



HAL
open science

Non-radial finite time blow-up for the fourth-order nonlinear Schrödinger equations

van Duong Dinh

► **To cite this version:**

van Duong Dinh. Non-radial finite time blow-up for the fourth-order nonlinear Schrödinger equations. Applied Mathematics Letters, In press, 10.1016/j.aml.2022.108084 . hal-03629255

HAL Id: hal-03629255

<https://hal.science/hal-03629255>

Submitted on 4 Apr 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NON-RADIAL FINITE TIME BLOW-UP FOR THE FOURTH-ORDER NONLINEAR SCHRÖDINGER EQUATIONS

VAN DUONG DINH

ABSTRACT. We revisit the finite time blow-up for the fourth-order Schrödinger equation with focusing inhomogeneous nonlinearity $-|x|^{-2}|u|^{\frac{4}{n}}u$. By exploiting localized virial estimates and spatial decay of the nonlinearity, we prove the finite time blow-up of non-radial solutions with negative energy. Our result is the first one dealing with the existence of non-radial blow-up solutions to the fourth-order Schrödinger equations.

1. INTRODUCTION

We consider the Cauchy problem for fourth-order nonlinear Schrödinger equations with focusing inhomogeneous nonlinearity

$$\begin{cases} i\partial_t u - \mu\Delta^2 u + \Delta u &= -|x|^{-2}|u|^{\frac{4}{n}}u, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u|_{t=0} &= u_0 \in H^2(\mathbb{R}^n), \end{cases} \quad (1.1)$$

where $n \geq 3$, $u : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{C}$, $u_0 : \mathbb{R}^n \rightarrow \mathbb{C}$, and $\mu > 0$. Equation (1.1) belongs to a class of general fourth-order nonlinear Schrödinger equations, namely

$$i\partial_t u - \mu\Delta^2 u + \Delta u = F(x, u)u, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \quad (1.2)$$

which was introduced by Karpman [13] and Karpman-Shagalov [14] in order to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The function $F(x, u)$ can be understood as a nonlinear potential affected by electron density (see e.g., [4]).

The local well-posedness for (1.1) was established by Cho-Ozawa-Wang [8] using a regularizing argument and Strichartz estimates. More precisely, there exist $T^* > 0$ and a unique solution

$$u \in C([0, T^*), H^2(\mathbb{R}^n)) \cap L^2_{\text{loc}}([0, T^*), W^{3, \frac{2n}{n-2}}(\mathbb{R}^n))$$

to (1.1). The maximal time of existence satisfies the blow-up alternative: either $T^* = \infty$ or if $T^* < \infty$, then

$$\lim_{t \nearrow T^*} \|u(t)\|_{H^2} = \infty.$$

In addition, there are conservation laws of mass and energy

$$M(u(t)) = \int |u(t, x)|^2 dx = M(u_0) \quad (\text{Mass})$$

and

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 + \mu |\Delta u(t, x)|^2 dx - \frac{n}{2n+4} \int |x|^{-2} |u(t, x)|^{2+\frac{4}{n}} dx = E(u_0). \quad (\text{Energy})$$

In the case of no harmonic term (Δu), equation (1.1) has a scaling invariance

$$u_\lambda(t, x) = \lambda^{\frac{n}{2}} u(\lambda^4 t, \lambda x), \quad \lambda > 0.$$

A direct computation shows that this scaling leaves the L^2 -norm of initial data invariant. Thus due to the mass conservation, equation (1.1) (without the Δu term) is usually called mass-critical.

After the pioneering work [8], the inhomogeneous biharmonic nonlinear Schrödinger equation has attracted considerable interest in the mathematical community (see e.g., [7, 12, 15] and references therein).

The main purpose of this paper is to study the finite time blow-up for (1.1). Before stating our result, let us recall previous works related to the finite time blow-up for (1.1) and more generally for (1.2).

