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Restricting directions for Kakeya sets

Anthony Gauvan

April 3, 2022

Abstract

We prove that the *Kakeya maximal conjecture* is equivalent to the Ω -*Kakeya maximal conjecture*. This completes a recent result in [2] where Keleti and Mathé proved that the *Kakeya conjecture* is equivalent to the Ω -*Kakeya conjecture*. Moreover, we improve concrete bound on the Hausdorff dimension of a Ω -Kakeya set : for any Bore set Ω in \mathbb{S}^{n-1} , we prove that if $X \subset \mathbb{R}^n$ contains for any $e \in \Omega$ a unit segment oriented along e then we have

$$d_X \geq \frac{6}{11}d_\Omega + 1$$

where d_E denotes the Hausdorff dimension of a set E .

1 Introduction

The *Kakeya* problem is a central question in harmonic analysis which can be formulated in different ways ; it is also related to *restriction* theory and arithmetic. A measurable set X in \mathbb{R}^n is said to be a *Kakeya* set if for any direction $e \in \mathbb{S}^{n-1}$ it contains a unit segment T_e oriented along e . The Kakeya conjecture concerns the Hausdorff dimension of Kakeya set X .

Conjecture 1 (Kakeya conjecture). *If X is a Kakeya set in \mathbb{R}^n then*

$$d_X = d_{\mathbb{S}^{n-1}} + 1 = n.$$

This conjecture has been proved by Davies in the plane in [1]. For $n \geq 3$, a vast amount of techniques have been developed in order to tackle this issue ; we invite the reader to look at [3] or [6] to see the extent of the techniques that might be deployed. Here, we will simply say that, specialists are able to prove that if X is a Kakeya set in \mathbb{R}^n then

$$d_X \geq \left(\frac{1}{2} + \epsilon_n\right)d_{\mathbb{S}^{n-1}} + 1$$

where $\epsilon_n > 0$ is a dimensional constant. There exists a more quantitative version of the Kakeya conjecture and we need to introduce the *Kakeya maximal operator* to state it. We define the *Kakeya maximal function*

$$K_\delta f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}_+$$

at scale $\delta > 0$ of a locally integrable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$K_\delta f(e) := \sup_{a \in \mathbb{R}^n} \frac{1}{|T_{e,\delta}(a)|} \int_{T_{e,\delta}(a)} |f|(x) dx$$

where $e \in \mathbb{S}^{n-1}$ and $T_{e,\delta}(a)$ stands for the tube in \mathbb{R}^n with center a , oriented along the direction e , of length 1 and radius δ . It appears that any quantitative information on $\|K_\delta\|_{\sigma,p}$ provides lower bound on the dimension of any Kakeya set X : for any $1 < p < \infty$ and $\beta > 0$ such that $n - \beta p > 0$, if we have

$$\|K_\delta\|_{\sigma,p} \lesssim_{n,p,\beta} \delta^{-\beta}$$

then the Hausdorff dimension of any Kakeya set in \mathbb{R}^n is at least $n - \beta p$. In regards of this fact, the following conjecture is called the *Kakeya maximal conjecture*, it is stronger than the Kakeya conjecture.

Conjecture 2 (Kakeya maximal conjecture). *For any $\epsilon > 0$ we have*

$$\|K_\delta\|_{\sigma,n} \lesssim_{n,\epsilon} \delta^{-\epsilon}.$$

In this text, we are concerned with a natural generalization of the Kakeya problem. Given an arbitrary Borel set of directions $\Omega \subset \mathbb{S}^{n-1}$, we say that a set X in \mathbb{R}^n is a Ω -Kakeya set if for any $e \in \Omega$ there exists a unit segment T_e oriented along e included in X . What can be said about the dimension of a Ω -Kakeya set ? The following conjecture seems plausible.

Conjecture 3 (Ω -Kakeya conjecture). *For any Borel set Ω in \mathbb{S}^{n-1} ; if X is a Ω -Kakeya set then*

$$d_X \geq d_\Omega + 1.$$

At least three questions can be asked. First, if we know that the Kakeya conjecture is true, can we say something about the Ω -Kakeya conjecture ? Secondly, can we state a *maximal* version of the Ω -Kakeya conjecture ? Lastly, if there exists a Ω -Kakeya maximal conjecture, what can we said about it given the Kakeya maximal conjecture ? Very recently, Keleti and Mathé gave a positive answer to the first question in [2].

