsection we prove that there is a unique solution to the following equation

$$\partial_t u + ((u + \overline{V}) \cdot \nabla)u + (u \cdot \nabla)\overline{V} = u^\perp + f \tag{2.6}$$

where $\overline{V}$ and f are the functions defined in (2.4) and (2.5).

Following the idea of [32, 39], we start by fixing $u \in \mathcal{C}_T H^{s+1}$ and solving the linearized equation:

$$\begin{cases} \partial_t \widetilde{u} + ((u + \overline{V}) \cdot \nabla)\widetilde{u} + (\widetilde{u} \cdot \nabla)\overline{V} = \widetilde{u}^\perp + \widetilde{f} \\ \widetilde{u}(0) = \widetilde{u}_0. \end{cases} \tag{2.7}$$

We have the following well-posedness theorem:

Theorem 2.9. *If s is an integer such that $s \geqslant 3$, $u \in \mathcal{C}_T H^{s+1}$, $\widetilde{u}_0 \in H^{s+1}$, μ with compact support and $\widetilde{f} \in L^1_T H^{s+1} \cap \mathcal{C}_T H^s$, then (2.7) has a solution $\widetilde{u} \in \mathcal{C}_T H^{s+1} \cap \mathcal{C}^1_T H^s$, unique in the space $\mathcal{C}_T H^1 \cap \mathcal{C}^1_T L^2$. Moreover, we have the following estimates:*

$$\|\widetilde{u}(t)\|_{H^{s+1}} \leqslant e^{CT(\|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}} + 1)} \left(\|\widetilde{u}_0\|_{H^{s+1}} + C \left\| \widetilde{f} \right\|_{L^1_T H^{s+1}} \right)$$

$$\|\partial_t \widetilde{u}(t)\|_{H^s} \leqslant C \left(\left\| \widetilde{f}(t) \right\|_{H^s} + (\|u\|_{L^\infty_T H^{s+1}} + \|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + 1) \|\widetilde{u}(t)\|_{H^{s+1}} \right).$$

Proof. The proof is a direct application of Theorem 1 of [32] which gives the well-posedness result and the estimates: We can rewrite (2.7) as

$$\partial_t \widetilde{u} + \sum_{i=1}^{2} A_i \partial_i \widetilde{u} + A_3 \widetilde{u} = f$$

where $A_i := (u_i + \overline{V}_i) I_2$ for $i \in \{1, 2\}$ and $A_3 := \begin{pmatrix} \partial_1 \overline{V}_1 & \partial_2 \overline{V}_1 + 1 \\ \partial_1 \overline{V}_2 - 1 & \partial_2 \overline{V}_2 \end{pmatrix}$.

To apply the theorem we need to prove the following:

(1) $A_i \in C_T L^2_{\mathrm{ul}}$ for $1 \leqslant i \leqslant 3$
(2) $\forall t \in [0, T]\ \|A_i(t)\|_{s+1,\mathrm{ul}} \leqslant K$ for $1 \leqslant i \leqslant 3$
(3) A_1 and A_2 symmetric
(4) $\widetilde{f} \in L^1_T H^{s+1} \cap C_T H^s$
(5) $\widetilde{u}_0 \in H^{s+1}$

where $K := \|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}_{\mathrm{ul}}} + C$. The three last points are automatically checked by the assumptions of the theorem. For the first point and the second point, since u is in $\mathcal{C}_T H^{s+1}$, we only need to prove that $\overline{V}$ is in H^{s+2}_{ul}, which is given by Corollary 2.8.

□

As in [32] and [39] we will use the previous estimates to apply a fixed point theorem $u \mapsto \widetilde{u}$ on Equation (2.7) to prove the well-posedness of the non-linear equation (2.6). Let us first recall that we have fixed $u_0 \in H^{s+1}$, $(\omega, \rho) \in X_T$ (where X_T is defined by (2.2)) and

$$R_0 := R[\rho_0 + \omega_0]$$

$$
\begin{aligned}
M_0 &:= \max(\|\rho_0\|_{H^s}, \|\omega_0\|_{H^s}, \|u_0\|_{H^{s+1}}) \\
V &:= -\nabla^\perp g * (\rho + \omega) \\
\overline{V} &:= -\left(\int \omega_0 + \rho_0\right) \nabla^\perp g * \chi \\
f &:= (\overline{V} - V)^\perp - \overline{V} \cdot \nabla \overline{V}.
\end{aligned}
$$

Then the well-posedness of (2.6) is given by the following theorem:

Theorem 2.10. *Let s be an integer such that $s \geqslant 3$, then*

(1) There exists $T^* = T^*(M_0, R_0) \leqslant T$ such that if $T_1 \leqslant T^*$, there is a unique solution $u \in C_{T_1} H^{s+1} \cap C^1_{T_1} H^s$ to (2.6), with

$$\|u\|_{L^\infty_{T_1} H^{s+1}} \leqslant 2M_0.$$

(2) Let u and u' be two solutions defined on $[0, T_1]$ with initial condition u_0 and forcing terms f and f', where

$$
\begin{aligned}
f' &:= (\overline{V} - V')^\perp - \overline{V} \cdot \nabla \overline{V} \\
V' &:= -\nabla^\perp g * (\rho' + \omega')
\end{aligned}
$$

and $(\rho', \omega') \in X_T$. Then we have

$$\left\|u - u'\right\|_{L^\infty_{T_1} H^r} \leqslant C e^{C(M_0, R_0) T_1} \left\|V - V'\right\|_{L^1_{T_1} H^r}$$

where $0 \leqslant r \leqslant s$.

Proof. Let $T_1 \leqslant T$. We will use a fixed-point method on the following subset of $\mathcal{C}_{T_1} L^2$:

$$
\widetilde{X}_{T_1} := \left\{ u \in L^\infty_{T_1} H^{s+1} \cap C_{T_1} H^s \,\middle|\, \|u\|_{L^\infty_{T_1} H^{s+1}} \leqslant 2M_0, u(0) = u_0, \right.
$$
$$
\left. \left\|u(t) - u(t')\right\|_{H^s} \leqslant \widetilde{L} |t - t'| \;\; \forall t, t' \in [0, T_1] \right\}
$$

where $\widetilde{L}$ depends only on M_0 and R_0 and c are constants to be fixed later. Let $u \in \widetilde{X}_{T_1}$ and $\widetilde{u}$ be the solution of (2.7) associated to u. By Theorem 2.9, for $t \leqslant T_1$, we have:

$$
\begin{aligned}
\|\widetilde{u}(t)\|_{H^{s+1}} &\leqslant e^{cT_1(\|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}} + 1)} \left(\|u_0\|_{H^s} + c \|f\|_{L^1_{T_1} H^{s+1}} \right) \\
&\leqslant e^{cT_1(C(R_0, M_0) + 2M_0 + 1)} (M_0 + cT_1 C(M_0, R_0))
\end{aligned}
$$

by Corollary 2.8. Thus we get

$$\|\widetilde{u}(t)\|_{H^{s+1}} \leqslant 2M_0$$

if T_1 is small enough. Moreover, using Corollary 2.8 again, we get

$$
\begin{aligned}
\|\partial_t \widetilde{u}(t)\|_{H^s} &\leqslant c \left(\|f(t)\|_{H^s} + (\|u\|_{L^\infty_T H^{s+1}} + \left\|\overline{V}\right\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + 1) \|\widetilde{u}\|_{H^{s+1}} \right) \\
&\leqslant c(C(M_0, R_0) + (2M_0 + C(M_0, R_0) + 1) 2M_0) =: \widetilde{L}
\end{aligned}
$$

Thus for all $T_1 \leqslant T^*$ we have built a map $\Psi : \widetilde{X}_{T_1} \longrightarrow \widetilde{X}_{T_1}$ such that $\Psi(u) = \widetilde{u}$, where $T^* = T^*(M_0, R_0)$. We will now show that Ψ is a contraction

for the induced distance on X_{T_1}. Let u and w be two elements of X_{T_1} and set $U := u - w$. Then $\widetilde{U} := \widetilde{u} - \widetilde{w}$ satisfies:

$$\partial_t \widetilde{U} + ((u + \overline{V}) \cdot \nabla)\widetilde{U} + (\widetilde{U} \cdot \nabla)\overline{V} = -(U \cdot \nabla)\widetilde{w} + \widetilde{U}^{\perp}.$$

Thus since $(U \cdot \nabla)\tilde{w} \in \mathcal{C}_T L^2 \cap L^1_T H^1$ we can apply Theorem 1 from [32] to have the following estimate:

$$\begin{aligned}
\left\|\widetilde{U}\right\|_{L^\infty_{T_1} L^2} &\leqslant e^{cT_1(\left\|\overline{V}\right\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}} + 1)} \left(0 + c\,\|(U \cdot \nabla)\widetilde{w}\|_{L^1_{T_1} L^2}\right) \\
&\leqslant e^{cT_1(C(M_0,R_0)+2M_0+1)} cT_1\, \|\nabla \widetilde{w}\|_{L^\infty_{T_1} L^\infty}\, \|U\|_{L^\infty_{T_1} L^2} \\
&\leqslant 4cM_0T_1 e^{cT_1(C(M_0,R_0)+2M_0+1)}\, \|U\|_{L^\infty_{T_1} L^2}
\end{aligned}$$

using 2.8 in the last inequality. Thus Ψ is a contraction if T is small enough, so since $\widetilde{X}_{T_1}$ is complete (this can be proved in the same way as the closedness of X_T which is proved in the beginning of section 2.4), it has a unique fixed point in $\widetilde{X}_{T_1}$, thus (2.6) has a unique solution for short time. Remark that the solution we find belongs to the space $L^\infty_{T_1} H^{s+1} \cap W^{1,\infty}_{T_1} H^s$. Let us justify that it also belongs to $\mathcal{C}_{T_1} H^{s+1} \cap \mathcal{C}^1_{T_1} H^s$:

Let $\varepsilon > 0$, $t_1, t_2 \in [0, T_1]$ and χ_n be a mollifier. We have:

$$\begin{aligned}
\|u(t_1) - u(t_2)\|_{H^{s+1}} \leqslant& \|\chi_n * (u(t_1) - u(t_2))\|_{H^{s+1}} \\
&+ \|[I_2 - \chi_n *](u(t_1) - u(t_2))\|_{H^{s+1}} \\
\leqslant& C_n \|u(t_1) - u(t_2))\|_{L^2} + \varepsilon
\end{aligned}$$

if n is big enough (see for example Theorem 4.22 of [9]). Thus since $u \in \mathcal{C}_{T_1} L^2$, if $|t_1 - t_2|$ is small enough,

$$\|u(t_1) - u(t_2)\|_{H^{s+1}} \leqslant 2\varepsilon$$

Thus $u \in \mathcal{C}_{T_1} H^{s+1}$. Moreover we have

$$\partial_t u = -((u + \overline{V}) \cdot \nabla)u - (u \cdot \nabla)\overline{V} + u^{\perp} + f$$

By assumption $f \in \mathcal{C}_{T_1} H^s$ and by the previous fixed point $u^{\perp} \in \mathcal{C}_{T_1} H^s$. Now using Claim (1) of Proposition 2.5, $\overline{V} \in \mathcal{C}_{T_1} H^{s+1}_{\mathrm{ul}}$, so since $s \geqslant 2$, we have

$$\begin{aligned}
((u + \overline{V}) \cdot \nabla)u &\in \mathcal{C}_{T_1} H^s \\
(u \cdot \nabla)\overline{V} &\in \mathcal{C}_{T_1} H^s
\end{aligned}$$

applying Lemma 2.9 of [32] which gives a sufficient condition to have the product of an $H^{s_1}_{\mathrm{ul}}$ and H^{s_2} function in H^r. Thus $u \in \mathcal{C}^1_{T_1} H^s$.