For the pure power-type nonlinearity, i.e., $F(x, u) = |u|^{p-1}$, the first blow-up results was given in a series of works of Baruch-Fibich-Mandelbaum [1–3], where they gave numerical simulations showing the existence of finite time blow-up solutions. In [5], Boulenger-Lenzmann provided rigorous analytical proofs for the existence of blow-up solutions. In particular, they proved that radial H^2 -solutions with negative energy blows up in finite time in both mass-critical and mass-supercritical regimes. In [6, 10], some finite time blow-up results were proved in the mass-supercritical regime for radial H^2 -solutions with non-negative energy. Still in the mass-supercritical regime, the author [9] showed the existence of finite time blow-up solutions with radial $\dot{H}^2 \cap \dot{H}^{\gamma_c}$ -data having negative energy, where γ_c is the critical regularity exponent.

We now turn to the case of focusing inhomogeneous nonlinearity (1.1) which is the main focus of this paper. In this case, we are only aware of the following blow-up result due to Cho-Ozawa-Wang [8].

2010 *Mathematics Subject Classification.* 35Q55; 35B44.

Key words and phrases. Fourth-order nonlinear Schrödinger equation; Blow-up.

Theorem 1.1 ([8]). *Let $\mu > 0$ and $u_0 \in H^2(\mathbb{R}^n)$ satisfy $E(u_0) < 0$. If one of the following conditions holds:*

- $n \geq 7$ and $|x|u_0 \in L^2$;
- $n \geq 5$ and u_0 is radial;

then the corresponding solution to (1.1) blows up in finite time, i.e., $T^ < \infty$.*

The proof of Theorem 1.1 is based on a virial argument in the same spirit of Glassey's work [11] on finite time blow-up for the standard nonlinear Schrödinger equation. However, due to the appearance of a biharmonic operator, a careful analysis is needed to derive a virial inequality which guarantees the finite time blow-up. More precisely, they proved that for sufficiently smooth solutions (e.g., $u \in C([0, T^*), H^3(\mathbb{R}^n))$), the virial quantity

$$\mathcal{V}(u(t)) := \left\langle (1 - 2\mu\Delta)^{-1/2}xu(t), (1 - 2\mu\Delta)^{-1/2}xu(t) \right\rangle$$

satisfies

$$\mathcal{V}(u(t)) \leq 16t^2E(u_0) + 4t \left(\operatorname{Im} \langle u_0, x \cdot \nabla u_0 \rangle + C(M(u_0))^{1+2/n} \right) + \mathcal{V}(u_0), \quad \forall t \in [0, T^*).$$

This inequality yields the finite time blow-up, i.e., $T^* < \infty$ provided that $E(u_0) < 0$. As the above inequality only holds for sufficiently smooth solutions and it is hard to expect such regularity for solutions of (1.1) with H^2 -initial data due to the singularity and lower power of nonlinearity, the authors implemented an approximation with regularized solutions to overcome the difficulty.

Note that in the above-mentioned works, the radial symmetry (or finite variance) plays a crucial role. To the best of our knowledge, there are no results concerning the finite time blow-up for general (non-radial and infinite variance) H^2 -solutions to (1.2) as well as (1.1) available in the literature. The main purpose of this note is to prove the following result.

Theorem 1.2. *Let $n \geq 3$, $\mu > 0$, and $u_0 \in H^2(\mathbb{R}^n)$ satisfy $E(u_0) < 0$. Then the corresponding solution to (1.1) blows up in finite time.*

Theorem 1.2 improves the one in [8] in several directions. First, we prove the existence of finite time blow-up solutions without radial symmetry and finite variance. Second, we show the finite time blow-up for all dimensions $n \geq 3$, where the local well-posedness for (1.1) is available. Our result is the first one dealing with the finite time blow-up for fourth-order nonlinear Schrödinger equations with general data.

The proof of Theorem 1.2 relies on a localized virial identity and a spatial decaying property of the nonlinearity. First, we derive a virial identity related to (1.1) using an idea of [5]. Second, by choosing a suitable cutoff function and using the spatial decay of nonlinearity, we carefully estimate each terms of the virial identity. Finally, an ODE argument allows us to show the finite blow-up.

This note is organized as follows. In Section 2, we derive a localized virial identity related to (1.1). Section 3 is devoted to the proof of the finite time blow-up given in Theorem 1.2.

