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# Towards weakly hyperbolic models for inaccessible pore volume in polymer flooding

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## Abstract

In the simulations of enhanced oil recovery by polymer flooding, it is widely acknowledged that the inaccessible pore volume (IPV) must be properly accounted for. The effect of IPV is to make the interstitial velocity of polymer molecules higher than that of water molecules. This acceleration is traditionally modeled by a velocity enhancement factor. The difficulty, however, lies in finding an adequate closure law for this factor that preserves well-posedness of the overall flow model. Previous works by Bartelds et al. (*Transp. Porous Med.* **26**, 75–88, 1997) and Hilden et al. (*Transp. Porous Med.* **114**, 65–86, 2016) have demonstrated that ill-posedness occurs for the popular constant enhancement factor and proposed some alternative physically motivated IPV laws that partially addressed the question of stability.

This paper is aimed at bringing new mathematical insights into the design of IPV laws that ensure weak hyperbolicity for a Buckley-Leverett type two-phase polymer flow model. To begin with, we show that by switching to Lagrangian coordinates, it is possible to derive several practical and meaningful sufficient conditions for local weak hyperbolicity. Next, special solutions to some of these sufficient conditions are shown to coincide with well-known IPV laws in the literature and then investigated more thoroughly. Finally, by patching together these piecewise sufficient conditions, we are in a position to develop original IPV laws that guarantee global weak hyperbolicity for the flow model, at least under mild restrictions.

## Keywords

polymer flooding, inaccessible pore volume, adsorption, system of conservation laws,  
weak hyperbolicity

## 1 Introduction

### 1.1 Context and objectives

Polymer flooding is an enhanced oil recovery (EOR) process by which a mixture of water and polymer is injected into an oil field [13, 31] in order to increase production. The viscosity of polymer reduces the mobility of the water, thus avoiding hydrodynamic instabilities of water-oil fronts [25, 28] and enabling oil to be better pushed, especially with a very viscous oil, which results in a higher extraction rate. Numerous experiments have shown that in such a mixture, the polymer molecules tend to travel faster than the water molecules. This effect, called *velocity enhancement* is essentially due to the *inaccessible pore volume* (IPV): in the porous medium, big

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polymer molecules cannot penetrate all the pores, unlike small water molecules. Having less volume to flood, the polymer molecules flow at an increased velocity [7, 20].

In most physical models, this acceleration effect is represented by an enhancement or IPV factor that measures the ratio between the polymer velocity and the water velocity. This factor  $\gamma \geq 1$  is to be expressed as an algebraic function of other state variables, such as the water saturation  $s$  and the polymer concentration  $c$  for a two-phase flow. The simplest instance of such a closure law  $\gamma(s, c)$  is a constant IPV factor  $\gamma \equiv 1 + \eta$ ,  $\eta \geq 0$ , which is commonly used in industrial codes. Unfortunately, it was shown [4, 15] that when combined with a Buckley-Leverett type one-dimensional two-phase polymer flow model [5, 26]

$$\partial_t(s) + \partial_x(f) = 0, \tag{1.1a}$$

$$\partial_t(sc + a) + \partial_x(fc\gamma) = 0, \tag{1.1b}$$

where the fractional flux  $f(s, c)$  and the adsorption  $a(c)$  are given functions, a constant IPV factor  $\gamma > 1$  yields an ill-posed system: there exist states  $(s, c)$  at which the characteristic speeds of (1.1) are conjugate complex numbers. The loss of weak hyperbolicity for system (1.1) at these states is a major impediment to the stability of linear perturbations [12, 22], which in turn prevents numerical simulations to be carried out without a blow-up in the polymer concentration at the water front [15].

Bartelds et al. [4] introduced the saturation-dependent *percolation* law  $\gamma(s) = s/(s - s_\bullet)$  in order to recover real characteristic speeds for  $s > s_\bullet$ , where  $s_\bullet$  is the inaccessible pore volume saturation. The shortcoming of the percolation law is that it requires  $s_\bullet < s_b$ , where  $s_b$  is the irreducible water saturation. For  $s_\bullet > s_b$ , Hilden et al. [15] advocated two models called *uniform polymer diffusion* and *non-uniform polymer diffusion* based on a less stringent notion of “acceptable” shocks instead of weak hyperbolicity. As shown by Hilden et al. [15] themselves and further analyzed by Romate and Guarnerio [27], these polymer diffusion models may violate hyperbolicity and end up giving unphysical polymer concentration values. A detailed review of the stability issues associated with various IPV factors can be found in [14, 33].

On the grounds of this state of the art, we believe that weak hyperbolicity of (1.1) must be the primary concern in the design of IPV laws. While we do not claim to have the definitive answer to the problem, we wish to put forward some new results and ideas. Unlike [4, 15], we will not resort to microscopic models as a starting point. Our approach relies on a mathematical study of the macroscopic model (1.1), although physical interpretation is not neglected. Note that we are focused only on weak hyperbolicity, i.e., the existence of two real eigenvalues of the Jacobian matrix of (1.1). We do not ask anything about the eigenvectors of this matrix and do not seek any stronger kind of hyperbolicity. This leaves open the possibility for resonance and its associated pathologies, but in our opinion, to successfully rule out ellipticity is already a big step forward.

## 1.2 Main results and outline

We first provide a precise mathematical statement of the problem in §2, where the most basic concepts are recalled. Emphasis is laid on the properties of the fractional flux, in relation with its specific dependence on the phase mobilities and the mobility reduction. Classical and more recent IPV laws are reviewed and discussed in from the standpoint of weak hyperbolicity.