Now let us prove the second point of our theorem: Let u and u' be two solutions associated to f_1 and f_2 defined on $[0, T_1]$ with $T_1 \leqslant T^*(M_0, R_0)$. Then $U := u - u'$ verifies:

$$\partial_t U + ((u + \overline{V}) \cdot \nabla)U + (U \cdot \nabla)(\overline{V} + u') = U^{\perp} + F$$

where $F := f - f'$. We can rewrite this equation as

$$\partial_t U + \sum_{i=1}^{2} A_i \partial_i U + BU = F$$

where $A_i := (u_i + \overline{V}_i)I_2$ and $B := \begin{pmatrix} \partial_1\overline{V}_1 + \partial_1 u_1' & 1 + \partial_2\overline{V}_1 + \partial_2 u_1' \\ \partial_1\overline{V}_2 + \partial_1 u_2' - 1 & \partial_2\overline{V}_2 + \partial_2 u_2' \end{pmatrix}$.
Then by Theorem 1 of [32], for any $0 \leqslant r \leqslant s$ we have:

$$\begin{aligned}
\|U\|_{L^\infty_{T_1}H^r} &\leqslant Ce^{cT_1(\|A_1\|_{L^\infty_{T_1}H^s_{\mathrm{ul}}}+\|A_2\|_{L^\infty_{T_1}H^s_{\mathrm{ul}}}+\|B\|_{L^\infty_{T_1}H^s_{\mathrm{ul}}})}\|F\|_{L^1_{T_1}H^r} \\
&\leqslant Ce^{c(\|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}}+M_0+1)T_1}\|F\|_{L^1_{T_1}H^r} \\
&\leqslant Ce^{c(C(M_0,R_0)+M_0+1)T_1}\left\|V-V'\right\|_{L^1_{T_1}H^r}
\end{aligned}$$

where we used Corollary 2.8 in the last inequality. □

2.3. **Continuity equations.** In this subsection we still fix $s \geqslant 3$, $u \in C_T H^{s+1} \cap C^1_T H^s$, $(\rho,\omega) \in (C_T H^s)^2$, $V := -\nabla^\perp g * (\rho+\omega)$, χ smooth with compact support such that $\int \chi = 1$, $\overline{V} := -\left(\int \omega_0 + \rho_0\right)\nabla^\perp g * \chi$, $v := u + \overline{V}$ and we consider the following continuity equations:

$$\begin{cases} \partial_t\widetilde{\omega} + \operatorname{div}(\widetilde{\omega}V) = 0 \\ \partial_t\widetilde{\rho} + \operatorname{div}(\widetilde{\rho}v) = 0 \end{cases} \tag{2.8}$$

with initial conditions (ρ_0,ω_0).

Theorem 2.11. *Let u,ρ,ω be as in the upper paragraph, there exists a solution $(\widetilde{\rho},\widetilde{\omega}) \in \mathcal{C}_T H^s \cap C^1_T H^{s-1}$ of (2.8), unique in $C_T L^2$. Moreover, we have the following estimates:*

$$\begin{aligned}
\|\widetilde{\rho}\|_{L^\infty_T H^s} &\leqslant \|\rho_0\|_{H^s} e^{cT\|u\|_{L^\infty_T H^s}} \exp\left(ce^{cT\|u\|_{L^\infty_T H^s}} T\|\nabla v\|_{L^\infty_T H^s}\right) \\
\|\widetilde{\omega}\|_{L^\infty_T H^s} &\leqslant \|\omega_0\| e^{cT\|\nabla V\|_{L^\infty_T H^s}} \\
\|\partial_t\widetilde{\omega}\|_{L^\infty_T H^{s-1}} &\leqslant C(1+R_T[\rho+\omega]^{\frac{1}{3}})\|\rho+\omega\|_{L^\infty_T H^s}\|\widetilde{\omega}\|_{L^\infty_T H^s} \\
\|\partial_t\widetilde{\rho}\|_{L^\infty_T H^{s-1}} &\leqslant C\left(\left|\int(\rho_0+\omega_0)\right| + \|u\|_{L^\infty_T H^{s+1}}\right)\|\widetilde{\rho}\|_{L^\infty_T H^s}.
\end{aligned}$$

Now let $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ be two solutions associated to two velocity fields $v_1 = u_1 + \overline{V}$ and $v_2 = u_2 + \overline{V}$ with same initial conditions, and $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ be two solutions associated to two velocity fields V_1 and V_2 with same initial conditions, then we have the following estimates:

$$\|\widetilde{\omega}_1 - \widetilde{\omega}_2\|_{L^\infty_T L^2} \leqslant CT\|V_1 - V_2\|_{L^\infty_T L^2}\|\widetilde{\omega}_2\|_{L^\infty_T H^3}$$

$$\|\widetilde{\rho}_1 - \widetilde{\rho}_2\|_{L^\infty_T L^2} \leqslant CT\|\widetilde{\rho}_2\|_{L^\infty_T H^3}\|v_2 - v_1\|_{L^\infty_T H^1} e^{cT\|u_1\|_{L^\infty_T H^3}}.$$

We will also give a general lemma to control the support of a compactly supported solution of a continuity equation:

Lemma 2.12. *If μ is the solution of the following continuity equation,*

$$\partial_t\mu + \operatorname{div}(\mu a) = 0$$

with $a \in C_T W^{1,\infty}$ and μ_0 with compact support, then μ has compact support and

$$R_T[\mu] \leqslant R[\mu_0] + T\|a\|_{L^\infty_T L^\infty}. \tag{2.9}$$

In order to prove the main theorem we will need the following result:

Lemma 2.13. *If a is a Lipschitz vector field, $\mu_0 \in L^2$ and $f \in L^1_T L^2$ then there exists a unique solution of the continuity equation*

$$\partial_t \mu + \operatorname{div}(\mu a) = f$$

in $\mathcal{C}_T L^2$. Moreover we have the following estimate

$$\|\mu(t)\|_{L^2} \leqslant \left(\|\mu_0\|_{L^2} + \int_0^t \|f(\tau)\|_{L^2} \, \mathrm{d}\tau \right) e^{c \int_0^t \|\operatorname{div}(a)(\tau)\|_{L^\infty} \mathrm{d}\tau}. \tag{2.10}$$

Proof of Lemma 2.13. The existence and uniqueness of the solution in $\mathcal{C}_T L^2$ can be obtained by Theorem 3.19 and Remark 3.20 of [2]. Moreover by Proposition 6 of [1], we know that for all t and almost every x we have

$$\begin{aligned} \mu(t, X(t,x)) =& \mu_0(x) \\ &+ \int_0^t \Big(\operatorname{div}(a)(s, X(s,x)) \mu(s, X(s,x)) + f(s, X(s,x)) \Big) \, \mathrm{d}s \end{aligned}$$

where X is the flow associated to a. Let us denote $\overline{h}(t,x) = h(t, X(t,x))$ for any function h. Taking the L^2 norm of the upper inequality we get

$$\|\overline{\mu}(t)\|_{L^2} \leqslant \|\mu_0\|_{L^2} + \left\|\overline{f}\right\|_{L^1_T L^2} + \int_0^t \|\operatorname{div}(a)(s)\|_{L^\infty} \|\overline{\mu}(s)\|_{L^2} \, .$$

Thus by Gronwall lemma,

$$\|\overline{\mu}(t)\|_{L^2} \leqslant (\|\mu_0\|_{L^2} + \left\|\overline{f}\right\|_{L^1_T L^2}) e^{\|\operatorname{div}(a)\|_{L^1_T L^\infty}}.$$

Now remark that for any L^2 function g,

$$\int |g(X(t,x))|^2 \, \mathrm{d}x = \int |JX^t(x)| |g(x)|^2 \, \mathrm{d}x \qquad \leqslant \|g\|^2_{L^2} \, e^{\|\operatorname{div}(a)\|_{L^1_T L^\infty}}$$

by inequality (7) of [1]. Using it for $\overline{\mu}$ and $\overline{f}$ we get inequality (2.10). □

Now we prove the main theorem of the section:

Proof of Theorem 2.11. Let us know use the previous lemma to prove the H^s bound on $\widetilde{\omega}$. Let α be a multi-index such that $|\alpha| \leqslant s$. Then, since V is divergent-free,

$$\partial_t \partial^\alpha \widetilde{\omega} + \operatorname{div}(V \partial^\alpha \widetilde{\omega}) = F^\alpha$$

where F^α is a combination of $\partial^{\alpha_1} V \cdot \partial^{\alpha_2} \nabla \omega$ with $|\alpha_1| + |\alpha_2| = s$, $|\alpha_2| \leqslant s-1$ and $|\alpha_1| \geqslant 1$. Thus by the upper estimate (2.10), since V is divergent-free, we have:

$$\|\partial^\alpha \widetilde{\omega}(t)\|_{L^2} \leqslant \left(\|\partial^\alpha \widetilde{\omega}_0\|_{L^2} + \int_0^t \|F^\alpha(\tau)\|_{L^2} \, \mathrm{d}\tau \right).$$

If $|\alpha_1| \leqslant s-1$, then

$$\begin{aligned} \|\partial^{\alpha_1} V \cdot \partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} &\leqslant \|\partial^{\alpha_1} V\|_{L^\infty} \|\partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} \\ &\leqslant \|\partial^{\alpha_1} V\|_{H^2} \|\partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} \\ &\leqslant \|\nabla V\|_{H^s} \|\widetilde{\omega}\|_{H^s} \, . \end{aligned}$$

If $|\alpha_1| = s$, then $\alpha_2 = 0$, thus

$$\begin{aligned} \|\partial^{\alpha_1} V \cdot \partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} &\leqslant \|\nabla V\|_{H^s} \|\nabla \widetilde{\omega}\|_{L^\infty} \\ &\leqslant \|\nabla V\|_{H^s} \|\nabla \widetilde{\omega}\|_{H^2} \end{aligned}$$

$$\leqslant \|\nabla V\|_{H^s} \|\widetilde{\omega}\|_{H^s}.$$

Thus

$$\|\partial^\alpha \widetilde{\omega}(t)\|_{L^2} \leqslant \left(\|\partial^\alpha \widetilde{\omega}_0\|_{L^2} + c\int_0^t \|\nabla V(\tau)\|_{H^s} \|\widetilde{\omega}(\tau)\|_{H^s}\, \mathrm{d}\tau \right).$$

Summing over all indices α, we get

$$\|\widetilde{\omega}(t)\|_{H^s} \leqslant \left(\|\widetilde{\omega}_0\|_{H^s} + c\int_0^t \|\nabla V(\tau)\|_{H^s} \|\widetilde{\omega}(\tau)\|_{H^s}\, \mathrm{d}\tau \right).$$

By Grönwall's lemma we get the first inequality of our theorem. Now we will prove the estimate on $\widetilde{\rho}$. For a multi-index α with $|\alpha| \leqslant s$ we also have

$$\partial_t \partial^\alpha \widetilde{\rho} + \operatorname{div}(v\partial^\alpha \widetilde{\rho}) = F^\alpha.$$

Because v is not divergent-free, F^α is now a combination of $\partial^{\alpha_1} v \partial^{\alpha_2}\widetilde{\rho}$ where $|\alpha_1| + |\alpha_2| = s+1$, $|\alpha_1| \geqslant 1$ and $|\alpha_2| \leqslant s$. If $|\alpha_1| \leqslant s-1$, we have

$$\begin{aligned}\|\partial^{\alpha_1} v \cdot \partial^{\alpha_2}\widetilde{\rho}\|_{L^2} &\leqslant \|\partial^{\alpha_1} v\|_{L^\infty} \|\partial^{\alpha_2}\widetilde{\rho}\|_{L^2} \\ &\leqslant \|\partial^{\alpha_1} v\|_{H^2} \|\partial^{\alpha_2}\widetilde{\rho}\|_{L^2} \\ &\leqslant \|\nabla v\|_{H^s} \|\widetilde{\rho}\|_{H^s}.\end{aligned}$$