2. LOCALIZED VIRIAL IDENTITY

In this section, we derive a localized virial identity related to (1.1) which plays a key element in our proof. To this end, we introduce

$$\phi(r) = \int_0^r \zeta(s) ds$$

with

$$\zeta(r) := \begin{cases} 2r & \text{if } 0 \leq r \leq 1, \\ 2r - 2(r-1)^6 & \text{if } 1 < r \leq 1 + 1/\sqrt[5]{6}, \\ \text{smooth and } \zeta'(r) < 0 & \text{if } 1 + 1/\sqrt[5]{6} < r < 10, \\ 0 & \text{if } r \geq 10. \end{cases}$$

Let $R > 0$ and define the radial function

$$\phi_R(x) = \phi_R(r) := R^2\phi(r/R), \quad r = |x|. \quad (2.1)$$

We collect some properties of ϕ_R in the following lemma.

Lemma 2.1. *We have*

$$\|\nabla^j \phi_R\|_{L^\infty} \lesssim R^{2-j}, \quad 0 \leq j \leq 6, \quad (2.2)$$

and

$$\operatorname{supp}(\nabla^j \phi_R) \subset \begin{cases} \{|x| \leq 10R\} & \text{if } j = 1, 2, \\ \{R \leq |x| \leq 10R\} & \text{if } 3 \leq j \leq 6, \end{cases} \quad (2.3)$$

and

$$\frac{\phi'_R(r)}{r} \leq 2, \quad \phi''_R(r) \leq 2, \quad \forall r \geq 0. \quad (2.4)$$

In addition, we have

$$\frac{\phi'_R(r)}{r} - \phi''_R(r) \geq 0, \quad \forall r \geq 0. \quad (2.5)$$

Proof. The properties (2.2)–(2.4) follow directly from the choice of ζ . Let us check (2.5). For $0 \leq r/R \leq 1$, (2.5) holds trivially as $\frac{\phi'_R(r)}{r} = \phi''_R(r) = 2$. For $1 < r/R \leq 1 + 1/\sqrt[5]{6}$, we have

$$\frac{\phi'_R(r)}{r} = \frac{\zeta(r/R)}{r/R} = 2 - 2\frac{(r/R - 1)^6}{r/R}$$

and

$$\phi''_R(r) = \zeta'(r/R) = 2 - 12(r/R - 1)^5.$$

Thus

$$\begin{aligned} \frac{\phi'_R(r)}{r} - \phi''_R(r) &= 12(r/R - 1)^5 - 2\frac{(r/R - 1)^6}{r/R} \\ &= 2(r/R - 1)^5 \left(6 - \frac{r/R - 1}{r/R} \right) \\ &= 2(r/R - 1)^5 \frac{5r/R + 1}{r/R} \geq 0. \end{aligned}$$

For $r/R > 1 + 1/\sqrt[5]{6}$, we have

$$\frac{\phi'_R(r)}{r} = \frac{\zeta(r/R)}{r/R} \geq 0, \quad \phi''_R(r) = \zeta'(r/R) \leq 0$$

which shows (2.5). \square

Let $u : [0, T^*) \times \mathbb{R}^n \rightarrow \mathbb{C}$ be a H^2 -solution to (1.1). We define the virial quantity

$$\mathcal{M}_{\phi_R}(t) := 2 \operatorname{Im} \int \nabla \phi_R \cdot \nabla u(t) \bar{u}(t) dx. \quad (2.6)$$

Lemma 2.2 (Localized virial identity). *Let $n \geq 3$ and $\mu > 0$. Then we have for all $t \in [0, T^*)$,*