Section §3 is devoted to the derivation of several practical sufficient conditions for weak hyperbolicity at a given state, which we call “local” weak hyperbolicity. The key ingredient to this endeavor is the Lagrangian framework, which greatly simplifies calculations. Sufficient conditions can then be obtained by various splittings of the discriminant of the characteristic polynomial into two parts, one of which is a perfect square. Aside from this usual path already followed by [4, 15], we expose another methodology inspired from traffic flow modeling, which amounts to specifying a velocity value that must lie in-between the two real eigenvalues. This viewpoint makes more physical sense and gives rise to even more sufficient conditions. Besides, we point out a crucial

connection between the model without adsorption ( $a \equiv 0$ ) and the model with adsorption ( $a \geq 0$ ). Sufficient conditions for the latter can then be inferred from those for the former by means of a change of variables.

A sufficient condition for weak hyperbolicity is a partial differential inequality on  $\gamma$  or a related function. Sometimes, an explicit solution can be determined. In §4, we study a few such special solutions and show that classical IPV laws, including percolation, can be recovered in this way. For the percolation law, we exhibit a change of variable that transforms the model without adsorption into that without IPV. This testifies to the fact that the mathematical behavior of the model without adsorption, equipped with the percolation law, is similar to that of a Keyfitz-Kranzer system [19], the Riemann problem of which was solved by Isaacson and Temple [17]. For the constant IPV law, we revisit the existence of elliptic states with a rigorous characterization of the frontier between the weak hyperbolic region and the elliptic one for the model without adsorption.

From the observation that a single sufficient condition can secure weak hyperbolicity only on a subdomain of  $(s, c) \in [0, 1] \times \mathbb{R}_+$ , it is natural to combine different sufficient conditions on different subdomains in the hope of building up an IPV law that is weakly hyperbolic globally, over the whole domain  $(s, c) \in [0, 1] \times \mathbb{R}_+$ . This is the program that we undertake in §5. The main obstacle to the feasibility of this construction is that, due to the matching conditions, the maximal amplitude of the IPV factor may be more or less limited. Nevertheless, there are ways to improve this construction.

## 2 Setting of the problem

We start by formulating the question of weak hyperbolicity for a prototype model in §2.1. Analytical results in the trivial no-IPV case ( $\gamma \equiv 1$ ) are recalled in §2.2, together with some notions that will be helpful in the sequel. The definition and the limit of existing non-trivial IPV laws are highlighted in §2.3.

### 2.1 Flow model with IPV in polymer flooding

We consider a two-phase (water-oil) immiscible incompressible polymer flow in a one-dimensional porous media. The derivation of its equations from a full three-dimensional model, detailed in [4, 9, 26], results in the system of conservation laws

$$\partial_t(s) + \partial_x(f) = 0, \quad (2.1a)$$

$$\partial_t(sc + a) + \partial_x(fc\gamma) = 0, \quad (2.1b)$$

where the unknowns  $(s, c)$  depend on  $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ . Here,  $s \in [0, 1]$  represents the water saturation (so that  $1 - s$  is the oil saturation) and  $c \in \mathbb{R}_+$  the polymer concentration in water. In (2.1), the adsorption  $a(\cdot)$  is a  $\mathcal{C}^1$ -function of  $c \in \mathbb{R}_+$  such that

$$a(0) = 0, \quad a' \geq 0, \quad (2.2)$$

hence  $a \geq 0$ .

The fractional flux  $f(\cdot, \cdot)$  is a  $\mathcal{C}^1$ -function of  $(s, c) \in [0, 1] \times \mathbb{R}_+ =: \Omega$  given by

$$f(s, c) = \frac{\Lambda_w(s)/R(c)}{\Lambda_o(1-s) + \Lambda_w(s)/R(c)}, \quad (2.3)$$

where the mobility reduction  $R(\cdot)$  is a  $\mathcal{C}^1$ -function of  $c \in \mathbb{R}_+$  subject to

$$R(0) = 1, \quad R' > 0, \quad \lim_{c \rightarrow +\infty} R(c) = +\infty. \quad (2.4)$$

Hence,  $R > 1$  for  $c > 0$ . These are the mathematical assumptions to be made on  $R$ . From the physical point of view, the polymer concentration must naturally be bounded above by a

maximal value representing the limit of dissolution in water. Likewise, there is maximal limit in the mobility reduction. However, from a mathematical point of view, it is necessary to allow the polymer concentration to go to infinity in order to cope with a possible model singularity.

The quantities  $\Lambda_w(s)$  and  $\Lambda_o(1-s)$  are respectively the water and oil mobilities<sup>1</sup>. Their definitions involve two parameters  $0 \leq s_b < s_\# \leq 1$  characterizing the porous medium:  $s_b \in [0, 1]$  is the irreducible water saturation, while  $s_{or} := 1 - s_\# \in [0, 1]$  is the residual oil saturation<sup>2</sup>. It is then assumed that  $\Lambda_w(\cdot)$  and  $\Lambda_o(\cdot)$  are  $\mathcal{C}^1$ -functions defined over  $[0, 1]$  such that

$$\Lambda_w = 0 \text{ on } [0, s_b], \quad \Lambda_w > 0 \text{ on } (s_b, 1], \quad \Lambda'_w \geq 0, \quad (2.5a)$$

$$\Lambda_o = 0 \text{ on } [0, 1 - s_\#], \quad \Lambda_o > 0 \text{ on } (1 - s_\#, 1] \quad \Lambda'_o \geq 0. \quad (2.5b)$$

