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From conformal correlators to analytic S-matrices: CFT₁/QFT₂

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ABSTRACT: We study families of one-dimensional CFTs relevant for describing gapped QFTs in AdS₂. Using the Polyakov bootstrap as our main tool, we explain how S-matrices emerge from the flat space limit of CFT correlators. In this limit we prove that the CFT OPE density matches that of a generalized free field, and that this implies unitarity of the S-matrix. We establish a CFT dispersion formula for the S-matrix, proving its analyticity except for singularities on the real axis which we characterize in terms of the CFT data. In particular positivity of the OPE establishes that any such S-matrix must satisfy extended unitarity conditions. We also carefully prove that for physical kinematics the S-matrix may be more directly described by a phase shift formula. Our results crucially depend on the assumption of a certain gap in the spectrum of operators. We bootstrap perturbative AdS bubble, triangle and box diagrams and find that the presence of anomalous thresholds in S-matrices are precisely signaled by an unbounded OPE arising from violating this assumption. Finally we clarify the relation between unitarity saturating S-matrices and extremal CFTs, establish a mapping between the dual S-matrix and CFT bootstraps, and discuss how our results help understand UV completeness or lack thereof for specific S-matrices.

KEYWORDS: AdS-CFT Correspondence, Conformal and W Symmetry, Nonperturbative Effects, Scattering Amplitudes

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1 Introduction

Recently, guided by conceptual and technical advances in the *conformal* bootstrap, the S-matrix bootstrap program [1] has been brought back to life [2–5]. In this approach to describing strongly coupled quantum field theories, we make elementary assumptions on the scattering processes, such as analyticity and unitarity, and use these to constrain typically low energy observables such as couplings and scattering lengths. While this approach has already led to many interesting results [5–29],¹ it is clear that one of the outstanding fundamental theoretical questions in this subject is how to rigorously justify and determine the precise analyticity properties of S-matrices. While hard-fought progress can be made starting from axiomatic approaches to QFT [31–33], we still find ourselves in the embarrassing position of being far from a satisfactory understanding of even the simplest case of two-to-two equal mass scattering.

One promising approach for dealing with this problem arises in the context of ‘rigid’ holography [2]. In this setup we consider a quantum field theory which is placed in an asymptotically AdS space, where metric fluctuations are assumed to be absent or suppressed, and study it using the standard tools of holography. It does not really matter how this placement is achieved — i.e. for which curvature couplings, choice of boundary conditions, and so on — as long as it *can* be done, in some way, for some sufficiently large AdS radius. This is because what we are really after is an understanding of the physics of the QFT for scales much smaller than the AdS radius, where such choices become irrelevant. Any such theory defines a set of boundary observables which behave in essentially every way as ordinary conformal correlators.² Recent work [34] (building on [35]) argued that for gapped QFTs the S-matrix is essentially directly determined from such correlators.³ Hence, we may hope to understand how the properties of S-matrices arise from those of CFT correlators, which are under better non-perturbative and analytic control [37–39].

The present work serves as the starting point for carrying out this program in the context of gapped 2d QFTs on AdS₂, which are described by families of 1d CFT correlators. This setup is technically simpler since it avoids dealing with the intricacies of the physics of spin, yet sufficiently rich to describe a wealth of interesting systems, including integrable field theories. This case also allows us to establish a detailed dictionary between the conformal and S-matrix bootstraps, via the language of extremal functionals [40–42], enlightening us on both of them.

In detail, we will perform an in depth study of how 2d S-matrices, describing two-to-two equal particle scattering, emerge from 1d conformal correlators of identical scalar operators. Our approach is purely CFT-centric: by this we mean that we do not *explicitly* consider QFTs in AdS₂, but rather we will show that *any* family of CFT correlators with a sufficiently large gap in the OPE *automatically* leads to an S-matrix satisfying all the expected properties. These properties include crossing symmetry, analyticity away from

¹See [30] for a recent overview.

²The CFTs lack a stress-tensor, but global Ward identities are still satisfied by correlators on conformally flat space.

³See also [36].

the real axis, and unitarity. We will see that the latter corresponds to the statement that CFT correlators in Euclidean kinematics approach generalized free fields in the flat space limit. We will establish the analyticity properties by deriving a dispersion formula for the S-matrix starting from one for CFT correlators. We fully characterise the singularities of such S-matrices in terms of the CFT data, and in particular we carefully prove a certain phase shift formula for the S-matrix in physical kinematics.

We test our results on concrete perturbative examples. In particular we show how the bubble, triangle and box diagrams in AdS₂ can be bootstrapped in the flat space limit, and find direct agreement of our formulae with Feynman amplitudes. These examples will also allow us to illustrate how our results may fail when the gap assumption is violated. In S-matrix language this gap is necessary to avoid the possibility of anomalous thresholds. In the CFT we explain that the gap is necessary to avoid unbounded OPE coefficients, arising from the existence of unitary solutions to crossing without identity. We show that for the triangle and box diagrams the appearance of anomalous thresholds precisely correlate with unboundedness of the OPE in the flat space limit.

With our non-perturbative link between scattering and conformal physics well established, we discuss the CFT description of unitarity-saturating S-matrices, such as those describing integrable QFTs. We argue such S-matrices arise from families of extremal CFT correlators, which contain a single tower of operators in the OPE with dimensions determined by the phase shift of the S-matrix, and naturally saturate various bounds on the CFT data. In particular we establish a general mapping between S-matrix and CFT bootstrap optimization problems. We discuss the implications of our construction to the understanding of UV completeness of S-matrices.

Let us now turn to a more detailed technical description of our main results.

Summary and outline.

Setting up the mapping. We will consider a family of CFT₁ correlators of four copies of the same field ϕ , and parameterized by its dimension Δ_ϕ :

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{\mathcal{G}(z)}{|x_{13}|^{2\Delta_\phi} |x_{24}|^{2\Delta_\phi}}, \quad z = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad x_{ij} = x_i - x_j. \quad (1.1)$$

We should have in mind the prototypical example where this family of correlators arises by taking a gapped 2d QFT in AdS₂ and considering AdS boundary observables in such a theory.⁴ In this case each correlator can be labeled by $\Delta_\phi \sim mR_{\text{AdS}}$, where m is the mass of some stable particle in the theory. The flat space limit corresponds to taking large AdS radius keeping physical masses fixed, and since scaling dimensions behave as $\Delta \sim MR$ we must take them all to be large. In particular we must send $\Delta_\phi \rightarrow \infty$. It was argued in [34] that in this limit one may extract the S-matrix of the QFT from the family of CFT correlators, more or less directly: the correlator becomes the S-matrix after a suitable identification of kinematic quantities. However, the limit is subtle and must be done carefully. The purpose of this work is to understand this procedure in detail.

⁴Conformal correlators are obtained by pushing AdS bulk insertions towards the AdS boundary [43].

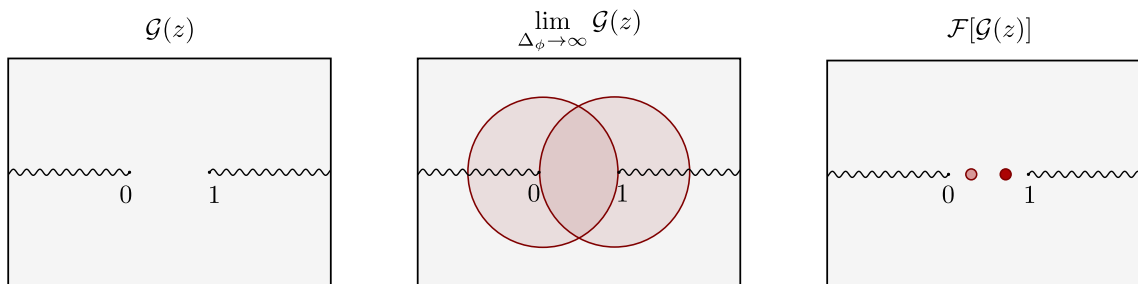


Figure 1. Schematic derivation of the flat space limit of the correlator. We start on the left with the conformal correlator $\mathcal{G}(z)$ which has branch points at $z = 0, 1$. We then take the limit when all dimensions are large; this limit is not well defined in certain kinematic regions as shown in the red blobs in the middle panel. Finally we consider the flat space limit where we analytically continue from the safe to the problematic regions to obtain the S-matrix; after this continuation we may encounter singularities in the S-matrix as shown in the right panel for the case of a single bound state pole.

The concrete mapping between correlators and S-matrices is defined as:

$$S(s) = \mathcal{F}[\mathcal{G}(z_s)], \quad z_s := 1 - s/4, \quad (1.2)$$

where the operator \mathcal{F} describes taking the flat space limit.⁵ The action of \mathcal{F} involves two steps. Firstly, we take the limit $\Delta_\phi \rightarrow \infty$ of the correlator in a kinematic region (to be found) where the limit is finite.⁶ Secondly, we must analytically continue the answer to the desired kinematic point. We illustrate this procedure schematically in figure 1. To see how this works in a simple example, consider the two families of correlators

$$\mathcal{G}^\pm(z) = \pm 1 + z^{-2\Delta_\phi} + (1-z)^{-2\Delta_\phi}. \quad (1.4)$$

These are boundary correlators for a free scalar or fermion field in AdS_2 , or generalized free fields, with the $+/-$ sign for the Boson/Fermion. Since these are free fields we expect the corresponding S-matrices to be ± 1 . To extract this from the correlator, we act with \mathcal{F} . First we go to a region where the limit Δ_ϕ exists:

$$\lim_{\Delta_\phi \rightarrow \infty} \mathcal{G}^\pm(z) = \begin{cases} \pm 1, & \text{if } |z| \geq 1 \ \& \ |1-z| \geq 1 \\ \infty, & \text{otherwise.} \end{cases} \quad (1.5)$$

Secondly, we analytically continue to all z . This step is trivial here, and we indeed have $S(s) = \mathcal{F}[\mathcal{G}] = \pm 1$. In general this second step is harder to perform, and we may encounter singularities hidden inside the “blobs” of figure 1.

⁵In our present 1d/2d context, our conventions are such that our S-matrix would match that of a 2-to-2 scattering process of identical particles of unit mass, written as

$$\text{out}(p_4, p_3 | p_1, p_2)_{\text{in}} = (2\pi)^2 4E_1 E_2 [\delta(\vec{p}_1 - \vec{p}_3)\delta(\vec{p}_2 - \vec{p}_4) + \delta(\vec{p}_1 - \vec{p}_4)\delta(\vec{p}_2 - \vec{p}_3)] S(s), \quad (1.3)$$

where $s = -(p_1 + p_2)^2$.

⁶Note that we are implicitly assuming that every CFT datum has a well defined limit, but which does not have to be finite.

Although the 2d QFT motivation described above is an important inspiration, we will want to remain agnostic about the origin of the family of correlators, and formulate everything purely in CFT language. In this sense, we can take equation (1.2) as a *definition* of a function $S(s)$, and ask about its properties depending on our assumptions about the CFT. Our goals are two-fold:

- Determine the analytic properties of $S(s)$ in the complex s -plane.
- Describe $S(s)$, and in particular its singularities, purely in terms of CFT data.

The flat space limit. To achieve these goals we need to characterize the action of \mathcal{F} . Our main tool for doing so is the Polyakov bootstrap [42, 44–47], which states that any CFT correlator has a representation of the form:

$$\mathcal{G}(z) = \mathcal{P}_0(z) + \sum_{\Delta \geq \Delta_0} a_\Delta \mathcal{P}_\Delta(z). \tag{1.6}$$

This is closely related to the usual OPE expansion (a_Δ are the OPE coefficients squared), but where conformal blocks are replaced by Polyakov blocks $\mathcal{P}_\Delta(z)$, which are essentially crossing symmetric sums of Witten exchange diagrams, as we review in section 2. In particular, their analyticity properties are the same as those of ordinary CFT correlators. Although in general Polyakov blocks are complicated functions, we will be able to compute them explicitly in the flat space limit, allowing us to describe the action of \mathcal{F} on each of them individually. For instance one finds for $\Delta_b < 2\Delta_\phi$ that⁷

$$\mathcal{F}[a_{\Delta_b} \mathcal{P}_{\Delta_b}(z_s)] \propto \frac{g_b^2}{s_b - s} + \frac{g_b^2}{s + s_b - 4}, \quad s_b = (\Delta_b/\Delta_\phi)^2, \quad g_b^2 \propto a_{\Delta_b}, \tag{1.7}$$

which illustrates that after analytic continuation one can find singularities that were not initially there.

Next, we must argue that the \mathcal{F} operation commutes with the infinite sum over states. This can be shown by deriving certain bounds on the OPE coefficients. To prove these bounds it will be necessary to make certain assumptions on the CFT spectrum. In practice it will be sufficient to demand that the lowest state in the OPE (above identity) should have dimension $\Delta_0 > \sqrt{2}\Delta_\phi$ as $\Delta_\phi \rightarrow \infty$. Such a condition is expected, since below some critical gap there exist unitary solutions to crossing without identity which can always be added with an arbitrarily large coefficient to any given CFT correlator. Remarkably, in S -matrix language this assumption translates into demanding that particle production must begin for $s > 2$, which is a sufficient condition for the absence of anomalous thresholds. We will investigate this link in several perturbative examples.

⁷The exact formulas are given in (3.10).

Under these assumptions, the sum over Polyakov blocks can be performed and becomes a dispersion formula for $S(s)$:

$$\begin{aligned}
 S(s) &= 1 + \int_{s_0}^{4m^2} ds' \widetilde{K}(s, s') \tilde{\rho}(s') - \int_{4m^2}^{\infty} ds' K(s, s') \rho(s') \tag{1.8} \\
 \rho(s) &= \lim_{\Delta_\phi \rightarrow \infty} \Delta_\phi^{-\alpha} \sum_{\Delta \in B_\alpha(s)} \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) 4 \sin^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_\phi) \right] \\
 \tilde{\rho}(s) &= \lim_{\Delta_\phi \rightarrow \infty} \Delta_\phi^{-\alpha} \sum_{\Delta \in B_\alpha(s)} \left(\frac{a_\Delta}{\tilde{a}_\Delta^{\text{free}}} \right).
 \end{aligned}$$

Here both K, \widetilde{K} are simple Cauchy-type kernels given in (3.19) and (3.29), and $a_\Delta^{\text{free}}, \tilde{a}_\Delta^{\text{free}}$ are both positive in the associated range of integration and determined by the OPE density of a generalized free field. The discontinuities $\rho, \tilde{\rho}$ are also completely specified as certain averages of the CFT data. Further details on these expressions will be given in the main text. The dispersion formula achieves our main goals: it proves that $S(s)$ is an analytic function in the complex- s plane with singularities determined by the CFT data.

Phase shift formula and extremality. It is of interest to have an expression of the S-matrix evaluated for physical kinematics:

$$S_{\text{phys}}(s) := \lim_{\epsilon \rightarrow 0^+} S(s + i\epsilon), \quad s > 4m^2. \tag{1.9}$$

To compute this we must take two limits on the CFT correlator in a specific order: first large Δ_ϕ and then small ϵ . In section 4 we will prove that these limits can be commuted. The CFT dispersion formula for $S(s)$ already guarantees validity of the real part of the phase shift formula. To complete the proof we will derive a dispersion relation for CFT correlators involving their imaginary part (as opposed to the double discontinuity) and then taking the flat space limit. Computing the physical S-matrix with the order of limits reversed leads to a particularly nice expression for the physical S-matrix known as the “phase shift formula”:

$$S_{\text{phys}}(s) = \lim_{\Delta_\phi \rightarrow \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) e^{-i\pi(\Delta - 2\Delta_\phi)} \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) \tag{1.10}$$

with $\hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s)$ approaching a delta function imposing $s = (\Delta/\Delta_\phi)^2$ whose explicit expression we give in (2.19).⁸ Thus for these kinematics the full S-matrix is a certain average of the CFT data. Importantly, the bounds on OPE coefficients mentioned previously guarantee unitarity of the S-matrix. Schematically, we have

$$a_\Delta \sim a_\Delta^{\text{free}} \quad \Rightarrow \quad |S_{\text{phys}}(s)| \leq 1 \tag{1.11}$$

⁸This formula was first proposed in [2] in higher dimensions. It was shown to follow from the OPE expansion of the correlator in [34], but it required assuming the commuting of limits. Our main result is thus that we prove this assumption is justified for the particular case of 1d CFTs.

The phase shift formula also allows us to better understand what unitarity saturation corresponds to in the CFT language: such S-matrices map onto CFT correlators whose OPE in the flat space limit is effectively (or exactly) described by a single tower of operators with dimensions described by the S-matrix phase shift.

Outline. The outline of this work is as follows. In section 2 we briefly review Polyakov blocks and their computation using master functionals [48] before studying them in the flat space limit. Section 3 is concerned with obtaining the action of \mathcal{F} . We first determine general bounds on the OPE density for states both above and below $2\Delta_\phi$ which acts as the physical threshold in S-matrix language. Using these bounds we obtain the flat space limit of the correlator in terms of a dispersion formula for the S-matrix. In section 4 we study the S-matrix directly in physical kinematics and prove that it satisfies a phase shift formula. Section 5 considers various perturbative checks of our formulae. In particular we study AdS bubble, triangle and box diagrams and show that they are correctly described by our formalism. In the regime where our CFT spectrum assumptions are violated, we find that the OPE decomposition of such diagrams contains a large unbounded component coinciding with the appearance of anomalous thresholds. In section 6 we study various consequences of our mapping between CFT correlators and S-matrices. We argue that any S-matrix which is a finite product of CDD factors may always be obtained as the flat space limit of certain families of CFT correlators. In particular we propose these families can always be chosen to be as extremal, i.e. as saturating a CFT bound. The section concludes by establishing a precise link between the functional based conformal bootstrap and the dual S-matrix bootstrap. We conclude with a discussion of the limitations and implications of our construction for understanding UV completeness of S-matrices, and future research directions. The paper is complemented by several technical appendices.

Note: while this paper was being concluded we became aware of [49] which discusses topics related to the present work.

2 Polyakov blocks in the flat space limit

The main goal of this section is to compute Polyakov blocks in the flat space limit. In the context of 1d CFTs, there are two simple kinds of Polyakov blocks \mathcal{P}_Δ^\pm , corresponding to bosonic/fermionic boundary operators with dimension Δ_ϕ exchanging a scalar/pseudoscalar state bulk field with dimension Δ in the AdS bulk. Polyakov blocks are crossing symmetric combinations of Witten exchange diagrams together with carefully chosen contact terms, and in principle we could use this to compute them. Instead, we will follow a different route by using their representation in terms of master functionals [48].

2.1 Master functionals

Consider the OPE expansion for a correlator

$$\mathcal{G}(z) = \sum_{\Delta \geq 0} a_\Delta G_\Delta(z|\Delta_\phi), \quad G_\Delta(z|\Delta_\phi) = z^{\Delta-2\Delta_\phi} {}_2F_1(\Delta, \Delta, 2\Delta, z), \quad (2.1)$$

where $a_\Delta := \lambda_{\phi\phi\mathcal{O}_\Delta}^2$ are the OPE coefficients squared. Crossing symmetry is the statement $\mathcal{G}(z) = \mathcal{G}(1-z)$, which using the OPE becomes:

$$\sum_{\Delta} a_\Delta F_\Delta(z|\Delta_\phi) = 0, \quad F_\Delta(z|\Delta_\phi) = G_\Delta(z|\Delta_\phi) - G_\Delta(1-z|\Delta_\phi). \quad (2.2)$$

We will often drop the explicit dependence on Δ_ϕ below. One way to extract information from the crossing equation is by defining suitable linear functionals [50]. An interesting class is defined by the following ansatz: [40–42]:

$$\Omega[F_\Delta] \equiv \Omega(\Delta) \equiv \frac{1}{2} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} dz f(z) F_\Delta(z) + \int_{\frac{1}{2}}^1 dz g(z) F_\Delta(z). \quad (2.3)$$

Acting with such functionals on the crossing equation leads to sum rules on the CFT data⁹

$$\Omega \left[\sum_{\Delta \geq 0} a_\Delta F_\Delta(z|\Delta_\phi) \right] = \sum_{\Delta \geq 0} a_\Delta \Omega(\Delta) = 0. \quad (2.4)$$

The bosonic/fermionic master functionals are families of functionals labeled by a cross-ratio w and denoted Ω_w^+, Ω_w^- . They were introduced in [48], where the reader can find further details on their definition and properties. They correspond to choices of kernels that will lead to particularly nice functional actions $\Omega(\Delta)$. Denoting these kernels f_w^\pm, g_w^\pm , they are defined by setting

$$g_w^\pm(z) = \hat{g}_w^\pm(z) \pm \delta(z-w), \quad \hat{g}_w^\pm(z) = \pm(1-z)^{2\Delta_\phi-2} f_w^\pm\left(\frac{z}{z-1}\right) \quad (2.5)$$

with $f_w^\pm(z) = f_w^\pm(1-z)$, as well as the gluing condition¹⁰

$$\mathcal{R}_z f_w^\pm(z) + g_w^\pm(z) + g_w^\pm(1-z) = 0, \quad \text{for } z \in (0, 1). \quad (2.7)$$

There are also certain boundary and analyticity conditions that we must impose to ensure the solution to these constraints is unique.

The reason we care about such functionals here is due to their intimate connection to Polyakov blocks $\mathcal{P}_\Delta(w)$. These blocks are the crossing symmetric sum of Witten exchange diagrams in AdS₂ [42, 44–46, 52]. As it turns out we have

$$\Omega_w^\pm(\Delta) = \mp \left[\mathcal{P}_\Delta^\pm(w) - G_\Delta(w|\Delta_\phi) \right]. \quad (2.8)$$

The functional sum rules associated to master functionals are therefore the statement of the Polyakov bootstrap:

$$\sum_{\Delta} a_\Delta \Omega_w^\pm(\Delta) = 0 \Leftrightarrow \mathcal{G}(w) = \sum_{\Delta} a_\Delta \mathcal{P}_\Delta^\pm(w) \quad (2.9)$$

⁹Under certain conditions, see [51].