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\phi_R}(t) &= \mu \int \Delta^3 \phi_R |u(t)|^2 dx - 2\mu \int \Delta^2 \phi_R |\nabla u(t)|^2 dx \\ &\quad + 8\mu \sum_{k,l,m} \int \partial_{lm}^2 \phi_R \partial_{kl}^2 \bar{u}(t) \partial_{mk}^2 u(t) dx - 4\mu \sum_{k,l} \int \partial_{kl}^2 \Delta \phi_R \partial_k \bar{u}(t) \partial_l u(t) dx \\ &\quad - \int \Delta^2 \phi_R |u(t)|^2 dx + 4 \sum_{k,l} \int \partial_{kl}^2 \phi_R \partial_k \bar{u}(t) \partial_l u(t) dx \\ &\quad - \frac{4}{n+2} \int \Delta \phi_R |x|^{-2} |u(t)|^{2+\frac{4}{n}} dx + \frac{2n}{n+2} \int \nabla \phi_R \cdot \nabla (|x|^{-2}) |u(t)|^{2+\frac{4}{n}} dx. \end{aligned} \quad (2.7)$$

Proof. We follow an idea of [5, Lemma 3.1]. We only provide formal computations, the desired identity follows from an approximation argument (see [5, Lemma 3.1] for more details). We rewrite $\mathcal{M}_{\phi_R}(t)$ as

$$\mathcal{M}_{\phi_R}(t) = \langle u(t), \Gamma_{\phi_R} u(t) \rangle,$$

where

$$\Gamma_{\phi_R} := -i(\nabla \phi_R \cdot \nabla + \nabla \cdot \nabla \phi_R).$$

Recall that if u solves the equation $i\partial_t u = Hu$, then

$$\frac{d}{dt} \langle u, Au \rangle = \langle u, [H, iA]u \rangle,$$

where $[H, A] = HA - AH$ is the commutator operator. Thanks to this identity, we have

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\phi_R}(t) &= \langle u(t), \mu[\Delta^2, i\Gamma_{\phi_R}]u(t) \rangle - \langle u(t), [\Delta, i\Gamma_{\phi_R}]u(t) \rangle - \langle u(t), [|x|^{-2}|u|^{\frac{4}{n}}, i\Gamma_{\phi_R}]u(t) \rangle \\ &=: \text{(I)} - \text{(II)} - \text{(III)}. \end{aligned}$$

We have from [5, (3.9)] that

$$[\Delta^2, i\Gamma_{\phi_R}] = 8 \sum_{k,l,m} \partial_{kl}^2 (\partial_{lm}^2 \phi_R) \partial_{mk}^2 + 4 \sum_{k,l} \partial_k (\partial_{kl}^2 \Delta \phi_R) \partial_l + 2 \sum_{k,l} \partial_k (\Delta^2 \phi_R) \partial_l + \Delta^3 \phi_R$$

which implies

$$\begin{aligned} \text{(I)} &= 8\mu \sum_{k,l,m} \int \partial_{lm}^2 \phi_R \partial_{kl}^2 \bar{u}(t) \partial_{mk}^2 u(t) dx - 4\mu \sum_{k,l} \int \partial_{kl}^2 \Delta \phi_R \partial_k \bar{u}(t) \partial_l u(t) dx \\ &\quad - 2\mu \int \Delta^2 \phi_R |\nabla u(t)|^2 dx + \mu \int \Delta^3 \phi_R |u(t)|^2 dx. \end{aligned}$$

We also have (see [5, (3.8)]) that

$$[\Delta, i\Gamma_{\phi_R}] = 4 \sum_{k,l} \partial_k (\partial_{kl}^2 \phi_R) \partial_l + \Delta^2 \phi_R$$

which gives

$$(II) = -4 \sum_{k,l} \int \partial_{kl}^2 \phi_R \partial_k \bar{u}(t) \partial_l u(t) dx + \int \Delta^2 \phi_R |u(t)|^2 dx.$$

Finally, we have

$$[|x|^{-2}|u(t)|^{\frac{4}{n}}, i\Gamma_{\phi_R}] = [|x|^{-2}|u(t)|^{\frac{4}{n}}, \nabla \phi_R \cdot \nabla + \nabla \cdot \nabla \phi_R] = -2\nabla \phi_R \cdot \nabla (|x|^{-2}|u(t)|^{\frac{4}{n}})$$

which yields

$$(III) = -2 \int \nabla \phi_R \cdot \nabla (|x|^{-2}|u(t)|^{\frac{4}{n}}) |u(t)|^2 dx.$$