Thus,  $\Lambda_w(s)$  vanishes for  $s \in [0, s_b]$  and increases with respect to  $s$ , while  $\Lambda_o(1-s)$  vanishes for  $s \in [s_\#, 1]$  and decreases with respect to  $s$ . A prominent example of mobility functions stems from the Brooks-Corey law [6] for relative permeabilities and reads

$$\Lambda_w(s) = \Lambda_w^m \left( \frac{s - s_b}{s_\# - s_b} \right)_+^{\alpha_w}, \quad \Lambda_o(1-s) = \Lambda_o^m \left( \frac{s_\# - s}{s_\# - s_b} \right)_+^{\alpha_o}, \quad (2.6)$$

where  $r_+ = \max(r, 0)$  stands for the positive part of  $r \in \mathbb{R}$  and  $(\Lambda_w^m, \Lambda_o^m) \in (\mathbb{R}_+^*)^2$  are two positive constants. The use of the positive part function enables us to extend the definition of the mobilities over  $s \in [0, 1]$  without any impact on the fractional flux. Note that to ascertain differentiability of  $\Lambda_w(\cdot)$  at  $s_b$  when  $s_b > 0$  and of  $\Lambda_o(\cdot)$  at  $s_\#$  when  $s_\# < 1$ , we must require

$$\alpha_w > 1, \quad \alpha_o > 1. \quad (2.7)$$

From now on, unless otherwise specified, it is taken for granted that

$$0 < s_b < s_\# < 1. \quad (2.8)$$

Thanks to the assumptions made on  $R$ ,  $\Lambda_w$  and  $\Lambda_o$ , the fractional flux  $f$  satisfies

$$f = 0 \text{ on } [0, s_b] \times \mathbb{R}_+, \quad f = 1 \text{ on } [s_\#, 1] \times \mathbb{R}_+. \quad (2.9)$$

Furthermore, on  $(s_b, s_\#) \times \mathbb{R}_+$ ,

$$0 < f < 1, \quad f_s > 0, \quad f_c < 0, \quad (2.10)$$

where  $f_s$  is the partial derivative with respect to  $s$  at fixed  $c$  and  $f_c$  is the partial derivative with respect to  $c$  at fixed  $s$ .

From the most general viewpoint, the velocity enhancement or IPV factor  $\gamma$  is continuous and piecewise differentiable function of  $(s, c) \in \Omega$ . The fact that its derivatives  $\gamma_s$  and  $\gamma_c$  are authorized to have jumps will make it easier for us define suitable IPV laws over the whole domain  $\Omega$ . However, it is capital to demand that the product  $fc\gamma$  be a  $\mathcal{C}^1$  function of  $(s, c) \in \Omega$ . From physical considerations, it is assumed that

$$\gamma \geq 1, \quad \gamma_s \leq 0. \quad (2.11)$$

The last inequality, which asserts that  $\gamma$  is a non-increasing function of  $s$  at fixed  $c$ , is a reasonable behavior.

Once the closure laws  $a$ ,  $R$ ,  $\Lambda_w$ ,  $\Lambda_o$ ,  $\gamma$  have been selected according to conditions (2.2), (2.4), (2.5), (2.11) and their regularity assumptions, system (2.1) can be cast under the abstract form

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0, \quad (2.12)$$

<sup>1</sup>The mobility of each phase is defined as the ratio between the relative permeability  $\kappa_r(\cdot)$  and the viscosity  $\mu$ . In other words,  $\Lambda_w(s) = \kappa_{r,w}(s)/\mu_w$  and  $\Lambda_o(s) = \kappa_{r,o}(s)/\mu_o$ . But this will not be needed for our study.

<sup>2</sup>In the literature, the notations  $s_{wi}$  and  $s_{or}$  are commonly used in place of  $s_b$  and  $1 - s_\#$ .

where  $\mathbf{w} = (s, sc + a)$  is the vector of conservative variables and  $\mathbf{f} = (f, fc\gamma)$  is the flux vector. The latter being a  $\mathcal{C}^1$  function of the former, it makes sense to speak of the Jacobian matrix  $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$  at each state  $\mathbf{w}$ . System (2.1) is said to be

- *hyperbolic* at state  $\mathbf{w}$  if the  $2 \times 2$  matrix  $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$  possesses two real eigenvalues and if it is real diagonalizable;
- *weakly hyperbolic* at state  $\mathbf{w}$  if the  $2 \times 2$  matrix  $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$  possesses two real eigenvalues; in case of a double eigenvalue, the eigenspace is not required to be of dimension two.

Hyperbolicity is a desirable feature of the model, insofar as it guarantees stability of linear perturbation around a steady state [12, 22]. The scope of this paper is, however, more modest. We wish to clarify conditions on the IPV law  $\gamma$  so that system (2.1) is weakly hyperbolic at a given state (local) and to attempt constructing IPV laws  $\gamma$  so that system (2.1) is weakly hyperbolic for all admissible states (global). In other words, we accept defective eigenvalues and the associated *resonance* phenomenon, which is known to appear in many celebrated models: pressureless gas [23], nozzle flow [1], Baer-Nunziato [3]. The situation we actually want to forbid is *ellipticity*, that is, the existence of states at which  $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$  possesses two conjugate complex eigenvalues with nonzero imaginary parts, since this gives rise to unphysical ill-posedness.

## 2.2 Preliminary notions and results

The unenhanced or no-IPV case, which corresponds to a constant  $\gamma = 1$ , has been extensively analyzed for  $[s_b, s_\#] = [0, 1]$ . In particular, the Riemann problem associated with

$$\partial_t(s) + \partial_x(f) = 0, \quad (2.13a)$$

$$\partial_t(sc + a) + \partial_x(fc) = 0, \quad (2.13b)$$

was solved by Johansen and Winther [18] for  $a' > 0$ ,  $a'' < 0$ , and by Matos et al. [24] for  $a' > 0$ ,  $a'' > 0$ . A straightforward calculation shows that the characteristic speeds of system (2.13) at any state  $\mathbf{w} = (s, sc + a)$  are always real and equal to

$$f_s(s, c) \quad \text{and} \quad \frac{f(s, c)}{s + a'(c)}. \quad (2.14)$$

This testifies to the global weak hyperbolicity of (2.13). Hyperbolicity is in fact *strict* (two distinct real eigenvalues) except on the set

$$f_s = f/(s + a'). \quad (2.15)$$

The no-adsorption subcase ( $a = 0$ ) of the no-IPV case has also received a lot of attention. Indeed, the system