¹⁰In this work we define:

$$\mathcal{R}_z f(z) \equiv \lim_{\epsilon \rightarrow 0^+} \frac{f(z+i\epsilon) + f(z-i\epsilon)}{2}, \quad \mathcal{I}_z f(z) \equiv \lim_{\epsilon \rightarrow 0^+} \frac{f(z+i\epsilon) - f(z-i\epsilon)}{2i}. \quad (2.6)$$

Equivalently, the same sum rules can be re-expressed as dispersion relations for CFT correlators. These take the form [48]

$$\begin{aligned}\bar{\mathcal{G}}(w) &= - \int_0^1 dz \hat{g}_w^+(z) d\text{Disc}^+ \bar{\mathcal{G}}(z) \\ \underline{\mathcal{G}}(w) &= + \int_0^1 dz \hat{g}_w^-(z) d\text{Disc}^- \underline{\mathcal{G}}(z)\end{aligned}\tag{2.10}$$

with

$$\bar{\mathcal{G}}(w) := \mathcal{G}(w) - \sum_{0 \leq \Delta \leq 2\Delta_\phi} a_\Delta \mathcal{P}_\Delta^+(w), \quad \underline{\mathcal{G}}(w) := \mathcal{G}(w) - \sum_{0 \leq \Delta \leq 2\Delta_\phi - 1} a_\Delta \mathcal{P}_\Delta^-(w)\tag{2.11}$$

and

$$d\text{Disc}^\pm \mathcal{G}(z) := \mathcal{G}(z) \mp (1-z)^{2\Delta_\phi} \mathcal{R}_z \mathcal{G}\left(\frac{z}{z-1}\right).\tag{2.12}$$

It follows from these results that if we know the master functional kernels we can compute the Polyakov blocks. While we do not have analytic expressions for the kernels in general, we do have them when Δ_ϕ becomes large:

$$f_w^\pm(z) \underset{\Delta_\phi \rightarrow \infty}{=} \pm \frac{2}{\pi} \sqrt{\frac{w(1-w)}{z(z-1)}} \frac{z-1/2}{(z-w)(z-1+w)}, \quad z > 1.\tag{2.13}$$

As a check, notice that this satisfies equation (2.7), which becomes

$$\pm \mathcal{R}_z f_w^\pm(z) \sim -\delta(w-z) - \delta(1-w-z)\tag{2.14}$$

since $\hat{g}_w(z)$ is exponentially suppressed relative to $f_w(z)$ at large Δ_ϕ .

After these preliminary remarks, we are now ready to compute Polyakov blocks in the flat space limit. But first let us point out that for the special case $\Delta = 0$ there is actually no computation to perform. This is because the Polyakov blocks $\mathcal{P}_0^\pm(w)$ are nothing but the generalized free boson/fermion correlators:¹¹

$$\mathcal{P}_0^\pm(w) = \mathcal{G}^\pm(z) = \pm 1 + z^{-2\Delta_\phi} + (1-z)^{-2\Delta_\phi}.\tag{2.15}$$

As for the other cases, we will be interested in the limit where both Δ and Δ_ϕ are large and the limit of the ratio Δ/Δ_ϕ is fixed and strictly different from two. It is convenient then to split the computation according to whether Δ is larger or smaller than ‘threshold’, which is $2\Delta_\phi$. This is because the master functional actions admit different representations depending on these two cases.

2.2 Polyakov blocks for $\Delta > 2\Delta_\phi$

While the action of the master functional is generally defined by equation (2.3), this is not necessarily the most convenient form. Under certain conditions in Δ it is possible to

¹¹This is shown in detail for instance in [42].

deform the contours in that definition to arrive at simpler expressions for the functional actions [41]. In this way, and using (2.8), we arrive at the following expressions:

$$\begin{aligned}
 \mathcal{P}_{\Delta}^{+}(w) &= -2 \sin^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_{\phi}) \right] \int_0^1 dz \hat{g}_w^{+}(z) G_{\Delta}(z|\Delta_{\phi}), & \Delta > 2\Delta_{\phi} \\
 \mathcal{P}_{\Delta}^{-}(w) &= +2 \cos^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_{\phi}) \right] \int_0^1 dz \hat{g}_w^{-}(z) G_{\Delta}(z|\Delta_{\phi}), & \Delta > 2\Delta_{\phi} - 1
 \end{aligned}
 \tag{2.16}$$

In passing notice that these expressions, together with validity of the Polyakov bootstrap, imply the dispersion relations (2.10).

Let us now take the large Δ_{ϕ} limit. Since we already know the functional kernels, we just need to study that limit for the conformal block. As it turns out, the exponential suppression of the \hat{g} kernels can be compensated by an exponential growth of the blocks. At large Δ we have [53]:

$$(1-z)^{2\Delta_{\phi}} G_{\Delta}(z|\Delta_{\phi}) \underset{\Delta \rightarrow \infty}{=} \left(\frac{1-z}{z} \right)^{2\Delta_{\phi}} \frac{[4\rho(z)]^{\Delta}}{\sqrt{1-\rho(z)^2}}, \quad \rho(z) := \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}. \tag{2.17}$$

By further taking $\Delta_{\phi} \rightarrow \infty$ one finds that for $\Delta > 2\Delta_{\phi}$ (with Δ/Δ_{ϕ} fixed) the conformal block is exponentially suppressed except in a narrow region around a specific value of z . This can be expressed in the following way. Let us define:

$$\mathcal{N}_{\Delta_{\phi}}(z, s) := \sqrt{\frac{\Delta_{\phi} s^3}{32\pi(s-4)}} \exp \left[-\frac{\Delta_{\phi} s^3}{32(s-4)} \left(z - \frac{s-4}{s} \right)^2 \right] \tag{2.18}$$

which is a Gaussian in the z variable centered at $z = (s-4)/s$ of width $O(\sqrt{\Delta_{\phi}})$, tending to a delta function in z . Alternatively, define

$$\hat{\mathcal{N}}_{\Delta_{\phi}}(\Delta, s) = \sqrt{\frac{2}{\pi \Delta_{\phi}(s-4)}} \exp \left[-2 \frac{(\Delta - \sqrt{s} \Delta_{\phi})^2}{\Delta_{\phi}(s-4)} \right], \tag{2.19}$$

which is also Gaussian but now in the Δ variable, with width $O(\sqrt{\Delta_{\phi}})$, and also approaching a delta function. We then have

$$\begin{aligned}
 (1-z)^{2\Delta_{\phi}} G_{\Delta}(z|\Delta_{\phi}) &\underset{\Delta, \Delta_{\phi} \rightarrow \infty}{\sim} \frac{16}{\Delta_{\phi} s_{\Delta}^{\frac{3}{2}}} \frac{\mathcal{N}_{\Delta_{\phi}}(z, s_{\Delta})}{a_{\Delta}^{\text{free}}}, & s_{\Delta} &\equiv \left(\frac{\Delta}{\Delta_{\phi}} \right)^2 \\
 &\underset{\Delta, \Delta_{\phi} \rightarrow \infty}{\sim} 2 \frac{\hat{\mathcal{N}}_{\Delta_{\phi}}(\Delta, \frac{4}{1-z})}{a_{\Delta}^{\text{free}}}
 \end{aligned}
 \tag{2.20}$$

where \sim means up to exponentially suppressed terms and a_{Δ}^{free} is the OPE density for a

generalized free field:¹²

$$a_{\Delta}^{\text{free}} = \frac{2\Gamma(\Delta)^2}{\Gamma(2\Delta - 1)} \frac{\Gamma(\Delta + 2\Delta_{\phi} - 1)}{\Gamma(2\Delta_{\phi})^2 \Gamma(\Delta - 2\Delta_{\phi} + 1)}. \quad (2.22)$$

Since in the limit of infinite Δ_{ϕ} the Gaussian becomes a delta function the integrals (2.16) trivialize and we get:

$$\mathcal{P}_{\Delta}^{\pm}(w|\Delta_{\phi}) \underset{\Delta_{\phi} \rightarrow \infty}{=} \mp \frac{4\sqrt{s_{\Delta}} \sin^2 \left[\frac{\pi}{2}(\Delta - 2\Delta_{\phi}) \right]}{\pi \Delta_{\phi}} \frac{\sqrt{w(1-w)}}{a_{\Delta}^{\text{free}}} \frac{z_{\Delta} - 1/2}{\sqrt{z_{\Delta}(z_{\Delta} - 1)} (z_{\Delta} - w)(w + z_{\Delta} - 1)}, \quad (2.23)$$

where $z_{\Delta} = 1 - s_{\Delta}/4$.

2.3 Polyakov blocks for $\Delta < 2\Delta_{\phi}$

Now we turn to the flat space limit of Polyakov blocks with dimension $\Delta < 2\Delta_{\phi}$. To remind us of this constraint we will sometimes denote such dimensions as Δ_b ('b' stands for bound state). The computation will be simpler to consider for general complex cross-ratio w — the special case where w is real is treated in appendix A but leads to results consistent with those here. For complex w the definition of the master functional action is almost exactly as before, but now the kernel g_w^{\pm} no longer contains the delta function piece. However, the kernel f_w^{\pm} still has a pair of poles at w and $1 - w$ with unit residue. The functional action for general Δ_{ϕ} and $\text{Re}[w] < \frac{1}{2}$ can be written as we have seen before in (2.3):

$$\Omega_w^{\pm}(\Delta_b) = \frac{1}{2} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} dz f_w^{\pm}(z) F_{\Delta_b}(z) + \int_{\frac{1}{2}}^1 dz g_w^{\pm}(z) F_{\Delta_b}(z) \quad (2.24)$$

Note that we can obtain the result for general values of w by deforming the contour of integration to avoid the poles in f_w^{\pm} . In the large Δ_{ϕ} limit we can safely ignore the exponentially suppressed contribution of g_w^{\pm} above if $\Delta_b < 2\Delta_{\phi}$. As for the f_w^{\pm} kernel, let us write

$$f_w(z) = \frac{\hat{f}_w(z)}{\sqrt{z(z-1)}}. \quad (2.25)$$

Then the functional action can be written as

$$\Omega_w^{\pm}(\Delta_b) = \frac{i}{2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dz \frac{\hat{f}_w(z)}{\sqrt{z(1-z)}} G_{\Delta_b}(1-z|\Delta_{\phi}) \quad (2.26)$$

Let us set $z_b = \frac{m_b^2}{4}$ with $m_b = \Delta_b/\Delta_{\phi}$. In the large Δ_{ϕ} limit with m_b fixed one finds by using the expressions of the previous subsection that the conformal block factor has a saddle

¹²By this we mean in particular that

$$\mathcal{G}^{\pm}(w) = G_0(w|\Delta_{\phi}) + \sum_{n=0}^{\infty} a_{\Delta_n^{\pm}}^{\text{free}} G_{\Delta_n^{\pm}}(w|\Delta_{\phi}), \quad \Delta_n^{\pm} = 2\Delta_{\phi} + 2n + \frac{1}{2} \mp \frac{1}{2}. \quad (2.21)$$

point at $z = z_b$. To do the integral we can therefore deform the contour of integration to the steepest descent contour, which is defined by the condition

$$\mathcal{I}_z \{2\sqrt{z_b} \log[4\rho(1-z)] - 2\log(1-z)\} = 0, \tag{2.27}$$

where $\rho(z)$ is defined in (2.17). However, we must be careful since when doing this we may cross the poles of f_w^\pm . Since there are two poles at w and $1-w$, this splits the complex- w plane into several regions, bounded by the contour above and its image under crossing, as shown in figure 2. We therefore find the result for the functional action is

$$\Omega_w^\pm(\Delta_b) \underset{\Delta_b, \Delta_\phi \rightarrow \infty}{=} \frac{m_b}{2\Delta_\phi} \frac{\mathcal{I}_z f_w^\pm(z_b)}{\tilde{a}_{\Delta_b}^{\text{free}}} + (\dots) \tag{2.28}$$

where the first term comes from the saddle point approximation, and we introduced

$$\tilde{a}_{\Delta_b}^{\text{free}} = \frac{a_{\Delta_b}^{\text{free}}}{2 \sin[\pi\Delta_\phi(2-m_b)]} \quad (\geq 0 \quad \text{for} \quad \sqrt{2}\Delta_\phi < \Delta_b < 2\Delta_\phi). \tag{2.29}$$

As for the pole contributions, represented as the (\dots) , it is easiest to write them in terms of the Polyakov block. Using the relation between the latter and the master functional actions we find:

$$\mathcal{P}_{\Delta_b}^\pm(w) \underset{\Delta, \Delta_b \rightarrow \infty}{=} \frac{m_b}{\pi\Delta_\phi \tilde{a}_{\Delta_b}^{\text{free}}} \frac{\sqrt{w(1-w)}}{\sqrt{z_b(1-z_b)}} \frac{z_b - 1/2}{(z_b - w)(w + z_b - 1)} + E_{\Delta_b}(w|\Delta_\phi) \tag{2.30}$$

with the crossing symmetric E_{Δ_b} satisfying:

$$E_{\Delta_b}(w|\Delta_\phi) = \begin{cases} G_{\Delta_b}(w|\Delta_\phi) & w \in \text{I} \\ 0 & w \in \text{II} \\ G_{\Delta_b}(1-w|\Delta_\phi) & w \in \text{III} \\ G_{\Delta_b}(w|\Delta_\phi) + G_{\Delta_b}(1-w|\Delta_\phi) & w \in \text{IV} \end{cases} \tag{2.31}$$

where the regions I through IV are shown pictorially in figure 2. In particular we have for real w

$$\begin{cases} w \in \text{I} & \Leftrightarrow 0 < w < \min(z_b, 1-z_b) \\ w \in \text{II} & \Leftrightarrow 1-z_b < w < z_b \\ w \in \text{III} & \Leftrightarrow \max(z_b, 1-z_b) < w < 1 \\ w \in \text{IV} & \Leftrightarrow z_b < w < 1-z_b \end{cases} \tag{2.32}$$

This completes our calculation. As we will see in the next section, the contributions to the flat space limit arising from these Polyakov blocks are obtained by throwing out $E_{\Delta_b}(w|\Delta_\phi)$.

3 Flat-space limit: general kinematics

In this section we will derive the flat space limit of the CFT correlator. For this to be possible we must have control over the behaviour of the OPE in the flat space limit. For states below $2\Delta_\phi$ such bounds already exist [2, 41], under certain assumptions which we will spell out. For states above $2\Delta_\phi$ we will prove new bounds, which imply that on average the OPE density of any CFT must universally match that of a generalized free field. With these results we will show that the Polyakov bootstrap translates into a dispersion formula for the S-matrix $S(s)$, establishing its analyticity properties in the complex s -plane.

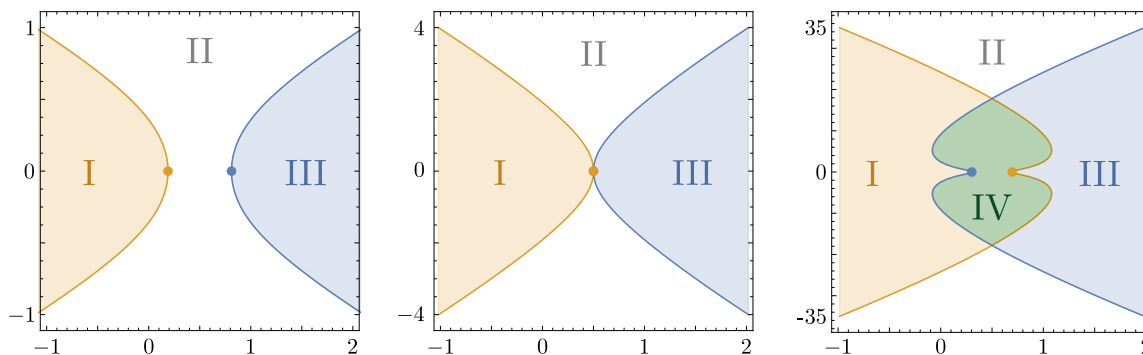


Figure 2. In the flat space limit Polyakov blocks become piecewise functions defined in generically four distinct regions. The boundaries of these regions are set by the conditions $\arg [G_{\Delta_b}(w|\Delta_\phi)] = 0 \cup \arg [G_{\Delta_b}(1-w|\Delta_\phi)] = 0$. We show these regions in the complex w plane for $\Delta_b = 1.8$ (left), $\Delta_b = \sqrt{2}$ (center) and $\Delta_b = 1.1$ (right).

3.1 OPE bounds

3.1.1 States below threshold

We would like to determine the regions where the single bound state contribution $a_{\Delta_b} \mathcal{P}_{\Delta_b}(w)$ is well defined in the flat space limit. In order to do this we need to have a bound on the OPE coefficient a_{Δ_b} (recall $\Delta_b < 2\Delta_\phi$). Such a bound was obtained analytically in [41] (following numerical bootstrap computations in [2], and similar results for the S-matrix bootstrap [3]). Let us briefly review some aspects of the derivation.

Suppose we have some CFT correlator whose OPE has support on the identity operator and a set \mathcal{S} of scaling dimensions. Then we can obtain bounds on OPE coefficients by constructing functionals with suitable positivity properties. Any functional ω leads to a sum rule:¹³

$$\omega(0) + \sum_{\Delta \in \mathcal{S}} a_{\Delta} \omega(\Delta) = 0. \quad (3.1)$$

Suppose now that the functional action satisfies the positivity constraints

$$\omega(\Delta) \geq 0 \quad \text{for all } \Delta \in \mathcal{S}. \quad (3.2)$$

Then the sum rule immediately implies the bounds:

$$\sum_{\Delta \in \mathcal{S}'} a_{\Delta} \omega(\Delta) \leq -\omega(0) \quad \text{for any } \mathcal{S}' \subset \mathcal{S} \quad \Rightarrow \quad a_{\Delta_b} \leq -\frac{\omega(0)}{\omega(\Delta_b)}, \quad \Delta_b \in \mathcal{S}. \quad (3.3)$$

Clearly, these bounds will in general depend on the set of states allowed on the OPE. We will find it useful to distinguish two particular sets of assumptions on this set:

- **Weak OPE condition:** the set \mathcal{S} does not contain any pair of states Δ_1, Δ_2 such that

$$s_{\Delta_1} = 4 - s_{\Delta_2}, \quad s_{\Delta} \equiv \left(\frac{\Delta}{\Delta_\phi} \right)^2. \quad (3.4)$$

¹³We have written ω instead of Ω to emphasize that the functional giving an optimal OPE coefficient is different from the master functional.

- **Strong OPE condition:** the set \mathcal{S} only contains states for which

$$s_\Delta > 2 \quad \Leftrightarrow \quad \Delta > \sqrt{2}\Delta_\phi. \quad (3.5)$$

To be more precise we have in mind imposing these conditions in the flat space limit, so that equations are meant to hold in the limit $\Delta_\phi \rightarrow \infty$. Clearly the strong condition implies the weaker one. For simplicity, we will for the most part work with the stronger assumption, although essentially all our results can be straightforwardly generalized for the weaker one. Note that violation of the weak OPE condition can completely undo most results in this work: a more detailed examination is conducted in section 5.5.

For CFTs satisfying the strong OPE condition, we can find an analytic functional satisfying the necessary positivity conditions in the flat space limit. It is called the sine-Gordon functional ω^{sG} and it can be obtained by choosing kernels [41]:

$$\begin{aligned} f_{z_b}^{sG}(z) &= \frac{2}{\pi} \frac{\sqrt{z_b(1-z_b)}}{\sqrt{z(z-1)}} \frac{z-1/2}{(z-z_b)(z-1+z_b)} \frac{1}{S_b^{sG}(s_z)}, \\ g_{z_b}^{sG}(z) &= (1-z)^{2\Delta_\phi-2} |f_{z_b}^{sG}(\frac{1}{1-z})|. \end{aligned} \quad (3.6)$$

with $s_z = 4(1-z)$ and the sine-Gordon S-matrix:

$$S_b^{sG}(s) = \frac{\sqrt{s(4-s)} + \sqrt{m_b^2(4-m_b^2)}}{\sqrt{s(4-s)} - \sqrt{m_b^2(4-m_b^2)}}, \quad (m_b^2 = 4z_b). \quad (3.7)$$

The action of the functional ω^{sG} implies the bound

$$\sum_{\sqrt{2}\Delta_\phi < \Delta_i < 2\Delta_\phi} \frac{m_i}{2\Delta_\phi} \left(\frac{a_{\Delta_i}}{\tilde{a}_{\Delta_i}^{\text{free}}} \right) \mathcal{I}_{z_i} f_{z_b}^{sG}(z_i) \leq 1, \quad z_i = \frac{1}{4} \left(\frac{\Delta_i}{\Delta_\phi} \right)^2, \quad (3.8)$$

and in particular

$$\frac{2}{\pi} \frac{m_b}{\Delta_\phi} \left(\frac{a_{\Delta_b}}{\tilde{a}_{\Delta_b}^{\text{free}}} \right) \leq \left(g_b^{sG} \right)^2, \quad \Delta_b = m_b \Delta_\phi < 2\Delta_\phi, \quad (3.9)$$

where $\left(g_b^{sG} \right)^2 = |2m_b^2(4-m_b^2)(m_b^2-2)^{-1}|$ is the squared cubic coupling coming from the sine-Gordon breathers' S-matrix. This result nicely makes contact with the flat space S-matrix bounds, since g_b^{sG} is the maximum cubic coupling for a bound state of mass m_b [54].

This result can be generalized to any spectrum \mathcal{S} satisfying the weaker OPE condition. In this case one can obtain a suitable functional by replacing the sine-Gordon S-matrix by a carefully chosen product of CDD poles. The point we wish to emphasize is that this construction only works under the weak condition on the OPE: otherwise the functional will not be positive. Technically this happens because the functional action below $2\Delta_\phi$ is essentially the value of the imaginary part of f evaluated at $z = 1 - s/4$, and this is always antisymmetric under $s \rightarrow 4 - s$.