Theorem 1 (Keleti-Mathé). *If the Kakeya conjecture is true then the Ω -Kakeya conjecture is also true.*

The proof of this Theorem relies on fine notions concerning Hausdorff and packing dimension and we invite the reader to look at [2] for more details.

2 Notations

We will work in the euclidean space \mathbb{R}^n with $n \geq 3$ endowed with the Lebesgue measure and the euclidean distance ; if U is a measure set in \mathbb{R}^n we denote by $|U|$ its n -dimensional Hausdorff measure and by $|U|_k$ its k -dimensional Hausdorff measure for $k < n$. Also we denote by d_U its Hausdorff dimension and by $\text{diam}(U)$ its diameter. We denote by σ the spherical surface measure on \mathbb{S}^{n-1} and μ will stand for a probability measure on \mathbb{S}^{n-1} ; we will denote by S_μ its support. The surface measure σ will usually not charge the support of μ *i.e.* we will have $\sigma(S_\mu) = 0$. We will see that we need to focus on the study of (μ, p) -norm of

$$K_\delta : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{S}^{n-1}, \mu)$$

where μ is an arbitrary measure on \mathbb{S}^{n-1} *i.e.* we will be interested in estimating the following quantity

$$\|K_\delta\|_{\mu,p} := \sup_{\|f\|_p \leq 1} \left(\int_{\mathbb{S}^{n-1}} (K_\delta f(e))^p d\mu \right)^{\frac{1}{p}}.$$

Hence the notation $\|K_\delta\|_{\mu,p}$ emphasizes the dependence on the probability measure μ set on the target space.

3 Results

We are going to formulate the appropriate *maximal* version of the Ω -Kakeya conjecture ; then we will prove that the Kakeya maximal conjecture implies the Ω -Kakeya maximal conjecture. In other words, *we prove the maximal analog to Theorem 1*. Our approach is close to the approach initiated by Mitsis in [5] ; here we work in higher dimension. We will start by proving the following Proposition.

Proposition 1. *Let μ be an arbitrary probability measure on \mathbb{S}^{n-1} and suppose we have $1 < p < \infty$ and $\beta > 0$ such that $n - \beta p > 0$. Suppose that we have*

$$\|K_\delta\|_{\mu,p} \lesssim_{n,p,\beta} \delta^{-\beta}.$$

In this case, for any Borel set of direction Ω containing the support S_μ of μ , the Hausdorff dimension of any Ω -Kakeya set X is at least $n - \beta p$.

In regards of this Proposition, we will call the following Conjecture the Ω -Kakeya maximal conjecture.

Conjecture 4 (Ω -Kakeya maximal conjecture). *Fix any probability μ defined on \mathbb{S}^{n-1} satisfying for some $d \in [0, n - 1]$*

$$[\mu]_d := \sup_{e \in \mathbb{S}^{n-1}, r > 0} \mu(B_{e,r}) r^{-d} \leq 1.$$

Then for any $\epsilon > 0$ we have

$$\|K_\delta\|_{\mu,n} \lesssim_{n,d,\epsilon} \delta^{\frac{(d+1)}{n} - (1+\epsilon)}.$$

Using Frostman's Lemma, one can easily checked that the Ω -Kakeya maximal conjecture implies the Ω -Kakeya conjecture. One of our main result is the following.

Theorem 2. *If the Kakeya maximal conjecture is true then the Ω -Kakeya maximal conjecture.*

In particular, since in the plane \mathbb{R}^2 we do have $\|K_\delta\|_{\sigma,2} \lesssim_\epsilon \delta^{-\epsilon}$ for any $\epsilon > 0$, this gives another proof of the Ω -Kakeya conjecture in the plane ; recall that this Theorem has also been established by Mitsis in [5].