$$\partial_t(s) + \partial_x(f) = 0, \quad (2.16a)$$

$$\partial_t(sc) + \partial_x(fc) = 0, \quad (2.16b)$$

belongs to a family of systems introduced by Keyfitz and Kranzer [19] in elasticity. System (2.16) is weakly hyperbolic with real characteristic speeds

$$f_s(s, c) \quad \text{and} \quad \frac{f(s, c)}{s}. \quad (2.17)$$

Resonance occurs when  $f_s = f/s$ . The Riemann problem associated with system (2.16) was solved by Isaacson and Temple [17] for  $[s_b, s_\#] = [0, 1]$ . The quantity

$$u = \frac{f}{s} \quad (2.18)$$

referred to as the water *interstitial* velocity, is of paramount importance in the general case. It can be verified that

$$u = 0 \text{ on } [0, s_b] \times \mathbb{R}_+, \quad u = 1/s \text{ on } [s_\sharp, 1] \times \mathbb{R}_+. \quad (2.19)$$

Note that  $u$  is well-defined at  $s = 0$  by continuity of  $u(s, c) = 0$  for  $s \in (s_b, s_\sharp)$ . Furthermore,

$$0 \leq u \leq 1/s, \quad u_c \leq 0 \quad (2.20)$$

with strict inequalities on  $(s_b, s_\sharp) \times \mathbb{R}_+$ .

The set  $f_s = f/s$ , which can then be equivalently characterized as  $u_s = 0$ , will turn out to play a very special role in the general case, especially in §5. This is why it is worth having a closer look at it. The following result gives a precise description of this set and does not seem to be well-known, at least to our knowledge.

**Proposition 2.1.** *Assume that the function*

$$\Upsilon(s) = -\frac{s\Lambda_o(1-s)}{\Lambda_w(s)} \quad (2.21)$$

*is increasing on  $(s_b, s_\sharp)$ , while its derivative  $\Upsilon'$  is decreasing on the same interval, with*

$$\lim_{s \downarrow s_b} \Upsilon'(s) = +\infty, \quad \lim_{s \uparrow s_\sharp} \Upsilon'(s) = 0. \quad (2.22)$$

*Then, the set of states  $(s, c) \in \Omega$  for which  $f_s = f/s$  consists of the vertical strip  $[0, s_b] \times \mathbb{R}_+$  and the transition curve*

$$s = (\Upsilon')^{-1}(1/R(c)) =: s_*(c). \quad (2.23)$$

*The latter lies entirely in the region  $(s_b, s_\sharp) \times \mathbb{R}_+$ , is increasing with respect to  $s \geq (\Upsilon')^{-1}(1) = s_*(0)$  and admits the vertical asymptote  $s = s_\sharp$  when  $c \rightarrow +\infty$ . Last but not least,*

$$u_s > 0 \text{ for } s_b < s < s_*(c), \quad u_s < 0 \text{ for } s_*(c) < s < s_\sharp. \quad (2.24)$$

*Proof.* It is obvious that for  $s \in [0, s_b]$ , since  $f = 0$  and  $u = 0$ , we have  $u_s = 0$  on  $[0, s_b]$ . Therefore, every state the strip  $[0, s_b] \times \mathbb{R}_+$  belongs to the sought-for set. On the other hand, for  $s \in [s_\sharp, 1]$ , since  $f = 1$  and  $u = 1/s$ , we have  $u_s = -1/s^2 \neq 0$ . Thus, no state in the strip  $[s_\sharp, 1] \times \mathbb{R}_+$  can be in the set.

We now focus on  $s \in (s_b, s_\sharp)$ . From definition (2.3), it follows that

$$\frac{sf_s}{f}(s, c) = \frac{s[\Lambda'_w(s)\Lambda_o(1-s) + \Lambda'_o(1-s)\Lambda_w(s)]}{\Lambda_w(s)\Lambda_o(1-s) + \Lambda_w^2(s)/R(c)}. \quad (2.25)$$

Solving the equation  $sf_s/f = 1$  for  $1/R(c)$ , we end up with

$$\begin{aligned} \frac{1}{R(c)} &= \frac{s[\Lambda'_w(s)\Lambda_o(1-s) + \Lambda'_o(1-s)\Lambda_w(s)] - \Lambda_w(s)\Lambda_o(1-s)}{\Lambda_w^2(s)} \\ &= \Upsilon'(s), \end{aligned} \quad (2.26)$$

where  $\Upsilon$  is the function defined by (2.21). Due to the assumptions,  $\Upsilon'$  is a decreasing function of  $s$ . Therefore, the reciprocal function  $(\Upsilon')^{-1}$  is well-defined as a decreasing function from  $(0, +\infty)$  to  $(s_b, s_\sharp)$ . Applying  $(\Upsilon')^{-1}$  to both sides of (2.26) yields (2.23). Since  $R$  is an increasing function,  $1/R$  is a decreasing function and its composition with the decreasing function  $(\Upsilon')^{-1}$  results in an increasing function  $s_* = (\Upsilon')^{-1} \circ (1/R)$ . When  $c \rightarrow +\infty$ , the last assumption of (2.4) implies  $1/R(c) \rightarrow 0$ , thus  $s_*(c) \rightarrow s_\sharp$ . Statement (2.24) is left to the readers.  $\square$

The behavior of  $s_*(\cdot)$  is depicted in Figure 1. In the literature, the existence and uniqueness of  $s_*(c)$  have to be postulated over a finite interval  $c \in [0, c^m]$ , and nothing can be said about its increasing or decreasing behavior. Here, Proposition 2.1 connects these properties to the function  $\Upsilon$ . It turns out that the requirements made of  $\Upsilon$  are met by the Brooks-Corey laws.