Now if $|\alpha_1| = s$ or $s+1$ (respectively $|\alpha_2| = 0$ or 1),

$$\begin{aligned}\|\partial^{\alpha_1} v \cdot \partial^{\alpha_2}\widetilde{\rho}\|_{L^2} &\leqslant \|\nabla v\|_{H^s} \|\partial^{\alpha_2}\widetilde{\rho}\|_{L^\infty} \\ &\leqslant \|\nabla v\|_{H^s} \|\partial^{\alpha_2}\widetilde{\rho}\|_{H^2} \\ &\leqslant \|\nabla v\|_{H^s} \|\widetilde{\rho}\|_{H^s}.\end{aligned}$$

Thus

$$\begin{aligned}\|\partial^\alpha \widetilde{\rho}(t)\|_{L^2} \leqslant \Big(&\|\partial^\alpha \widetilde{\rho}_0\|_{L^2} \\ &+ c\int_0^t \|\nabla v(\tau)\|_{H^s} \|\widetilde{\rho}(\tau)\|_{H^s}\, \mathrm{d}\tau \Big) e^{c\int_0^T \|\operatorname{div}(v)\|_{L^\infty}(\tau)\,\mathrm{d}\tau}.\end{aligned}$$

Summing over all indices α, we get

$$\begin{aligned}\|\widetilde{\rho}(t)\|_{H^s} &\leqslant \left(\|\widetilde{\rho}_0\|_{H^s} + c\int_0^t \|\nabla v(\tau)\|_{H^s} \|\widetilde{\rho}(\tau)\|_{H^s}\, \mathrm{d}\tau \right) e^{c\int_0^T \|\operatorname{div}(v)\|_{L^\infty}(\tau)\,\mathrm{d}\tau} \\ &\leqslant \left(\|\widetilde{\rho}_0\|_{H^s} + c\int_0^t \|\nabla v(\tau)\|_{H^s} \|\widetilde{\rho}(\tau)\|_{H^s}\, \mathrm{d}\tau \right) e^{c\int_0^T \|u(\tau)\|_{H^s}\,\mathrm{d}\tau}\end{aligned}$$

because $\operatorname{div}(v) = \operatorname{div}(u)$. The corresponding estimate follows by Grönwall's lemma.

Now let us bound the time derivatives of $\widetilde{\omega}$ and $\widetilde{\rho}$. Take α a multi-index with $|\alpha| \leqslant s-1$, then

$$\partial_t \partial^\alpha \widetilde{\omega} = -\partial^\alpha (V\cdot\nabla\widetilde{\omega}) = \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1,\alpha_2} \partial^{\alpha_1} V \cdot \nabla \partial^{\alpha_2}\widetilde{\omega}.$$

Moreover,

$$\begin{aligned}\|\partial^{\alpha_1} V \cdot \nabla\partial^{\alpha_2}\widetilde{\omega}\|_{L^2} &\leqslant \|\partial^{\alpha_1} V\|_{L^\infty} \|\nabla\partial^{\alpha_2}\widetilde{\omega}\|_{L^2} \\ &\leqslant C(\|V\|_{L^\infty} + \|\nabla V\|_{H^s}) \|\widetilde{\omega}\|_{H^s}.\end{aligned}$$

Now by Claim (3) of Proposition 2.5,

$$\|V\|_{L^\infty} \leqslant CR_T[\rho+\omega]^{\frac{1}{3}} \|\rho+\omega\|_{H^1}$$

and $\|\nabla V\|_{H^s} \leqslant C\|\rho+\omega\|_{H^s}$. Thus we have our estimate.

Let us do the same kind of computations for $\widetilde{\rho}$:

$$\partial_t \partial^\alpha \widetilde{\rho} = \partial^\alpha(\operatorname{div}(u)\widetilde{\rho} + u\cdot\nabla\widetilde{\rho} + \overline{V}\cdot\nabla\widetilde{\rho}).$$

If $|\alpha_1+\alpha_2| = s-1$,

$$\begin{aligned}\|\partial^{\alpha_1} u \cdot \partial^{\alpha_2}\nabla\widetilde{\rho}\|_{L^2} &\leqslant \|\partial^{\alpha_1}u\|_{L^\infty}\left\|\partial^{\alpha_2}\widetilde{\nabla}\rho\right\|_{L^2}\\ &\leqslant \|u\|_{H^{s+1}}\|\widetilde{\rho}\|_{H^s}.\end{aligned}$$

We do the same estimates for every term composing $\partial^\alpha(\operatorname{div}(u)\widetilde{\rho})$, except for

$$\begin{aligned}\|\partial^\alpha \operatorname{div}(u)\widetilde{\rho}\|_{L^2} &\leqslant \|u\|_{H^s}\|\widetilde{\rho}\|_{L^\infty}\\ &\leqslant \|u\|_{H^{s+1}}\|\widetilde{\rho}\|_{H^s}.\end{aligned}$$

Now for the third term, if $|\alpha_1+\alpha_2| = s-1$,

$$\begin{aligned}\left\|\partial^{\alpha_1}\overline{V}\cdot\nabla\partial^{\alpha_2}\widetilde{\rho}\right\|_{L^2} &\leqslant \left\|\partial^{\alpha_1}\overline{V}\right\|_{L^\infty}\|\nabla\partial^{\alpha_2}\widetilde{\rho}\|_{L^2}\\ &\leqslant C\left|\int(\rho_0+\omega_0)\right|\|\nabla g * \partial^{\alpha_1}\chi\|_{L^\infty}\|\widetilde{\rho}\|_{H^s}\\ &\leqslant C\left|\int(\rho_0+\omega_0)\right|\|\widetilde{\rho}\|_{H^s}\end{aligned}$$

by Claim (3) of Proposition 2.5. Thus we have the estimate we wanted to prove.

Now let us prove the last point of our theorem. Substracting the two continuity equations satisfied by $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$, we have

$$\partial_t(\widetilde{\omega}_1-\widetilde{\omega}_2) + \operatorname{div}(V_1(\widetilde{\omega}_1-\widetilde{\omega}_2)) = (V_2-V_1)\cdot\nabla\widetilde{\omega}_2.$$

Using estimate (2.10), we have

$$\begin{aligned}\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{L^\infty_T L^2} &\leqslant c\int_0^T \|(V_1-V_2)\cdot\nabla\widetilde{\omega}_2\|_{L^2}(\tau)\,\mathrm{d}\tau\\ &\leqslant CT\|V_1-V_2\|_{L^\infty_T L^2}\|\widetilde{\omega}_2\|_{H^3}.\end{aligned}$$

Now we prove the last estimate we need for $\widetilde{\rho}_1-\widetilde{\rho}_2$:

$$\partial_t(\widetilde{\rho}_1-\widetilde{\rho}_2) + \operatorname{div}(v_1(\widetilde{\rho}_1-\widetilde{\rho}_2)) = (v_2-v_1)\cdot\nabla\widetilde{\rho}_2 + \operatorname{div}(v_2-v_1)\widetilde{\rho}_2.$$

We can bound the second term the same way that we did for the previous one:

$$\begin{aligned}\|(v_2-v_1)\cdot\nabla\widetilde{\rho}_2 + \operatorname{div}(v_2-v_1)\widetilde{\rho}_2\|_{L^2} &\leqslant \|v_1-v_2\|_{L^2}\|\nabla\widetilde{\rho}_2\|_{L^\infty}\\ &\quad + \|\operatorname{div}(v_1-v_2)\|_{L^2}\|\widetilde{\rho}_2\|_{L^\infty}\\ &\leqslant 2\|v_1-v_2\|_{H^1}\|\widetilde{\rho}_2\|_{H^3}.\end{aligned}$$

Thus by (2.10),

$$\begin{aligned}\|\widetilde{\rho}_1-\widetilde{\rho}_2\|_{L^\infty_T L^2} &\leqslant CT\|\widetilde{\rho}_2\|_{L^\infty_T H^3}\|v_2-v_1\|_{L^\infty_T H^1} e^{cT\|\operatorname{div}(v_1)\|_{L^\infty_T H^2}}\\ &\leqslant CT\|\widetilde{\rho}_2\|_{L^\infty_T H^3}\|v_2-v_1\|_{L^\infty_T H^1} e^{cT\|u_1\|_{L^\infty_T H^3}}\end{aligned}$$

because $\operatorname{div}(v_1) = \operatorname{div}(u_1)$. □

Now let us prove lemma 2.12:

Proof of Lemma 2.12. Solving the continuity equation by characteristics, we see that

$$\operatorname{supp}(\mu(t)) = \psi^t(\operatorname{supp}(\mu(0)))$$

where ψ is the flow associated to a. Moreover, for $x \in \operatorname{supp}\mu_0$,

$$\begin{aligned}|\psi^t(x)| &\leqslant |\psi^0(x)| + |\psi^t(x) - \psi^0(x)| \\ &\leqslant |x| + \left|\int_0^t a(\tau, \psi^\tau(x))\,\mathrm{d}\tau\right| \\ &\leqslant R[\mu_0] + T\,\|a\|_{L^\infty_T L^\infty}\,.\end{aligned}$$

Taking the supremum for all x in $\operatorname{supp}(\mu_0)$, we get (2.9). □

2.4. **Monokinetic spray System.** In this section we prove the well-posedness result of system (1.4), that is Theorem 2.1:

Proof of Theorem 2.1. Let $(\rho_0, \omega_0) \in H^s$, $u_0 \in H^{s+1}$ and χ be a smooth function with compact support such that $\int \chi = 1$. We recall that we have defined

$$M_0 := \max(\|\rho_0\|_{H^s}, \|\omega_0\|_{H^s}, \|u_0\|_{H^{s+1}}),\ R_0 := R[\rho_0 + \omega_0]$$

and

$$\begin{aligned}X_T := \Big\{ &(\omega,\rho) \in L^\infty_T H^s \cap C_T H^{s-1} \Big| \omega(0) = \omega_0, \rho(0) = \rho_0, \\ &\|\rho\|_{L^\infty_T H^s} \leqslant 2M_0, \|\omega\|_{L^\infty_T H^s} \leqslant 2M_0, R_T[\rho+\omega] \leqslant 2R_0, \\ &\forall t \in [0,T], \int (\rho(t)+\omega(t)) = \int (\rho_0+\omega_0), \\ &\forall t,t' \in [0,T], \big\|\rho(t)-\rho(t')\big\|_{H^{s-1}} \leqslant L|t-t'|, \\ &\big\|\omega(t)-\omega(t')\big\|_{H^{s-1}} \leqslant L|t-t'|, \Big\}\end{aligned}$$

with $L > 0$ that we will fix later. Let us justify that X_T is the complete metric space for the distance

$$d((\rho_1,\omega_1),(\rho_2,\omega_2)) := \|\rho_1-\rho_2\|_{L^\infty_T L^2} + \|\omega_1-\omega_2\|_{L^\infty_T L^2}\,.$$

It is sufficient to prove that X_T is closed in $(L^\infty_T L^2)^2$. Let us consider a sequence of functions (ρ_N, ω_N) in X_T and $(\rho,\omega) \in (L^\infty_T L^2)^2$ such that

$$d((\rho_N,\omega_N),(\rho,\omega)) \underset{N\to\infty}{\longrightarrow} 0$$

and prove that $(\rho,\omega) \in X_T$. By Banach-Alaoglu's theorem, since H^s is a Hilbert space, for almost every time there exists a subsequence $\rho_{\varphi_t(n)}(t)$ that converges weakly in H^s. Thus by uniqueness of the limit in weak L^2 $\rho_{\varphi_t(n)}(t)$ converges weakly to $\rho(t)$ for almost every $t \in [0,T]$. By lower semi-continuity of the H^s norm we get that

$$\|\rho\|_{L^\infty_T H^s} \leqslant 2M_0. \tag{2.11}$$

By the same kind of argument we can prove that

$$\|\omega\|_{L^\infty_T H^s} \leqslant 2M_0. \tag{2.12}$$

and that for all $t, t' \in [0,T]$

$$\begin{aligned} \left\|\rho(t)-\rho(t')\right\|_{H^{s-1}} &\leqslant L|t-t'| \\ \left\|\omega(t)-\omega(t')\right\|_{H^{s-1}} &\leqslant L|t-t'|. \end{aligned} \tag{2.13}$$

As a consequence ρ and ω are continuous in time with value in H^{s-1} and thus

$$\begin{aligned} \omega(0) &= \omega_0 \\ \rho(0) &= \rho_0. \end{aligned} \tag{2.14}$$

Moreover for all $t \in [0,T]$,

$$\int \mathbf{1}_{B(0,2R_0)}(\rho_N^2(t)+\omega_N^2(t)) \underset{N\to+\infty}{\longrightarrow} \int \mathbf{1}_{B(0,2R_0)}(\rho^2(t)+\omega^2(t)) = 0$$

by strong convergence in L^2. Thus ρ and ω have compact support and

$$R[\rho+\omega] \leqslant 2R_0. \tag{2.15}$$

Finally, compact support and convergence in L^2 implies convergence in L^1 so we get that for every $t \in [0,T]$,

$$\int (\rho(t)+\omega(t)) = \int (\rho_0+\omega_0). \tag{2.16}$$

Inequalities (2.11), (2.12), (2.13), (2.14), (2.15) and (2.16) gives us that $(\rho,\omega) \in X_T$, so X_T is closed in $L^\infty_T L^2$.