Using the identity

$$\nabla (|x|^{-2}|u|^{2+\frac{4}{n}}) = \frac{n+2}{2} \nabla (|x|^{-2}|u|^{\frac{4}{n}}) |u|^2 - \frac{n}{2} \nabla (|x|^{-2}) |u|^{2+\frac{4}{n}},$$

we get

$$\nabla (|x|^{-2}|u|^{\frac{4}{n}}) |u|^2 = \frac{2}{n+2} \nabla (|x|^{-2}|u|^{2+\frac{4}{n}}) + \frac{n}{n+2} \nabla (|x|^{-2}) |u|^{2+\frac{4}{n}}.$$

Thus we obtain

$$(III) = \frac{4}{n+2} \int \Delta \phi_R |x|^{-2} |u(t)|^{2+\frac{4}{n}} dx - \frac{2n}{n+2} \int \nabla \phi_R \cdot \nabla (|x|^{-2}) |u(t)|^{2+\frac{4}{n}} dx.$$

Collecting the above identities, we prove the result. \square

Lemma 2.3. *Let $n \geq 3$ and $\mu > 0$. Then we have for all $t \in [0, T^*)$,*

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\phi_R}(t) &= \mu \int \Delta^3 \phi_R |u(t)|^2 dx - 2\mu \int \Delta^2 \phi_R |\nabla u(t)|^2 dx \\ &\quad - 4\mu \sum_{k,l} \int \partial_{kl}^2 \Delta \phi_R \partial_k \bar{u}(t) \partial_l u(t) dx - \int \Delta^2 \phi_R |u(t)|^2 dx \\ &\quad + 8\mu \int \frac{\phi'_R}{r} \sum_k |\nabla \partial_k u(t)|^2 + \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) \sum_k |x \cdot \nabla \partial_k u(t)|^2 dx \\ &\quad + 4 \int \frac{\phi'_R}{r} |\nabla u(t)|^2 + \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) |x \cdot \nabla u(t)|^2 dx \\ &\quad - \frac{4}{n+2} \int \left(\phi''_R + (2n-1) \frac{\phi'_R}{r} \right) |x|^{-2} |u(t)|^{2+\frac{4}{n}} dx. \end{aligned} \tag{2.8}$$

Proof. As ϕ_R is radial, we use the fact that

$$\partial_{kl}^2 \phi_R = \left(\delta_{kl} - \frac{x_k x_l}{r^2} \right) \frac{\phi'_R}{r} + \frac{x_k x_l}{r^2} \phi''_R$$

to have

$$\begin{aligned} \sum_{k,l} \partial_{kl}^2 \phi_R \partial_k \bar{u}(t) \partial_l u(t) &= \sum_{k,l} \frac{\phi'_R}{r} \delta_{kl} \partial_k \bar{u}(t) \partial_l u(t) + \sum_{k,l} \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) x_k x_l \partial_k \bar{u}(t) \partial_l u(t) \\ &= \frac{\phi'_R}{r} |\nabla u(t)|^2 + \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) |x \cdot \nabla u(t)|^2. \end{aligned}$$

We also have

$$\begin{aligned} \sum_{k,l,m} \partial_{lm}^2 \phi_R \partial_{kl}^2 \bar{u}(t) \partial_{mk}^2 u(t) &= \sum_{k,l,m} \frac{\phi'_R}{r} \delta_{lm} \partial_{kl}^2 \bar{u}(t) \partial_{mk}^2 u(t) + \sum_{k,l,m} \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) x_l x_m \partial_{kl}^2 \bar{u}(t) \partial_{mk}^2 u(t) \\ &= \frac{\phi'_R}{r} \sum_k |\nabla \partial_k u(t)|^2 + \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3} \right) \sum_k |x \cdot \nabla \partial_k u(t)|^2. \end{aligned}$$

Furthermore, as $\Delta \phi_R = \phi''_R + (n-1) \frac{\phi'_R}{r}$, we have

$$\Delta \phi_R + \frac{n}{2} \nabla \phi_R \cdot \nabla (|x|^{-2}) |x|^2 = \phi''_R + (2n-1) \frac{\phi'_R}{r}.$$

From (2.7) and the above identities, we obtain (2.8). \square

3. FINITE TIME BLOW-UP

We are now able to show our main result given in Theorem 1.2.