Theorem 3. *For any Borel set Ω in \mathbb{S}^1 , if X is a Ω -Kakeya set then*

$$d_X \geq d_\Omega + 1.$$

At this point, it is interesting to note Theorems 1 and Theorem 2 *cannot provide partial result* to the Ω -Kakeya conjecture. For example, say we can prove that if X is a Kakeya set then we have

$$d_X \geq \frac{3}{4}(n - 1) + 1.$$

In this situation, *we cannot use Theorems 1 and 2* - neither their methods of proof - to show that for any Ω -Kakeya set Y , we have

$$d_Y \geq \frac{3}{4}d_\Omega + 1.$$

Hence, in order to obtain further partial result on the Ω -Kakeya conjecture, we are going to employ Bourgain's arithmetic argument in order to prove the following Theorem.

Theorem 4. *For any Borel set Ω in \mathbb{S}^{n-1} and any Ω -Kakeya set X in \mathbb{R}^n we have*

$$d_X \geq \frac{6}{11}d_\Omega + 1.$$

In [7], Venieri proved that if Ω is d -Alfhors regular then a Ω -Kakeya set X has Hausdorff dimension greater than $\frac{d+2}{2} + \frac{1}{2}$. Theorem 4 strengthen this result since it gives better estimate for large n and also since we do not make assumption concerning the set of direction Ω .

4 Proof of Proposition 1

We let μ be an arbitrary probability measure on \mathbb{S}^{n-1} and suppose we have $1 < p < \infty$ and $\beta > 0$ such that $n - \beta p > 0$. We also suppose that we have

$$\|K_\delta\|_{\mu,p} \lesssim_{n,p,\beta} \delta^{-\beta}.$$

We fix then an arbitrary Borel set of directions Ω which contains S_μ and we let X included in \mathbb{R}^n be a Ω -Kakeya set ; we are going to prove that we have

$$d_X \geq n - \beta p.$$

Fix an arbitrary $\alpha \in (0, n - \beta p)$. Consider a covering of X by balls $B_i = B(x_i, r_i)$ such that $r_i < 1$ for any $i \in \mathcal{I}$. We will show that we have

$$\sum_{i \in \mathcal{I}} r_i^\alpha \gtrsim_\alpha 1$$

which gives $d_X \geq \alpha$. For $e \in \Omega$, let $T_e \subset X$ be a unit segment oriented along the direction e ; for $k \geq 1$ we order the balls B_i by their radii defining

$$\mathcal{I}_k = \left\{ i \in \mathcal{I} : r_i \simeq \frac{1}{2^k} \right\}.$$

We also define

$$\Omega_k = \left\{ e \in \Omega : \left| T_e \cap \bigcup_{i \in \mathcal{I}_k} B_i \right|_1 \geq \frac{1}{2k^2} \right\}.$$

It is not difficult to show that we have $\Omega = \bigcup_{k \geq 1} \Omega_k$. We are going to fatten a little bit every segment T_e in order to deal with tubes. We define for $k \geq 1$ the set

$$Y'_k = \bigcup_{i \in \mathcal{I}_k} B(x_i, 2r_i).$$

For $e \in \Omega$, by simple geometry we have the following inequality

$$|T_{e,2^{-k}} \cap Y'_k| \gtrsim \frac{1}{k^2} |T_{e,2^{-k}}|.$$

Hence for any $e \in \Omega_k$ we have $K_{2^{-k}} \mathbb{1}_{Y'_k}(e) \gtrsim \frac{1}{k^2}$. Using our hypothesis on K_δ , we obtain

$$\mu(\Omega_k) \lesssim \mu \left(\left\{ K_{2^{-k}} \mathbb{1}_{Y'_k} \geq \frac{1}{2k^2} \right\} \right) \lesssim_{n,p,\beta} k^{2p} 2^{k\beta p} |Y'_k|.$$

Since we have $|Y'_k| \lesssim_n 2^{-kn} \#\mathcal{I}_k$ it follows that $\mu(\Omega_k) \leq k^{2p} 2^{-k(n-\beta p)} \#\mathcal{I}_k$. We have selected $\alpha < n - \beta p$ and so we have by polynomial comparison $k^{2p} 2^{-k(n-\beta p)} \lesssim_\alpha 2^{-k\alpha}$. Hence we have

$$\sum_{i \in \mathcal{I}} r_i^\alpha \geq \sum_k 2^{-\alpha k} \#\mathcal{I}_k \gtrsim_\alpha \sum_k \mu(\Omega_k) \geq \mu(\Omega) = 1$$

since Ω contains the support S_μ of μ .