After this review, let us assume the strong condition on the OPE so that the bounds derived above are valid. The bound (3.9) now allows us to show that the Polyakov block

obtained in equation (2.30) contains a finite piece in the flat space limit. We will shortly prove that this is the only relevant such piece, so that:

$$\mathcal{F} \left[a_{\Delta_b} \mathcal{P}_{\Delta_b}^\pm(w) \right] = \frac{1}{4} \sqrt{\frac{w(1-w)}{z_b(1-z_b)}} \left(\frac{g_b^2}{z_b-w} + \frac{g_b^2}{z_b-1+w} \right), \quad g_b^2 = \frac{2}{\pi} \frac{m_b}{\Delta_\phi} \left(\frac{a_{\Delta_b}}{\tilde{a}_{\Delta_b}^{\text{free}}} \right). \quad (3.10)$$

with the effective coupling bounded from above, $g_b^2 \leq (g_b^{sG})^2$. To prove that this is the indeed the only contribution, we must show that the flat space limit of $a_{\Delta_b} E_{\Delta_b}(w|\Delta_\phi)$ is zero. For example, in region III we have the product $a_{\Delta_b} G_{\Delta_b}(1-w|\Delta_\phi)$, which gives

$$\begin{aligned} \lim_{\Delta_\phi \rightarrow \infty} a_{\Delta_b} |G_{\Delta_b}(1-w|\Delta_\phi)| &\leq \lim_{\Delta_\phi \rightarrow \infty} \left(g_b^{sG} \right)^2 \frac{2\Delta_\phi}{\pi m_b} \tilde{a}_{\Delta_b}^{\text{free}} |G_{\Delta_b}(1-w|\Delta_\phi)|, \\ &\leq \lim_{\Delta_\phi \rightarrow \infty} (\text{factor}) \times \left| \frac{(1-m_b^2/4)^2 (2+m_b)^{m_b} (1-\sqrt{w})^{m_b}}{(1-w)^2 (2-m_b)^{m_b} (1+\sqrt{w})^{m_b}} \right|^{\Delta_\phi}, \end{aligned} \quad (3.11)$$

$$(\text{factor}) = \left(g_b^{sG} \right)^2 \times 8 \frac{\sqrt{\Delta_\phi}}{\pi^{\frac{3}{2}}} m_b^{-1/2} (2+m_b)^{-3/2} (2-m_b)^{-1/2} \left| \frac{\sqrt{w}+1}{w^{1/4}} \right|.$$

From the large Δ_ϕ limit we get zero or infinity depending on the regions where the expression inside the absolute value in the second line is less or bigger than one. The final result in the complex plane is shown in figure 3. The “bad” regions where we get divergences are always contained in the ones for the identity block, and always cover a subset of the real line $1 < z < 2$ (which in S-matrix language is part of the physical values of the center of mass energy $4 < s < 8$).¹⁴ Since outside these bad regions the large Δ_ϕ limit of these factors, and therefore of $a_{\Delta_b} E_{\Delta_b}$, is equal to zero, the analytic continuation is trivial and the flat space limit of a single Polyakov block with $\Delta < 2\Delta_\phi$ is indeed given by (3.10).

We thus conclude that the flat space limit of a single state below threshold (3.10) gives precisely the crossing symmetric bound state pole in our normalization.

3.1.2 States above threshold

To understand the contributions of states above $2\Delta_\phi$ we have to obtain a bound on the ratio $a_\Delta/a_\Delta^{\text{free}}$. Although such bounds have appeared before for finite Δ_ϕ [42], they were derived for large Δ and fixed Δ_ϕ , whereas here we are interested in the limit where both are large and of the same order of magnitude.

We begin by considering the quantity:

$$\begin{aligned} N_z &\equiv (1-z)^{2\Delta_\phi} \left[\mathcal{G}(z) - \sum_{0 \leq \Delta \leq 2\Delta_\phi} a_\Delta G_\Delta(z|\Delta_\phi) \right] \\ &\stackrel{\Delta_\phi \rightarrow \infty}{=} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi} \left(\Delta, \frac{4}{1-z} \right), \end{aligned} \quad (3.12)$$

¹⁴These bad regions match the ones corresponding to the presence of an AdS Landau diagram [34].

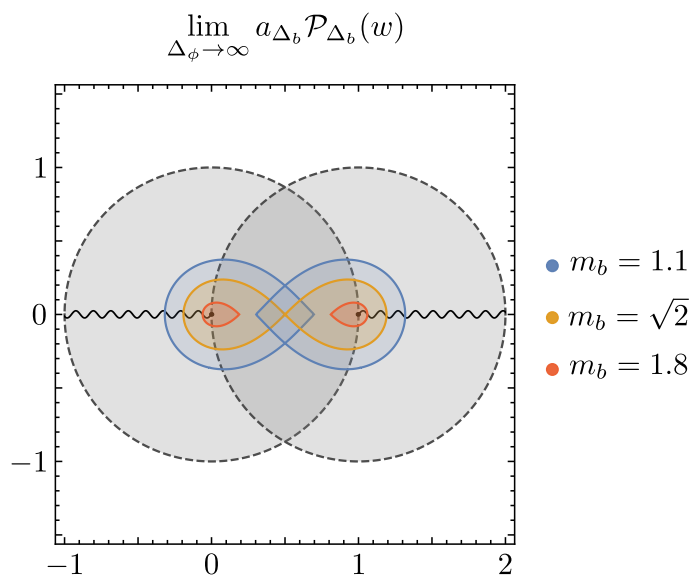


Figure 3. Large Δ_ϕ limit of a single state below threshold $\Delta_b = m_b \Delta_\phi < 2\Delta_\phi$. Each color represents different bound state masses. The limit is well defined for complex values of w outside the shaded regions and diverges inside. The divergent regions (and their crossing symmetric images) start at $w = m_b^2/4$ ($w = 1 - m_b^2/4$) and always include a subset of the physical line $w > 1$ ($w < 0$); they overlap whenever $m_b < \sqrt{2}$. These regions lie always inside the divergent region for the identity given by the two discs depicted in gray.

where $\hat{\mathcal{N}}_{\Delta_\phi}$ was defined in equation (2.19). We see that in the flat space limit, the Euclidean correlator (with suitable subtractions) knows about the average OPE density around $\Delta = 2\Delta_\phi/\sqrt{1-z}$, where the average is taken in a region of width $\sqrt{\Delta_\phi}$. To obtain a bound on N_z , we invoke the following results proven by one of us in [48]:

$$\mathcal{G}^-(z) + \sum_{0 < \Delta \leq 2\Delta_\phi} a_\Delta \mathcal{P}_\Delta^-(z) \leq \mathcal{G}(z) \leq \mathcal{G}^+(z) + \sum_{0 < \Delta \leq 2\Delta_\phi} a_\Delta \mathcal{P}_\Delta^+(z), \quad 0 < z < 1. \quad (3.13)$$

Importantly, these bounds hold for any Δ_ϕ . They imply that for $0 < z < 1$

$$(1-z)^{2\Delta_\phi} (\mathcal{G}^-(z) - G_0(z|\Delta_\phi)) + R^-(z) \leq N_z \leq (1-z)^{2\Delta_\phi} (\mathcal{G}^+(z) - G_0(z|\Delta_\phi)) + R^+(z), \quad (3.14)$$

where

$$R^\pm(z) = (1-z)^{2\Delta_\phi} \sum_{0 < \Delta \leq 2\Delta_\phi} a_\Delta [\mathcal{P}_\Delta^\pm(z) - G_\Delta(z|\Delta_\phi)]. \quad (3.15)$$

Let us now make the same assumptions on the spectrum as for the previous subsection, i.e. the strong condition on the OPE. In this case:

$$R^\pm(z) \underset{\Delta_\phi \rightarrow \infty}{=} 0. \quad (3.16)$$

This follows from the computation of Polyakov blocks in section 2.3 and the results from the previous subsection. The point is that even though the combination $a_\Delta [\mathcal{P}_\Delta(z) - G_\Delta(z|\Delta_\phi)]$

may diverge (as depicted in the bad regions of figure 3), the extra exponential decay of the factor $(1-z)^{2\Delta_\phi}$ is enough to counterbalance the divergence and get a vanishing result.

Using the expressions for \mathcal{G}^\pm , we conclude from 3.14 that:

$$\lim_{\Delta_\phi \rightarrow \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) = 1, \quad s \in (4, \infty) \quad (3.17)$$

Physically this means that, in the flat space limit, any CFT satisfying our assumptions on the spectrum must have an OPE density above $2\Delta_\phi$ which, in a precise sense, is the same as that of a free field “on average”.¹⁵

As we will later see, this result will tell us that the S -matrix for physical kinematics $s > 4m^2$ must satisfy the unitarity condition $|S(s)| \leq 1$.

3.2 A dispersion formula for S

We will now use the expressions derived in the previous subsection to determine the flat space limit of the correlator. In what follows we focus on the bosonic Polyakov blocks and drop the \pm labels. Let us first consider the contributions coming from Polyakov blocks above threshold. We have

$$\bar{\mathcal{G}}(z) = \sum_{\Delta > 2\Delta_\phi} a_\Delta \mathcal{P}_\Delta(z) \stackrel{\Delta_\phi \rightarrow \infty}{=} - \sum_{\Delta > 2\Delta_\phi} \frac{8\sqrt{s\Delta}}{\Delta_\phi} K(s_z, s_\Delta) \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \sin^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_\phi) \right], \quad (3.18)$$

with

$$K(s, s') = \frac{2}{\pi} \frac{\sqrt{s(4-s)}}{\sqrt{s'(s'-4)}} \frac{s' - 2}{(s' - s)(s + s' - 4)}. \quad (3.19)$$

The idea now is to take advantage of the fact that the summand splits into two factors, one of which is varying slowly with Δ , another which is varying fast. Let us define

$$\rho_\alpha(s) := \lim_{\Delta_\phi \rightarrow \infty} \Delta_\phi^{-\alpha} \sum_{\Delta \in B_\alpha(s)} \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) 4 \sin^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_\phi) \right], \quad (3.20)$$

with

$$B_\alpha(s) := \{ \Delta : \sqrt{s}\Delta_\phi \leq \Delta < \sqrt{s}\Delta_\phi + \Delta_\phi^\alpha \}, \quad (3.21)$$

a bin in dimension space of size Δ_ϕ^α . Setting $\alpha < 1$, a simple calculation¹⁶ yields

$$\bar{\mathcal{G}}(z) \stackrel{\Delta_\phi \rightarrow \infty}{=} - \int_4^\infty ds' K(s_z, s') \rho_\alpha(s'). \quad (3.24)$$

¹⁵Note for a free field we have indeed

$$\sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) = 2 \sum_{n=0}^\infty \hat{\mathcal{N}}_{\Delta_\phi}(2\Delta_\phi + 2n, s) = \int_{2\Delta_\phi}^\infty d\Delta \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) \stackrel{\Delta_\phi \rightarrow \infty}{=} 1,$$

which explains the awkward factor of two.

¹⁶Here’s the simple calculation. We write the overall sum in (3.18) into a sum over bins as

$$\lim_{\Delta_\phi \rightarrow \infty} \sum_{\Delta > 2\Delta_\phi} a_\Delta \mathcal{P}_\Delta(z) \sim \sum_{k=0}^\infty \frac{2\sqrt{s_k}}{\Delta_\phi^{1-\alpha}} \rho_\alpha(s_k) K(s_z, s_k), \quad (3.22)$$

It is clear that the end result must be independent of the particular choice of α . Hence all ρ_α , at least for $\alpha > 1/2$ converge to the same density in the sense of distributions. In fact, we must have

$$\rho_\alpha(s) \sim \rho(s) \equiv \mathcal{R}_z \left[\lim_{\Delta_\phi \rightarrow \infty} \bar{\mathcal{G}}(z_s) \right] < \infty, \quad (3.25)$$

where \sim means equality as distributions. In section 4 we will show that we may safely commute the limit with \mathcal{R}_z , giving us another expression for $\rho(s)$. Crucially, the bounds on the OPE density derived in the previous subsection imply

$$0 \leq \rho_\alpha(s) \leq 2, \quad 1/2 < \alpha < 1. \quad (3.26)$$

We conclude that $\bar{\mathcal{G}}(z)$ has a finite flat space limit for any complex z , with its values determined by formula (3.24).

Moving on, let us consider the contributions of Polyakov blocks below threshold. These contributions are written as

$$\mathcal{F} \left[\sum_{0 \leq \Delta \leq 2\Delta_\phi} a_\Delta \mathcal{P}_\Delta(w) \right] = 1 + \sum_{0 < \Delta \leq 2\Delta_\phi} \mathcal{F}[a_\Delta \mathcal{P}_\Delta(w)]. \quad (3.27)$$

On the righthand side we have separated out the contribution of the identity and made the assumption that the flat space limit commutes with the sum over states. This is justified under the assumptions we made on the OPE. Proceeding, let us write (3.10) as:

$$\mathcal{F}[a_\Delta \mathcal{P}_\Delta(w)] = \frac{2\sqrt{s_\Delta}}{\Delta_\phi} \left(\frac{a_\Delta}{\tilde{a}_\Delta^{\text{free}}} \right) \tilde{K}(s_w, s_\Delta), \quad \Delta < 2\Delta_\phi, \quad (3.28)$$

with

$$\tilde{K}(s, s') = \frac{2}{\pi} \frac{\sqrt{s(4-s)}}{\sqrt{s'(4-s')}} \frac{s' - 2}{(s' - s)(s + s' - 4)}. \quad (3.29)$$

We can now distinguish between two situations. In the first, a state in the OPE with dimension Δ_b sits isolated, i.e. the nearest states to lie at a distance which scales as Δ_ϕ . In this case we can write this expression as

$$\mathcal{F}[a_{\Delta_b} \mathcal{P}_{\Delta_b}(w)] = \pi g_b^2 \tilde{K}(s_w, s_b), \quad g_b^2 \equiv \frac{2}{\pi} \frac{\sqrt{s_{\Delta_b}}}{\Delta_\phi} \left(\frac{a_{\Delta_b}}{\tilde{a}_{\Delta_b}^{\text{free}}} \right). \quad (3.30)$$

The effective coupling g_b^2 is guaranteed to be finite thanks to the bound reviewed in section 3.1.1. In a more general situation we again introduce a density

$$\tilde{\rho}_\alpha(s) = \lim_{\Delta_\phi \rightarrow \infty} \Delta_\phi^{-\alpha} \sum_{\Delta \in B_\alpha(s)} \left(\frac{a_\Delta}{\tilde{a}_\Delta^{\text{free}}} \right). \quad (3.31)$$

with $s_k = \left(\frac{2\Delta_\phi + k\Delta_\phi^\alpha}{\Delta_\phi} \right)^2$. To convert the sum to an integral, note:

$$dk = \frac{\Delta_\phi^{1-\alpha}}{2} \frac{ds_k}{\sqrt{s_k}} \quad \Rightarrow \quad \sum_k \frac{2\sqrt{s_k}}{\Delta_\phi^{1-\alpha}} \rightarrow \int_4^\infty ds'. \quad (3.23)$$

For an isolated bound state this would give a singular density which should be interpreted as a delta function. The dependence on α is again essentially irrelevant and we can drop it. In this way we obtain

$$\mathcal{F} \left[\sum_{0 \leq \Delta \leq 2\Delta_\phi} a_\Delta \mathcal{P}_\Delta(w) \right] = 1 + \int_{s_0}^4 ds' \widetilde{K}(s_w, s') \tilde{\rho}(s'). \quad (3.32)$$

Note that $\tilde{\rho}(s)$ is non-negative by construction, but it may become arbitrarily large, as in the delta function example above. However, this is not true for the integrated density. Indeed, the bound (3.8) can be translated as:

$$\int_{s_0}^4 ds' \frac{\widetilde{K}(s_b, s')}{-S_b^{sG}(s')} \tilde{\rho}(s') \leq 1. \quad (3.33)$$

This implies that integrating the density against any smooth function of s yields a finite result. Indeed, for real s it implies:

$$\int_{s_0}^4 ds' \widetilde{K}(s, s') \tilde{\rho}(s') \leq \frac{\sqrt{s_0(4-s_0)} + \sqrt{s(4-s)}}{\sqrt{s_0(4-s_0)} - \sqrt{s(4-s)}}, \quad 4-s_0 < s < s_0. \quad (3.34)$$

From this result it follows finiteness of the integral for any complex s away from the integration contour, by analyticity of the kernel \widetilde{K} .

We have now obtained expressions for the flat space limit of states both below and above threshold. Putting them together we obtain the following *CFT dispersion formula for S* :

$$\begin{aligned} S(s) &= 1 + \int_{s_0}^4 ds' \widetilde{K}(s, s') \tilde{\rho}(s') - \int_4^\infty ds' K(s, s') \rho(s') \quad (3.35) \\ \rho(s) &= \lim_{\Delta_\phi \rightarrow \infty} \Delta_\phi^{-\alpha} \sum_{\Delta \in B_\alpha(s)} \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) 4 \sin^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_\phi) \right] \\ \tilde{\rho}(s) &= \lim_{\Delta_\phi \rightarrow \infty} \Delta_\phi^{-\alpha} \sum_{\Delta \in B_\alpha(s)} \left(\frac{a_\Delta}{\tilde{a}_\Delta^{\text{free}}} \right). \end{aligned}$$

This formula provides an expression for the flat space limit of the CFT correlator anywhere on the complex plane. An equivalent way of obtaining it would be in terms of the correlator itself. Indeed, a different way of stating our results is that

$$\mathcal{G}(w) \underset{\Delta_\phi \rightarrow \infty}{=} \frac{1}{w^{2\Delta_\phi}} + \frac{1}{(1-w)^{2\Delta_\phi}} + \sum_{\Delta_0 \leq \Delta_b \leq 2\Delta_\phi} a_{\Delta_b} E_{\Delta_b}(w|\Delta_\phi) + S(s_w), \quad (3.36)$$

with $S(s)$ defined by (3.35) above. This expression makes clear that $\mathcal{F}[\mathcal{G}] = S(s)$.

The CFT dispersion formula explicitly provides the desired analytic continuation of $S(s)$, which is initially defined only for sufficiently large s , to the entire complex plane. In particular it establishes that $S(s)$ is analytic everywhere except on the real axis. In turn, its singularities can be obtained directly from this expression and are computable in terms of the dual CFT data. Note that defining

$$T(s) := 2\sqrt{s(4-s)} (S(s) - 1), \quad (3.37)$$

then we have

$$\begin{aligned} \rho(s) &= \frac{\mathcal{I}_s T(s)}{2\sqrt{s(s-4)}}, & s > 4, \\ \tilde{\rho}(s) &= \frac{\mathcal{I}_s T(s)}{2\sqrt{s(4-s)}}, & s < 4 \end{aligned} \tag{3.38}$$

Then the bounds $0 \leq \rho(s) \leq 2$ are consistent with unitarity of the S-matrix, while $\tilde{\rho}(s) \geq 0$ is usually called “extended unitarity” [1, 55, 56]. To the best of our knowledge the latter has not been established non-perturbatively in QFT, but here we see it follows simply from the definition of $\tilde{\rho}$ in terms of the CFT OPE.

To reiterate, really what we have learned from this construction are the analyticity properties of S . It would of course be straightforward to arrive at a dispersion relation for $S(s)$ of the form above, interpreting $\rho, \tilde{\rho}$ as discontinuities of S in the relevant cuts, if we had assumed analyticity to begin with. Here we have instead begun with the CFT correlator and its analyticity properties, which are well established, and ended up proving those of $S(s)$.

4 Flat space limit: physical kinematics

The goal of this section is to determine and justify a simple expression for the S-matrix in physical kinematics, the phase shift formula. An expression very similar to ours was first argued to hold for general holographic QFTs in [2]. The present derivation is essentially the same as in [34], but simpler because it is specialized to $d = 1$. However, as we explain, this derivation cannot be rigorously justified unless certain limits commute, and we will therefore have to prove that this is the case.

4.1 Phase shift formula

We are interested in determining the S-matrix in physical kinematics. That is:

$$S_{\text{phys}}(s_z) := \lim_{\epsilon \rightarrow 0^+} \mathcal{F}[\mathcal{G}(z - i\epsilon)], \quad z < 0. \tag{4.1}$$

Ignoring for the moment the issue of analytic continuation, this involves taking the two limits — flat space, and going to physical kinematics — in a definite order. Suppose that the limits commute, so that we can write instead:

$$S_{\text{phys}}(s_z) \stackrel{!}{=} \lim_{\Delta_\phi \rightarrow +\infty} \lim_{\epsilon \rightarrow 0^+} \mathcal{G}(z - i\epsilon), \quad \text{for } s_z > s_0 > 4. \tag{4.2}$$

Proving this is non-trivial and will be justified in the next subsection. For now let us take it as given, and see what expression we may obtain for S_{phys} . Using the OPE we have

$$S_{\text{phys}}(s_z) = \lim_{\Delta_\phi \rightarrow +\infty} \sum_{\Delta \geq 0} a_\Delta \frac{G_\Delta\left(\frac{z}{z-1}\right)}{(-z)^{2\Delta_\phi}} e^{-i\pi(\Delta - 2\Delta_\phi)}. \tag{4.3}$$

for $z < z_0$ which we determine as follows. Separate contributions below and above threshold, so that

$$S_{\text{phys}}(s_z) = \lim_{\Delta_\phi \rightarrow +\infty} \left[\sum_{\Delta < 2\Delta_\phi} a_\Delta \frac{G_\Delta\left(\frac{z}{z-1}\right)}{(-z)^{2\Delta_\phi}} e^{-i\pi(\Delta-2\Delta_\phi)} + \sum_{\Delta > 2\Delta_\phi} a_\Delta \frac{G_\Delta\left(\frac{z}{z-1}\right)}{(-z)^{2\Delta_\phi}} e^{-i\pi(\Delta-2\Delta_\phi)} \right], \quad (4.4)$$

and let us assume the strong condition on the OPE, so that the bounds on OPE coefficients reviewed in section 3.1.1 are satisfied. Then it is not hard to see that all terms in the first sum are exponentially suppressed for $z < -1$, by the same logic used in section 3.1.2. So $z_0 = -1$ and the analytic continuation of S_{phys} for all $z < 0$ is simply

$$S_{\text{phys}}(s_z) = \lim_{\Delta_\phi \rightarrow +\infty} \sum_{\Delta > 2\Delta_\phi} a_\Delta \frac{G_\Delta\left(\frac{z}{z-1}\right)}{(-z)^{2\Delta_\phi}} e^{-i\pi(\Delta-2\Delta_\phi)}. \quad (4.5)$$

Using expression (2.20) we get the phase shift formula:

$$S_{\text{phys}}(s) = \lim_{\Delta_\phi \rightarrow +\infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) e^{-i\pi(\Delta-2\Delta_\phi)}. \quad (4.6)$$

This result allows us to express the physical S-matrix directly in terms of the CFT data. As promised, the bounds on the OPE density derived in section 3.1.2 nicely translate into unitarity of the S-matrix:

$$|S_{\text{phys}}(s)| \leq \lim_{\Delta_\phi \rightarrow +\infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) = 1. \quad (4.7)$$

To justify the phase shift formula, we must show that the flat space and physical kinematics limits commute, i.e.:

$$\lim_{\epsilon \rightarrow 0^+} S(s + i\epsilon) = S_{\text{phys}}(s), \quad (4.8)$$

where the right hand side is understood to be computed by the phase shift formula. We will prove this in two steps, first by showing this relation holds when taking the real part, and then when taking the imaginary part.