Proof of Theorem 1.2. The proof is divided into two steps.

Step 1. A localized virial estimate. We first show that for all $t \in [0, T^*)$,

$$\frac{d}{dt} \mathcal{M}_{\phi_R}(t) = 32E(u(t)) - 8\|\nabla u(t)\|_{L^2}^2 + CR^{-2} + CR^{-2}\|\nabla u(t)\|_{L^2}^2 \quad (3.1)$$

for some constant $C > 0$ depending only on μ and $M(u_0)$. Thanks to (2.8), we have for all $t \in [0, T^*)$,

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\phi_R}(t) &= 16\mu\|\Delta u(t)\|_{L^2}^2 + 8\|\nabla u(t)\|_{L^2}^2 - \frac{16n}{n+2} \int |x|^{-2}|u(t)|^{2+\frac{4}{n}} dx \\ &\quad - 8\mu \int \left(2 - \frac{\phi'_R}{r}\right) \sum_k |\nabla \partial_k u(t)|^2 dx + 8\mu \int \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3}\right) \sum_k |x \cdot \nabla \partial_k u(t)|^2 dx \\ &\quad - 4 \int \left(2 - \frac{\phi'_R}{r}\right) |\nabla u(t)|^2 dx + 4 \int \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3}\right) |x \cdot \nabla u(t)|^2 dx \\ &\quad + \frac{4}{n+2} \int \left((2 - \phi''_R) + (2n-1) \left(2 - \frac{\phi'_R}{r}\right)\right) |x|^{-2}|u(t)|^{2+\frac{4}{n}} dx \\ &\quad + \mu \int \Delta^3 \phi_R |u(t)|^2 dx - 2\mu \int \Delta^2 \phi_R |\nabla u(t)|^2 dx \\ &\quad - 4\mu \sum_{k,l} \int \partial_{kl}^2 \Delta \phi_R \partial_k \bar{u}(t) \partial_l u(t) dx - \int \Delta^2 \phi_R |u(t)|^2 dx \\ &= 32E(u(t)) - 8\|\nabla u(t)\|_{L^2}^2 + E_1(u(t)) + E_2(u(t)) + E_3(u(t)) + E_4(u(t)), \end{aligned}$$

where

$$\begin{aligned} E_1(u(t)) &= -8\mu \int \left(2 - \frac{\phi'_R}{r}\right) \sum_k |\nabla \partial_k u(t)|^2 dx + 8\mu \int \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3}\right) \sum_k |x \cdot \nabla \partial_k u(t)|^2 dx \\ &\quad - 4 \int \left(2 - \frac{\phi'_R}{r}\right) |\nabla u(t)|^2 dx + 4 \int \left(\frac{\phi''_R}{r^2} - \frac{\phi'_R}{r^3}\right) |x \cdot \nabla u(t)|^2 dx \end{aligned}$$

and

$$\begin{aligned} E_2(u(t)) &= \frac{4}{n+2} \int \left((2 - \phi''_R) + (2n-1) \left(2 - \frac{\phi'_R}{r}\right)\right) |x|^{-2}|u(t)|^{2+\frac{4}{n}} dx, \\ E_3(u(t)) &= -2\mu \int \Delta^2 \phi_R |\nabla u(t)|^2 dx - 4\mu \sum_{k,l} \int \partial_{kl}^2 \Delta \phi_R \partial_k \bar{u}(t) \partial_l u(t) dx, \\ E_4(u(t)) &= \mu \int \Delta^3 \phi_R |u(t)|^2 dx - \int \Delta^2 \phi_R |u(t)|^2 dx. \end{aligned}$$

From (2.4) and (2.5), we see that $E_1(u(t)) \leq 0$.

From (2.2) and the conservation of mass, we have

$$E_4(u(t)) \lesssim (\mu\|\Delta^3 \phi_R\|_{L^\infty} + \|\Delta^2 \phi_R\|_{L^\infty}) \|u(t)\|_{L^2}^2 \lesssim R^{-2},$$

where the implicit constant depends only on μ and $M(u_0)$.