Now let us build a contraction $X_T \longrightarrow X_T$. For $(\rho,\omega) \in X_T$ fixed, we have defined

- $V := -\nabla^\perp g * (\rho+\omega)$
- $\overline{V} := -\left(\int \rho_0+\omega_0\right)\nabla^\perp g * \chi$
- $f := (V-\overline{V})^\perp - (\overline{V}\cdot\nabla)\overline{V}$.

By Corollary 2.8, $f \in L^\infty_T H^{s+1} \cap \mathcal{C}_T H^s$. Let T_1 be sufficiently small so that Theorem 2.10 can be applied and u be the solution of (2.6) given by this theorem, $v = u + \overline{V}$, and $(\widetilde{\rho},\widetilde{\omega})$ be the solution of (2.8) given by Theorem 2.11. According to Theorem 2.10, the smallness of T_1 depends on M_0 and R_0. Now let us justify that for small enough $T_2 \leqslant T_1$, we have $(\widetilde{\rho},\widetilde{\omega}) \in X_{T_2}$. By Theorem 2.10, we have the following estimates:

$$\|\widetilde{\rho}\|_{L^\infty_{T_1} H^s} \leqslant \|\rho_0\|_{H^s}\, e^{cT_1\|u\|_{L^\infty_{T_1} H^s}} \exp\left(ce^{cT_1\|u\|_{L^\infty_{T_1} H^s}} T_1 \|\nabla v\|_{L^\infty_{T_1} H^s}\right)$$

$$\|\widetilde{\omega}\|_{L^\infty_{T_1} H^s} \leqslant \|\omega_0\|\, e^{cT_1\|\nabla V\|_{L^\infty_{T_1} H^s}}.$$

Remark that

$$\|\nabla V\|_{L^\infty_{T_1} H^s} \leqslant C\,\|\rho+\omega\|_{L^\infty_T H^s} \leqslant 4CM_0$$

by Claim (2) of Proposition 2.5. Moreover by Claim (2) of Proposition 2.5 and Theorem 2.10,

$$\begin{aligned} \|\nabla v\|_{L^\infty_{T_1} H^s} &\leqslant C(\|u\|_{L^\infty_{T_1} H^{s+1}} + \left\|\nabla\overline{V}\right\|_{L^\infty_{T_1} H^s}) \\ &\leqslant C(2M_0 + CR_0M_0) \end{aligned}$$

$$\leqslant C(1+R_0)M_0.$$

Thus $\|\widetilde{\rho}\|_{L^\infty_{T_2}H^s} \leqslant 2M_0$ and $\|\widetilde{\omega}\|_{L^\infty_{T_2}H^s} \leqslant 2M_0$ if $T_2 \leqslant T_1$ and T_2 small enough with respect to M_0 and R_0. Now, by Lemma 2.12 and Claim (3) of Proposition 2.5, if $0 \leqslant t \leqslant T_2$ we have

$$\begin{aligned}
R[\widetilde{\rho}(t)+\widetilde{\omega}(t)] &\leqslant R[\widetilde{\rho}(t)] + R[\widetilde{\omega}(t)] \\
&\leqslant R_0 + t(\|v\|_{L^\infty} + \|V\|_{L^\infty}) \\
&\leqslant R_0 + t(\|u\|_{H^s} + \left\|\overline{V}\right\|_{L^\infty} + \|V\|_{L^\infty}) \\
&\leqslant R_0 + t(2M_0 + C\left|\int \rho_0+\omega_0\right| + CR_{T_2}[\rho+\omega]^{\frac{1}{3}}\|\rho+\omega\|_{H^1}) \\
&\leqslant R_0 + T_2(2M_0 + 2CR_0M_0 + 4CR_0^{\frac{1}{3}}M_0) \\
&\leqslant 2R_0
\end{aligned}$$

if T_2 is small enough with respect to R_0 and M_0. By Theorem 2.11, we have:

$$\begin{aligned}
\|\partial_t\widetilde{\omega}\|_{L^\infty_{T_2}H^{s-1}} &\leqslant C(1+R_{T_2}[\rho+\omega]^{\frac{1}{3}})\|\rho+\omega\|_{L^\infty_{T_2}H^s}\|\widetilde{\omega}\|_{L^\infty_{T_2}H^s} \\
&\leqslant C(1+(2R_0)^{\frac{1}{3}})4M_02M_0 \\
\|\partial_t\widetilde{\rho}\|_{L^\infty_{T_2}H^{s-1}} &\leqslant C\left(\left|\int \rho_0+\omega_0\right| + \|u\|_{L^\infty_{T^2}H^{s+1}}\right)\|\widetilde{\rho}\|_{L^\infty_{T_2}H^s} \\
&\leqslant C\big(2R_0M_0+2M_0\big)2M_0.
\end{aligned}$$

Choosing L large enough (with respect to M_0 and R_0), we have

$$\begin{aligned}
\|\partial_t\widetilde{\omega}\|_{L^\infty_{T_2}H^{s-1}} &\leqslant L \\
\|\partial_t\widetilde{\rho}\|_{L^\infty_{T_2}H^{s-1}} &\leqslant L.
\end{aligned}$$

Thus we have built a map $\Phi : (\rho,\omega) \mapsto (\widetilde{\rho},\widetilde{\omega})$ such that $\Phi(X_{T_2}) \subset X_{T_2}$. We will now prove that Φ is a contraction for the $L^\infty_{T_2}L^2$ norm.

Let $(\rho_1,\omega_1),(\rho_2,\omega_2) \in X_{T_2}$, $(\widetilde{\rho}_1,\widetilde{\omega}_1) = \Phi(\rho_1,\omega_1)$ and $(\widetilde{\rho}_2,\widetilde{\omega}_2) = \Phi(\rho_2,\omega_2)$. By Theorem 2.11, we have

$$\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{L^\infty_{T_2}L^2} \leqslant CT_2\|V_1-V_2\|_{L^\infty_{T_2}L^2}\|\widetilde{\omega}_2\|_{L^\infty_{T_2}H^3}$$

and

$$\|\widetilde{\rho}_1-\widetilde{\rho}_2\|_{L^\infty_{T_2}L^2} \leqslant CT_2\|\widetilde{\rho}_2\|_{L^\infty_{T_2}H^3}\|v_2-v_1\|_{L^\infty_{T_2}H^1}e^{cT\|u_1\|_{L^\infty_{T_2}H^3}}.$$

Moreover, by Theorem 2.10:

$$\|v_2-v_1\|_{L^\infty_{T_2}H^1} = \|u_2-u_1\|_{L^\infty_{T_2}H^1} \leqslant Ce^{C(M_0,R_0)T_2}\|V_1-V_2\|_{L^\infty_{T_2}H^1}$$

Thus

$$\begin{aligned}
\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{L^\infty_{T_2}L^2} &+ \|\widetilde{\rho}_1-\widetilde{\rho}_2\|_{L^\infty_{T_2}L^2} \\
&\leqslant 2CT_2M_0\|V_1-V_2\|_{L^\infty_{T_2}L^2} \\
&\quad + 2CT_2M_0e^{2cT_2M_0}Ce^{C(M_0,R_0)T_2}\|V_1-V_2\|_{L^\infty_{T_2}H^1} \\
&\leqslant C(M_0,R_0,T)T_2\|V_1-V_2\|_{L^\infty_{T_2}H^1}
\end{aligned}$$

for any $T_2 \leqslant T_1 \leqslant T$. Moreover,

$$\begin{aligned}
&\|V_1 - V_2\|_{L^\infty_{T_2} H^1} \\
&\leqslant \|V_1 - V_2\|_{L^\infty_{T_2} L^2} + \|\nabla(V_1 - V_2)\|_{L^\infty_{T_2} L^2} \\
&\leqslant C(1 + (4R_0 + (4R_0)^3)^{\frac{1}{2}}) \, \|\rho_1 + \omega_1 - \rho_2 - \omega_2\|_{L^\infty_{T_2} L^2}
\end{aligned}$$

by Claims (2) and (6) of Proposition 2.5. Thus Φ is a contraction if T_2 is small enough (with respect to M_0 and R_0), so it has a unique fixed point $(\rho, \omega) \in X_{T_2}$. □

3. Mean-field limit

In this section we prove Proposition 1.8, Theorem 1.9, Proposition 1.11 and Proposition 1.14. Let us begin by proving Proposition 1.8.

3.1. **Proof of Proposition 1.8.** For $0 < \eta < 1$ we define

$$g^{(\eta)}(x) = \begin{cases} -\dfrac{1}{2\pi} \ln(\eta) & \text{if } |x| \leqslant \eta \\ g(x) & \text{if } |x| \geqslant \eta \end{cases}$$

and we denote $\delta_y^{(\eta)}$ the uniform probability measure on the circle of center y and radius η. We have the following lemma:

Lemma 3.1. *For any $0 < \eta < 1$ and $y \in \mathbb{R}^2$,*

$$\int g(x - z) \, \mathrm{d}\delta_y^{(\eta)}(z) = g^{(\eta)}(x - y)$$

Proof. By a change of variable we may assume that $y = 0$. The function

$$f(x) = \int_{\partial B(0,\eta)} g(x - z) \, \mathrm{d}\delta_0^{(\eta)}(z)$$

is locally bounded and satisfies $\Delta f = -\delta_0^{(\eta)} = \Delta g^{(\eta)}$. Now if $|x| \geqslant \eta$, we have

$$\begin{aligned}
\int_{\partial B(0,\eta)} g(x - z) \, \mathrm{d}\delta_0^{(\eta)}(z) - g^{(\eta)}(x) &= \int_{\partial B(0,\eta)} (g(x - z) - g(x)) \, \mathrm{d}\delta_0^{(\eta)}(z) \\
&= \int_{\partial B(0,\eta)} g\left(\frac{x}{|x|} - \frac{z}{|x|}\right) \mathrm{d}\delta_0^{(\eta)}(z) \\
&\underset{|x| \to \infty}{\longrightarrow} \int_{\partial B(0,\eta)} -\frac{1}{2\pi} \ln(1) = 0
\end{aligned}$$

by dominated convergence theorem. Therefore $f - g^{(\eta)}$ is a harmonic bounded function so it is constant. Since $f(z) = g(\eta) = g^{(\eta)}(z)$ for any z of norm η, we get that $f = g^{(\eta)}$. □

Integrating by parts, since $\int \omega - \omega_N = 0$, we have

$$\|\nabla g * (\omega - \omega_N)\|_{L^2}^2 = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x - y)(\omega - \omega_N)(x)(\omega - \omega_N)(y) \, \mathrm{d}x \, \mathrm{d}y. \tag{3.1}$$