4.2 Commuting limits: real part

We begin by recalling the dispersion relation (2.10) for the subtracted correlator:

$$\bar{\mathcal{G}}(w) = - \int_0^1 dz \hat{g}_w^+(z) d\text{Disc}^+ \bar{\mathcal{G}}(z), \quad (4.9)$$

which holds for any Δ_ϕ . Taking the large Δ_ϕ limit we already know that the lefthand side becomes

$$\lim_{\Delta_\phi \rightarrow \infty} \bar{\mathcal{G}}(z_s) = S(s) - 1 - \int_{s_0}^4 ds' \tilde{K}(s, s') \tilde{\rho}(s'), \quad (4.10)$$

since it is the sum of Polyakov blocks above $2\Delta_\phi$. On the other hand, the righthand side can be written as

$$\lim_{\Delta_\phi \rightarrow \infty} \int_0^1 dz \hat{g}_w^+(z) d\text{Disc}^+ \bar{\mathcal{G}}(z) = \int_4^\infty ds' K(s, s') \rho_{\text{phys}}(s'), \quad (4.11)$$

with¹⁷

$$\rho_{\text{phys}}(s) := \lim_{\Delta_\phi \rightarrow \infty} (1-z)^{2\Delta_\phi} d\text{Disc}^+ \bar{\mathcal{G}}(z) \Big|_{z=\frac{s-4}{s}}. \quad (4.12)$$

This means that the dispersion relation for $\bar{\mathcal{G}}$ directly translates into a dispersion relation for $S(s)$. But we already had such a relation, and since they involve the same kernels K, \tilde{K} they must agree. More precisely, this implies that as distributions we must have

$$\rho_\alpha(s) \sim \rho_{\text{phys}}(s). \quad (4.13)$$

Now let us compute ρ_{phys} directly. Using:

$$\lim_{\Delta_\phi \rightarrow \infty} (1-z)^{2\Delta_\phi} d\text{Disc}^+ \bar{\mathcal{G}}(z) = \lim_{\Delta_\phi \rightarrow \infty} (1-z)^{2\Delta_\phi} d\text{Disc}^+ \left[\mathcal{G}(z) - \sum_{\Delta \leq 2\Delta_\phi} a_\Delta G_\Delta(z) \right], \quad (4.14)$$

together with the OPE and the unitarity bound, we can find

$$\begin{aligned} \rho_{\text{phys}}(s) &= \lim_{\Delta_\phi \rightarrow +\infty} \sum_{\Delta > 2\Delta_\phi} \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) 4 \sin^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_\phi) \right] \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) \\ &= \text{Re} [1 - S_{\text{phys}}(s)] \end{aligned} \quad (4.15)$$

where on the last line we use the phase shift formula for S_{phys} . We see that indeed as distributions $\rho_\alpha(s) \sim \rho_{\text{phys}}(s)$ when $\alpha > 1/2$, as can be seen by splitting the sum over states in the latter into bins of size $\sqrt{\Delta_\phi}$. Hence we have proven:

$$\mathcal{R}_s[1 - S(s)] = \text{Re}[1 - S_{\text{phys}}(s)], \quad (4.16)$$

which establishes that the phase shift formula holds for the real part.

4.3 Commuting limits: imaginary part

The previous derivation makes clear that if we are to prove the phase shift formula for the imaginary part, we must write a dispersion relation for CFT correlators which will depend on the imaginary part of \mathcal{G} . Such a dispersion relation has appeared before in the literature [57], but as we explain it suffers from some minor issues concerning subtractions which we clarify and settle here.

The idea is to use Cauchy's formula for the correlator,

$$\mathcal{G}(z) = \oint \frac{dz'}{2\pi i} \frac{\mathcal{G}(z')}{z' - z}, \quad (4.17)$$

¹⁷Recall from (2.5) the relation between the functional kernels $\hat{g}_w^+(z) = (1-z)^{2\Delta_\phi-2} f_w^+(\frac{z}{z-1})$. A simple change of variables $z \rightarrow \frac{z}{z-1}$ along with the identification $f_{w=1-s/4}^+(z = 1-s'/4) = -K(s, s')$ leads to (4.11).

and now deform the contour to pick up the discontinuities of \mathcal{G} . However, there are two issues. Firstly, we have to worry about contributions from $z = \infty$. Secondly, we may get divergences from the contribution of the contour close to $z = 0, 1$. To deal with these issues we must implement subtractions. The cleanest way to do this is to define a subtracted correlator $\tilde{\mathcal{G}}(z) := \mathcal{G}(z) - \sum_{\Delta \leq 2\Delta_\phi + 1} a_\Delta \mathcal{P}_\Delta(z)$. This has improved soft behaviour near $z = 0, 1$ as compared to $\bar{\mathcal{G}}(z)$ and leads to a dispersion relation

$$\tilde{\mathcal{G}}(w) = - \int_1^\infty \frac{dz}{\pi} \frac{w(1-w)}{z(1-z)} \frac{2z-1}{(w-z)(w+z-1)} \mathcal{I}_z \tilde{\mathcal{G}}(z). \quad (4.18)$$

which holds for any CFT. For the purpose at hand however, it is more convenient to work with a slightly different dispersion relation, where we improve the behaviour near $z = \infty$ instead. Let us set¹⁸

$$C := \lim_{z \rightarrow \infty} \bar{\mathcal{G}}(z). \quad (4.19)$$

Then we can write the dispersion relation

$$\bar{\mathcal{G}}(w) = C - \int_1^\infty \frac{dz}{\pi} \frac{2z-1}{(w-z)(w+z-1)} \mathcal{I}_z \bar{\mathcal{G}}(z). \quad (4.20)$$

This holds for arbitrary Δ_ϕ . Using the CFT dispersion formula for $s > 4$ we have:

$$\begin{aligned} \mathcal{I}_s \left[\lim_{\Delta_\phi \rightarrow \infty} \bar{\mathcal{G}}(z_s) \right] &= \mathcal{I}_s \left[S(s) - 1 - \int_{s_0}^4 ds' \tilde{K}(s, s') \tilde{\rho}(s') \right] \\ &= \mathcal{I}_s S(s) - \int_{s_0}^4 ds' \mathcal{I}_s \tilde{K}(s, s') \tilde{\rho}(s'). \end{aligned} \quad (4.21)$$

Alternatively, we can compute the same quantity directly using our new dispersion relation. Matching the two we get:

$$\mathcal{I}_s S(s) = \lim_{\Delta_\phi \rightarrow \infty} \mathcal{I}_s \bar{\mathcal{G}}(z_s) + \int_{s_0}^4 ds' \mathcal{I}_s \tilde{K}(s, s') \tilde{\rho}(s'). \quad (4.22)$$

Our goal will be to prove that the righthand side is the same as $\text{Im } S_{\text{phys}}$. Consider first the case where there are no states below threshold other than the identity. Using the OPE, a computation which by now should be familiar gives

$$\begin{aligned} \lim_{\Delta_\phi \rightarrow \infty} \mathcal{I}_s \bar{\mathcal{G}}(z_s) &= \sum_{\Delta \geq 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \sin [\pi(\Delta - 2\Delta_\phi)] \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, z_s) \\ &= \text{Im } S_{\text{phys}}(s) \end{aligned} \quad (4.23)$$

as desired. To finish the proof, we must show that states below threshold do not change the result. This will follow if we can show that:

$$\lim_{\Delta_\phi \rightarrow \infty} \mathcal{I}_s \left[\sum_{0 < \Delta \leq 2\Delta_\phi} a_\Delta [\mathcal{P}_\Delta(z_s) - G_\Delta(1 - z_s)] \right] = - \int_{s_0}^4 ds' \mathcal{I}_s \tilde{K}(s, s') \tilde{\rho}(s'), \quad s > 4. \quad (4.24)$$

¹⁸Note that any 1d CFT correlator is bounded by a constant at $z = \infty$, see e.g. [51].

The meaning of this equation is that the phase shift formula for Polyakov blocks below $2\Delta_\phi$ (which will appear on the left side of this equation) should give the same answer as the flat space limit of those blocks as computed in sections 2.3 and 3 (the right side of the equation). That is, we must check the limits commute by hand for these blocks. We show this by direct computation in appendix B, which completes the proof.

5 Perturbative checks

In this section we check our flat space prescription in a few perturbative examples. Our starting point is the theory of generalized free bosons ϕ_1 with dimension Δ_1 . Throughout this section we will focus on the four point correlator of this field, and so we will set $\Delta_\phi \equiv \Delta_1$. We will also refer to the mass of the corresponding AdS bulk field as m_1 but we will set this to one, so that e.g. $\Delta_2/\Delta_1 \equiv m_2/m_1 = m_2$.

5.1 General remarks

Before any perturbation we have the generalized free boson correlator which has the following s -channel conformal block decomposition:

$$\mathcal{G}_0(w) = 1 + \sum_n a_{[11]_n}^{\text{free}} G_{[11]_n}(w) = \mathcal{P}_0(w), \tag{5.1}$$

with the double trace dimensions $[11]_n = 2\Delta_1 + 2n$. We then perturb the theory by coupling ϕ_1 to another generalized free boson ϕ_2 with conformal dimension Δ_2 . The four point function of ϕ_1 then admits a perturbative expansion of the form

$$\begin{aligned} \mathcal{G}(w) = & \mathcal{G}_0(w) + g^2 \sum_n \left(a_{[11]_n}^{(1)} G_{[11]_n}(w|\Delta_1) + a_{[11]_n}^{\text{free}} \gamma_{[11]_n}^{(1)} \partial G_{[11]_n}(w|\Delta_1) \right) + \\ & + g^2 \sum_n a_{[22]_n}^{(1)} G_{[22]_n}(w|\Delta_1) + \dots, \end{aligned} \tag{5.2}$$

where $\partial G \equiv \partial_\Delta G$. In the first line we have the first order corrections to the dimensions and OPE coefficients of double trace operators $[\phi_1\phi_1]_n$ and in the second line the contributions coming from the introduction of ϕ_2 .¹⁹ We can write (5.2) alternatively in terms of Polyakov blocks as follows

$$\begin{aligned} \mathcal{G}(w) = & \mathcal{P}_0(w) + g^2 \sum_n \left(a_{[11]_n}^{(1)} \mathcal{P}_{[11]_n}(w) + a_{[11]_n}^{\text{free}} \gamma_{[11]_n}^{(1)} \partial \mathcal{P}_{[11]_n}(w) \right) + \\ & + g^2 \sum_n a_{[22]_n}^{(1)} \mathcal{P}_{[22]_n}(w) + \dots \end{aligned} \tag{5.3}$$

Using the Polyakov blocks obtained from the free master functionals for ϕ_1 many of the terms above are zero

$$\mathcal{G}(w) = \mathcal{P}_0(w) + g^2 \left[\sum_n a_{[22]_n}^{(1)} \mathcal{P}_{[22]_n}(w) + a_{[11]_0}^{\text{free}} \gamma_{[11]_0}^{(1)} \partial \mathcal{P}_{[11]_0}(w) \right] + \dots \tag{5.4}$$

¹⁹We are assuming there is no coupling of the form $g\phi_1^2\phi_2$ so that only double traces appear in (5.2). One can consider particle ϕ_1 and ϕ_2 transforming differently under a Z_2 symmetry so that this cubic interaction is not allowed.

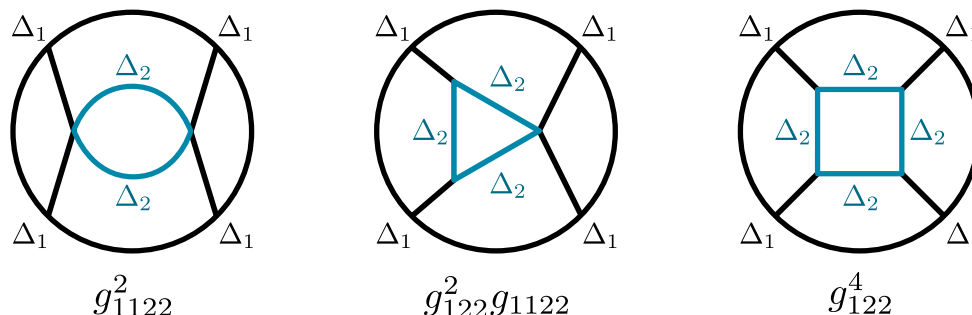


Figure 4. Bubble, triangle and box Witten diagrams considered in the main text.

This is because the Polyakov blocks have double zeros at the double trace dimensions $\Delta = [11]_n$. For the bosonic blocks this is true for every n except for $n = 0$, with the last term amounting to a four-point contact interaction in AdS_2 , i.e. a D-function $D_{\Delta_1\Delta_1\Delta_1\Delta_1}(z)$ [58–60].

So far we have kept the discussion fairly general regarding the type of couplings between ϕ_1 and ϕ_2 . We now restrict to cubic $g_{122}\phi_1\phi_2\phi_2$ and quartic $g_{1122}\phi_1\phi_1\phi_2\phi_2$ interactions, where at one loop level we encounter bubble, triangle and box diagrams each associated with different powers of g_{122} and g_{1122} (see figure 4)

$$\mathcal{G}(w) = \mathcal{G}_0(w) + g_{1122}^2 \mathcal{G}^{\text{bubble}}(w) + g_{1122}g_{122}^2 \mathcal{G}^{\text{triangle}}(w) + g_{122}^4 \mathcal{G}^{\text{box}}(w) + \dots \quad (5.5)$$

The idea is to illustrate in these specific examples how the flat space amplitude is recovered from the CFT data using our dispersion relation (3.35). In other words, we want to exhibit how the weighted sum of the Polyakov blocks for a specific diagram becomes the interacting part of the amplitude

$$g \sum_n a_{[22]_n}^{\text{diagram}} \mathcal{P}_{[22]_n}^{\text{diagram}}(w) \xrightarrow{\mathcal{F}} \frac{T^{\text{diagram}}(s)}{2\sqrt{s(4-s)}} = g \int ds' \rho^{\text{diagram}}(s') K(s, s'), \quad (5.6)$$

and the OPE coefficients the spectral density

$$\left(\frac{a_{[22]_n}^{\text{diagram}}}{a_{[22]_n}^{\text{free},11}} \right) \xrightarrow{\mathcal{F}} \rho^{\text{diagram}}(s) = \frac{\mathcal{I}_s T(s)}{2\sqrt{s(s-4)}}. \quad (5.7)$$

as given in (3.38).

Before we proceed to the flat space limit of the one-loop diagrams, let us discuss two building blocks needed to extract $a_{[22]_n}^{\text{diagram}}$. Given the quartic g_{1122} and cubic g_{122} couplings, the building blocks for the one-loop diagrams are the contact and exchange Witten diagrams, with associated coefficients $a_{[22]_n}^{\text{contact},1122}$ and $a_{[22]_n}^{\text{t,exchange}}$. These coefficients appear in the well known expansions of tree-level diagrams in terms of (s-channel) conformal

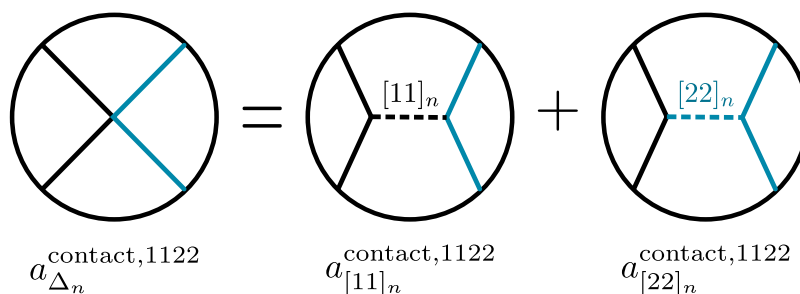


Figure 5. Pictorial representation of equation (5.8) showing the OPE coefficients of the double trace operators $[11]_n$ and $[22]_n$ appearing in the s -channel conformal block decomposition of the contact diagram written in (5.8).

blocks. For the 1122 contact diagram we have²⁰

$$\mathcal{G}^{\text{contact},1122}(w) = D_{\Delta_1\Delta_1\Delta_2\Delta_2}(w) = \sum_{n=0}^{\infty} a_{[11]_n}^{\text{contact},1122} G_{[11]_n}(w) + \sum_{n=0}^{\infty} a_{[22]_n}^{\text{contact},1122} G_{[22]_n}(w), \quad (5.8)$$

which we represent pictorially in figure 5. Here $G_{\Delta}^{11,22}$ are appropriate conformal blocks for a 1122 correlator, and the coefficient $a_{[22]_n}^{\text{contact},1122}$ was computed in [60, 61] and reads up to a normalization

$$a_{[22]_n}^{\text{contact},1122} = \lambda_{11[22]_n}^{(1)} \lambda_{22[22]_n}^{(0)} \propto \begin{cases} \frac{\sqrt{\pi}(-1)^n \Gamma(-n+\Delta_1-\Delta_2) \Gamma(n+\Delta_2)^4 \Gamma(n+\Delta_1+\Delta_2-\frac{1}{2}) \Gamma(n+2\Delta_2-\frac{1}{2})}{2n! \Gamma(\Delta_1)^2 \Gamma(\Delta_2)^2 \Gamma(2(n+\Delta_2)) \Gamma(2n+2\Delta_2-\frac{1}{2})}, & 2\Delta_2 + 2n < 2\Delta_1, \\ \frac{\pi^{\frac{3}{2}} \Gamma(n+\Delta_2)^4 \Gamma(n+\Delta_1+\Delta_2-\frac{1}{2}) \Gamma(n+2\Delta_2-\frac{1}{2})}{2n! \sin[\pi(\Delta_1-\Delta_2)] \Gamma(1+n-\Delta_1+\Delta_2) \Gamma(\Delta_1)^2 \Gamma(\Delta_2)^2 \Gamma(2(n+\Delta_2)) \Gamma(2n+2\Delta_2-\frac{1}{2})}, & 2\Delta_2 + 2n > 2\Delta_1. \end{cases} \quad (5.9)$$

Note that the zeroth order OPE coefficients $\lambda_{22[22]_n}^{(0)}$ appear in the generalized free field correlator $\mathcal{G}^{2222}(z)$ where the external particles are ϕ_2 , so that we have the relation

$$a_{[22]_n}^{\text{free}} \Big|_{\Delta_{\phi}=\Delta_2} \equiv a_{[22]_n}^{\text{free},22} = \left(\lambda_{22[22]_n}^{(0)} \right)^2. \quad (5.10)$$

Now we review the second building block we use in our examples: the expansion of the t -exchange diagram in s -channel blocks. It is given by

$$\mathcal{G}^{\text{t-exchange}}(w) = \sum_{n=0}^{\infty} a_{[11]_{\bar{n}}}^{\text{t-exchange}} G_{[11]_{\bar{n}}}^{11,22}(w) + \sum_{n=0}^{\infty} a_{[22]_{\bar{n}}}^{\text{t-exchange}} G_{[22]_{\bar{n}}}^{11,22}(w), \quad (5.11)$$

where the double traces contain both parity even and odd states which we implement with the notation $[jj]_{\bar{n}} = 2\Delta_j + n$. The coefficients $a_{[jj]_{\bar{n}}}^{\text{t-exchange}}$ can be obtained in terms of $a_{[jj]_{\bar{n}}}^{\text{contact}}$ by means of the recursion relation of [61].²¹

²⁰The relevant expressions and flat space limit for the equal dimensions contact term are discussed in appendix B.

²¹The schematic form of the recursion relation is

$$\mu_{\bar{n}-1} a_{\bar{n}-1}^{\text{t-exch}} + \nu_{\bar{n}} a_{\bar{n}}^{\text{t-exch}} + \rho_{\bar{n}+1} a_{\bar{n}+1}^{\text{t-exch}} = a_{n=2\bar{n}}^{\text{contact}}, \quad (5.12)$$

where the recursion coefficients μ, ν, ρ are rational functions of $\Delta_1, \Delta_2, [22]_{\bar{n}}$. For explicit expressions in more general scenarios see section 4.2 of [61].

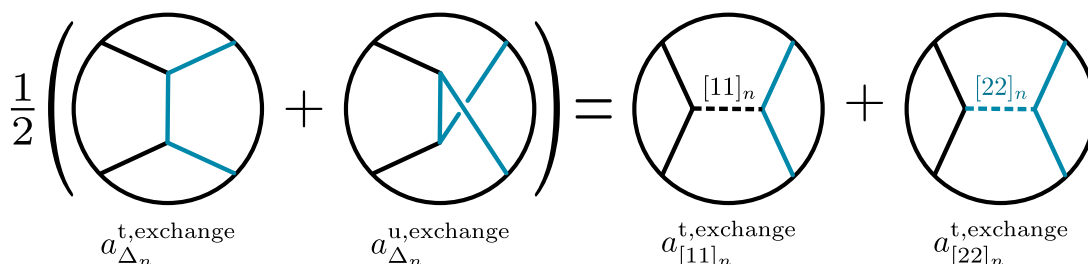


Figure 6. The sum of t - and u -exchange Witten diagrams has an expansion in terms of purely parity even OPE coefficients.