5 Proof of Theorem 2

We are going to prove Theorem 2 proving the following estimate.

Theorem 5. *Fix $1 < p < \infty$ and let μ be a probability on \mathbb{S}^{n-1} satisfying $[\mu]_d \leq 1$ for some $0 \leq d \leq n - 1$. In this case we have for any $\delta > 0$*

$$\|K_\delta\|_{\mu,p} \lesssim_{n,d,p} \delta^{-\frac{n-(d+1)}{p}} \|K_\delta\|_{\sigma,p}.$$

This estimate comes from the fact that a function $K_\delta f$ is almost δ -discrete. Observe that this estimate is not possible in general since the surface measure σ typically does not charge the support S_μ of the measure μ i.e. $\sigma(S_\mu) = 0$. The following Lemma is a manifestation of the idea that we should not define the orientation of an object more precisely than its eccentricity.

Lemma 1. *For any $\delta > 0$ and any directions $e_1, e_2 \in \mathbb{S}^{n-1}$ satisfying $|e_1 - e_2| \leq \delta$ we have*

$$K_\delta f(e_1) \simeq_n K_\delta f(e_2)$$

for any locally integrable function f .

Proof. This comes from the fact that there is a dimensional constant $a_n > 1$ such that if we have two tubes $T_{e_1,\delta}, T_{e_2,\delta}$ with $|e_1 - e_2| < \delta$ then one can find $\vec{t} \in \mathbb{R}^n$ such that

$$\vec{t} + \frac{1}{a_n} T_{e_1,\delta} \subset T_{e_2,\delta} \subset \vec{t} + a_n T_{e_1,\delta}.$$

□

We can then relate the (σ, p) -norm of $K_\delta f$ with a discrete sum over a family $e_{\delta, \mathbb{S}^{n-1}} \subset \mathbb{S}^{n-1}$ which is δ -separated and maximal for this property.

Lemma 2. *For f locally integrable and any family $e_{\delta, \mathbb{S}^{n-1}}$ of \mathbb{S}^{n-1} which is maximal and δ -separated, we have*

$$\|K_\delta f\|_{\sigma,p}^p \simeq_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_\delta f(e)^p \delta^{n-1}.$$

Proof. On one hand we have

$$\int_{\mathbb{S}^{n-1}} K_\delta f(e)^p d\sigma(e) \lesssim_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_\delta f(e)^p \sigma(B(e, \delta)) \simeq_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_\delta f(e)^p \delta^{n-1}.$$

On the other hand we have

$$\int_{\mathbb{S}^{n-1}} K_\delta f(e)^p d\sigma(e) \gtrsim_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_\delta f(e)^p \sigma(B(e, \frac{\delta}{2})) \simeq_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_\delta f(e)^p \delta^{n-1}$$

which concludes. \square

We can now prove Theorem 5.

Proof. Fix $\delta > 0$ and consider a family $(e_k)_{k \leq m} \subset S_\mu$ which is δ -separated and whose cardinal is maximal ; this implies that we have

$$S_\mu \subset \bigcup_{k \leq m} B_{e_k, 2\delta}.$$

For f in $L^p(\mathbb{R}^n)$ we have then

$$\int_{S_\mu} K_\delta f(e)^p d\mu(e) \lesssim_n \sum_{k \leq m} \int_{B_{e_k, 2\delta}} K_\delta f(e_k)^p \mu(B_{e_k, 2\delta}) \lesssim_{n,d} \sum_{k \leq m} K_\delta f(e_k)^p \delta^d$$

using Lemma 1 and the fact that $[\mu]_d \leq 1$. Now we complete the family $(e_k)_{k \leq m}$ into a δ -separated family $e_{\delta, \mathbb{S}^{n-1}}$ which is maximal in \mathbb{S}^{n-1} . We have then