For a more detailed justification of such integrations by parts we refer to [50, Equality (1.23)]. Therefore we only need to justify Inequality (1.10) to get that $\mathcal{H} \geqslant 0$. For that purpose we define

$$\rho_N^{(\eta)} := \frac{1}{N}\sum_{i=1}^N \delta_{q_i}^{(\eta)}.$$

We have

$$\begin{aligned}
&\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho+\omega-\rho_N-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)\\
&\qquad= \iint_{(\mathbb{R}^2\times\mathbb{R}^2)} g(x-y)(\rho+\omega-\rho_N^{(\eta)}-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)\\
&\qquad\quad+\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\,\mathrm{d}\rho_N(x)\,\mathrm{d}\rho_N(y)-\,\mathrm{d}\rho_N^{(\eta)}(x)\,\mathrm{d}\rho_N^{(\eta)}(y))\\
&\qquad\quad+2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)} g(x-y)(\omega-\omega_N+\rho)(x)\,\mathrm{d}x\,\mathrm{d}(\rho_N^{(\eta)}-\rho_N)(y)\\
&\qquad= L_1+L_2+L_3.
\end{aligned}$$

Integrating by parts the first line we find that

$$L_1 = \int |\nabla g * (\rho+\omega-\rho_N^{(\eta)}-\omega_N)|^2 \geqslant 0.$$

For the second line, by Lemma 3.1 we have

$$L_2 = \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}\int_{\mathbb{R}^2}(g(q_i-q_j)-g^{(\eta)}(q_i-y))\,\mathrm{d}(\delta_{q_i}+\delta_{q_i}^{(\eta)})(y)\,\mathrm{d}y$$

This quantity have been bounded in [41, Inequality 2.14] so we get

$$L_2 \geqslant -\frac{C}{N}\sum_{i=1}^N \eta_i^2.$$

Finally,

$$|L_3| \leqslant C|g*(\omega-\omega_N+\rho)|_{C^\gamma}\eta^\gamma$$

so by Morrey's inequality (see for example [9, Theorem 9.12]) and Hardy-Littlewood-Sobolev inequality (see for example [2, Theorem 1.7]) for some $p>2$ we have

$$\begin{aligned}
|L_3| &\leqslant C\,\|\nabla g*(\omega-\omega_N+\rho)\|_{L^p}\,\eta^\gamma\\
&\leqslant C\,\|\omega-\omega_N+\rho\|_{L^{\frac{2p}{p+2}}}\\
&\leqslant C(\|\omega\|_{L^1\cap L^\infty}+\|\omega_N\|_{L^1\cap L^\infty}+\|\rho\|_{L^1\cap L^\infty})\eta^\gamma.
\end{aligned}$$

We get Inequality (1.10) by taking $\eta = N^{-1}$.

3.2. **Proof of Theorem 1.9.** We want to compute the derivative of the functional $\mathcal{H}_N = T_1+T_2+T_3+T_4+T_5+T_6+T_7$ defined in (1.11). We will denote $\alpha := \omega+\rho$ and $\alpha_N := \omega_N+\rho_N$.

$$T_1 := \frac{1}{N}\sum_{i=1}^N |v(q_i)-p_i|^2$$

$$
\begin{aligned}
T_2 &:= \int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\alpha_N(t,\,\mathrm{d}x)\alpha_N(t,\,\mathrm{d}y)\\
T_3 &:= \int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\alpha(t,x)\alpha(t,y)\,\mathrm{d}x\,\mathrm{d}y\\
T_4 &:= -2\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\alpha(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y)\\
T_5 &:= \|\omega(t)-\omega_N(t)\|_{L^2}^2\\
T_6 &:= \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\\
T_7 &= BN^{-\gamma}.
\end{aligned}
$$

Claim 3.2. For every $t\in[0,T]$, we have

$$
\begin{aligned}
\frac{\mathrm{d}T_1}{\mathrm{d}t} =& -\frac{2}{N}\sum_{i=1}^N \nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}\\
&-\frac{2}{N}\sum_{i=1}^N p_i\cdot\left(\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N \nabla g(q_i-q_j)+\nabla g*(\omega_N-\alpha)(q_i)\right)\\
&+2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\rho_N(t,\,\mathrm{d}x)(\alpha_N-\alpha)(t,\,\mathrm{d}y)\\
=:& \,T_{1,1}+T_{1,2}+T_{1,3}.
\end{aligned}
$$

Proof of Claim 3.2. Since $v\in\mathcal{C}^1([0,T]\times\mathbb{R}^2,\mathbb{R}^2)$ we can compute

$$
\begin{aligned}
\frac{\mathrm{d}T_1}{\mathrm{d}t} =& \frac{2}{N}\sum_{i=1}^N (v(q_i)-p_i)\cdot(\partial_t v(q_i)+(p_i\cdot\nabla)v(q_i)-\dot{p}_i)\\
=& \frac{2}{N}\sum_{i=1}^N (v(q_i)-p_i)\cdot\Big(-(v\cdot\nabla)v(q_i)+(v-V)^\perp(q_i)\\
&+(p_i\cdot\nabla)v(q_i)-p_i^\perp+\nabla g*\omega_N(q_i)+\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N\nabla g(q_i-q_j)\Big)\\
=& \frac{2}{N}\sum_{i=1}^N (v(q_i)-p_i)\cdot\Big(((p_i-v(t,q_i))\cdot\nabla)v(q_i)+(v(q_i)-p_i)^\perp\\
&+\nabla g*(\omega_N-\alpha)(q_i)+\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N\nabla g(q_i-q_j)\Big)\\
=& -\frac{2}{N}\sum_{i=1}^N \nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{N}\sum_{i=1}^{N}(v(q_i)-p_i)\cdot\left(\nabla g*(\omega_N-\alpha)(q_i)+\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla g(q_i-q_j)\right)\\
=&-\frac{2}{N}\sum_{i=1}^{N}\nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}\\
&-\frac{2}{N}\sum_{i=1}^{N}p_i\cdot\left(\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla g(q_i-q_j)+\nabla g*(\omega_N-\alpha)(q_i)\right)\\
&+\frac{2}{N}\sum_{i=1}^{N}v(q_i)\cdot\left(\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla g(q_i-q_j)+\nabla g*(\omega_N-\alpha)(q_i)\right)\\
=&-\frac{2}{N}\sum_{i=1}^{N}\nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}\\
&-\frac{2}{N}\sum_{i=1}^{N}p_i\cdot\left(\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla g(q_i-q_j)+\nabla g*(\omega_N-\alpha)(q_i)\right)\\
&+2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}v(t,x)\cdot\nabla g(x-y)\rho_N(t,\mathrm{d}x)\,\mathrm{d}(\alpha_N-\alpha)(t,y)\\
=&T_{1,1}+T_{1,2}+T_{1,3}.
\end{aligned}
$$

□

In the incoming computations, we will find some terms which look like $T_{1,2}$, that is, terms depending on p_i (which will cancel out) or like $T_{1,3}$, that is terms of the form:

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}A(t,x)\cdot\nabla g(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

with A a smooth vector field (for example v or V) and μ, ν some signed finite measures (for example α or $\rho-\rho_N$). We will finish our computations grouping all terms corresponding to the same vector field A. Let us now compute the time derivative of T_2. Notice that the energy

$$\mathcal{E}_N=\frac{1}{N}\sum_{i=1}^{N}|p_i|^2+\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}g(x-y)\,\mathrm{d}\alpha_N(t,x)\,\mathrm{d}\alpha_N(t,y)$$

of System (1.3) is constant in time (for more details see [33, Proposition 5.1]). Thus we have

$$T_2=\mathcal{E}_N-\frac{1}{N}\sum_{i=1}^{N}|p_i|^2$$

and

$$\begin{aligned}\frac{\mathrm{d}T_2}{\mathrm{d}t} &= -\frac{2}{N}\sum_{i=1}^{N} p_i \cdot \left(p_i^{\perp} - \nabla g * \omega_N(q_i) - \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \nabla g(q_i - q_j)\right)\\ &= \frac{2}{N}\sum_{i=1}^{N} p_i \cdot \left(\nabla g * \omega_N(q_i) + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \nabla g(q_i - q_j)\right) \qquad (3.2)\\ &=: T_{2,1}.\end{aligned}$$

Let us compute the time derivative of the third term:

Claim 3.3. $T_3 \in W^{1,\infty}([0,T])$ and for almost every $t \in [0,T]$, we have

$$\begin{aligned}\frac{\mathrm{d}T_3}{\mathrm{d}t} &= 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} v(t,x)\cdot\nabla g(x-y)\rho(t,x)\alpha(t,y)\,\mathrm{d}x\,\mathrm{d}y.\\ &=: T_{3,1}.\end{aligned}$$

Proof of Claim 3.3. Let $(g_\eta)_{0<\eta<1}$ be a family of smooth functions such that

- $g_\eta(x) = g(x) \quad \text{if} \quad |x| \geqslant \eta$,
- $|g_\eta(x)| \leqslant |g(x)|$,
- $|\nabla g_\eta(x)| \leqslant \dfrac{C}{|x|}$.
- $g_\eta(-x) = g_\eta(x)$.

For $0 \leqslant s,t \leqslant T$ and $0 < \eta < 1$, we have

$$T_3(t) - T_3(s) = \iint g(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y.$$

Remark that

$$\begin{aligned}&\iint |g_\eta(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))|\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad\leqslant \iint |g(x-y)|(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))|\,\mathrm{d}x\,\mathrm{d}y < +\infty\end{aligned}$$

because α has compact support. Thus by dominated convergence theorem we have

$$T_3(t) - T_3(s) = \lim_{\eta\to 0}\iint g_\eta(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y. \qquad (3.3)$$

Since g_η is smooth and α has compact support, we have by (1.8) that

$$\int g_\eta(x-y)(\alpha(t,x) - \alpha(s,x))\,\mathrm{d}x = \int_s^t \int (\rho v + \omega V)(\tau,x)\cdot\nabla g_\eta(x-y)\,\mathrm{d}x\,\mathrm{d}\tau$$

Since $(\rho v + \omega V)\cdot * \nabla g_\eta \in L^\infty([0,T],\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}))$, we get from the upper equation that $g_\eta * \alpha \in W^{1,\infty}([0,T],\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}))$ and that for almost every $t \in [0,T]$,

$$\partial_t(g_\eta * \alpha) = -(\rho v + \omega V)\cdot * \nabla g_\eta.$$

Thus we can use $g_\eta * \alpha$ as a test function in (1.8) to get

$$\iint g_\eta(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y$$

$$
\begin{aligned}
&= -\int_s^t \int ((\rho v + \omega V)\cdot *\nabla g_\eta)\alpha + \int_s^t \int (\rho v + \omega V)\cdot \nabla (g_\eta * \alpha) \\
&= -\int_s^t \iint (\rho v + \omega V)(\tau, x)\cdot \nabla g_\eta(y-x)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&+ \int_s^t \iint (\rho v + \omega V)(\tau, x)\cdot \nabla g_\eta(x-y)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&= 2\int_s^t \iint (\rho v + \omega V)(\tau, x)\cdot \nabla g_\eta(x-y)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}
$$

Remark that for almost any $\tau \in [0,T]$, we have

$$
\begin{aligned}
&\iint |\nabla g_\eta(x-y)\cdot(\rho v + \omega V)(\tau, x)\alpha(\tau, y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant \iint \frac{1}{|x-y|}|(\rho v + \omega V)(\tau, x)|\alpha(\tau, y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant \|\rho v + \omega V\|_{L^\infty_T L^1} \sup_{\tau\in[0,T]}\sup_{x\in\mathbb{R}^2} \int \frac{|\alpha(\tau, y)|}{|x-y|}\,\mathrm{d}y \\
&\quad \leqslant (\|\rho\|_{L^\infty_T L^1}\|v\|_{L^\infty_T L^\infty} + \|\omega\|_{L^\infty_T L^1}\|V\|_{L^\infty_T L^\infty}) R_T[\alpha]\,\|\alpha\|_{L^\infty_T L^\infty}\,.
\end{aligned}
$$

where the last inequality follows from the proof of Claim (3) of Proposition 2.5. Thus by dominated convergence theorem,

$$
\begin{aligned}
&\iint g_\eta(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \xrightarrow[\eta\to 0]{} 2\int_s^t \iint (\rho v + \omega V)(\tau, x)\cdot \nabla g(x-y)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}
$$