However, since we are dealing with scalar particles (as opposed to pseudo scalars), we are only interested in parity even OPE coefficients $a_{[22]_n}$. This can be simply fixed by noting the relation between t - and u -exchange OPE coefficients: $a_{[jj]_{\bar{n}}}^{u\text{-exchange}} = (-1)^j a_{[jj]_{\bar{n}}}^{t\text{-exchange}}$, so that by taking the sum of t - and u -exchange Witten diagrams we have an expansion purely in terms of parity even states. The last point is displayed in figure 6.

The one-loop coefficients $a_{[22]_n}^{1\text{-loop diagram}}$ can be computed in terms of the tree level data discussed above $a_{[22]_n}^{\text{contact}}$, $a_{[22]_n}^{t\text{-exchange}}$. In the following sections we explain how to extract this one-loop coefficients and compute the flat space limit of each diagram.

5.2 Bubble diagram

Let us start with the crossing symmetric sum of bubble diagrams with external dimensions Δ_1 and internal dimensions Δ_2 (see first diagram in figure 4). According to (5.4), its Polyakov block expansion reads

$$\mathcal{G}^{\text{bubble}}(w) = \sum_{n=0}^{\infty} a_{[22]_n}^{\text{bubble}} \mathcal{P}_{[22]_n}(w) + \text{contact}, \tag{5.13}$$

where $[22]_n = 2\Delta_2 + 2n$ indicates as usual the double trace dimensions of ϕ_2 with even parity. All we have to do now is to understand what the coefficients $a_{[22]_n}^{\text{bubble}}$ are. The idea is that we can get this data from lower loop level data in the same spirit as one does for amplitudes using the optical theorem. At the level of Witten diagrams, one can see this procedure as cutting the intermediate bulk-to-bulk propagators giving rise to lower loop level diagrams glued together conformally as nicely explained in [62]. For the bubble diagram at hand we want to cut the two internal propagators and isolate the double trace contributions of ϕ_2 . The cutting and gluing procedure implies the following relation between the bubble and contact OPE coefficients:

$$a_{[22]_n}^{\text{bubble}} = \left(\lambda_{11[22]_n}^{\text{contact}} \right)^2. \tag{5.14}$$

To express $a_{[22]_n}^{\text{bubble}}$ in terms of $a_{[22]_n}^{\text{contact}}$ appearing in (5.9) we need to divide by $\left(\lambda_{22[22]_n}^{(0)} \right)^2 = a_{[22]_n}^{\text{free},22}$

$$a_{[22]_n}^{\text{bubble}} = \left(a_{[22]_n}^{\text{contact}} \right)^2 / a_{[22]_n}^{\text{free},22}. \tag{5.15}$$

This relation is depicted in figure 7.

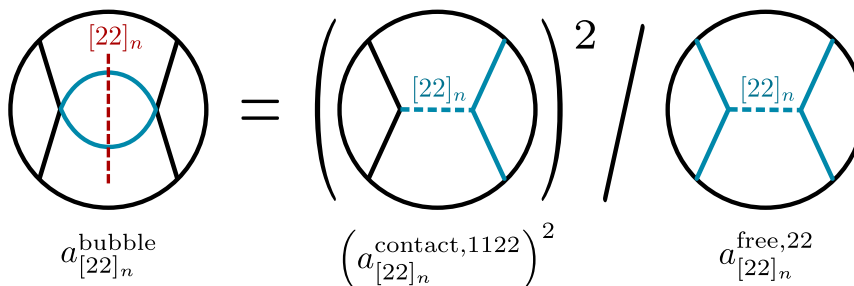


Figure 7. The OPE coefficients of the double trace operators $[22]_n$ appearing in the s -channel bubble diagram can be extracted from the ones appearing in the conformal block expansion of the Witten contact diagram $a_{[22]_n}^{\text{contact},1122}$ and $a_{[22]_n}^{\text{free},22}$ as illustrated.

We would like to check that in the flat space limit the OPE density matches the spectral density of the flat space bubble amplitude. We will consider the less trivial case where the internal dimensions are smaller than the external ones $\Delta_2 < \Delta_1$ (or $m_2 < m_1$ in flat space amplitude), so that there is the two-particle cut for m_2 starting before the one for m_1 . Normalizing the external mass to one $m_1 = 1$, we are instructed to set

$$\frac{\Delta_2}{\Delta_1} = m_2, \quad \frac{[22]_n}{\Delta_1} = \sqrt{s}, \tag{5.16}$$

and compute the CFT dispersion densities as in (3.35). In this case the averaging procedure is trivial and we get

$$\begin{aligned} \rho_{\text{CFT}}^{\text{bubble}}(s) &= \lim_{\Delta_1 \rightarrow \infty} \left(\frac{a_{[22]_n}^{\text{bubble}}}{a_{[22]_n}^{\text{free},11}} \right) = \frac{1}{4\sqrt{s(s-4)}\sqrt{s(s-4m_2^2)}}, \quad (s > 4), \\ \tilde{\rho}_{\text{CFT}}^{\text{bubble}}(s) &= \lim_{\Delta_1 \rightarrow \infty} \left(\frac{a_{[22]_n}^{\text{bubble}}}{\tilde{a}_{[22]_n}^{\text{free},11}} \right) = \frac{1}{4\sqrt{s(4-s)}\sqrt{s(s-4m_2^2)}}, \quad (s < 4), \end{aligned} \tag{5.17}$$

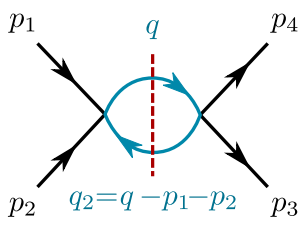
where we have chosen a particular convenient overall normalization in (5.9).²²

Now we would like to compare our CFT density (5.17) with the spectral density of the flat space amplitude. From (5.6) we need the discontinuity of the amplitude. We could compute the full amplitude first and then take the discontinuity, but we will take a shortcut and get the density directly from Cutkosky rules [55].²³ (This method is especially useful when considering more complicated diagrams like the massive box below, where computing the exact analytic amplitude is difficult.) In principle we should consider all possible bubble diagrams where we take all permutations of the external legs, but the discontinuity (or imaginary part) for $s > 4m_2^2$ is non-zero only for the diagram below. By replacing the internal propagators with momenta q and $q_2 = q - p_1 - p_2$ with delta functions that put

²²This was chosen to match with the exact density computer for the amplitude further below. We could of course also match the overall normalization factor by carefully choosing the normalization of AdS propagators and couplings appropriately [2].

²³See for example section 6.3.4 of [63] for details.

the particles on shell we find the discontinuity of the amplitude



$$\begin{aligned}
 \mathcal{I}_s T^{\text{bubble}}(s) &= \\
 &= \frac{(-i)^0}{2} \int \frac{d^2 q}{(2\pi)^2} (2\pi)^2 \theta(q^0) \theta(-q_2^0) \delta(q^2 - m_2^2) \delta(q_2^2 - m_2^2) \\
 &= \frac{1}{2} \int_{\pm} \frac{d|\vec{q}|}{8E|\vec{q}|} \delta(|\vec{q}| - \sqrt{E^2 - m_2^2}) = \frac{1}{2\sqrt{s(s - 4m_2^2)}} \tag{5.18}
 \end{aligned}$$

where we have chosen the center of mass frame to do the computation so $q = (E, \vec{q})$ where $2E = \sqrt{s}$ is the total energy. The \pm indicates we need to sum over the two possible signs for the spatial component of the internal momenta.

From here we see that indeed we have a match between the flat space CFT density and the spectral density in the amplitude below and above threshold

$$\rho^{\text{bubble}}(s) = \frac{\mathcal{I}_s T^{\text{bubble}}(s)}{2\sqrt{s(s - 4)}} = \frac{1}{4\sqrt{s(s - 4)}\sqrt{s(s - 4m_2^2)}} = \rho_{\text{CFT}}^{\text{bubble}}(s), \tag{5.19}$$

$$\tilde{\rho}^{\text{bubble}}(s) = \frac{\mathcal{I}_s T^{\text{bubble}}(s)}{2\sqrt{s(4 - s)}} = \frac{1}{4\sqrt{s(4 - s)}\sqrt{s(s - 4m_2^2)}} = \tilde{\rho}_{\text{CFT}}^{\text{bubble}}(s). \tag{5.20}$$

Finally, the constant term ambiguity in (5.13) can be fixed by comparing to the full s, t, u crossing symmetric flat space amplitude which includes a constant term coming from the $s \rightarrow u = 0$ term.

5.3 Triangle diagram

Let us now consider the term associated with $g_{122}^2 g_{1122}$ in the expansion (5.5) which is the crossing symmetric sum of the triangle Witten diagrams.

Taking again the Polyakov block expansion (5.4) we have

$$\mathcal{G}^{\text{triangle}}(w) = \sum_{n=0}^{\infty} a_{[22]_n}^{\text{triangle}} \mathcal{P}_{[22]_n}(w) + \text{contact}, \tag{5.21}$$

where the sum is over the double trace operators of ϕ_2 with even parity. Same as for the bubble diagram, we can extract the coefficients $a_{[22]_n}^{\text{triangle}}$ from the tree level diagrams we generate when cutting internal propagators in a way such that we have two external legs on the right and two on the left of the cut. In this case we are left with a t -exchange and (1122) contact diagrams. The relation between OPE coefficients is then:

$$a_{[22]_n}^{\text{triangle}} = \lambda_{11[22]_n}^{t,\text{exchange}} \times \lambda_{11[22]_n}^{\text{contact}} = \left(a_{[22]_n}^{t,\text{exchange}} \times a_{[22]_n}^{\text{contact}} \right) / a_{[22]_n}^{\text{free},22}. \tag{5.22}$$

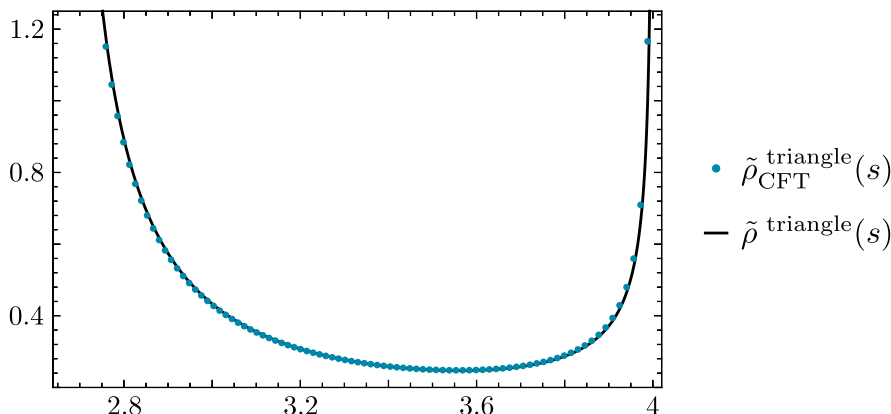


Figure 8. Comparison between the amplitude’s spectral density below threshold $\tilde{\rho}^{\text{triangle}}(s)$ and large dimension ($\Delta_1 = 500$) numerical CFT density $\tilde{\rho}_{\text{CFT}}^{\text{triangle}}(s)$ for $m_2^2 = 2/3$.

This time one of the OPE coefficients $a_{[22]_n}^{\text{t,exchange}}$ is not available in closed form, but it can be computed via the recursion relations of [61] sketched in (5.12). This fact prevents us from computing the CFT density

$$\tilde{\rho}_{\text{CFT}}^{\text{triangle}}(s) = \lim_{\Delta_1 \rightarrow \infty} \left(\frac{a_{[22]_n}^{\text{triangle}}}{\tilde{a}_{[22]_n}^{\text{free,11}}} \right), \quad s = \left(\frac{[22]_n}{\Delta_1} \right)^2 < 4 \quad (5.23)$$

analytically, so we will resort to a numerical comparison by solving the recursion relations for large enough dimensions. The result for $\Delta_1 = 500$ and $\Delta_2^2 = 2/3\Delta_1^2$ is shown in blue dots in figure 8.

To compare with the flat space amplitude, let us extract the spectral density of the triangle diagram using the Cutkosky rule. Taking the s -channel discontinuity we find

$\mathcal{I}_s T^{\text{triangle}}(s) =$

$$\begin{aligned}
 &= \frac{(-i)}{2} \int \frac{d^2q}{(2\pi)^2} \frac{i}{(q-p_1)^2 - m_2^2 + i\epsilon} (2\pi)^2 \theta(q^0) \theta(-q_2^0) \delta(q^2 - m_2^2) \delta(q_2^2 - m_2^2) \\
 &= \frac{1}{2} \int_{\pm} \frac{d|\vec{q}|}{8E|\vec{q}|} \frac{\delta(|\vec{q}| - \sqrt{E^2 - m_2^2})}{((q-p_1)^2 - m_2^2 + i\epsilon)} = \frac{s-2}{2(1+m_2^2(s-4))\sqrt{s(s-4m_2^2)}}. \quad (5.24)
 \end{aligned}$$

where the convention is the same as the bubble case. Using (5.7), we then get the exact density below threshold

$$\tilde{\rho}^{\text{triangle}}(s) = \frac{s-2}{4(1+m_2^2(s-4))\sqrt{s(4-s)}\sqrt{s(s-4m_2^2)}}. \quad (5.25)$$

In figure 8 we plot both the CFT and flat space densities and see that we get a perfect match. This is the case also for the density above threshold as well as for any value of m_2 such that $m_2^2 > 1/2$. We will see in section 5.5 what happens for internal masses below this value.

5.4 Box diagram

Finally, we have the crossing symmetric sum of box diagram from the term with g_{122}^4 in (5.4). The procedure for the comparison is the same as for the triangle diagram. The Polyakov expansion reads

$$\mathcal{G}^{\text{box}}(w) = \sum_{n=0}^{\infty} a_{[22]n}^{\text{box}} \mathcal{P}_{[22]n}(w) + \text{contact}. \quad (5.26)$$

By cutting the box in two, we see that the coefficients $a_{[22]n}^{\text{box}}$ are given by the square of the (parity even) t -exchange diagram OPE coefficients

$$a_{[22]n}^{\text{box}} = (\lambda_{11[22]n}^{\text{t,exchange}})^2 = \left(a_{[22]n}^{\text{t,exchange}} \right)^2 / a_{[22]n}^{\text{free},22}. \quad (5.27)$$

As for the triangle, these coefficients are not available in closed form, so once again we perform a numerical comparison with the flat space density.

To compare with the flat space computation, we take the imaginary part or discontinuity of the amplitude for $s > 4m_2^2$. As in the previous examples, we should consider all possible box diagrams but only the box and twisted box below have non zero discontinuity in this region. One can also make sense of this particular combination recalling that the even parity coefficients $a_{[22]n}^{\text{t,exchange}}$ are extracted from the sum of t - and u -exchange Witten diagrams as in figure 6. This means that with the cutting and gluing procedure we have precisely the sum of box and twisted box diagrams as below. Using Cutkosky rules we arrive at

$$\begin{aligned} \mathcal{I}_s \left[T^{\text{box}}(s) + T^{\text{twisted box}}(s) \right] &= \text{Diagram 1} + \text{Diagram 2} \\ &= \frac{(-i)^0}{2} \int \frac{d^2q}{(2\pi)^2} \frac{i}{(q-p_1)^2 - m_2^2 + i\epsilon} \frac{-i}{(q-p_4)^2 - m_2^2 - i\epsilon} (2\pi)^2 \theta(q^0) \theta(-q_2^0) \delta(q^2 - m_2^2) \delta(q_2^2 - m_2^2) \\ &\quad + (p_4 \leftrightarrow p_3) \\ &= \frac{1}{2} \int_{\pm} \frac{d|\vec{q}|}{8E|\vec{q}|} \frac{\delta\left(|\vec{q}| - \sqrt{E^2 - m_2^2}\right)}{\left((q-p_1)^2 - m_2^2 + i\epsilon\right)\left((q-p_4)^2 - m_2^2 - i\epsilon\right)} + (p_4 \leftrightarrow p_3) \\ &= \frac{(s-2)^2}{2(1+m_2^2(s-4))^2 \sqrt{s(s-4m_2^2)}} \end{aligned} \quad (5.28)$$

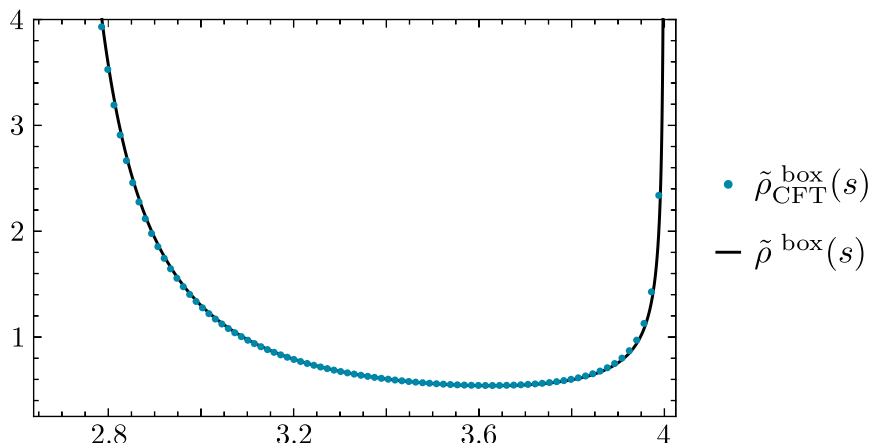


Figure 9. Comparison between the amplitude’s spectral density below threshold $\tilde{\rho}^{\text{box}}(s)$ and large dimension ($\Delta_1 = 500$) numerical CFT density $\tilde{\rho}_{\text{CFT}}^{\text{box}}(s)$ for $m_2^2 = 2/3$.

and the corresponding spectral density is:

$$\tilde{\rho}^{\text{box}}(s) = \frac{(s - 2)^2}{4(1 + m_2^2(s - 4))^2 \sqrt{s(4 - s)} \sqrt{s(s - 4m_2^2)}}. \tag{5.29}$$

In figure 9 we show the match between the expression above and the CFT density obtained numerically for $\Delta_1 = 500$ and $m_2^2 = 2/3$. As for the triangle diagram, this match persists for any $m_2^2 > 1/2$. For $m_2^2 < 1/2$ we encounter anomalous thresholds which are the topic of the next section.

5.5 Anomalous thresholds

So far we have considered ranges of masses such that we avoid anomalous thresholds. The latter are singularities in the amplitude which do not have a direct interpretation in terms of intermediate physical states. One can obtain them by computing the associated Landau diagrams where internal particles are on-shell (see e.g. [1]). In higher spacetime dimensions, these singularities are typically branch points whereas in two dimensions we encounter poles (a famous example are the Coleman-Thun double poles in the sine-Gordon model [64]). For the diagrams discussed above we have anomalous thresholds for the triangle and box when the internal mass has values $\frac{m_1^2}{4} < m_2^2 < \frac{m_1^2}{2}$.²⁴ As can be checked — e.g. from direct computation of the Feynman integral or Landau equations — the leading singularity of the triangle diagram is a simple pole

$$T_{\text{anom}}^{\text{triangle}}(s) \sim \frac{\pi m_1}{m_2^2 \sqrt{4m_2^2 - m_1^2}} \frac{1}{s - \left(4m_1^2 - \frac{m_1^4}{m_2^2}\right)}, \quad \frac{m_1^2}{4} < m_2^2 < \frac{m_1^2}{2}. \tag{5.30}$$

²⁴We shall not consider $m_2^2 < \frac{m_1^2}{4}$, as particle with mass m_1 would become unstable ($m_1 > 2m_2$) in this region. For the sake of clarity we write all m_1 factors in this discussion.

For the box diagram we have that the leading singularity is a double pole at the crossing symmetric point, occurring only at specific value $m_2^2 = m_1^2/2$

$$T_{\text{anom}}^{\text{box, leading}}(s) \sim \frac{4\pi}{(s - 2m_1^2)^2}, \quad m_2^2 = \frac{m_1^2}{2}, \quad (5.31)$$

and the sub-leading singularity when reducing²⁵ the box diagram to the triangle above

$$T_{\text{anom}}^{\text{box, subleading}}(s) = T_{\text{anom}}^{\text{triangle}}(s), \quad \frac{m_1^2}{4} < m_2^2 < \frac{m_1^2}{2}. \quad (5.32)$$

In all these examples, anomalous thresholds are present when $m_2^2 < 1/2$, which implies that the two-particle production cut begins at $s_0 < 2$. Equivalently, in CFT language, the OPE contains a tower of states with dimensions $[22]_n$ beginning below $\sqrt{2}\Delta_\phi$. But this violates our assumptions on the CFT spectrum spelled out in 3.1.1, where they were required in order to ensure boundedness of the OPE. It is tempting therefore to conjecture that this unboundedness is related with anomalous behaviour in the S-matrix. The amplitudes above provide then an excellent opportunity for understanding this.