$$\sum_{k \leq m} K_\delta f(e_k)^p \delta^d \leq \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_\delta f(e)^p \delta^d \simeq \delta^{d+1-n} \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_\delta f(e)^p \delta^{n-1} \simeq_n \delta^{d+1-n} \|K_\delta f\|_{\sigma,p}^p$$

using the previous lemma. \square

We can now prove Theorem 2 *i.e.* we can prove that the Kakeya maximal conjecture implies to the Ω -Kakeya maximal conjecture. This simply comes from the fact that the ϵ -loss can be easily transferred thanks to Theorem 5.

Proof. Fix any probability μ defined on \mathbb{S}^{n-1} satisfying for some $d \in [0, n-1]$, $[\mu]_d \leq 1$. Thanks to Theorem 5, if the Kakeya maximal conjecture is true then we have

$$\|K_\delta\|_{\mu,n} \lesssim_{n,d} \delta^{\frac{(d+1)}{n}-1} \|K_\delta\|_{\sigma,n} \lesssim_{n,d,\epsilon} \delta^{\frac{(d+1)}{n}-(1+\epsilon)}$$

i.e. the Ω -Kakeya maximal conjecture is true. \square

6 Proof of Theorem 4

The proof of Theorem 4 follows Bourgain's arithmetic argument for the classic Kakeya problem ; this method relies on the following two results. The first one allow us to give an upper bound on the *difference set* $A - B$.

Theorem 6 (Sum-difference Theorem). *Fix any $\delta > 0$ and suppose that A, B are finite subset of $\delta\mathbb{Z}^n$ such that $\#A, \#B \leq N$. If $G \subset A \times B$ satisfies*

$$\#\{a + b : (a, b) \in G\} \lesssim N$$

then we have $\#\{a - b : (a, b) \in G\} \leq N^{2-\frac{1}{\delta}}$.

The second Theorem needed is due to Heath-Brown [8] : it states that if S is a large subset of $\{0, \dots, M\}$ for M large enough then S contains an arithmetic progression of length 3.

Theorem 7 (Heath-Brown). *There exists an integer M_0 such that if $M > M_0$ is a integer and if S is a subset of $\{0, \dots, M\}$ such that*

$$\#S \geq \frac{M}{\log(M)^c}$$

then S contains a subset of the form $\{m, m+m', m+2m'\} \subset S$. Here $c > 0$ is an absolute constant.

For the sake of clarity, we have decomposed the proof of Theorem 4 in two steps. We will denote by $C(A, \delta)$ the smallest number of balls of radius δ needed to cover the set A .

Decomposition of the Ω -Kakeya set

To begin with, we may suppose that Ω is contained in a small spherical cap ; concretely we suppose that for any $e = (e_1, \dots, e_n) \in \Omega$ we have $e_n > \frac{1}{2}$. We fix then $d < d_\Omega$ arbitrarily close and we use Frostman's Lemma to obtain an probability μ such that $S_\mu \subset \Omega$ and also $[\mu]_d \leq 1$. We consider then a Ω -Kakeya set X and we suppose that X is contained in $[0, 1]^n$. For any $e \in \Omega$ we will denote by T_e a unit segment oriented along e contained in X . Finally we fix $s > d_X$ and we will prove that we have

$$s \geq \frac{6}{11}d + 1.$$

We fix $\epsilon \in (0, 1)$ arbitrarily small and we let $\eta \in (0, 1)$ such that defining $\delta_k = 2^{-2^{\eta k}}$ we have for any $k \geq 1$,

$$\delta_k^{s+\epsilon} \leq \delta_{k+1}^s.$$

Since $s > d_X$, for arbitrary large k_0 , we can cover X by a countable collection of balls $\{B_i\}_{i \in \mathcal{I}}$ and such that for any $i \in \mathcal{I}$, we have $\text{diam}(B_i) < \delta_{k_0}$ and $\sum_{i \in \mathcal{I}} \text{diam}(B_i)^s < 1$. In addition, we take k_0 so large that we have

$$k_0^4 \max(\delta_{k_0}^\eta, \delta_{k_0}^{\eta d}) < 1.$$

We denote by Y the union of the balls $\{B_i\}_{i \in \mathcal{I}}$ *i.e.*

$$Y := \bigcup_{i \in \mathcal{I}} B_i$$

and for $k \geq k_0$ we will denote by $\mathcal{I}_k := \{i \in \mathcal{I} : \delta < \text{diam}(B_i) \leq \delta\}$ and also $Y_k := \bigcup_{i \in \mathcal{I}_k} B_i$. We can control the size of $\#\mathcal{I}_k$.