Combining the upper limit with (3.3) we get that

$$
T_3(t) - T_3(s) = 2\int_s^t \iint (\rho v + \omega V)(\tau, x)\cdot \nabla g(x-y)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau.
$$

Remark that since $\nabla g * \alpha = -V^\perp$, we have

$$
\int_s^t \iint (\omega V)(\tau, x)\cdot \nabla g(x-y)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau = 0.
$$

Finally, we get

$$
T_3(t) - T_3(s) = 2\int_s^t \iint v(\tau, x)\cdot \nabla g(x-y)\rho(\tau, x)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y
$$

which ends the proof of 3.3 for almost every $t \in [0,T]$. □

Now for the fourth term, we have:

Claim 3.4.

$$
\begin{aligned}
\frac{\mathrm{d}T_4}{\mathrm{d}t} =& 2\iint V(t,x)\cdot \nabla g(x-y)(\omega_N - \omega)(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\
&- 2\iint v(t,x)\cdot \nabla g(x-y)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\
&- \frac{2}{N}\sum_{i=1}^N p_i \cdot \nabla g * \alpha(q_i)
\end{aligned}
$$

$$=: T_{4,1} + T_{4,2} + T_{4,3}.$$

Proof of Claim 3.4. Recall that

$$\begin{aligned} T_4 =& -2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\alpha(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ =& -2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}\,\mathrm{d}y \\ & -\frac{2}{N}\sum_{i=1}^N \int_{\mathbb{R}} g(x-q_i(t))\alpha(t,x)\,\mathrm{d}x. \end{aligned}$$

Using g_η in the same way we did for the previous claim, one can prove that T_4 is $W^{1,\infty}$ and that for almost every $t \in (0,T)$,

$$\begin{aligned} \frac{\mathrm{d}T_4}{\mathrm{d}t} =& \Big(-2 \iint \nabla g(x-y)\cdot(V(t,x)\omega(t,x)+\rho(t,x)v(t,x))\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ & + 2\iint \nabla g(x-y)\cdot V_N(t,y)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y\Big) \\ & + \frac{2}{N}\sum_{i=1}^N p_i \cdot \int \nabla g(x-q_i)\alpha(t,x)\,\mathrm{d}x \\ =& A_1 + A_2. \end{aligned}$$

Let us compute each term. For the second term in A_1, remark that:

$$\begin{aligned} & \iint \nabla g(x-y)\cdot V_N(t,y)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\iiint \nabla g(x-y)\cdot\nabla^\perp g(y-z)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z) \\ &= \iiint \nabla^\perp g(x-y)\cdot\nabla g(y-z)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z) \\ &= \iint \left(-\int \nabla^\perp g(y-x)\alpha(t,x)\,\mathrm{d}x\right)\cdot\nabla g(y-z)\omega_N(t,y)\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z) \\ &= \iint V(t,y)\cdot\nabla g(y-z)\omega_N(t,y)\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z) \\ &= \iint V(t,x)\cdot\nabla g(x-y)\omega_N(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y). \end{aligned}$$

It follows that

$$\begin{aligned} A_1 =& 2\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ & -2\iint v(t,x)\cdot\nabla g(x-y)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ =& T_{4,1} + T_{4,2}. \end{aligned}$$

For the second term:

$$A_2 = -\frac{2}{N}\sum_{i=1}^N p_i \cdot \int \nabla g(q_i-x)\alpha(t,x)\,\mathrm{d}x$$

$$= -\frac{2}{N}\sum_{i=1}^{N} p_i \cdot \nabla g * \alpha(q_i)$$
$$= T_{4,3}.$$

□

We now need to differentiate the fifth term with respect to time.

Claim 3.5. T_5 is Lipschitz and for almost every $t \in [0,T]$ we have:

$$\frac{\mathrm{d}T_5}{\mathrm{d}t} = -2\int \nabla\omega \cdot (V - V_N)(\omega - \omega_N)$$
$$=: T_{5,1}.$$

Proof of Claim 3.5. Let $(\chi_\eta)_{\eta>0}$ be a sequence of mollifiers with compact support and set $\omega_N^\eta = \chi_\eta * \omega_N$. For $t, s \in [0,T]$, we have

$$T_5(t) - T_5(s) = \int |\omega(t) - \omega_N(t)|^2 - \int |\omega(s) - \omega_N(s)|^2$$
$$= \lim_{\eta\to 0} \int |\omega(t) - \omega_N^\eta(t)|^2 - \int |\omega(s) - \omega_N^\eta(s)|^2$$
$$= \lim_{\eta\to 0} \int_s^\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int |\omega(\tau) - \omega_N^\eta(\tau)|^2\right) \mathrm{d}\tau$$

Now,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\int |\omega(\tau) - \omega_N^\eta(\tau)|^2 =& 2\int (\omega - \omega_N^\eta)\,\mathrm{div}(\omega V - \chi_\eta * (\omega_N V_N)) \\
=& 2\int (\omega - \omega_N^\eta)\,\mathrm{div}(\omega(V - V_N) + (\omega - \omega_N^\eta)V_N) \\
&+ 2\int (\omega - \omega_N^\eta)\,\mathrm{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) \\
=& 2\int (\omega - \omega_N^\eta)\nabla\omega \cdot (V - V_N) \\
&+ 2\int (\omega - \omega_N^\eta)\nabla(\omega - \omega_N^\eta) \cdot V_N \\
&+ 2\int \omega\,\mathrm{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) \\
&- 2\int \omega_N^\eta\,\mathrm{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)).
\end{aligned}$$

For the first term, remark that for any $1 < p < 2$, we have $V_N \in L^p_{\mathrm{loc}}$ and $\omega_N^\eta \underset{\eta\to 0}{\longrightarrow} \omega_N$ in L^q where $\frac{1}{p} + \frac{1}{q} = 1$. Thus since $\omega - \omega_N$ has compact support and $\nabla\omega \in L^\infty$, we have

$$2\int (\omega - \omega_N^\eta)\nabla\omega \cdot (V - V_N) \underset{\eta\to 0}{\longrightarrow} 2\int (\omega - \omega_N)\nabla\omega \cdot (V - V_N).$$

Remark also that the second term cancels out because V_N is divergent-free:

$$\int (\omega - \omega_N^\eta)V_N \cdot \nabla(\omega - \omega_N^\eta) = -\frac{1}{2}\int V_N \cdot \nabla|\omega - \omega_N^\eta|^2 = 0$$

We will now prove that the two last term tends to zero. For the third term, we integrate by parts to get:

$$2\int \omega \operatorname{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) = -2\int \nabla\omega \cdot (\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N))$$

Since $\omega_N^\eta V_N \underset{\eta\to 0}{\longrightarrow} \omega_N V_N$, $\chi_\eta * (\omega_N V_N) \underset{\eta\to 0}{\longrightarrow} \omega_N V_N$ in L^1 and $\nabla\omega \in L^\infty$, we have

$$2\int \nabla\omega \cdot (\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) \underset{\eta\to 0}{\longrightarrow} 0.$$

For the last term, since all the q_i are outside of the support of ω_N (see Remark 1.3), they are also outside of the support of ω_N^η if η is small enough. Thus we have:

$$V_N \in W^{1,p}(\operatorname{supp}\omega_N^\eta)$$

for any $2 < p < +\infty$. By the commutator estimate of DiPerna and Lions in [16] (see [14, Lemma 2.2] for more details) we get

$$[V_N, \chi_\eta *]\omega_N \underset{\eta\to 0}{\longrightarrow} 0 \qquad \text{in } L^1_{\text{loc}}.$$

Since ω_N^η is uniformly bounded in L^∞, we obtain

$$\int \omega_N^\eta [V_N, \chi_\eta *]\omega_N \underset{\eta\to 0}{\longrightarrow} 0.$$

which ends the proof of Claim 3.5. □

For the sixth term:

Claim 3.6. T_6 Lipschitz and for almost every $t \in [0,T]$ we have

$$\begin{aligned}\frac{\mathrm{d}T_6}{\mathrm{d}t} =& 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot V(t,x)(\omega-\omega_N)(t,x)(\omega-\omega_N)(t,y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2\int_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot \nabla^\perp g * (\omega-\omega_N)(t,x)\omega_N(t,x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y).\end{aligned}$$

Proof of Claim 3.6. Using g_η in the same way we did for Claim 3.3, one can prove that T_6 is $W^{1,\infty}$ and that for almost every $t \in (0,T)$,

$$\begin{aligned}\frac{\mathrm{d}T_6}{\mathrm{d}t} =& 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot (V\omega - V_N\omega_N)(t,x)(\omega-\omega_N)(t,y)\,\mathrm{d}x\,\mathrm{d}y \\ =& 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot V(t,x)(\omega-\omega_N)(t,x)(\omega-\omega_N)(t,y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot (V-V_N)(t,x)\omega_N(t,x)(\omega-\omega_N)(t,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Since $V - V_N = -\nabla^\perp g * (\omega - \omega_N + \rho - \rho_N)$ we get

$$\begin{aligned}&\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot (V-V_N)(t,x)\omega_N(t,x)(\omega-\omega_N)(t,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot \nabla^\perp g * (\omega-\omega_N)(t,x)\omega_N(t,x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y)\end{aligned}$$

which ends the proof of Claim 3.6. □

Remark that all terms depending on p_i (coming from the equations of Claim 3.2, Claim 3.4 and Equation (3.2)) cancels out, that is

$$T_{1,2} + T_{2,1} + T_{4,3} = 0.$$

Now let us group all terms of the form

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

coming from the equations of Claims 3.2, 3.3 and 3.4:

$$\begin{aligned}
&T_{1,3} + T_{3,1} + + T_{4,2}\\
=&2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)(\rho_N\otimes(\alpha_N-\alpha)-\rho\otimes\alpha_N+\rho\otimes\alpha)(\,\mathrm{d}x\,\mathrm{d}y)\\
=&2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\rho-\rho_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
=&2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&-2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)(\omega-\omega_N)(t,x)\,\mathrm{d}x\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
=&\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}(v(t,x)-v(t,y))\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&+2\int v^\perp(t,x)\cdot(V-V_N)(t,x)(\omega-\omega_N)(t,x)\,\mathrm{d}x
\end{aligned}$$

because

$$\begin{aligned}
&2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&=\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,y)\cdot\nabla g(y-x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\\
&=\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}(v(t,x)-v(t,y))\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y).
\end{aligned}$$

Let us do the same for V (there is only one term, coming from the equations of Claim 3.4):

$$\begin{aligned}
T_{4,1} &= 2\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y)\\
&= 2\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\,\mathrm{d}x\,\mathrm{d}(\alpha-\alpha_N)(t,y)
\end{aligned}$$

because

$$\begin{aligned}
&\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\alpha(t,y)\,\mathrm{d}x\,\mathrm{d}y\\
&=\int V(t,x)(\omega_N-\omega)(t,x)\cdot V^\perp(t,x)\,\mathrm{d}x\\
&=0.
\end{aligned}$$

Thus

$$T_{4,1} = -2 \iint V^\perp(t,x) \cdot (V - V_N)(t,x)(\omega(t,x) - \omega_N(t,x))\,\mathrm{d}x.$$

Putting all terms together, we obtain

(3.4)

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t} =& -\frac{2}{N}\sum_{i=1}^{N} \nabla v(q_i) : (v(q_i) - p_i)^{\otimes 2} \\
&+ \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (v(t,x) - v(t,y)) \cdot \nabla g(x-y)(\alpha - \alpha_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\
&+ \int_{\mathbb{R}^2} A \cdot (V - V_N)(\omega - \omega_N) \\
&+ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot V(t,x)(\omega - \omega_N)(t,x)(\omega - \omega_N)(t,y)\,\mathrm{d}x\,\mathrm{d}y \\
&+ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot \nabla^\perp g * (\omega - \omega_N)(t,x)\omega_N(t,x)\,\mathrm{d}x\,\mathrm{d}(\rho - \rho_N)(y) \\
=:& R_1 + R_2 + R_3 + R_4 + R_5
\end{aligned}$$

with $A = 2(v^\perp - V^\perp - \nabla\omega)$. In order to control R_3, we will need the following result:

Lemma 3.7. *If $A \in W^{1,\infty}$, then there exists $\lambda > 0$ and a constant C depending only on $\|A\|_{W^{1,\infty}}$ such that*

$$\begin{aligned}
\left|\int A \cdot (V - V_N)(\omega - \omega_N)\right| \leqslant C\Bigg(&\mathcal{F}(Q_N,\rho) + \|\omega - \omega_N\|_{L^2}^2 \\
&+ \iint g(x-y)(\omega - \omega_N)(x)(\omega - \omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y + N^{-\lambda}\Bigg)
\end{aligned}$$

where $\mathcal{F}$ is the functionnal defined by (1.9).