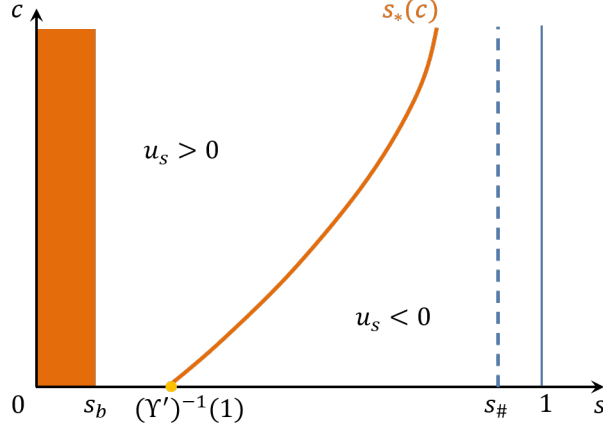


Figure 1: States for which  $u_s = 0$  (in orange) the  $(s, c)$ -plane.

**Proposition 2.2.** *For the Brooks-Corey laws (2.6), the conditions on  $\Upsilon$  hold true if*

$$\alpha_o \geq \alpha_w > 1. \quad (2.27)$$

*Proof.* Plugging (2.6) into (2.21), we get the explicit expression

$$\Upsilon(s) = -(s_{\#} - s_b)^{\alpha_w - \alpha_o} \frac{\Lambda_o^m}{\Lambda_w^m} \frac{s(s_{\#} - s)^{\alpha_o}}{(s - s_b)^{\alpha_w}} \quad (2.28)$$

in the Brooks-Corey case. It is convenient to put

$$\sigma = \frac{s - s_b}{s_{\#} - s_b} \in [0, 1], \quad S = s_{\#} - s_b > 0,$$

so that  $s = S\sigma + s_b$  and

$$\Upsilon(s) = -\frac{\Lambda_o^m}{\Lambda_w^m} (S\sigma + s_b) \frac{(1 - \sigma)^{\alpha_o}}{\sigma^{\alpha_w}} = -\frac{\Lambda_o^m}{\Lambda_w^m} (1 - \sigma)^{\alpha_o} (S\sigma^{-\alpha_w + 1} + s_b\sigma^{-\alpha_w}). \quad (2.29)$$

A brute force calculation shows that

$$\Upsilon'(s) = \frac{\Lambda_o^m}{\Lambda_w^m} S (1 - \sigma)^{\alpha_o - 1} \sigma^{-\alpha_w - 1} \left\{ s_b \alpha_w + [s_b(\alpha_o - \alpha_w) + S(\alpha_w - 1)]\sigma + S(\alpha_o - \alpha_w + 1)\sigma^2 \right\}. \quad (2.30)$$

Plainly,  $\Upsilon'(s) > 0$  for all  $s \in (s_b, s_{\#})$  if  $\alpha_o \geq \alpha_w \geq 1$ . Moreover, it is easily seen that

$$\lim_{s \downarrow s_b} \Upsilon'(s) = +\infty, \quad \lim_{s \uparrow s_{\#}} \Upsilon'(s) = 0.$$

To tell more about the monotonicity of  $\Upsilon'$ , a further differentiation leads to

$$\Upsilon''(s) = -\frac{\Lambda_o^m}{\Lambda_w^m} S^2 (1 - \sigma)^{\alpha_o - 2} \sigma^{-\alpha_w - 2} \left\{ (\alpha_o - 1)\sigma P(\sigma) + (1 - \sigma)Q(\sigma) \right\} \quad (2.31)$$

with

$$P(\sigma) = s_b \alpha_w + [s_b(\alpha_o - \alpha_w) + S(\alpha_w - 1)]\sigma + S(\alpha_o - \alpha_w + 1)\sigma^2, \quad (2.32a)$$

$$Q(\sigma) = s_b \alpha_w (\alpha_w + 1) + \alpha_w [s_b(\alpha_o - \alpha_w) + S(\alpha_w - 1)]\sigma + S(\alpha_w - 1)(\alpha_o - \alpha_w + 1)\sigma^2. \quad (2.32b)$$

If  $\alpha_o \geq \alpha_w \geq 1$ , then  $P > 0$  and  $Q > 0$  for all  $s \in (s_b, s_{\#})$ . Consequently,  $\Upsilon'' < 0$ .  $\square$



### 2.3 Standard IPV laws and their limits

The mathematically pleasant IPV law  $\gamma = 1$  is not in agreement with physical observations, because this means that the large size of the polymer molecules does not influence their path in a porous medium. For realistic reservoir simulations, an IPV factor  $\gamma \geq 1$  is mandatory. Below we enumerate the IPV laws available in the literature.

1. By far, the most frequently used IPV law is the constant IPV

$$\gamma(s, c) = 1 + \eta, \quad \eta > 0, \quad (2.33)$$

due to its simplicity. Many physical arguments have been raised [4, 15] against (2.33). From a purely mathematical standpoint, the best result to date [4, 15] is that when  $\eta > 0$  is small enough, then the states  $(s_*(c), c)$  on the transition curve of the no-IPV law become elliptic. In §4.1, we will shed more light on this question.

2. Bartelds et al. [4] worked out the saturation-dependent percolation law

$$\gamma(s, c) = \frac{s}{s - s_\bullet}, \quad s > s_\bullet, \quad (2.34)$$

in which  $s_\bullet \in (0, s_\#)$  is a new parameter called *IPV saturation*. The advantage of (2.34) is that it ensures weak hyperbolicity for system (2.1). In §4.3, we will show that for the no-adsorption case, system (2.1) equipped with (2.34) is in fact equivalent to the no-adsorption no-IPV model (2.16). This brings a new understanding to the percolation law. Now, the shortcoming of (2.34) is that it holds only for  $s > s_\bullet$ . For the percolation law to be valid and remain bounded on  $(s_b, s_\#)$ , it is necessary that  $s_\bullet < s_b$ . But the latter condition is not always physically observed.