Claim 1. *We have $\#\mathcal{I}_k \delta_k^{s+\epsilon} < 1$.*

Proof. By definition of \mathcal{I}_k and since we have $\delta_k^{s+\epsilon} \leq \delta_{k+1}^s$ and $\sum_{i \in \mathcal{I}} \text{diam}(B_i)^s < 1$, we obtain

$$\#\mathcal{I}_k \delta_k^{s+\epsilon} \leq \#\mathcal{I}_k \delta_{k+1}^s \leq \sum_{i \in \mathcal{I}_k} \text{diam}(B_i)^s < 1.$$

□

step 1 : refinement to a single scale

We wish to work at a single scale with respect to this covering. Hence we are going to exhibit a $k \geq k_0$ such that there is a specific subset $\Omega_k \subset \Omega$ adapted to the covering $\{B_i\}_{i \in \mathcal{I}_k}$: on one hand Ω_k is large and on the other hand, for any $e \in \Omega_k$, the unit segment $T_e \subset X$ is well covered by the balls $\{B_i\}_{i \in \mathcal{I}_k}$.

Claim 2. *There exists $k \geq k_0$ and $\Omega_k \subset \Omega$ such that for any $e \in \Omega_k$,*

$$|T_e \cap Y_k|_1 \geq \frac{1}{k^2}$$

and also $\mu(\Omega_k) > \frac{1}{k^2}$.

Proof. If this is not the case, then for any $k \geq k_0$ we have

$$\mu \left(\left\{ e \in \Omega : |T_e \cap Y_k|_1 \geq \frac{1}{k^2} \right\} \right) \leq \frac{1}{k^2}.$$

Hence we have

$$\mu \left(\left\{ e \in \Omega : \exists k \geq k_0, |T_e \cap Y_k|_1 \geq \frac{1}{k^2} \right\} \right) \leq \sum_{k \geq k_0} \frac{1}{k^2} < \mu(\Omega)$$

and so there is $e \in \Omega$ such that $|T_e \cap Y_k|_1 < \frac{1}{k^2}$ for any $k \geq k_0$. This is not possible since Y covers X and so T_e in particular. \square

step 2 : slicing \mathbb{R}^n at two scales

We fix such a k and we let $\delta := \delta_k$. Recall that since $k \geq k_0$ and that we can choose k_0 arbitrarily large, the same is true for k *i.e.* the integer k can be chosen arbitrarily large. Also observe that by definition we have

$$k \simeq \log \log(\delta^{-\eta}).$$

Now we fix two integers $N, M \in \mathbb{N}$ such that

$$(N, M) \simeq (\delta^{\eta-1}, \delta^{-\eta}).$$

We are going to slice \mathbb{R}^n at two different scales (δ and δ^η) along the vector $(0, \dots, 1)$. Precisely for $j \leq N$ and $m \leq M$, we define

$$A_{j,m} := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : j\delta + mN\delta \leq x_n \leq (j+1)\delta + mN\delta\}$$

and $A_j := \bigcup_{m \leq M} A_{j,m}$.