Proof. Let us fix $I = \int A \cdot (V - V_N)(\omega - \omega_N)$, then

$$\begin{aligned}
I =& -\iint A(x) \cdot \nabla^\perp g(x-y)(\omega - \omega_N)(x)\,\mathrm{d}(\omega + \rho - \omega_N - \rho_N)(y)\,\mathrm{d}x \\
=& \frac{1}{2}\iint (A^\perp(x) - A^\perp(y)) \cdot \nabla g(x-y)(\omega - \omega_N)(x)(\omega - \omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y \\
&- \int \nabla g * [\cdot A^\perp(\omega - \omega_N)](y)\,\mathrm{d}(\rho - \rho_N)(y) \\
=:& I_1 + I_2.
\end{aligned}$$

By [50, Lemma 4.3],

$$I_1 = c\int \nabla A^\perp : [g * (\omega - \omega_N), g * (\omega - \omega_N)]$$

where for $i,j \in \{1,2\}$ and h regular enough,

$$[h,h]_{i,j} = 2\partial_i h \partial_j h - |\nabla h|^2 \delta_{i,j}.$$

Hence

$$|I_1| \leqslant C\,\|\nabla A\|_{L^\infty}\,\|\nabla g * (\omega - \omega_N)\|_{L^2}^2\,.$$

Therefore by (3.1) we have

$$|I_1| \leqslant C \|\nabla A\|_{L^\infty} \iint g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.5}$$

Now denote

$$\xi_N := -\nabla g * [\cdot A^\perp(\omega-\omega_N)].$$

We can write

$$I_2 = \int \xi_N(y)(\rho-\rho_N)(\,\mathrm{d}y).$$

Using Proposition 1.5 (proved in [50]), we get that for any $0<\theta<1$, there exists constants $C,\lambda>0$ such that

$$\begin{aligned}|I_2| \leqslant & C|\xi_N|_{C^{0,\theta}} N^{-\lambda} \\ & + C\|\nabla\xi_N\|_{L^2}\left(\mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\right)^{\frac{1}{2}}.\end{aligned}$$

By Morrey's inequality (see for example [9, Theorem 9.12]) and Proposition 2.7, for some $p>2$ depending only on θ, we have

$$\begin{aligned}|\xi_N|_{C^{0,\theta}} &\leqslant C\left\|\nabla^2 g * [\cdot A^\perp(\omega-\omega_N)]\right\|_{L^p} \\ &\leqslant C\|A(\omega-\omega_N)\|_{L^p} \\ &\leqslant C\|A\|_{L^\infty}\|\omega-\omega_N\|_{L^p} \\ &\leqslant C\|A\|_{L^\infty}\left(\left\|\omega^0\right\|_{L^p} + \left\|\omega_N^0\right\|_{L^p}\right)\end{aligned}$$

by Remark 1.3. Therefore by Assumption (1.12),

$$|\xi_N|_{C^{0,\theta}} \leqslant C.$$

where C is independent of N. Now, by Proposition 2.7,

$$\left\|\nabla^2 g * [\cdot A^\perp(\omega-\omega_N)]\right\|_{L^2} \leqslant \|A\|_{L^\infty}\|\omega-\omega_N\|_{L^2}.$$

Thus we obtain the inequality we wanted to prove. □

Let us get back to the expression of $\dfrac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t} = R_1+R_2+R_3+R_4+R_5$ given by (3.4). We have

$$|R_1| \leqslant \frac{2\|\nabla v\|_{L^\infty}}{N}\sum_{i=1}^N |v(q_i)-p_i|^2 \leqslant 2\|\nabla v\|_{L^\infty}\mathcal{H}_N. \tag{3.6}$$

For the second term,

$$\begin{aligned}R_2 =& \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (v(t,x)-v(t,y))\cdot\nabla g(x-y)(\omega-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (v(t,x)-v(t,y))\cdot\nabla g(x-y)(\rho-\rho_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\ &+ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (v(t,y)-v(t,x))\cdot\nabla g(x-y)(\omega-\omega_N)(x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y) \\ =:& R_{2,1}+R_{2,2}+R_{2,3}.\end{aligned}$$

We can bound $R_{2,1}$ as we did to obtain Inequality (3.5) and we get

$$|R_{2,1}| \leqslant C \|\nabla v\|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.7}$$

Using Proposition 1.7 (proved in [50]) with $\mu = \rho \in L^\infty$, we get

$$R_{2,2} \leqslant C \|v\|_{W^{1,\infty}} \left(\mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-\lambda}\right). \tag{3.8}$$

Now,

$$R_{2,3} = \int \chi_N \,\mathrm{d}(\rho-\rho_N)$$

with $\chi_N = -2v\cdot\nabla g * (\omega-\omega_N) + 2\nabla g * (\cdot v(\omega-\omega_N))$. Using Proposition 1.5, we get that

$$\begin{aligned}|R_{2,3}| \leqslant & C\Bigg(|\chi_N|_{\mathcal{C}^{0,\theta}} N^{-\lambda} + \|\nabla\chi_N\|_{L^2}\Bigg(\mathcal{F}(Q_N,\rho)\\ &+ (1+\|\rho\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\Bigg)^{\frac{1}{2}}\Bigg).\end{aligned}$$

Now by Morrey's inequality (see for example [9, Theorem 9.12]), Hardy-Littlewood-Sobolev inequality (see for example [2, Theorem 1.7]) and Proposition 2.7, for some $p>2$ we have

$$\begin{aligned}|\chi_N|_{\mathcal{C}^{0,\theta}} \leqslant & C(\|\nabla v\|_{L^\infty}\|\nabla g*(\omega-\omega_N)\|_{L^p} + \|v\|_{L^\infty}\left\|\nabla^2 g*(\omega-\omega_N)\right\|_{L^p}\\ &+ \left\|\nabla^2 g*(\cdot v(\omega-\omega_N))\right\|_{L^p})\\ \leqslant & C\|v\|_{W^{1,\infty}}(\|\omega-\omega_N\|_{L^p} + \|\omega-\omega_N\|_{L^{\frac{2p}{p+2}}})\\ \leqslant & C\|v\|_{W^{1,\infty}}(\|\omega-\omega_N\|_{L^1} + \|\omega-\omega_N\|_{L^\infty})\\ \leqslant & C\|v\|_{W^{1,\infty}}(\left\|\omega^0\right\|_{L^1} + \left\|\omega_N^0\right\|_{L^1} + \left\|\omega^0\right\|_{L^\infty} + \left\|\omega_N^0\right\|_{L^\infty})\end{aligned}$$

by Remark 1.3. Therefore by Assumption (1.12),

$$|\chi_N|_{C^{0,\theta}} \leqslant C.$$

Moreover, using Proposition 2.7 and Equation (3.1), we have

$$\begin{aligned}\|\nabla\chi_N\|_{L^2} \leqslant & \|\nabla v\|_{L^\infty}\|\nabla g*(\omega-\omega_N)\|_{L^2} + \|v\|_{L^\infty}\left\|\nabla^2 g*(\omega-\omega_N)\right\|_{L^2}\\ &+ \left\|\nabla^2 g*(\cdot v(\omega-\omega_N))\right\|_{L^2}\\ \leqslant & C\|v\|_{W^{1,\infty}}\Bigg(\Bigg(\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\frac{1}{2}}\\ &+ \|\omega-\omega_N\|_{L^2}\Bigg).\end{aligned}$$

Therefore

$$\begin{aligned}|R_{2,3}| \leqslant & C\|v\|_{W^{1,\infty}}\Bigg(\|\omega-\omega_N\|_{L^2}^2\\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\\ &+ \mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-\lambda}\Bigg).\end{aligned} \tag{3.9}$$

Combining inequalities (3.7), (3.8) and (3.9) we find that

$$|R_2| \leqslant C\,\|v\|_{W^{1,\infty}}\,(\mathcal{H}_N + \mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-\lambda}). \tag{3.10}$$

Now using Lemma 3.7, since V, v and $\nabla\omega$ are in L^∞,

$$\begin{aligned}|R_3| \leqslant & C\Big(\mathcal{F}(Q_N,\rho) + \|\omega-\omega_N\|_{L^2}^2 + N^{-\lambda} \\ & + \iint g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\Big).\end{aligned} \tag{3.11}$$

We can bound R_4 as we did to obtain Inequality (3.5) (with $A = V$) and we get

$$|R_4| \leqslant C\,\|\nabla V\|_{L^\infty} \iint g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.12}$$

Finally we have

$$R_5 = \int \nabla g * (\cdot u_N)\,\mathrm{d}(\rho-\rho_N)$$

with $u_N = -\omega_N\nabla^\perp g * (\omega-\omega_N)$. Using Proposition 1.5 we get

$$\begin{aligned}|R_5| \leqslant & C\Big(|\nabla g * (\cdot u_N)|_{\mathcal{C}^{0,\theta}} N^{-\lambda} + \left\|\nabla^2 g * (\cdot u_N)\right\|_{L^2}\Big(\mathcal{F}(Q_N,\rho) \\ & + (1+\|\rho\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\Big)^{\frac{1}{2}}\Big)\end{aligned}$$

Using Morrey's inequality (see for example [9, Theorem 9.12]), Proposition 2.7 and Hardy-Littlewood-Sobolev inequality (see for example [2, Theorem 1.7]), we get that for some $p > 2$,

$$\begin{aligned}|\nabla g * (\cdot u_N)|_{\mathcal{C}^{0,\theta}} &\leqslant C\left\|\nabla^2 g * (\cdot u_N)\right\|_{L^p} \\ &\leqslant C\,\|u_N\|_{L^p} \\ &\leqslant C\,\|\omega_N\|_{L^\infty}\,\|\nabla g * (\omega-\omega_N)\|_{L^p} \\ &\leqslant C\,\|\omega_N\|_{L^\infty}\,\|\omega-\omega_N\|_{L^{\frac{2p}{p+2}}} \\ &\leqslant C\,\|\omega_N\|_{L^\infty}\,(\|\omega-\omega_N\|_{L^1} + \|\omega-\omega_N\|_{L^\infty}) \\ &\leqslant C\left\|\omega_N^0\right\|_{L^\infty}(\left\|\omega^0\right\|_{L^1} + \left\|\omega_N^0\right\|_{L^1} + \left\|\omega^0\right\|_{L^\infty} + \left\|\omega_N^0\right\|_{L^\infty})\end{aligned}$$

by Remark 1.3. Therefore by Assumption (1.12),

$$|\nabla g * (\cdot u_N)|_{\mathcal{C}^{0,\theta}} \leqslant C.$$

Now using Proposition 2.7 again, we get

$$\begin{aligned}\left\|\nabla^2 g * (\cdot u_N)\right\|_{L^2} &\leqslant C\,\|u_N\|_{L^2} \\ &\leqslant C\,\|\omega_N\|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

by Equation (3.1). Therefore by Assumption (1.12),

$$\begin{aligned}|R_5| \leqslant & C\Big(\mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-\lambda} \\ & + \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\Big)\end{aligned} \tag{3.13}$$