We also have from (2.2) that

$$E_3(u(t)) \lesssim R^{-2} \|\nabla u(t)\|_{L^2}^2.$$

Finally, using (2.3), we have $\text{supp} \left\{ \left(2 - \frac{\phi''_R}{r}\right) + (2n-1) \left(2 - \frac{\phi'_R}{r}\right) \right\} \subset \{|x| \geq R\}$. Thus, using (2.2) and the standard Gagliardo-Nirenberg inequality, we get

$$E_2(u(t)) \lesssim R^{-2} \|u(t)\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}} \lesssim R^{-2} \|\nabla u(t)\|_{L^2}^2 \|u(t)\|_{L^2}^{\frac{4}{n}} \lesssim R^{-2} \|\nabla u(t)\|_{L^2}^2.$$

Collecting the above estimates, we prove (3.1).

Step 2. An ODE argument. We will show the finite time blow-up under the assumption that $E(u_0) < 0$. Assume by contradiction that $T^* = \infty$. By the energy conservation, we have from (3.1) that

$$\frac{d}{dt} \mathcal{M}_{\phi_R}(t) \leq 32E(u_0) - 8\|\nabla u(t)\|_{L^2}^2 + CR^{-2} + CR^{-2}\|\nabla u(t)\|_{L^2}^2, \quad \forall t \in [0, \infty).$$

As $E(u_0) < 0$, we take $R > 0$ sufficiently large so that

$$\frac{d}{dt} \mathcal{M}_{\phi_R}(t) \leq 16E(u_0) - 4\|\nabla u(t)\|_{L^2}^2, \quad \forall t \in [0, \infty). \quad (3.2)$$

Integrating this inequality, we have

$$\begin{aligned}\mathcal{M}_{\phi_R}(t) &= \mathcal{M}_{\phi_R}(0) + \int_0^t \frac{d}{ds} \mathcal{M}_{\phi_R}(s) ds \\ &\leq |\mathcal{M}_{\phi_R}(0)| + 16E(u_0)t, \quad \forall t \geq 0.\end{aligned}$$

This shows that $\mathcal{M}_{\phi_R}(t) \leq 0$ for all $t \geq t_0 := \frac{|\mathcal{M}_{\phi_R}(0)|}{16|E(u_0)|}$. Integrating (3.2) over $[t_0, t]$, we get

$$\begin{aligned}\mathcal{M}_{\phi_R}(t) &= \mathcal{M}_{\phi_R}(t_0) + \int_{t_0}^t \frac{d}{ds} \mathcal{M}_{\phi_R}(s) ds \\ &\leq -4 \int_{t_0}^t \|\nabla u(s)\|_{L^2}^2 ds, \quad \forall t \geq t_0.\end{aligned}$$

On the other hand, we have

$$|\mathcal{M}_{\phi_R}(t)| \leq \|\nabla \phi_R\|_{L^\infty} \|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2} \leq C(R, M(u_0)) \|\nabla u(t)\|_{L^2}, \quad \forall t \geq 0$$

which implies

$$\mathcal{M}_{\phi_R}(t) \leq -A \int_{t_0}^t |\mathcal{M}_{\phi_R}(s)|^2 ds, \quad \forall t \geq t_0$$

for some constant $A = A(R, M(u_0)) > 0$. Now set

$$Z(t) := \int_{t_0}^t |\mathcal{M}_{\phi_R}(s)|^2 ds, \quad t \geq t_0.$$

We see that $Z(t)$ is non-decreasing, non-negative, and satisfies

$$Z'(t) = |\mathcal{M}_{\phi_R}(t)|^2 \geq A^2 Z(t), \quad \forall t \geq t_0.$$

For $t_1 > t_0$, we integrate this inequality over $[t_1, t]$ to get

$$Z(t) \geq \frac{Z(t_1)}{1 - A^2 Z(t_1)(t - t_1)}, \quad \forall t \geq t_1.$$

In particular, $Z(t) \rightarrow \infty$ as $t \nearrow t^* := t_1 + \frac{1}{A^2 Z(t_1)}$. Thus $\mathcal{M}_{\phi_R}(t) \leq -AZ(t) \rightarrow -\infty$ as $t \nearrow t^*$. Therefore the solution cannot exist for all $t \geq 0$, hence it must blow up in finite time. \square