Claim 3. *For any $e \in \Omega_k$ and $j \leq N$, we have $|T_e \cap A_j|_1 \simeq M\delta \simeq \frac{1}{N}$.*

Proof. The claim comes from the fact that we have supposed that for any $e = (e_1, \dots, e_n) \in \Omega$ we have $e_n > \frac{1}{2}$. \square

By definition of Ω_k , we also have the following estimate

$$\frac{1}{k^4} \leq \frac{\mu(\Omega_k)}{k^2} \leq \int_{\Omega_k} |Y_k \cap T_e|_1 d\mu(e) = \sum_{j \leq N} \int_{\Omega_k} |Y_k \cap T_e \cap A_j|_1 d\mu(e).$$

We define then the subset $J \subset \{0, \dots, N\}$ as

$$J = \{j \leq N : \int_{\Omega_k} |Y_k \cap T_e \cap A_j|_1 \geq \frac{1}{2Nk^4}\}.$$

The following claim states that this set J is not too small in $\{0, \dots, N\}$.

Claim 4. *We have $\#J \gtrsim \delta^\eta N$*

Proof. The proof comes from a reverse Markov inequality. □

Now for each $j \in J$, we extract a subset $\Omega_{k,j}$ from Ω_k in the same fashion that we have extracted Ω_k from Ω . The proof is the same than for Claim 2.

Claim 5. *For any $j \in J$, there exists $\Omega_{k,j} \subset \Omega_k$ such that for any $e \in \Omega_{k,j}$,*

$$|T_e \cap Y_k \cap A_j|_1 > \frac{|T_e \cap A_j|_1}{4k^4}$$

and also $\mu(\Omega_{k,j}) > \frac{\mu(\Omega_k)}{k^2}$.

step 3 : conclusion

Observe that for $|j - j'| > 2$, the sets $Y_k \cap A_j$ and $Y_k \cap A_{j'}$ are separated by a distance greater than 2δ . Hence, on one hand we have

$$\sum_{j \in J} C(Y_k \cap A_j, \delta) \lesssim C(Y_k, \delta) \lesssim \delta^{-s-\epsilon}.$$

On the other hand, suppose that for any $j \in J$ we have

$$C(Y_k \cap A_j, \delta) \gtrsim \delta^{\frac{6}{11}(2\eta-1)d}.$$

In this case we obtain

$$\sum_{j \in J} C(Y_k \cap A_j, \delta) \gtrsim \#J \times \delta^{\frac{6}{11}(2\eta-1)d} \gtrsim \delta^{\frac{6}{11}(2\eta-1)d + (2\eta-1)d}.$$

Since δ is small enough, we get $\frac{6}{11}(1 - 2\eta)d - 2\eta + 1 \leq s + \epsilon$. Taking ϵ and η arbitrarily small we conclude that

$$s \geq \frac{6}{11}d + 1.$$

Lower bound for $C(Y_k \cap A_j, \delta)$

Hence we are left to prove that we have, for any $j \in J$, the following bound

$$C(Y_k \cap A_j, \delta) \gtrsim \delta^{\frac{6}{11}(2\eta-1)d}.$$

We will start by applying Heath-Brown's Theorem and we will use thereafter the Sum-difference Theorem to obtain a bound on $C(Y_k \cap A_j, \delta)$.

step 1 : application of Heath-Brown's Theorem

For any $j \in J$ and any $e \in \Omega_{k,j}$, we consider the following subset of $\{0, \dots, M\}$

$$K(e, j) := \{m \leq M : Y_k \cap T_e \cap A_{j,m} \neq \emptyset\}.$$

The following claim states that $K(e, j)$ contains a lot of element in $\{0, \dots, M\}$.

Claim 6. *We have $\#K(e, j) \gtrsim \frac{M}{\log \log(M)}$.*

Proof. We have

$$\#K(e, j) \times \delta \gtrsim |Y_k \cap T_e \cap A_j|_1 > \frac{1}{4Nk^4}.$$

Hence

$$\#K(e, j) \gtrsim \frac{M}{k^4} \simeq \frac{M}{\log \log(M)}.$$

□

Since we can take M arbitrary large and that $K(e, j)$ is quite large in $\{0, \dots, M\}$, we are able to exhibit arithmetic progressions of three terms in $K(e, j)$ using the Theorem of Heath-Brown.