Combining inequalities (3.6), (3.10), (3.11), (3.12) and (3.13) we get

$$\left|\frac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t}\right| \leqslant C\left(\mathcal{H}_N + \mathcal{F}(Q_N,\rho) + N^{-\beta}\right).$$

for some $\beta > 0$. We are only remained to bound $\mathcal{F}(Q_N,\rho)$ by $\mathcal{H}_N$. Let us write

$$\begin{aligned}\mathcal{F}(Q_N,\rho) &= \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)(\omega+\rho-\omega_N-\rho_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)\\ &\quad - \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)(\omega-\omega_N)(x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y)\\ &\leqslant \mathcal{H}_N + 0 + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)(\omega-\omega_N)(x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y).\end{aligned}$$

To bound the upper integral, we use Proposition 1.5 to get that

(3.14)

$$\begin{aligned}&2\left|\int g*(\omega-\omega_N)\,\mathrm{d}(\rho-\rho_N)\right|\\ &\qquad\leqslant C\Bigg(|g*(\omega-\omega_N)|_{\mathcal{C}^{0,\theta}}N^{-\lambda} + 2C\,\|\nabla g*(\omega-\omega_N)\|_{L^2}\\ &\qquad\quad\times\left(\mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\right)^{\frac{1}{2}}\Bigg)\\ &\qquad\leqslant C|g*(\omega-\omega_N)|_{\mathcal{C}^{0,\theta}}N^{-\lambda} + C\,\|\nabla g*(\omega-\omega_N)\|_{L^2}^2\\ &\qquad\quad+\frac{1}{2}\left(\mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\right).\end{aligned}$$

Now by Morrey's inequality (see for example [9, Theorem 9.12]) and Hardy-Littlewood-Sobolev inequality (see for example [2, Theorem 1.7]), for some $p > 2$,

$$\begin{aligned}|g*(\omega-\omega_N)|_{\mathcal{C}^{0,\theta}} &\leqslant C\,\|\nabla g*(\omega-\omega_N)\|_{L^p}\\ &\leqslant C\,\|\omega-\omega_N\|_{L^{\frac{2p}{p+2}}}\\ &\leqslant C\left\|\omega^0\right\|_{L^{\frac{2p}{p+2}}} + \left\|\omega_N^0\right\|_{L^{\frac{2p}{p+2}}}\\ &\leqslant C(\left\|\omega^0\right\|_{L^1\cap L^\infty} + \left\|\omega_N^0\right\|_{L^1\cap L^\infty})\\ &\leqslant C\end{aligned} \tag{3.15}$$

by Assumption (1.12). Using (3.1) we also have

$$\|\nabla g*(\omega-\omega_N)\|_{L^2}^2 \leqslant \mathcal{H}_N.$$

Therefore

$$\mathcal{F}(Q_N,\rho) \leqslant CN^{-\lambda} + C\mathcal{H}_N + \frac{1}{2}\mathcal{F}(Q_N,\rho)$$

for some for some $\lambda > 0$, hence

$$\mathcal{F}(Q_N,\rho) \leqslant C(\mathcal{H}_N + N^{-\lambda}). \tag{3.16}$$

It follows that

$$\frac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t} \leqslant C\left(\mathcal{H}_N + N^{-\beta}\right).$$

Applying Grönwall's lemma we get

$$\mathcal{H}_N(t) \leqslant (\mathcal{H}_N(0) + CN^{-\beta})e^{CT},$$

that is Inequality (1.15).

3.3. **Proof of Proposition 1.11.** Let φ be a smooth test function with compact support in $\mathbb{R}^4$. We have:

$$\begin{aligned}
&\int \varphi(x,\xi)\left(\frac{1}{N}\sum_{i=1}^N \delta_{(q_i,p_i)} - \rho\otimes\delta_{\xi=v(x)}\right)(\mathrm{d}x\,\mathrm{d}\xi)\\
=&\frac{1}{N}\sum_{i=1}^N[\varphi(q_i,p_i)-\varphi(q_i,v(q_i))]\\
&+\frac{1}{N}\sum_{i=1}^N\varphi(q_i,v(q_i)) - \int\varphi(x,v(x))\rho(x)\,\mathrm{d}x\\
=:& T_1+T_2.
\end{aligned}$$

Let us bound T_1:

$$\begin{aligned}
|T_1| &\leqslant \frac{1}{N}\sum_{i=1}^N \|\varphi(q_i,\cdot)\|_{W^{1,\infty}}\,|p_i - v(q_i)|\\
&\leqslant \|\varphi\|_{H^5}\,\frac{1}{N}\sum_{i=1}^N |p_i - v(q_i)|
\end{aligned}$$

by Sobolev embedding. Using Cauchy-Schwarz inequality we get

$$|T_1| \leqslant C\,\|\varphi\|_{W^{1,\infty}}\left(\frac{1}{N}\sum_{i=1}^N |p_i - v(q_i)|^2\right)^{\frac{1}{2}}. \tag{3.17}$$

For the second term:

$$\begin{aligned}
|T_2| &= \left|\int\varphi(x,v(x))(\rho-\rho_N)(\,\mathrm{d}x)\right|\\
&\leqslant \|\varphi\circ(I_d,v)\|_{H^2}\,\|\rho-\rho_N\|_{H^{-2}}\,.
\end{aligned}$$

Let $f := (I_d, v)$, then

$$\|\varphi\circ f\|_{H^2} \leqslant C(1+\|\nabla v\|^2_{W^{1,\infty}})\sup_{0\leqslant k\leqslant 2}\left\|\nabla^k\varphi\circ f\right\|_{L^2}. \tag{3.18}$$

Now let $\psi := \partial^\alpha\varphi$ for some multi-index α of length $k\in\{0,1,2\}$, then

$$\begin{aligned}
\|\psi\circ f\|^2_{L^2} &= \int|\psi(x,v(t,x))|^2\,\mathrm{d}x\\
&\leqslant \sup_y\int|\psi(x,y)|^2\,\mathrm{d}x\\
&\leqslant \|F\|_{W^{3,1}}
\end{aligned}$$

where $F(y) := \int |\psi(x,y)|^2 \,\mathrm{d}x$ (see for example [9, Corollary 9.13]). Remark that

$$\begin{aligned}|\partial_{y_i} F(y)| &= 2\left|\int \partial_{y_i}\psi(x,y)\psi(x,y)\,\mathrm{d}x\right| \\ &\leqslant \int |\partial_{y_i}\psi(x,y)|^2\,\mathrm{d}x + \int |\psi(x,y)|^2\,\mathrm{d}x.\end{aligned}$$

Doing the same computations for all derivatives of F of order less or equal than three and integrating in y gives us

$$\|F\|_{W^{3,1}} \leqslant C\,\|\psi\|^2_{H^3}\,.$$

From (3.18) and the upper equation we get

$$\|\varphi\circ f\|_{H^2} \leqslant C(1+\|\nabla v\|^2_{W^{1,\infty}})\,\|\varphi\|_{H^5}\,. \tag{3.19}$$

Now, by [46, Proposition 3.10] (which is a refined version of Proposition 1.5), we have

$$\|\rho-\rho_N\|_{H^{-2}} \leqslant C(|\mathcal{F}(Q_N,\rho)|^{\frac12} + N^{-\frac12}|\ln(N)|^{\frac12} + (1+\|\rho\|_{L^\infty})N^{-\frac12}).$$

Using Assumption (1.16) we can bound $\mathcal{F}(Q_N,\rho)$ as in (3.16) to get

$$|\mathcal{F}(Q_N,\rho)| \leqslant C(\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N) + N^{-\lambda}) \tag{3.20}$$

and therefore

$$\|\rho-\rho_N\|_{H^{-2}} \leqslant C(\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N)^{\frac12} + (1+\|\rho\|_{L^\infty})N^{-\lambda})$$

for some $\lambda>0$. Combining the upper inequality with (3.17) and (3.19) we get that

$$\left|\int \varphi(x,\xi)\left(\frac1N\sum_{i=1}^N \delta_{(q_i,p_i)} - \rho\otimes\delta_{\xi=v(x)}\right)\right|$$

$$\leqslant C(1+\|\nabla v\|^2_{W^{1,\infty}})\,\|\varphi\|_{H^5}\left(\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N)^{\frac12} + C(1+\|\rho\|_{L^\infty})N^{-\beta}\right)^{\frac12}$$

for some $\beta>0$. Thus we get (1.17). It follows from this estimate that if

$$\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N) \underset{N\to\infty}{\longrightarrow} 0$$

then

$$\frac1N\sum_{i=1}^N \delta_{(q_i,p_i)} \underset{H^{-5}}{\longrightarrow} \rho\otimes\delta_{\xi=v(t,x)}.$$

By equality (3.1) we also have

$$\|\nabla g * (\omega_N-\omega)\|^2_{L^2} + \|\omega-\omega_N\|^2_{L^2} \underset{N\to+\infty}{\longrightarrow} 0$$

Now remark that for any $\mu\in L^2$,

$$\begin{aligned}\|\nabla g*\mu\|^2_{L^2} &= C\left\|\widehat{\nabla g}\widehat{\mu}\right\|^2_{L^2} \\ &= C\int \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi \\ &= C\,\|\mu\|^2_{\dot H^{-1}}\end{aligned} \tag{3.21}$$

and therefore

$$\omega_N - \omega \underset{N\to+\infty}{\longrightarrow} 0 \text{ in } L^2 \cap \dot{H}^{-1}.$$

Finally, using Inequality (3.20), we have

$$\mathcal{F}(Q_N, \rho) \underset{N\to+\infty}{\longrightarrow} 0$$

and therefore by [46, Proposition 3.10] we get that for any $a < -1$,

$$\rho_N \underset{N\to+\infty}{\longrightarrow} \rho \text{ in } H^a$$

which concludes the proof of Proposition 1.11.

3.4. **Proof of Proposition 1.14.** We have

$$\begin{aligned}\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} & g(x-y)(\rho_0+\omega_0-\rho_{N,0}-\omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)\\ =&\mathcal{F}(Q_{N,0},\rho_0)+\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)\\ &-2\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})(x)\,\mathrm{d}x\,\mathrm{d}(\rho_0-\rho_{N,0})(y)\end{aligned}$$

It is proved in Theorem 1.1 of [18] that the weak-$*$ convergence of $\rho_{N,0}$ to ρ_0 and the convergence of

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g(q_i^0-q_j^0)$$

to

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\rho_0(x)\rho_0(y)\,\mathrm{d}x\,\mathrm{d}y$$

ensures that

$$\mathcal{F}(Q_{N,0},\rho_0) \underset{N\to+\infty}{\longrightarrow} 0. \tag{3.22}$$

Using (3.1), (3.21) and the convergence of $\omega_{N,0}$ to ω_0 in $\dot{H}^{-1}$ we have that

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \underset{N\to+\infty}{\longrightarrow} 0.$$

Using inequalities (3.14) and (3.15) we have

$$\begin{aligned}&\left|\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})(x)\,\mathrm{d}x\,\mathrm{d}(\rho_0-\rho_{N,0})(y)\right|\\ \leqslant& C\Bigg(\left(\|\omega_0\|_{L^1\cap L^\infty}+\|\omega_{N,0}\|_{L^1\cap L^\infty}\right)N^{-\lambda}\\ &+\|\nabla g*(\omega_0-\omega_{N,0})\|_{L^2}\left(\mathcal{F}(Q_{N,0},\rho_0)+(1+\|\rho_0\|_{L^\infty})N^{-1}+\frac{\ln(N)}{N}\right)^{\frac{1}{2}}\Bigg).\end{aligned}$$

Using Assumption (1.18), equations (3.1), (3.21), (3.22) and the convergence of $\omega_{N,0}$ to ω_0 in $\dot{H}^{-1}$ we get that

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})(x)\,\mathrm{d}x\,\mathrm{d}(\rho_0-\rho_{N,0})(y) \underset{N\to+\infty}{\longrightarrow} 0$$

which ends the proof of Proposition 1.14.

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