3. The uniform diffusion law

$$\gamma(s, c) = \begin{cases} \frac{s}{\Lambda_w(s)} \frac{\Lambda_w(s) - \Lambda_w(s_\bullet)}{s - s_\bullet} & \text{if } s_\bullet > s_b \text{ and } s > s_\bullet, \\ \frac{s}{s - s_\bullet} & \text{if } s_\bullet \leq s_b \text{ and } s > s_\bullet, \\ 0 & \text{if } s \leq s_\bullet, \end{cases} \quad (2.35)$$

and the non-uniform diffusion law

$$\gamma(s, c) = \begin{cases} \left[ 1 - \frac{s_b}{s_\bullet} \frac{\Lambda_w(s_\bullet)}{\Lambda_w(s)} \right] \frac{s}{s - s_b} & \text{if } s_\bullet > s_b \text{ and } s > s_\bullet, \\ \frac{s}{s - s_\bullet} & \text{if } s_\bullet \leq s_b \text{ and } s > s_\bullet, \\ 1 & \text{if } s \leq s_\bullet, \end{cases} \quad (2.36)$$

proposed later by Hilden et al. [15] in order to tackle the difficult case  $s_\bullet > s_b$ , as well as to extend  $\gamma$  to the whole interval  $s \in [0, 1]$ . In place of weak hyperbolicity, the authors opted for a less demanding but more handy notion of stability for shock fronts. While numerical simulations demonstrate a usually better behavior, the remaining elliptic regions may still cause trouble, as pointed out by the authors themselves and confirmed by Romate and Guarnerio [27]. Besides, note that the condition  $\gamma_s \leq 0$  is not always satisfied.

Our ultimate goal is to devise a physically acceptable IPV law that ensures weakly hyperbolic for (2.1) globally over  $\Omega$ . To achieve this purpose in §5, an imperative prerequisite is to study the sufficient conditions that do so locally, i.e., at a given state. This is the subject of §3.

### 3 Sufficient conditions for local weak hyperbolicity

We first set up in §3.1 a framework which mimics compressible hydrodynamics and where the calculation of characteristic polynomials becomes simpler. Next, we follow in §3.2 the usual technique of splitting the discriminant in order to derive some first sufficient conditions. These conditions are given in §3.3 a new interpretation in terms of control of eigenvalues, which then appears as a more productive technique.

#### 3.1 Transformation to Lagrange coordinates

The velocity  $u$  introduced in (2.18) enables us to reformulate system (2.6) under a form that bears more resemblance to hydrodynamics, namely,

$$\partial_t(s) + \partial_x(su) = 0, \quad (3.1a)$$

$$\partial_t(sc + a) + \partial_x(scu + q) = 0, \quad (3.1b)$$

in which we propose to call

$$q = (\gamma - 1)scu \quad (3.2)$$

the *IPV deviation*. Instead of working with the conservative variables  $(s, sc + a)$ , which is awkward because an inversion of  $sc + a$  is needed to compute  $c$ , it is more judicious to take  $(s, c)$  as primary variables. System (3.1) can then be put under the quasilinear form

$$\begin{bmatrix} 1 & 0 \\ c & s + a' \end{bmatrix} \partial_t \begin{bmatrix} s \\ c \end{bmatrix} + \begin{bmatrix} u + su_s & su_c \\ c(u + su_s) + q_s & s(u + cu_c) + q_c \end{bmatrix} \partial_x \begin{bmatrix} s \\ c \end{bmatrix} = 0. \quad (3.3)$$

This leads us to consider the characteristic polynomial

$$\wp^{\text{Eul}}(\zeta) = \begin{vmatrix} u + su_s - \zeta & su_c \\ c(u + su_s) + q_s - \zeta c & s(u + cu_c) + q_c - \zeta(s + a') \end{vmatrix}, \quad (3.4)$$

where the superscript “Eul” reminds us that we are in the Eulerian coordinates  $(x, t)$ . Multiplying the first row of (3.4) by  $c$  and subtracting the product from the second row, we obtain

$$\wp^{\text{Eul}}(\zeta) = \begin{vmatrix} u + su_s - \zeta & su_c \\ q_s & su + q_c - \zeta(s + a') \end{vmatrix}. \quad (3.5)$$

The Eulerian eigenvalues are the roots of  $\wp^{\text{Eul}}$ . To answer the question of how to make them real, it is advisable to not directly deal with  $\wp^{\text{Eul}}$ .

Equation (3.1a), rearranged as  $\partial_t(s) = \partial_x(-su)$ , can be seen as the Schwarz condition for cross partial derivatives. As a consequence, there exists a function  $m$  of  $(x, t)$  such that  $dm = s dx - su dt$ . In other words,

$$\partial_x m = s, \quad \partial_t m = -su. \quad (3.6)$$

We refer to  $m$  as the Lagrangian abscissa. By a change of referential from  $(x, t)$  to  $(m, t)$ , system (3.1) becomes

$$\partial_t(\tau) - \partial_m(u) = 0, \quad (3.7a)$$

$$\partial_t(c + a\tau) + \partial_m(q - au) = 0, \quad (3.7b)$$

with

$$\tau = \frac{1}{s}. \quad (3.8)$$

The full procedure is detailed in [9, 29, 32]. In compressible hydrodynamics,  $\tau$  is the specific volume, while  $s$  (often denoted by  $\rho$ ) is the density. The utmost interest of Lagrangian coordinates

is to suppress the nonlinearities  $(su, scu)$  associated with the Eulerian representation and to keep only the genuine nonlinearities  $(u, q)$  intrinsic to the problem.