Our first comment is that we do not believe that this unboundedness is merely an artifact of the flat space limit. That is, we believe the OPE is genuinely unbounded (and not just parametrically larger than our bounds) whenever our gap assumption fails. A rigorous proof that this is the case when $\Delta_0 \leq \frac{4}{3}\Delta_\phi$ follows from considering the following family of functions:

$$\mathcal{B}_\alpha(z) = \frac{1}{[z(1-z)]^{(2-\alpha)\Delta_\phi}} \quad (5.33)$$

This function is clearly crossing symmetric, and it admits a conformal block decomposition with a leading operator of dimension $\Delta_0 = \alpha\Delta_\phi$ (i.e. without identity). But furthermore the OPE is *positive* whenever $\Delta_0 \leq \frac{4}{3}\Delta_\phi$. For instance [65]:²⁶

$$\mathcal{B}_{\frac{4}{3}}(z) = \sum_{n=0}^{\infty} b_n G_{\frac{4}{3}\Delta_\phi + 2n}(z|\Delta_\phi), \quad (5.34)$$

with $b_n = \frac{\left(\frac{4\Delta_\phi}{3}\right)_{2n}^2 {}_3F_2\left(-2n, \frac{2\Delta_\phi}{3}, 2n + \frac{8\Delta_\phi}{3} - 1; \frac{4\Delta_\phi}{3}, \frac{4\Delta_\phi}{3}; 1\right)}{(2n)! \left(2n + \frac{8\Delta_\phi}{3} - 1\right)_{2n}} > 0.$

This means that given a general unitary CFT correlator \mathcal{G} with $\Delta_0 \leq \frac{4}{3}\Delta_\phi$, we can obtain a new one by doing $\mathcal{G} \rightarrow \mathcal{G} + \lambda \mathcal{B}_{\frac{4}{3}}$ with arbitrarily large positive λ . This establishes there exist CFT correlators for which quantities like $\tilde{\rho}$ and ρ are not in general finite, at least when the gap is below $\frac{4}{3}\Delta_\phi$. This simple example explains the general mechanism establishing that it is not possible to bound the OPE whenever the gap is below some critical value: the existence of unitary correlators without identity whose overall coefficient

²⁵When writing the amplitude in terms of Feynman parameters, the reduced diagrams come from some of these Feynman parameters vanishing, effectively setting the length of the associated internal leg to zero.

²⁶See also appendix B of [66] for a related discussion.

may therefore become arbitrarily large.²⁷ Can we improve our proof for any gap below $\sqrt{2}\Delta_\phi$? Unfortunately, an exploration of generalized free field correlators of composite operators (where the above function arises) does not seem to lead to solutions with a gap higher than $\frac{4}{3}\Delta_\phi$. Perhaps we did not try hard enough, or perhaps such solutions must necessarily be interacting. Let us proceed assuming such solutions do exist, to avoid an (in our view) artificial separation between gaps below $\frac{4}{3}\Delta_\phi$ and above it.

Let us test our conjecture linking OPE unboundedness and anomalous behaviour on the example of the box diagram, in the region $\frac{m_1^2}{4} < m_2^2 < \frac{m_1^2}{2}$. In this case, the correct scattering amplitude is obtained by taking

$$\tilde{\rho}^{\text{box}}(s) \rightarrow \tilde{\rho}^{\text{box}}(s) + \tilde{\rho}_{\text{anom}}^{\text{box}}(s), \quad \frac{m_1^2}{4} < m_2^2 < \frac{m_1^2}{2}, \quad (5.35)$$

$$\tilde{\rho}_{\text{anom}}^{\text{box}}(s) = -\frac{i\pi s}{4(s-4)}\delta\left(s - \frac{1}{m_2^2}\right). \quad (5.36)$$

That is, the naive analytic continuation of the density to $m_2^2 < \frac{m_1^2}{2}$ fails, as we must add a term with a delta function reproducing the simple pole in (5.32) and (5.30). We should point out that, in fact, the density is not uniquely defined in the region $(s_0, 4 - s_0)$. This is because

$$\int_{s_0}^4 ds' \widetilde{K}(s, s') \delta\tilde{\rho}(s') = 0, \quad \delta\tilde{\rho}(s) = \delta\tilde{\rho}(4 - s), \quad \delta\tilde{\rho}(s > 4 - s_0) = 0 \quad (5.37)$$

owing to the antisymmetry property $\widetilde{K}(s, s') = -\widetilde{K}(s, 4 - s')$.

With this caveat in mind, we can compare the expected density with the one computed from the OPE data in the CFT. This is shown in figure 10. We have chosen to set $\Delta_0 = \frac{4}{3}\Delta_\phi$ for simplicity, but the figure looks similar for other choices. We see the density computed from the CFT only matches the one obtained from the amplitude above some critical value of s . In our explorations we find that it varies very little with Δ_ϕ , but it depends strongly on Δ_0 . Also, this value is always above $4 - s_0$, so that the ambiguity in the definition of $\tilde{\rho}$ cannot help us cure the mismatch. Furthermore, we find the anomalous pole always lies in the region where there is a mismatch. More strikingly, it seems that the anomalously large piece of the CFT density closely resembles the density computed from function \mathcal{B} , that is

$$\tilde{\rho}_{\text{CFT}}^{\mathcal{B}}(s) = \lim_{\Delta_\phi \rightarrow \infty} \frac{b_n}{\tilde{a}_{\frac{4}{3}\Delta_\phi + 2n}^{\text{free}}}, \quad s = \left(\frac{\frac{4}{3}\Delta_\phi + 2n}{\Delta_\phi}\right)^2. \quad (5.38)$$

These results suggest that anomalous thresholds are indeed directly correlated with unboundedness of the OPE. They also suggest that this unboundedness could follow from the appearance of a large component in the CFT OPE which satisfies crossing symmetry by itself. Therefore, it seems that to obtain the correct flat space limit of the full correlator this piece must be first subtracted and its limit handled separately. Presumably this will

²⁷A general argument that this must be the case is as follows: if there is no bound on a particular OPE coefficient, then it must be possible to construct unitary families of correlators \mathcal{G}_λ where that OPE coefficient is some large number λ . Then $\partial_\lambda \mathcal{G}_\lambda|_{\lambda=\infty}$ is a unitarity solution to crossing without identity.

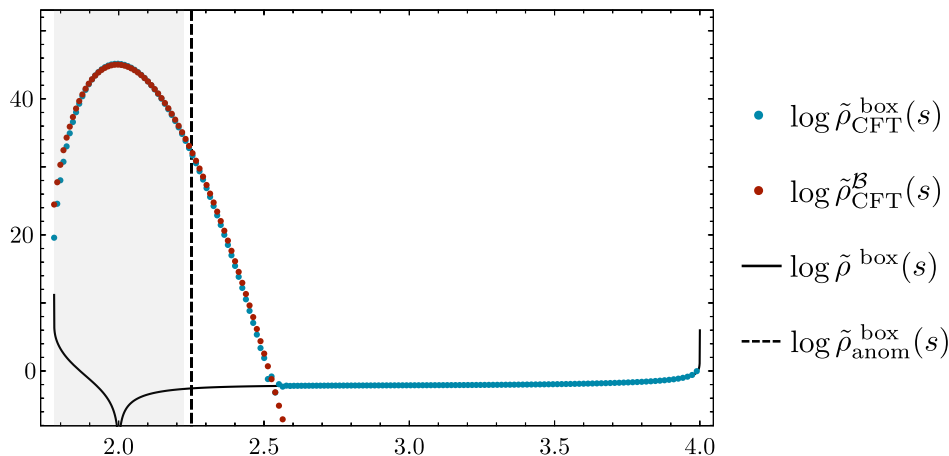


Figure 10. Densities $\tilde{\rho}$ in the presence of anomalous thresholds, in the case $2\Delta_2 = \frac{4}{3}\Delta_\phi \Leftrightarrow m_2^2 = \frac{4}{9}$. The densities $\tilde{\rho}_{\text{CFT}}^{\text{B}}, \tilde{\rho}_{\text{CFT}}^{\text{box}}$ are computed with $\Delta_\phi = 500$. Above a critical $s \approx 2.5$ the CFT and exact densities agree, but below this value the mismatch is exponential (getting worse with increasing Δ_ϕ). The anomalous pole contribution is shown as a dashed line. The ambiguity in $\tilde{\rho}$ is only present in the shaded band.

lead to the appearance of new, anomalous, contributions to the CFT density. Finally, note that these results are consistent with the picture in [34]: the unbounded piece of the OPE should map into the “wrong” saddles corresponding to AdS Landau diagrams.

6 Extremality

We have established a mapping from certain families of CFT correlators and S -matrices. In this section we will explore general properties of this mapping and how they lead to a detailed link between extremal S -matrices, extremal CFTs, and bootstrap problems in both contexts.

6.1 Extremality and the flat space limit

Information loss. Our basic formula states that

$$S(s) = \mathcal{F}[\mathcal{G}(z_s)]. \tag{6.1}$$

The S -matrix may be computed for physical kinematics via the phase shift formula (4.6), and elsewhere on the complex plane from the dispersion relation (3.35). Our definition of the operation \mathcal{F} is linear, so that given two distinct CFT families $\mathcal{G}_1, \mathcal{G}_2$ matching with S -matrices S_1 and S_2 , we can build an entire family of new solutions:

$$\begin{aligned} S_{1+2}(s) &= xS_1(s) + (1-x)S_2(s) = x\mathcal{F}[\mathcal{G}_1(z_s)] + (1-x)\mathcal{F}[\mathcal{G}_2(z_s)] \\ &= \mathcal{F}[x\mathcal{G}_1(z) + (1-x)\mathcal{G}_2(z)], \quad x \in [0, 1] \end{aligned} \tag{6.2}$$

The constraint $x \in [0, 1]$ simultaneously guarantees that the identity operator appears in the sum of CFT correlators with unit coefficient, and that the unitarity condition $|S_{1+2}| \leq 1$

holds from those of S_1, S_2 . More interestingly, starting from any two S -matrices, their product is also an S -matrix. This is ensured by

$$\mathcal{F}[\mathcal{G}_1(z_s)\mathcal{G}_2(z_s)] = \mathcal{F}[\mathcal{G}_1(z_s)]\mathcal{F}[\mathcal{G}_2(z_s)] \tag{6.3}$$

which follows from elementary properties of the limit and of analytic continuation. This property is a bit surprising from the way we have effectively constructed the \mathcal{F} operation, which requires the OPE, and we clarify why it is true in appendix C.

These properties imply that the \mathcal{F} operation loses information. For instance, we can take any CFT correlator and multiply it by the generalized free field correlator, the flat space limit will be unaffected since $\mathcal{F}[\mathcal{G}^+] = 1$. Or we could start with a correlator and add to it the sum of generalized free fermion and boson correlators, whose flat space limits ($S = \pm 1$) add up to zero. In both cases the OPE structure of the CFT correlator before and after multiplication/addition is very different, but in the flat space limit these differences are subleading and get washed out. Thus, distinct families of CFT correlators can nevertheless lead to the same S -matrix. This is of course in line with our general expectations: placing a QFT in AdS for specific choices of boundary conditions and curvature couplings will lead to specific families of CFTs, but any such choice must nevertheless have the same flat space limit.

Extremality. A particularly interesting class of 2d S -matrices are those which saturate unitarity for physical kinematics, i.e. $|S(s)| = 1$ for $s > 4$, which we will call *extremal*. This is generally found to be the case for S -matrices saturating bootstrap bounds (see e.g. [13]), and it is also true for integrable models [67, 68]. Extremal S -matrices can be expressed as products of CDD factors, which describe zeros or poles, and take the form:

$$S_{s_b}^{\text{pole}}(s) = \frac{\sqrt{s(4-s)} + \sqrt{s_b(4-s_b)}}{\sqrt{s(4-s)} - \sqrt{s_b(4-s_b)}}, \quad S_{s_r}^{\text{zero}}(s) = \frac{\sqrt{s(4-s)} - \sqrt{s_r(4-s_r)}}{\sqrt{s(4-s)} + \sqrt{s_r(4-s_r)}}. \tag{6.4}$$

A general extremal S -matrix is thus written as

$$|S| = 1 \quad \Rightarrow \quad S(s) = \prod_i S_{s_i}^{\text{pole}}(s) \prod_j S_{s_j}^{\text{zero}}(s). \tag{6.5}$$

Let us examine what the condition $|S| = 1$ means for the CFT data. Using the phase shift formula we have

$$S(s) = \lim_{\Delta_\phi \rightarrow \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) e^{-i\pi(\Delta - 2\Delta_\phi)} \tag{6.6}$$

with

$$\lim_{\Delta_\phi \rightarrow \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) = 1. \tag{6.7}$$

Recall that the Gaussian $\mathcal{N}_{\Delta_\phi}$ implies that in these sums the only states which contribute lie inside a narrow window of width $\sqrt{\Delta_\phi}$ centered around $\Delta = \sqrt{s}\Delta_\phi$. The only way in

which the unitarity condition can be saturated is if in (6.6) the phases remain coherent in any such window. More precisely, this should be true of the phases of the states which contribute predominantly to the OPE (see appendix C). Such states should be described by a single tower of operators with dimensions $\Delta_n = 2\Delta_\phi + 2n + \gamma_n(\Delta_\phi)$ satisfying:

$$\gamma_n(\Delta_\phi) \underset{\substack{n, \Delta_\phi \rightarrow \infty \\ n/\Delta_\phi \text{ fixed}}}{\rightarrow} \gamma(s), \quad s := \lim_{\substack{n, \Delta_\phi \rightarrow \infty \\ n/\Delta_\phi \text{ fixed}}} \left(\frac{2\Delta_\phi + 2n}{\Delta_\phi} \right)^2 \quad (6.8)$$

That is, the anomalous dimensions can be promoted to a slowly varying function of s in the large Δ_ϕ limit. In this case not only is unitarity saturated but we can explicitly describe the S-matrix in terms of the spectrum of the CFT:

$$S(s) = \lim_{\Delta_\phi \rightarrow \infty} \sum_{\Delta > 2\Delta_\phi} 2 \left(\frac{a_\Delta}{a_\Delta^{\text{free}}} \right) \hat{\mathcal{N}}_{\Delta_\phi}(\Delta, s) e^{-i\pi \gamma_n(\Delta_\phi)} = e^{-i\pi \gamma(s)}, \quad \text{if } |S| = 1. \quad (6.9)$$

At this point, we would like to pose two questions. The first is: can we construct families of CFT correlators which in the flat space limit describe any S-matrix of the form (6.5)? In the following two subsections we will formulate and solve bootstrap optimization problems whose optimal solutions are CFT correlators which map to single CDD pole and CDD zero factors. Thanks to the product property (6.3), the answer to this question is therefore affirmative.

The second question is, can such families be chosen to be extremal: that is, that the OPE in the corresponding correlators contains a single tower of operators not effectively but exactly, even away from the strict $\Delta_\phi \rightarrow \infty$ limit. Tensor products and sums of correlators do not satisfy this property, but extremal correlators do arise as optimal solutions of bootstrap problems [42, 69]. Therefore, here we also believe the answer is affirmative: in section 6.4 we will show that there is a direct mapping between large classes of S-matrix and CFT bootstrap problems. This means that the associated optimal solutions are also mapped into each other, thus establishing a link between extremal S-matrices and families of extremal correlators.

6.2 Bootstrapping the CDD pole

We will begin by describing an extremal correlator which in the flat space limit describes the CDD pole. Although it is already known that this correlator can be obtained by maximizing the OPE coefficient of a state corresponding to the pole [2, 41], here we will follow a different route based on master functionals, by maximizing the correlator.²⁸

Consider then the problem of maximizing the value of a CFT correlator, $\mathcal{G}(w)$, whose spectrum starts at some dimension $\Delta_0 > \sqrt{2}\Delta_\phi$, i.e. it satisfies the strong OPE condition. A bound may be obtained by constructing a functional Ω_w^{int} satisfying the properties

$$\Omega_w^{\text{int}}(\Delta) \geq G_\Delta(w|\Delta_\phi), \quad \text{for all } \Delta \geq \Delta_0, \quad (6.10)$$

²⁸The correlator maximization problem with a gap Δ_0 is expected to be extremized by the same correlator which maximizes the OPE coefficient at Δ_0 [48].

since acting with such a functional on the crossing equation is easily seen to lead to

$$\mathcal{G}(w) \leq \mathcal{P}_0^{\text{int}}(w|\Delta_\phi), \quad \mathcal{P}_\Delta^{\text{int}}(w) := G_\Delta(w|\Delta_\phi) - \Omega_w^{\text{int}}(\Delta). \quad (6.11)$$

If $\mathcal{P}_0^{\text{int}}(w)$ matches a physical CFT correlator $\mathcal{P}_0^{\text{int}}(w) = \mathcal{G}^{\text{int}}(w)$ then the bound is optimal and saturated by that correlator. Note that we should think of $\mathcal{P}_\Delta^{\text{int}}(w)$ as a kind of interacting version of ordinary Polyakov blocks, and the superscript serves to remind us of this fact.

Such a functional can be defined in the same way as the master functionals Ω_w^\pm of section 2.1, but with the important difference that for general Δ_ϕ the kernels g_w^{int} and f_w^{int} aren't simply related. However, for $\Delta_\phi \rightarrow \infty$ we can choose:

$$g_w^{\text{int}}(z) = \hat{g}_w^{\text{int}}(z) + \delta(w - z), \quad \hat{g}_w^{\text{int}}(z) \underset{\Delta_\phi \rightarrow \infty}{\sim} (1 - z)^{2\Delta_\phi - 2} |f_w^{\text{int}}(\frac{1}{1-z})|, \quad (6.12)$$

In the limit of large Δ_ϕ the remaining constraint on f_w^{int} becomes

$$\mathcal{R}_z f_w^{\text{int}}(z) \sim -\delta(z - w) - \delta(1 - z - w) \quad \text{for } z \in (0, 1) \quad (6.13)$$

which can be solved as

$$f_w^{\text{int}}(z) = 4K(s_w, s_z) \frac{S^{\text{int}}(s_w)}{S^{\text{int}}(s_z)} + \dots \quad (6.14)$$

where we assumed $S^{\text{int}}(s)$ that has the analyticity properties of an S-matrix, but that it does not have zeros for any complex s . The corrections shown as \dots depend on the details of S^{int} , and would allow us to relax the absence of zeros, but we will not need to write them out explicitly for the time being. Note that with $S^{\text{int}} = \pm 1$ we recover the master functionals f_w^\pm .

The sum rule for the functional Ω_w^{int} can be stated as validity of the interacting Polyakov bootstrap:

$$\Omega_w^{\text{int}}(0) + \sum_{\Delta \geq \Delta_0} a_\Delta \Omega_w^{\text{int}}(\Delta) = 0 \quad \Leftrightarrow \quad \mathcal{G}(w) = \mathcal{P}_0^{\text{int}}(w) + \sum_{\Delta \geq \Delta_0} a_\Delta \mathcal{P}_\Delta^{\text{int}}(w) \quad (6.15)$$

The computation of these interacting Polyakov blocks is essentially identical to the free case, and we will give the result below. This equation translates into a dispersion relation for the CFT, which we will write directly in the flat space limit in terms of the S-matrix:

$$S(s) = S^{\text{int}}(s) - S^{\text{int}}(s) \int_{s_0}^{\infty} ds' \mathcal{I}_{s'} \left[\tilde{K}(s, s') \left(1 - \frac{S(s')}{S^{\text{int}}(s')} \right) \right] \quad (6.16)$$

The bound can now be obtained as follows. The positivity conditions on Ω^{int} are the statement that

$$\mathcal{P}_\Delta^{\text{int}}(w) \leq 0, \quad \Delta \geq \Delta_0. \quad (6.17)$$

In the dispersion relation this is the constraint that the integral is positive. Indeed if that's the case the dispersion relation immediately implies the bound $S(s_w) \leq S^{\text{int}}(s_w)$. We split the integral into two pieces:

$$S^{\text{int}}(s_w) \int_{s_0}^4 ds' \tilde{K}(s_w, s') \mathcal{I}_{s'} \left[-\frac{S(s')}{S^{\text{int}}(s')} \right] + S^{\text{int}}(s_w) \int_4^{\infty} ds' K(s_w, s') \mathcal{R}_{s'} \left[1 - \frac{S(s')}{S^{\text{int}}(s')} \right] \quad (6.18)$$

The second term on the right represents the contributions above threshold. Setting

$$|S^{\text{int}}(s)| = 1, \quad s > 4 \tag{6.19}$$

and using $|S(s)| \leq 1$, then the integral will be manifestly non-negative as long as $S^{\text{int}}(s_w) \geq 0$. As for the first term, notice that $\mathcal{I}_s S(s) = \tilde{\rho}(s) \geq 0$, and hence we can make those contributions positive as well by demanding that $S^{\text{int}}(s)$ is real and negative for $s_0 < s < 4$.