ACKNOWLEDGMENTS

This work was supported in part by the European Union's Horizon 2020 Research and Innovation Programme (Grant agreement CORFRONMAT No. 758620, PI: Nicolas Rougerie). The author would like to express his deep gratitude to his wife - Uyen Cong for her encouragement and support. The author also would like to thank the reviewers for their helpful comments and suggestions.

REFERENCES

- [1] G. Baruch and G. Fibich, Singular solutions of the L^2 -supercritical biharmonic nonlinear Schrödinger equation, *Nonlinearity* 24 (2011), no. 6, 1843–1859. [1](#)
- [2] G. Baruch, G. Fibich, and E. Mandelbaum, Ring-type singular solutions of the biharmonic nonlinear Schrödinger equation, *Nonlinearity* 23 (2010), no. 11, 2867–2887. [1](#)
- [3] G. Baruch, G. Fibich, and E. Mandelbaum, Singular solutions of the biharmonic nonlinear Schrödinger equation, *SIAM J. Appl. Math.* 70 (2010), no. 8, 3319–3341. [1](#)
- [4] L. Bergé, Soliton stability versus collapse, *Phys. Rev. E*, 62 (2000), no. 3, 3071–3074. [1](#)
- [5] T. Boulenger and E. Lenzmann, Blowup for biharmonic NLS, *Ann. Sci. Éc. Norm. Supér.* 50 (2017), no. 3, 503–544. [1](#), [2](#), [3](#)
- [6] D. Bonheure, J. B. Castéras, T. Gou, and L. Jeanjean, Strong instability of ground states to a fourth order Schrödinger equation, *Int. Math. Res. Not. IMRN*, 17 (2019), 5299–5315. [1](#)
- [7] L. Campos and C. M. Guzmán, Scattering for the non-radial inhomogeneous biharmonic NLS equation, preprint available at <https://arxiv.org/abs/2107.12359>, 2021. [1](#)
- [8] Y. Cho, T. Ozawa, and C. Wang, Finite time blowup for the fourth-order NLS, *Bull. Korean Math. Soc.*, 53 (2016), no. 2, 615–640. [1](#), [2](#)
- [9] V. D. Dinh, On blowup solutions to the focusing intercritical nonlinear fourth-order Schrödinger equation, *J. Dynam. Differential Equations* 31 (2019), no. 4, 1793–1823. [1](#)
- [10] V. D. Dinh, Dynamics of radial solutions for the focusing fourth-order nonlinear Schrödinger equations, *Nonlinearity* 34 (2021), no. 2, 776–821. [1](#)
- [11] R. T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, *J. Math. Phys.*, 18 (1977), no. 9, 1794–1797. [2](#)
- [12] C. M. Guzmán and A. Pastor, On the inhomogeneous biharmonic nonlinear Schrödinger equation: local, global and stability results, *Nonlinear Anal. Real World Appl.*, 56 (2020), 103174. [1](#)
- [13] V. I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrödinger-type equations, *Phys. Rev. E* 53(1996), no. 2, 1336–1339. [1](#)
- [14] V. I. Karpman and A. G. Shagalov, Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion, *Phys. D*, 144 (2000), no. 1-2, 194–210. [1](#)
- [15] T. Saanouni, Energy scattering for radial focusing inhomogeneous bi-harmonic Schrödinger equations, *Calc. Var. Partial Differ. Equ.*, 60 (2021), no. 3, 1–27. [1](#)

(V. D. Dinh) ECOLE NORMALE SUPÉRIEURE DE LYON & CNRS, UMPA (UMR 5669), FRANCE AND DEPARTMENT OF MATHEMATICS, HO CHI MINH CITY UNIVERSITY OF EDUCATION, 280 AN DUONG VUONG, HO CHI MINH CITY, VIETNAM

Email address: contact@duongdinh.com