From the quasilinear form

$$\begin{bmatrix} 1 & 0 \\ a & 1 + a'\tau \end{bmatrix} \partial_t \begin{bmatrix} \tau \\ c \end{bmatrix} + \begin{bmatrix} -u_\tau & -u_c \\ q_\tau - au_\tau & q_c - (au)_c \end{bmatrix} \partial_m \begin{bmatrix} \tau \\ c \end{bmatrix} = 0 \quad (3.9)$$

of (3.7), we define the Lagrangian characteristic polynomial

$$\wp^{\text{Lag}}(\lambda) = \begin{vmatrix} -u_\tau - \lambda & -u_c \\ q_\tau - au_\tau - \lambda a & q_c - (au)_c - \lambda(1 + a'\tau) \end{vmatrix} \quad (3.10)$$

Multiplying the first row of (3.10) by  $a$  and subtracting the product from the second row, we obtain

$$\wp^{\text{Lag}}(\lambda) = \begin{vmatrix} -u_\tau - \lambda & -u_c \\ q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix}. \quad (3.11)$$

After expansion and simplification,

$$\wp^{\text{Lag}}(\lambda) = (1 + a'\tau)\lambda^2 + [(1 + a'\tau)u_\tau - (q_c - a'u)]\lambda + q_\tau u_c - (q_c - a'u)u_\tau. \quad (3.12)$$

The roots of  $\wp^{\text{Lag}}$  are related to those of  $\wp^{\text{Eul}}$  in the following fashion.

**Lemma 3.1.** *For all  $\lambda \in \mathbb{C}$ , we have*

$$\wp^{\text{Lag}}(\lambda) = s\wp^{\text{Eul}}(u + \tau\lambda). \quad (3.13)$$

*Proof.* Inserting  $\zeta = u + \tau\lambda$  into (3.5) and remembering that  $\tau s = 1$ , we get

$$\wp^{\text{Eul}}(u + \tau\lambda) = \begin{vmatrix} su_s - \tau\lambda & su_c \\ q_s & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix}. \quad (3.14)$$

Using now  $su_s = -\tau u_\tau$  and  $q_s = -\tau^2 q_\tau$ , we successively have

$$\begin{aligned} \wp^{\text{Eul}}(u + \tau\lambda) &= \begin{vmatrix} -\tau u_\tau - \tau\lambda & su_c \\ -\tau^2 q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix} \\ &= (-s) \begin{vmatrix} \tau^2 u_\tau + \tau^2 \lambda & -u_c \\ -\tau^2 q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix} \\ &= (-s)(-\tau^2) \begin{vmatrix} -u_\tau - \lambda & -u_c \\ q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix} \\ &= \tau \wp^{\text{Lag}}(\lambda). \end{aligned}$$

Multiplying both sides by  $s$ , we arrive at (3.13).  $\square$

Since the relation between the Lagrangian eigenvalue  $\lambda$  and the Eulerian eigenvalue  $\zeta = u + \tau\lambda$  involves real numbers  $u$  and  $\tau$ , requiring that  $\wp^{\text{Eul}}$  has real roots is tantamount to requiring that  $\wp^{\text{Lag}}$  has real roots.

## 3.2 Splitting of the discriminant

The most elementary way to guarantee that the quadratic polynomial  $\wp^{\text{Lag}}$  has two real roots is to make sure that its discriminant is non-negative, which leads to a partial differential inequality. The idea is to write the discriminant as the sum of a perfect square and a residual term, the sign of which must be controlled.

**Theorem 3.1.** *Let  $g = (\gamma - 1)sc$  so that  $q = gu$ . If any of the following (non equivalent) conditions is satisfied at state  $(s, c) \in \Omega$ , then system (2.6) is weakly hyperbolic at this state:*

$$[\text{SC0}] \quad (g_c - a')u_\tau - g_\tau u_c \geq 0, \quad (3.15)$$

$$[\text{SC1}] \quad q_\tau \geq 0, \quad (3.16)$$

$$[\text{SC2}] \quad (1 + a'\tau)g_\tau - g(g_c - a') \geq 0. \quad (3.17)$$

*Proof.* By virtue of (3.12), the discriminant of  $\wp^{\text{Lag}}$  reads

$$\Delta = [(1 + a'\tau)u_\tau - (q_c - a'u)]^2 + 4(1 + a'\tau)[(q_c - a'u)u_\tau - q_\tau u_c] \quad (3.18)$$

Then, [SC0] stems from requiring that the second summand is non-negative, that is,

$$(q_c - a'u)u_\tau - q_\tau u_c \geq 0, \quad (3.19)$$

as  $1 + a'\tau > 0$ . Substituting  $gu$  for  $q$ , the above inequality becomes

$$(g_c u - a'u)u_\tau - g_\tau u u_c \geq 0 \quad (3.20)$$

after cancellation of  $g u_c u_\tau$ . Since  $u \geq 0$ , it is clear that (3.15) implies (3.19).

Applying the identity  $(A - B)^2 + 4AB = (A + B)^2$  with  $A = (1 + a'\tau)u_\tau$  and  $B = q_c - a'u$  to (3.18), we come up with

$$\Delta = [(1 + a'\tau)u_\tau + (q_c - a'u)]^2 - 4(1 + a'\tau)q_\tau u_c, \quad (3.21)$$

which is another splitting of the discriminant. Then, [SC1] stems from requesting that the second summand is non-negative, that is,

$$-u_c q_\tau \geq 0. \quad (3.22)$$

Since  $u_c \leq 0$ , it is obvious that (3.16) implies (3.22).