We are nearly done. The constraints on S^{int} imply that it can be written as a product of CDD poles, with pole positions $s_p \leq s_0$ and chosen such that both $S^{\text{int}}(s_w) \geq 0$ and $S^{\text{int}}(s_0 \leq s \leq 4) \leq 0$. Any S^{int} of this form gives a valid upper bound, but the optimal such bound is obtained by choosing an isolated CDD pole at precisely s_0 , since in this case it can be saturated an S-matrix satisfying our assumptions. We conclude

$$S(s) \leq S_{s_0}^{\text{pole}}(s). \tag{6.20}$$

To wrap up, let us check that the CFT extremal correlator $\mathcal{P}_0^{\text{int}}$ which saturates our bound is related to the extremal S-matrix in the way we expect, namely $\mathcal{F}[\mathcal{P}_0^{\text{int}}] = S^{\text{int}}$. Although below strictly speaking we have in mind the case $S^{\text{int}} = S_{s_0}^{\text{pole}}$, we will stick to the ‘int’ notation since most steps in our computation hold more generally. We will therefore compute:

$$\mathcal{P}_0^{\text{int}}(w) = G_0(w|\Delta_\phi) - \Omega_w^{\text{int}}(0) \tag{6.21}$$

To obtain $\Omega^{\text{int}}(0)$ we use the fact that it is a valid functional, and so the corresponding sum rule must be satisfied by any CFT correlator. In particular it must be satisfied by a generalized free field. Therefore

$$\Omega_w^{\text{int}}(0) = - \sum_{n=0}^{\infty} a_{\Delta_n}^{\text{free}} \Omega_w^{\text{int}}(\Delta_n) = 1 + \frac{1}{(1-w)^{2\Delta_\phi}} + \sum_{n=0}^{\infty} a_{\Delta_n}^{\text{free}} \mathcal{P}_{\Delta_n}^{\text{int}}(w) \tag{6.22}$$

with $\Delta_n = 2\Delta_\phi + 2n$. To compute the interacting Polyakov blocks, we use that in this case the full interacting master functional is written

$$f_w^{\text{int}}(z) = 4K(s_w, s_z) \frac{S^{\text{int}}(s_w)}{S^{\text{int}}(s_z)} + f_{z_0}^{sG}(z) E_{\Delta_0}(w|\Delta_\phi) \tag{6.23}$$

with $s_0 = 4z_0$ and f^{sG} the sine-Gordon functional (3.6). The last term is chosen to insure that $\mathcal{P}_{\Delta_0}^{\text{int}}(z) = 0$. The computation of the interacting Polyakov blocks is now almost exactly the same as the one we did for the free case in sections 2 and 3. In particular we find

$$\sum_{n=0}^{\infty} a_{\Delta_n}^{\text{free}} \mathcal{P}_{\Delta_n}^{\text{int}}(w) = S^{\text{int}}(s) \int_4^\infty ds' K(s, s') \mathcal{R}_{s'} \left[1 - \frac{1}{S^{\text{int}}(s')} \right] + a_{\Delta_0}^{sG} E_{\Delta_0}(w|\Delta_\phi) \tag{6.24}$$

where we used $\sum_{n=0}^{\infty} a_{\Delta_n}^{\text{free}} \omega^{sG}(\Delta_n) = -\omega^{sG}(0) = a_{\Delta_0}^{sG}$. Evaluating the first line and putting everything together we find

$$\mathcal{G}^{\text{int}}(w) \equiv \mathcal{P}_0^{\text{int}}(w) = \frac{1}{w^{2\Delta_\phi}} + \frac{1}{(1-w)^{2\Delta_\phi}} + a_{\Delta_0}^{sG} E_{\Delta_0}(w|\Delta_\phi) + S^{\text{int}}(s_w). \tag{6.25}$$

In this way we see that the flat space limit of the extremal correlator for the conformal bootstrap problem is indeed the extremal S-matrix for the corresponding S-matrix bootstrap problem.

6.3 Bootstrapping the CDD zero

We will now show that there is an extremal CFT that leads to the CDD zero S-matrix in the flat space limit.

Let us introduce a functional Ω_w^∂ whose kernels are given by

$$f_w^\partial(z) = 4K(s_w, s_z) \frac{d_w}{S_{s_w}^{\text{zero}}(s_z)}. \quad (6.26)$$

with

$$d_w := \left. \frac{d}{dz} S_{s_w}^{\text{zero}}(s_z) \right|_{z=w} = \frac{2w-1}{4w(1-w)}. \quad (6.27)$$

To satisfy the gluing condition we need to set

$$g_w^\partial(z) = (1-z)^{2\Delta_\phi-2} |f_w^\partial(\frac{1}{1-z})| + \delta'(w-z). \quad (6.28)$$

Let us set

$$\mathcal{P}_\Delta^\partial(w) \equiv \partial_w G_\Delta(w|\Delta_\phi) - \Omega_w^\partial(\Delta). \quad (6.29)$$

Note that the $f_w^\partial, g_w^\partial$ kernels above are *not* derivatives of the master functional kernels f_w^+, g_w^+ , and accordingly the $\mathcal{P}_\Delta^\partial$ are *not* derivatives of ordinary Polyakov blocks. The sum rule of Ω_w^∂ now yields

$$\mathcal{G}'(w) = \mathcal{P}_0^\partial(w) + \sum_{\Delta_0 \leq \Delta_b \leq 2\Delta_\phi} a_\Delta \mathcal{P}_{\Delta_b}^\partial(w) - d_w \int_4^\infty ds K(s_w, s) \mathcal{R}_s \left[1 - \frac{S(s)}{S_{s_w}^{\text{zero}}(s)} \right] \quad (6.30)$$

It is easy to compute

$$\begin{aligned} \mathcal{P}_0^\partial(w) &= \partial_w \left[\frac{1}{w^{2\Delta_\phi}} + \frac{1}{(1-w)^{2\Delta_\phi}} \right] + d_w \\ a_{\Delta_b} \mathcal{P}_{\Delta_b}^\partial(w) &= \partial_w E_{\Delta_b}(w) + 2 \frac{m_b}{\Delta_\phi} \widetilde{K}(s_w, m_b^2) \frac{d_w}{S_{s_w}(m_b^2)} \end{aligned} \quad (6.31)$$

Positivity now implies

$$\mathcal{G}'(w) \leq \partial_w \left[\frac{1}{w^{2\Delta_\phi}} + \frac{1}{(1-w)^{2\Delta_\phi}} \right] + d_w, \quad \frac{1}{2} < w < \frac{1}{4} \left(\frac{\Delta_0}{\Delta_\phi} \right)^2 \quad (6.32)$$

with optimality achieved when $S(s) = S_{s_w}^{\text{zero}}(s)$. We conclude that the CDD zero S-matrix arises from the flat space limit of the family of correlators which saturates an upper bound on the *derivative* of the correlator at a point. This nicely ties in with the S-matrix derivation in [6].

An apparent puzzle is how to characterize such CFTs from the point of view of the OPE, since they have no bound states. In appendix D we argue that these CFTs arise as a deformation of the generalized free boson where the leading scalar dimension Δ_0 is pushed parametrically close to the maximal gap $2\Delta_\phi + 1$.

6.4 Dual S-matrix and conformal bootstraps

The goal of this section is to explain how the S-matrix bootstrap in its dual formulation [13, 18, 26] is related to the conformal bootstrap written in the language of functionals.

S-matrix bootstrap. We begin in the S-matrix picture. We want to solve the optimization problem:

$$\max \mathcal{F}_P^{\text{S-mat}}, \quad \mathcal{F}_P^{\text{S-mat}} := \int_{s_0}^4 ds \tilde{c}(s)\tilde{\rho}(s) + \int_4^\infty ds c(s)\rho(s) \quad (6.33)$$

with

$$S(s) = 1 + \int_{s_0}^4 ds' \tilde{K}(s, s') \tilde{\rho}(s') - \int_4^\infty ds' K(s, s') \rho(s'), \quad |S(s^+)| \leq 1 \quad (6.34)$$

The primal variables are $\rho, \tilde{\rho} \geq 0$ and c, \tilde{c} are some chosen cost functions. To get an upper bound we will introduce dual variables to get a quantity larger than \mathcal{F}_P :

$$\mathcal{F}_P^{\text{S-mat}} \leq \mathcal{F}_D^{\text{S-mat}} + \int_4^\infty ds [|k(s)| - \mathcal{R}_s(k(s)S(s))] + \int_{s_0}^4 ds \tilde{k}(s)\tilde{\rho}(s) \quad (6.35)$$

where by assumption:

$$\tilde{k}(s) \geq 0 \quad \text{for } s \in (s_0, 4). \quad (6.36)$$

Note that positivity of the first term follows from unitarity of the S-matrix. To get a bound valid for any S-matrix, we first write $S(s)$ in terms of the $\rho, \tilde{\rho}$ using (6.34). We then impose conditions on $k(s), \tilde{k}(s)$ to eliminate all dependence on the primal variables from the right hand side of the above. (An example will be given further below cf. (6.58)). The result is:

$$\mathcal{F}_D^{\text{S-mat}} := \int_4^\infty ds [|k(s)| - \mathcal{R}_s k(s)], \quad (6.37)$$

and by construction we have

$$\max \mathcal{F}_P^{\text{S-mat}} \leq \min \mathcal{F}_D^{\text{S-mat}}. \quad (6.38)$$

Optimality is achieved if²⁹

$$\begin{cases} \tilde{k}(s) = 0 \text{ or } \tilde{\rho}(s) = 0, & s_0 < s < 4 \\ S_{\text{ext}}(s) = |k(s)|/k(s), & s > 4 \end{cases} \quad (6.39)$$

where S_{ext} stands for the optimal (extremal) S-matrix.

²⁹Note that in the region $s > 4$, optimality can also be attained through $k(s) = 0$ as numerically observed in [13]. Here we do not consider this possibility but focus on unitarity saturating S-matrices.

CFT bootstrap. Now consider the following CFT bootstrap optimization problem:

$$\max \mathcal{F}_P^{\text{CFT}}, \quad \mathcal{F}_P^{\text{CFT}} = \sum_{\Delta \geq \Delta_0} a_\Delta \mu_\Delta \quad (6.40)$$

with

$$\mathcal{G}(z) = G_0(z|\Delta_\phi) + \sum_{\Delta \geq \Delta_0} a_\Delta G_\Delta(z|\Delta_\phi), \quad \Delta_0 \geq \sqrt{2}\Delta_\phi. \quad (6.41)$$

The primal variables are now the OPE coefficients $a_\Delta \geq 0$, and μ_Δ are again some chosen cost functions. For example, OPE maximization would correspond to $\mu_\Delta = \delta_{\Delta, \Delta_b}$, and correlator minimization to $\mu_\Delta = -G_\Delta(w)$. The gap assumption may be replaced by the weak OPE condition spelled out in section 3.1.1, and as we already know, it plays the role of unitarity for the S-matrix problem since it implies the OPE is bounded.

To obtain a bound we will again add a positive quantity to the primal objective. Let us introduce a functional satisfying:

$$\Omega(0) + \sum_{\Delta \geq \Delta_0} a_\Delta \Omega(\Delta) = 0, \quad \text{and} \quad \Omega(\Delta) \geq \mu_\Omega(\Delta) \quad \text{for} \quad \Delta \geq \Delta_0. \quad (6.42)$$

To get an upper bound we consider

$$\mathcal{F}_P^{\text{CFT}} \leq \mathcal{F}_P^{\text{CFT}} + \sum_{\Delta \geq \Delta_0} a_\Delta [\Omega(\Delta) - \mu_\Omega(\Delta)]. \quad (6.43)$$

We now constrain our functional such that $\mu_\Omega = \mu$. Using (6.42) we define

$$\mathcal{F}_D^{\text{CFT}} := -\Omega(0), \quad (6.44)$$

from which we obtain an upper bound

$$\max \mathcal{F}_P^{\text{CFT}} \leq \min \mathcal{F}_D^{\text{CFT}}. \quad (6.45)$$

Optimality is achieved if

$$a_\Delta = 0 \quad \text{or} \quad \Omega(\Delta) = \mu(\Delta), \quad \Delta \geq \Delta_0. \quad (6.46)$$

Mapping the problems. We have presented these optimization problems in a suggestive way which makes clear that they are closely related. Let us now make the link more precise starting from the CFT problem in the flat space limit.

First, it is clear that since $\rho, \tilde{\rho}$ are directly related to OPE coefficients, the CFT correlator may also be expressed in terms of those variables. It is also clear that it is possible to choose CFT cost functions μ_Δ which will give $\mathcal{F}_P^{\text{S-mat}} = \mathcal{F}_P^{\text{CFT}}$.³⁰ Let us set

$$\hat{\mathcal{G}}(z) = \mathcal{G}(z) - \sum_{\Delta \leq 2\Delta_\phi} a_\Delta G_\Delta(z|\Delta_\phi). \quad (6.48)$$

³⁰In detail, this is achieved by setting

$$\begin{aligned} \mu_\Delta &= \frac{2\sqrt{s_\Delta}}{\Delta_\phi \bar{a}_\Delta^{\text{free}}} \tilde{c}(s_\Delta), & \Delta_0 \leq \Delta < 2\Delta_\phi \\ \mu_\Delta &= \frac{2\sqrt{s_\Delta}}{\Delta_\phi a_\Delta^{\text{free}}} 4 \sin^2 \left[\frac{\pi}{2} (\Delta - 2\Delta_\phi) \right] c(s_\Delta), & \Delta \geq 2\Delta_\phi. \end{aligned} \quad (6.47)$$

Then the phase shift formula and the bounds of section 3.1.2 can be written:

$$\begin{aligned} \widehat{\mathcal{G}}(z) &\underset{\Delta_\phi \rightarrow \infty}{=} S(s_z), & z < 0 \\ (1-z)^{2\Delta_\phi} \widehat{\mathcal{G}}(z) &\underset{\Delta_\phi \rightarrow \infty}{=} 1, & z \in (0, 1) \end{aligned} \tag{6.49}$$

We now parameterize our functional in terms of kernels f, g as usual. Inspired by previous examples, we will set

$$g(z) = (1-z)^{2\Delta_\phi-2} |f(\frac{1}{1-z})| + \delta g(z), \quad z \in (0, 1), \tag{6.50}$$

with $\delta g(z)$ introducing extra freedom and assumed to not scale with Δ_ϕ , so that

$$\mathcal{R}_z f(z) \sim -\delta g(z) - \delta g(1-z) \tag{6.51}$$

Using the definition of the functional action we can find

$$\begin{aligned} \sum_{\Delta \geq 2\Delta_\phi} a_\Delta [\Omega(\Delta) - \mu_\Omega(\Delta)] &= \int_{-\infty}^0 dz \left\{ |f(z)| - \mathcal{R}_z[f(z)\widehat{\mathcal{G}}(z)] \right\} \\ \sum_{\Delta_0 \leq \Delta < 2\Delta_\phi} a_\Delta [\Omega(\Delta) - \mu_\Omega(\Delta)] &= \int_{z_0}^1 dz [\mathcal{I}_z f(z) \tilde{\rho}(4-s_z)] \end{aligned} \tag{6.52}$$

where the $\mu_\Omega(\Delta)$ are in general non-zero, since computing just $\Omega(\Delta)$ leads to more contributions other than those shown on the RHS. For instance on the first line we should have

$$\sum_{\Delta \geq 2\Delta_\phi} a_\Delta \mu_\Omega(\Delta) = \int_0^1 dz \delta g(z) \widehat{\mathcal{G}}(z) \tag{6.53}$$

It is now clear we should identify:

$$k(s) = 4f(z_s), \quad \tilde{k}(s) = 4\mathcal{I}_z f(z_s) \tag{6.54}$$

In particular, this leads to

$$\mathcal{F}_D^{\text{S-mat}} = \int_4^\infty ds \left(|k(s)| - \mathcal{R}_s k(s) \right) = \int_{-\infty}^0 dz \left(|f(z)| - \mathcal{R}_z f(s) \right) = \mathcal{F}_D^{\text{CFT}}. \tag{6.55}$$

which completes our mapping between the two bootstrap problems.

For completeness, and to conclude, let us point out that it is straightforward to generalize these constructions to enforce specific constraints on S-matrices/correlators, such as presence of states with definite couplings. In this case we should set the costs of such states to be zero, and since they do not need to be eliminated we do not to constrain the associated dual variables. For instance, in CFT bootstrap language, this leads to a modified dual objective of the form

$$\mathcal{F}_D^{\text{CFT}} = -\Omega(0) - \sum_i a_{\Delta_i} \Omega(\Delta_i) = \int_{-\infty}^0 dz [|f(z)| - \mathcal{R}_z f(s)] + \sum_i \mathcal{I}_z f(z_i) \tilde{\rho}_i. \tag{6.56}$$

where the $\tilde{\rho}_i$ are given and fixed.

Example: correlator maximization. Let us now illustrate the above map between the CFT functionals and the dual S-matrix bootstrap problem through the example of the correlator maximization problem considered in section 6.2. In the flat space limit, the corresponding S-matrix bootstrap problem becomes bounding the amplitude evaluated at a point $S(s_w)$ with $s_w \in (4 - s_0, s_0)$. In particular, in this case the primal objective (6.33) is given by the cost functions

$$F_P^{\text{S-mat}} = S(s_w) - 1 \quad \Rightarrow \quad \begin{cases} \tilde{c}(s') = \tilde{K}(s_w, s') \geq 0, & s' \in (s_0, 4) \\ c(s') = -K(s_w, s') \leq 0, & s' > 4 \end{cases} \quad (6.57)$$

as can be seen by directly evaluating (6.34) at $s = s_w$. The standard dual formulation gives the minimization problem with the dual functional (6.37) where the dual variables k, \tilde{k} are subject to the constraints:

$$\begin{aligned} \mathcal{R}_s k(s) &= -c(s) - \frac{2}{\pi} \int_4^\infty ds' \frac{\sqrt{s'(s'-4)}}{\sqrt{s(s-4)}} \frac{s-2}{(s-s')(s'+s-4)} \mathcal{I}_s k(s'), \quad s > 4, \\ -\tilde{k}(s) &= \tilde{c}(s) - \frac{2}{\pi} \int_4^\infty ds' \frac{\sqrt{s'(s'-4)}}{\sqrt{s(4-s)}} \frac{s-2}{(s-s')(s'+s-4)} \mathcal{I}_s k(s'), \quad s_0 < s < 4. \end{aligned} \quad (6.58)$$

which arise by demanding decoupling of the primal variables. Plugging in the cost functions, we can solve these constraints by setting

$$\begin{aligned} k(s) &= K(s_w, s) - \frac{2}{\pi} \int_4^\infty ds' \frac{\sqrt{s'(s'-4)}}{\sqrt{s(s-4)}} \frac{s-2}{(s-s')(s'+s-4)} \text{Im}k(s'), \\ \tilde{k}(s) &= \mathcal{I}_s k(s). \end{aligned} \quad (6.59)$$

Notice this is consistent with (6.54). We also have

$$\mathcal{R}_s k(s) = -\delta(s - s_w) - \delta(4 - s - s_w). \quad (6.60)$$

which is the same as equation (6.13) satisfied by $f_w^{\text{int}}(z)$, again in agreement with (6.54). It follows that we can also solve for k in the same way, by setting

$$k(s) = K(s_w, s) \frac{S^{\text{int}}(s_w)}{S^{\text{int}}(s)}. \quad (6.61)$$

The analyticity properties of $k(s)$ imply that the function $S^{\text{int}}(s)$ should be crossing-symmetric and meromorphic with branch points at $s = 0$ and $s = 4$. Recall (6.39), i.e., optimality implies $S_{\text{ext}}(s) = S^{\text{int}}(s)$ with $|S^{\text{int}}(s^+)| = 1$ and $S^{\text{int}}(s)$ can then be chosen to be a product of CDD poles. Now it is the positivity constraint (6.36) which forces these poles to be located below s_0 , and choosing without loss of generality $S^{\text{int}}(s_w) > 0$, we can compute the dual objective (6.37) as:

$$F_D^{\text{S-mat}} = S^{\text{int}}(s_w) \int_4^\infty ds K(s_w, s) \left(1 - \mathcal{R}_s \left[\frac{1}{S^{\text{int}}(s)} \right] \right) = S^{\text{int}}(s_w) - 1. \quad (6.62)$$

Optimality is achieved when $F_D^{\text{S-mat}} = F_P^{\text{S-mat}}$, i.e. when $S(s_w) = S^{\text{int}}(s_w)$, which happens if both are given by a single CDD pole at $s = s_0$, as before in section 6.2.

7 Discussion and Outlook

In this work we have studied the detailed relation between families of 1d CFT correlators and 2d S-matrices. Our main results are a derivation of the analyticity properties of such S-matrices under certain gap assumptions, and a characterization of their singularity structure in terms of the CFT data.

It would be important to clarify for which S-matrices do our results apply. The starting point of our construction is that the S-matrix arises from a gapped QFT. Can any such QFTs be placed in an AdS space of sufficiently large radius? This is certainly the case for any Lagrangian QFT, but more generally it seems hard to imagine an obstruction given that correlators in any such theory decay exponentially at scale separations larger than the gap. For instance, it seems we could define the correlators of the theory in the following manner: starting from local QFT correlation functions, we push fields away from each other distances much larger than the inverse mass gap while keeping the effects of curvature negligible. At this scale correlators factorize into products of two point functions given by free massive propagators. We can then “complete” each such propagator to an AdS bulk-to-bulk propagator by adding a small correction. At this point we can move around operators to arbitrarily large distances, thus leading to fully well defined correlators in AdS.

Conversely, we can hope to use our results to show that some S-matrices *cannot* arise from a gapped QFT. Indeed, many if not most 2d S-matrices consistent with crossing, analyticity and unitarity have a non-trivial UV behaviour, which can be understood in the context of generalized $T\bar{T}$ deformations [70–73], and it would be great to understand if we can use our results to investigate if these theories can arise from gapped QFTs in AdS. One idea would be to show that such S-matrices do not arise from QFTs with local observables: in practice one would need to show that it would not be possible to construct local operators from such a theory’s S-matrix, i.e. by “inverting” LSZ. In CFT language we would have to show that the bulk reconstruction problem [74] would not be solvable given the CFT data implied by the S-matrix.

A different kind of argument relies on the fact that our construction implicitly acts as an UV completion for S-matrices. Consider an S-matrix of the form:

$$S_{T\bar{T}}(s) = e^{-\ell^2 \sqrt{s(4-s)}} \tag{7.1}$$

which arises in the context of $T\bar{T}$ deformations. One of the peculiarities of this S-matrix is that the associated density oscillates indefinitely with s :

$$\rho_{T\bar{T}}(s) = 2 \sin^2 \left(\ell^2 \sqrt{s(s-4)} \right). \tag{7.2}$$

But now recall that in CFT language the density ρ is given by the double discontinuity of the correlator,

$$\rho(s) = \lim_{\Delta_\phi \rightarrow \infty} \left[(1-z)^{2\Delta_\phi} \text{dDisc}^+ \bar{\mathcal{G}}(z) \right] \Big|_{z=\frac{s-4}{s}}. \tag{7.3}$$

Generally we would expect therefore that $d\rho(s)/ds$ should instead decay with some power related to the Regge behaviour of the correlator [48]. To make this expectation precise, we can use the bounds on the correlator and its derivative derived in sections 6.2 and 6.3 to find

$$\frac{d}{ds}\rho(s) \leq \frac{4\Delta_\phi}{s-4} \quad \text{for } s > 8, \tag{7.4}$$

where the bound holds up to exponentially small corrections in Δ_ϕ . We should understand this expression as telling us that Δ_ϕ acts as a hidden UV cut-off, beyond which we must have $\rho(s)$ decay to a constant. S-matrices arising from a UV complete QFT must therefore satisfy this constraint above some scale no larger than Δ_ϕ . This makes sense: $\Delta_\phi = mR_{\text{AdS}}$ can also be thought of as an IR cutoff, and hence there should be no reason why high energy processes should know about it. As an example, notice that any S-matrix expressed as an arbitrary large (but finite) product of CDD factors satisfies this property.³¹ In contrast, for general S-matrices with an essential singularity at infinity such as $S_{T\bar{T}}$, the indefinite oscillations in the density imply that there have to be corrections at a UV scale $s \sim \Delta_\phi/\ell^2$ if they are to arise from a CFT₁/QFT₂ system. Physically, one way of understanding how an S-matrix can possibly fail this UV constraint is if it arises from a gravitational theory. In this case there is indeed UV/IR mixing: we cannot scatter particles with arbitrarily large energies while ignoring the arbitrarily large but finite size of the AdS box, since eventually we can create black holes whose size grows with the center of mass energy. Thus our argument suggests that essential singularities signal the presence of gravitational physics, in agreement with the logic of [35].