Claim 7. *For any $j \in J$ and $e \in \Omega_{k,j}$, the set $K(e, j)$ contains a subset of the form*

$$\{m, m + m', m + 2m'\} \subset K(e, j).$$

Hence for any $j \in J$ and any $e \in \Omega_{k,j}$, there exists $a_e, b_e \in Y_k(n\delta) \cap T_e(n\delta) \cap A_j \cap \delta\mathbb{Z}^n$ such that

$$\frac{a_e + b_e}{2} \in Y_k(n\delta) \cap T_e(n\delta) \cap A_j \cap \delta\mathbb{Z}^n$$

and a_e, b_e belong to different sets $A_{j,m}$. Observe also that the sets $A_{j,m}$ for different indices m are at least distant of $\simeq \delta^\eta$. Thus if δ is small enough we have $|a_e - b_e| \gtrsim \delta^\eta$. Finally, we consider the sets $A = \{a_e : e \in \Omega_{k,j}\}$, $B = \{b_e : e \in \Omega_{k,j}\}$ and

$$G := \{(a_e, b_e) : e \in \Omega_{k,j}\} \subset A \times B.$$

Recall that we have

$$A, B \subset \delta\mathbb{Z}^n.$$

step 2 : upper bound for $\#\{a - b : (a, b) \in G\}$

We are going to give an upper bound and lower bound on $\#\{a - b : (a, b) \in G\}$. Observe that the cardinal of A, B and $\{a + b : (a, b) \in G\}$ is controlled by the covering number $C(Y_k \cap A_j, \delta)$. Hence a direct application of the Sum-difference Theorem yields an upper bound on $\#\{a - b : (a, b) \in G\}$

Claim 8. *We have $\#\{a - b : (a, b) \in G\} \lesssim C(Y_k \cap A_j, \delta)^{\frac{11}{6}}$.*

step 3 : lower bound for $\#\{a - b : (a, b) \in G\}$

Finally we are also able to provide a lower bound on $\#\{a - b : (a, b) \in G\}$ using the measure μ .

Claim 9. *We have $\delta^{(2\eta-1)d} \lesssim \#\{a - b : (a, b) \in G\}$.*

Proof. Since a_e and b_e are in the $n\delta$ -neighbourhood of T_e and $|a_e - b_e| > \frac{\delta^\eta}{2}$ for $e \in \Omega_{k,j}$, it follows that balls roughly of radius $\delta^{1-\eta}$ centred at the unit vectors $\frac{a_e - b_e}{|a_e - b_e|}$ (for $e \in \Omega_{k,j}$) cover $\Omega_{k,j}$. As $\mu(\Omega_{k,j}) > \frac{1}{4k^4}$, this implies

$$\#(A - B) \gtrsim \frac{\delta^{(\eta-1)d}}{k^4} \gtrsim \delta^{(2\eta-1)d}.$$

□

Hence for any $j \in J$, we have

$$C(Y_k \cap A_j, \delta) \gtrsim \delta^{\frac{6}{11}(2\eta-1)d}.$$

This concludes the proof of Theorem 4.

References

- [1] R.O. Davies, Some remarks on the Kakeya problem, Proc. Cambridge Phil. Soc. 69 (1971), 417–421
- [2] T. Keleti, A Mathé, *Equivalences between different forms of the Kakeya conjecture and duality of Hausdorff and packing dimensions for additive complements.* arXiv preprint arXiv:2203.15731, 2022.
- [3] N. Katz, T. Tao. Recent progress on the Kakeya conjecture. arXiv preprint math/0010069, 2000.
- [4] P. Mattila, Fourier Analysis and Hausdorff Dimension (Cambridge Studies in Advanced Mathematics).
- [5] T. Mitsis, Norm estimates for the Kakeya maximal function with respect to general measures. Real Anal. Exchange, 27(2):563–572, 2001/02.
- [6] K. Nets Hawk, I. Laba, and T. Tao. “An Improved Bound on the Minkowski Dimension of Besicovitch Sets in \mathbb{R}^3 .” Annals of Mathematics 152, no. 2 (2000): 383–446.
- [7] L. Venieri ; Dimension estimates for Kakeya sets defined in an axiomatic setting, Annales Academiae Scientiarum Fennicae Mathematica Dissertationes (2017) 161:1-73
- [8] D. R. Heath-brown ; Integer sets containing no arithmetic progressions, J. London Math. Soc,1987