There are several open questions and directions of research to pursue in the future. The most important is to clarify the singularity structure in the presence of anomalous thresholds. In the CFT these appear when the gap in the OPE is sufficiently small, causing a catastrophic loss of control in the OPE data. We have conjectured that a resolution could lie in an understanding of unitary solutions to crossing without identity and in particular their flat space limit. It would be nice to understand what this means in practice, and possible links to the observations based on the Mellin amplitude prescription made in [34], for which the 1d Mellin amplitudes of [75] might prove useful.

An exciting direction to explore is scattering processes of higher numbers of particles, with the most interesting being three-to-three. Firstly because analyticity properties in this case are poorly understood, and secondly because of our map between extremal CFTs and extremal S-matrices. Integrability can be formulated in terms of the Yang-Baxter equation which expresses three-to-three scattering processes in terms of two-to-two, and it would be nice to understand what these conditions mean for the CFT data.

It would also be interesting to generalize our study to CFT correlators of distinct fields, which map onto S-matrices describing distinguishable particles. A special and rich set of examples corresponds to S-matrices with global symmetry. In this case not only

³¹It is important to point out however that there should be extra consistency conditions in order for the S-matrix to describe a UV complete theory. For the example of CDD factors at hand, one such condition goes along the lines of not having more resonances than bound states in the theory (see [72, 73]).

have S-matrix bootstrap studies been performed, but also CFT ones. Some of the S-matrix components now have “left” cuts which are constrained by unitarity, and it would be interesting to understand how these are determined by the CFT data. There are also S-matrices saturating bounds with intricate analytic structure. It would be nice to understand their CFT interpretation.

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A Polyakov blocks for $\Delta < 2\Delta_\phi$: real argument

In this subsection we perform the computation of Polyakov blocks with dimension $\Delta < 2\Delta_\phi$ for the special case where the cross-ratio w takes real values. The computation follows in the footsteps of a similar one in [41]. Again, we set $\Delta = \Delta_b$. After a change of variables we can rewrite a general functional action in terms of a single conformal block

$$\Omega(\Delta_b) = - \left[\int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} dz \frac{f(z)}{2} - \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}} dz \frac{f(1-z)}{2} + \int_{\frac{1}{2}}^1 dz g(z) - \int_0^{\frac{1}{2}} dz g(1-z) \right] G_{\Delta_b}(1-z|\Delta_\phi). \tag{A.1}$$

In the large Δ_ϕ limit with m_b fixed the conformal block factor has a saddle point at $z = z_b \equiv m_b^2/4$ with steepest descent direction now running along the imaginary axis. We will therefore modify shift our contour, making use of the gluing condition (2.7) to obtain

$$\Omega(\Delta_b) = - \left[\int_{z_b}^{z_b+i\infty} dz \frac{f(z)}{2} - \int_{z_b-i\infty}^{z_b} dz \frac{f(1-z)}{2} + \int_{z_b}^1 dz g(z) - \int_0^{z_b} dz g(1-z) \right] G_{\Delta_b}(1-z|\Delta_\phi). \tag{A.2}$$

Now we restrict to master functionals $\Omega_w^\pm(\Delta_b)$. The first two integrals in the above equation can be performed directly by the steepest descent method. Since the poles of f_w^\pm lie on the real axis we don’t have to worry about the steepest descent contour crossing them. Instead those contributions now effectively appear through the last two terms above, since $g_w^\pm(z) = \hat{g}_w^\pm(z) \pm \delta(w-z)$, where the kernel $\hat{g}_w(z)$ is exponentially suppressed with respect to $f_w(z)$. That leads us to the expression

$$\Omega_w^\pm(\Delta_b) \underset{\Delta_b, \Delta_\phi \rightarrow \infty}{=} \frac{m_b \mathcal{I}_z f_w^\pm(z_b)}{\Delta_\phi \tilde{a}_{\Delta_b}^{\text{free}}} \mp \left[\int_{z_b}^1 dz \delta(z-w) - \int_0^{z_b} dz \delta(z-1+w) \right] G_{\Delta_b}(1-z|\Delta_\phi), \tag{A.3}$$

where

$$\tilde{a}_{\Delta_b}^{\text{free}} = \frac{a_{\Delta_b}^{\text{free}}}{2 \sin[\pi \Delta_\phi(2 - m_b)]} \quad (\geq 0 \quad \text{for} \quad \sqrt{2}\Delta_\phi < \Delta_b < 2\Delta_\phi). \quad (\text{A.4})$$

The last two terms with delta functions in (A.3) evaluate to different combinations of individual conformal blocks, depending on the possible configurations between the saddle point z_b and the master functional parameter w . Using the relation between the Polyakov blocks and the master functional actions we find:

$$\mathcal{P}_{\Delta_b}^\pm(w) \underset{\Delta, \Delta_b \rightarrow \infty}{=} \frac{m_b}{\pi \Delta_\phi \tilde{a}_{\Delta_b}^{\text{free}}} \frac{\sqrt{w(1-w)}}{\sqrt{z_b(4-z_b)}} \frac{z_b - 1/2}{(z_b - w)(w + z_b - 1)} + E_{\Delta_b}(w|\Delta_\phi) \quad (\text{A.5})$$

with the crossing symmetric E_{Δ_b} satisfying:

$$E_{\Delta_b}(w|\Delta_\phi) = \begin{cases} G_{\Delta_b}(w|\Delta_\phi) & 0 < w < \min(z_b, 1 - z_b) \\ 0 & 1 - z_b < w < z_b \\ G_{\Delta_b}(1 - w|\Delta_\phi) & \max(z_b, 1 - z_b) < w < 1 \\ G_{\Delta_b}(w|\Delta_\phi) + G_{\Delta_b}(1 - w|\Delta_\phi) & z_b < w < 1 - z_b \end{cases} \quad (\text{A.6})$$

This is in agreement with the results in the main text.

B The phase shift formula for Polyakov blocks

The goal of this section is to show that the phase shift formula applied to Polyakov blocks leads to the same result as their flat space limit computed in section 2.2. Consider then

$$[\mathcal{P}_\Delta(z) - G_\Delta(z)] = - \sum_{n=0}^{\infty} [\alpha_n(\Delta) G_{\Delta_n}(z_s) + \beta_n(\Delta) \partial_\Delta G_{\Delta_n}(z_s)] \quad (\text{B.1})$$

We want to prove that

$$\lim_{\Delta_\phi \rightarrow \infty} \lim_{\epsilon \rightarrow 0} [\mathcal{P}_\Delta(z_s) - G_\Delta(z_s)] = ig^2 \frac{\sqrt{s(s-4)}}{\sqrt{s_\Delta(4-s_\Delta)}} \frac{2s_\Delta - 4}{(s - s_\Delta)(s - 4 + s_\Delta)} \quad (\text{B.2})$$

The sum over states localizes on those Δ_n satisfying $\Delta_n \sim \Delta_\phi \sqrt{1-z}$. This means that to compute the above we need to determine the functional actions in the limit where $n, \Delta_\phi \rightarrow \infty$ with fixed ratio. We will do this relying on fact that we can write the Polyakov block as a sum of Witten exchange diagrams:

$$\mathcal{P}_\Delta(z) = W_\Delta^{(S)}(z) + W_\Delta^{(T)}(z) + W_\Delta^{(U)}(z) + n(\Delta)\mathcal{C}(z) \quad (\text{B.3})$$

with some suitably chosen $n(\Delta)$ and $\mathcal{C}(z)$ the Ψ^4 contact term in AdS₂. We have

$$\mathcal{C}(z) = \sum_{n=0}^{\infty} [a_n G_{\Delta_n}(z) + b_n \partial_\Delta G_{\Delta_n}(z)] \quad (\text{B.4})$$

with

$$b_n = a_{\Delta_n}^{\text{free}} \frac{(2n)!(\Delta_\phi)_n^4 (4\Delta_\phi - 1)_{2n}}{2(n!)^2 (2\Delta_\phi)_n^2 (2\Delta_\phi)_{2n}^2}, \quad a_n = \frac{1}{2} \partial_n b_n \quad (\text{B.5})$$

We begin by applying the phase shift formula to the contact term. We begin by rearranging

$$\mathcal{C}(z) = \sum_{n=0}^{\infty} \left\{ \left(\frac{a_n}{a_{\Delta_n}^{\text{free}}} - \frac{\partial_n a_{\Delta_n}^{\text{free}}}{2a_{\Delta_n}^{\text{free}}} \right) a_{\Delta_n}^{\text{free}} G_{\Delta_n}(z) + \frac{b_n}{a_{\Delta_n}^{\text{free}}} \partial_{\Delta} [a_{\Delta_n}^{\text{free}} G_{\Delta_n}(z)] \right\} \quad (\text{B.6})$$

In the flat space limit the second set of terms dominate. Using the asymptotic form of conformal blocks (2.20) we find

$$\mathcal{C}(z_s) \sim \frac{-i\pi}{\sqrt{s(s-4)}}, \quad (\text{B.7})$$

with $s > 4$, up to an irrelevant overall constant. The s -channel exchange diagram is simply related to the contact diagram by the action of the Casimir operator (see e.g [61]). We have

$$W_{\Delta}^{(S)}(z_s) \sim \sum_{n=0}^{\infty} \frac{\beta_n^{(S)}(\Delta)}{a_{\Delta_n}^{\text{free}}} \partial_{\Delta} [a_{\Delta_n}^{\text{free}} G_{\Delta_n}(z)] \quad (\text{B.8})$$

with $\beta_n^{(S)} \sim b_n / [\Delta(\Delta - 1) - \Delta_n(\Delta_n - 1)]$. Including the precise normalisation and taking the flat space limit one finds

$$W_{\Delta}^{(S)}(z_s) = -i \frac{2\sqrt{s_{\Delta}}}{\pi \Delta_{\phi} \tilde{a}_{\Delta}^{\text{free}}} \frac{\sqrt{s_{\Delta}(4-s_{\Delta})}}{\sqrt{s(s-4)}} \frac{1}{s-s_{\Delta}}. \quad (\text{B.9})$$

Doing the same computation for the other channels is not trivial, since the OPE coefficients are not known in closed form for general n, Δ_{ϕ} . However, we do know that whatever the phase shift formula gives has to be a crossing symmetric expression, so we can simply sum this result over images. This leaves the overall contact term to be fixed. Its coefficient is determined by demanding that in the OPE expansion of \mathcal{P}_{Δ} we have $\beta_0(\Delta) = 0$. In the flat space limit this means that we must tune the contact term such that $\mathcal{P}_{\Delta}(z_s)$ is suppressed at threshold, $s \rightarrow 4$. Doing this leads to the result:

$$\tilde{a}_{\Delta}^{\text{free}} (\mathcal{P}_{\Delta}(z_s) - G_{\Delta}(z_s)) = -i \frac{2\sqrt{s_{\Delta}}}{\Delta_{\phi}} \frac{2}{\pi} \frac{\sqrt{s(s-4)}}{\sqrt{s_{\Delta}(4-s_{\Delta})}} \frac{s_{\Delta} - 2}{(s-s_{\Delta})(4-s-s_{\Delta})}. \quad (\text{B.10})$$

This agrees on the nose with the limit of expression (3.28).

C Products of CFT correlators and S-matrices

Let us adress an apparent puzzle. Consider two extremal CFT correlators leading to two S -matrices saturating unitarity. The product property (6.3) then guarantees that the product CFT correlator will also lead to an extremal S -matrix simply given by the product of the previous two. But this is surprising, since the product CFT correlator is not “extremal”, in the sense that it will not only the two towers of operators in \mathcal{G}_1 and \mathcal{G}_2 , but also new operators arising from their tensor product. So naively these three towers of states, each of which have different anomalous dimensions, should interfere with each other and lead to $|S| < 1$. The solution as we is that there is a single subset of these operators which dominates the OPE in the flat space limit.

First note that if we say that $\mathcal{G} = \mathcal{G}_1\mathcal{G}_2$ corresponds to a given Δ_ϕ , then we must assign $x\Delta_\phi$ to \mathcal{G}_1 and $(1-x)\Delta_\phi$ to \mathcal{G}_2 . For simplicity let us set $x = 1/2$. There are then three towers of (non-identity) states with dimensions

$$\begin{aligned}\Delta_{1,n} &= \Delta_\phi + 2n + \gamma_1(n) \\ \Delta_{2,m} &= \Delta_\phi + 2m + \gamma_2(m) \\ \Delta_{nm,p} &= \Delta_{1,n} + \Delta_{2,m} + 2p\end{aligned}\tag{C.1}$$

Then the statement is that the first two towers actually give subleading contributions to the OPE in the flat space limit, while the third tower is then equivalent to a single tower of states with dimensions $\Delta_{12} = 2\Delta_\phi + 2p + \gamma_1(s) + \gamma_2(s)$, leading to the expected result $S(s) = e^{-i\pi[\gamma_1(s)+\gamma_2(s)]}$.

Let us see how the OPE coefficients of each tower compare to a_Δ^{free} , introducing the explicit dependence on Δ_ϕ . Then at a given Δ the first two towers give contributions of the form $a_\Delta^{\text{free}}(\Delta_\phi/2)/a_\Delta^{\text{free}}(\Delta_\phi)$ which is exponentially suppressed. The last tower has OPE coefficients

$$a_{\Delta_{1,n}}^{\text{free}}(\Delta_\phi/2)a_{\Delta_{2,m}}^{\text{free}}(\Delta_\phi/2) \times \lambda(\Delta_1, \Delta_2, p)\tag{C.2}$$

with $n + m + p \sim (\Delta - 2\Delta_\phi)/2$ and the λ coefficients appear in

$$G_{\Delta_1}(z)G_{\Delta_2}(z) = \sum_{p=0}^{\infty} \lambda(\Delta_1, \Delta_2, p)G_{\Delta_1+\Delta_2+2p,z}(z),\tag{C.3}$$

and are given explicitly by

$$\lambda(\Delta_1, \Delta_2, p) = \frac{2^{-4p}\Gamma\left(p + \frac{1}{2}\right) (\Delta_1)_p (\Delta_2)_p \left(\Delta_1 + \Delta_2 + \frac{1}{2}\right)_{p-1} (\Delta_1 + \Delta_2 + 1)_{2p-1}}{\sqrt{\pi}\Gamma(p+1) \left(\Delta_1 + \frac{1}{2}\right)_p \left(\Delta_2 + \frac{1}{2}\right)_p \left(\Delta_1 + \Delta_2 + \frac{1}{2}\right)_{2p-1} (\Delta_1 + \Delta_2 + 1)_{p-1}}\tag{C.4}$$

We begin by noting that λ decreases exponentially with p even for p, Δ_1, Δ_2 large, so the dominant contribution to the sum comes from $p \sim 0$. We are left with a sum over n, m , but it is now easy to show

$$\lim_{\Delta_\phi \rightarrow \infty} \frac{a_{\Delta_{1,n}}^{\text{free}}(\Delta_\phi/2)a_{\Delta-\Delta_{1,n}}^{\text{free}}(\Delta_\phi/2)}{a_\Delta^{\text{free}}} = \begin{cases} 1 & \Delta_{1,n} = \Delta/2 \\ 0 & \text{otherwise} \end{cases}\tag{C.5}$$

and so the dominant contributions come from states with $n \sim m$ and $p = 0$, i.e. with

$$\Delta = 2\Delta_\phi + 2n + \gamma_1(s) + \gamma_2(s).\tag{C.6}$$

D Extremal CFTs for the CDD zero

There is a simple one parameter family of extremal CFT correlators obtained as a deformation of the generalized free boson.³² These correlators saturate an upper bound on

³²See [42] and [69] for analytic and numerical studies.

the OPE coefficient of an operator sitting at the gap $\Delta_0 = 2\Delta_\phi + g$, where $g < 1$, so that each correlator is labeled by both g and Δ_ϕ , i.e. $\mathcal{G}(z) \equiv \mathcal{G}_g(z|\Delta_\phi)$. One finds both numerically and analytically in perturbation theory that the spectrum of the correlator \mathcal{G}_g varies smoothly as we dial g away from zero, where the correlator matches the generalized free boson. By this we mean that not only no new states appear beyond those already contained in the generalized free solution, but furthermore that the dimensions of the states vary continuously with g . The deformation is relevant, in the sense that for any fixed $g < 1$, anomalous dimensions of operators eventually decay to zero at high energies, but as $g \rightarrow 1$ the spectrum approaches that of a generalized free fermion. Another way of putting it is that the same family can in principle be obtained as an irrelevant deformation of the latter solution.

We will argue that in the limit $\Delta_\phi \rightarrow \infty$ this family of extremal CFTs maps onto the family of S-matrices described by a single CDD factor. This factor will be a zero or a pole depending on g . For definiteness we will focus on the former case, and explain the relationship to the latter in due course. That is, we claim:

$$S^{\text{CFT}}(s) = S_{s_0}^{\text{zero/pole}}(s), \quad S^{\text{CFT}}(s) \equiv \mathcal{F}[\mathcal{G}_g(z_s)], \quad (\text{D.1})$$

where g is related to s_0 and must be taken to scale with Δ_ϕ in a precise way. The picture is that for $g/\Delta_\phi < 0$ and fixed for large Δ_ϕ we recover CDD pole, where as for $0 < g < 1$, we have a CDD zero, where in particular a finite position of the zero requires $1 - g \sim 1/\sqrt{\Delta_\phi}$.

We can check this mapping perturbatively when $g \ll 1$. In this limit the CFT is a small deformation of the free boson solution, and is described by a free boson in AdS_2 with a small quartic contact interaction. The interaction shifts the dimensions of double trace operators in the correlator, but does not introduce any new states to the theory. The anomalous dimensions of operators are given by [42]

$$\gamma_n(\Delta_\phi) := \Delta_n - 2\Delta_n^B = 2g \frac{(2n)!(\Delta_\phi)_n^4 (4\Delta_\phi - 1)_{2n}}{2(n!)^2 (2\Delta_\phi)_n^2 (2\Delta_\phi)_{2n}} + O(g^2) \quad (\text{D.2})$$

This leads to

$$\gamma(s) = \frac{4g\sqrt{2}}{\sqrt{\pi\Delta_\phi}\sqrt{s(s-4)}} + O(g^2) \quad (\text{D.3})$$

from which we can determine the S-matrix S^{CFT} . Expanding it we find a perfect match with $S_{s_0}^{\text{zero}}$ if we equate

$$\sqrt{s_0(4-s_0)} = \sqrt{\frac{2\pi}{\Delta_\phi}} g. \quad (\text{D.4})$$

which implies that in this regime s_0 is parametrically close to threshold. Negative g requires continuing s_0 around the cut beginning at $s = 4$, which turns the CDD zero into a CDD pole. Hence, in a certain sense, the second sheet of the S-matrix is the region $2\Delta_\phi < \Delta < 2\Delta_\phi + 1$. The function $\gamma(s)$ has an interesting form. The CFT computation which leads to (D.2) requires that all anomalous dimensions are small. But this is true even when g

is not parametrically small, as long as we are not too close to threshold $s \sim 4$. Hence, for finite g we should expect that

$$\gamma(s) = \frac{c(g)}{\sqrt{\Delta_\phi s(s-4)}}, \quad \sqrt{s_0(4-s_0)} = \frac{c(g)}{\sqrt{\Delta_\phi}} \quad (\text{D.5})$$

The function $c(g)$ captures the non-perturbative dynamics which are only relevant for $\Delta - 2\Delta_\phi = O(1)$ in the flat space limit. Unfortunately we do not have information about this function beyond leading order.³³ However, we expect that for any fixed $g < 1$, $c(g)$ remains finite, and hence the CFT family maps onto a single CDD zero parametrically close to threshold.

Let us now assume the correspondence between S^{CFT} and $S_{s_0}^{\text{zero}}$ is true to see what can be learned about the mapping between s_0 and g . Since g is the anomalous dimension of operators close to threshold we impose the condition

$$S_{s_0}^{\text{zero}}(s = 4 + \epsilon) = g \quad (\text{D.6})$$

where $\epsilon = \epsilon(g)$ should be parametrically small. In fact, we should expect $\epsilon = O(\Delta_\phi^{-1})$. A simple calculation yields

$$\sqrt{s_0(4-s_0)} = \sqrt{\epsilon(g)} \tan\left(\frac{\pi g}{2}\right) \quad (\text{D.7})$$

Consistency with our results at finite $g < 1$ determines

$$\sqrt{\Delta_\phi \epsilon(g)} \tan\left(\frac{\pi g}{2}\right) = c(g). \quad (\text{D.8})$$

For small g this yields $\epsilon(g) = 8\pi/\Delta_\phi$, i.e. $\Delta_\phi \epsilon(g)$ is order one as expected. Assuming this remains true for any g we see that to have a finite s_0 requires taking g parametrically close to unity, $1 - g = O(\Delta_\phi^{-1/2})$. In this regime $c(g)$ becomes large and perturbation theory breaks down.

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³³In reference [42] anomalous dimensions were computed up to cubic order in g but only for fixed $\Delta_\phi = 1$.

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