An even more subtle splitting of the discriminant can be worked out by first substituting  $gu$  for  $q$  in (3.21) to have

$$\Delta = [(1 + a'\tau)u_\tau - a'u + g_c u + g u_c]^2 - 4(1 + a'\tau)(g_\tau u + g u_\tau)u_c \quad (3.23)$$

and next by applying the identity  $(A + B)^2 = (A - B)^2 + 4AB$  with  $A = (1 + a'\tau)u_\tau - a'u + g_c u$  and  $B = g u_c$  to obtain

$$\Delta = [(1 + a'\tau)u_\tau - a'u + g_c u - g u_c]^2 - 4[(1 + a'\tau)g_\tau - g(g_c - a')]u u_c \quad (3.24)$$

after some algebraic cancellations. Then, [SC2] results from imposing that the second summand is non-negative, that is,

$$-u u_c [(1 + a'\tau)g_\tau - g(g_c - a')] \geq 0. \quad (3.25)$$

Since  $u \geq 0$  and  $u_c \leq 0$ , it is plain that (3.17) entails (3.25).  $\square$

**REMARK 3.1.** We stress out that the three sufficient conditions of Theorem 3.1 are not equivalent to each other, since they are associated with three different splittings of the discriminant. It is also worth mentioning that similar calculations were carried out in [4, 15] using Eulerian coordinates, which made them a little more cumbersome.  $\square$

**REMARK 3.2.** [SC1] is appealing in that it does not involve the adsorption  $a$ . However, it does involve the velocity  $u$ . In contrast, [SC2] is attractive in that it does not involve the velocity  $u$ . It seems to involve the adsorption rate  $a'$ , but the dependence with respect to  $a'$  can be removed in the following way. Rewriting (3.17) as

$$g_\tau - g g_c + a'(\tau g)_\tau \geq 0, \quad (3.26)$$

we observe that  $(\tau g)_\tau = [(\gamma - 1)c]_\tau = \gamma_\tau = -s^2 \gamma_s$ . Therefore, if there can be found an IPV law  $\gamma$  such that

$$[\text{SC2}'] \quad g_\tau - g g_c \geq 0, \quad \gamma_s \leq 0, \quad (3.27)$$

then (3.26) will be automatically satisfied. We recall that the condition  $\gamma_s \leq 0$  is physically natural and was already postulated in (2.11).  $\square$

### 3.3 Control of eigenvalues with respect to a prescribed velocity

The splitting of the discriminant, such as presented in §3.2, does not help us perceive the actual significance of each sufficient condition. More intuition about the sufficient conditions can be gained by adhering to following principle.

**Lemma 3.2.** *If  $\wp^{\text{Lag}}(\lambda_*) \leq 0$  for some  $\lambda_* \in \mathbb{R}$ , then  $\wp^{\text{Lag}}$  has two real roots  $\lambda_- \leq \lambda_+$  and  $\lambda_*$  lies in between them, i.e.,  $\lambda_- \leq \lambda_* \leq \lambda_+$ .*

*Proof.* By (3.12),  $\wp^{\text{Lag}}$  is a strictly convex quadratic polynomial, with  $\lim_{\lambda \rightarrow \pm\infty} \wp^{\text{Lag}}(\lambda) = +\infty$ . If  $\wp^{\text{Lag}}(\lambda_*) < 0$  for some  $\lambda_* \in \mathbb{R}$ , then according to the intermediate value theorem, there exist  $\lambda_- \in (-\infty, \lambda_*)$  and  $\lambda_+ \in (\lambda_*, \infty)$  such that  $\wp^{\text{Lag}}(\lambda_-) = \wp^{\text{Lag}}(\lambda_+) = 0$ . We thus have two distinct real roots with  $\lambda_- < \lambda_* < \lambda_+$ .

If  $\wp^{\text{Lag}}(\lambda_*) = 0$  for some  $\lambda_* \in \mathbb{R}$ , then  $\lambda_*$  is a real root. Since  $\wp^{\text{Lag}}$  has real coefficients, the other root is also real and we still have  $\lambda_- \leq \lambda_* \leq \lambda_+$ .  $\square$

**Theorem 3.2.** *If any of the conditions in Table 1 is satisfied at state  $(s, c) \in \Omega$ , then system (2.6) is weakly hyperbolic at this state. In such as case, the two Lagrangian eigenvalues enclose  $\lambda_*$ , and the two Eulerian eigenvalues enclose  $\zeta_* = u + \tau\lambda_*$ .*

#	practical expression	$\lambda_*$	$\zeta_*$
[SC0]	$(g_c - a')u_\tau - g_\tau u_c \geq 0$	0	$u$
[SC1]	$q_\tau \geq 0$	$-u_\tau$	$f_s$
[SC2]	$(1 + a'\tau)g_\tau - g(g_c - a') \geq 0$	$\frac{(g_c - a')u}{1 + a'\tau}$	$\frac{(\gamma c)_c f}{s + a'}$
[SC3]	$(1 + a'\tau)(q - au)_\tau - a(q - au)_c \geq 0$	$-u_\tau + \frac{au_c}{1 + a'\tau}$	$f_s + \frac{au_c}{s + a'}$
[SC4]	$(1 + a'\tau)(g_c u_\tau - g_\tau u_c) - a'(gu)_c \geq 0$	$-\frac{a'u}{1 + a'\tau}$	$\frac{f}{s + a'}$

Table 1: Sufficient conditions for weak hyperbolicity.

*Proof.* For each row of Table 1, insert the value of  $\lambda_*$  in the third column into (3.12) to compute  $\wp^{\text{Lag}}(\lambda_*)$ . After algebraic cancellations, we get

$$\begin{aligned} \wp^{\text{Lag}}(0) &= u[u_c g_\tau - u_\tau (g_c - a')], \\ \wp^{\text{Lag}}(-u_\tau) &= u_c q_\tau, \\ \wp^{\text{Lag}}\left(\frac{(g_c - a')u}{1 + a'\tau}\right) &= uu_c \left[ g_\tau - g \frac{g_c - a'}{1 + a'\tau} \right], \\ \wp^{\text{Lag}}\left(-u_\tau + \frac{au_c}{1 + a'\tau}\right) &= u_c \left[ (q - au)_\tau - \frac{a(q - au)_c}{1 + a'\tau} \right], \\ \wp^{\text{Lag}}\left(-\frac{a'u}{1 + a'\tau}\right) &= u \left[ g_\tau u_c - g_c u_\tau + \frac{a'}{1 + a'\tau} (gu)_c \right]. \end{aligned}$$

Since  $1 + a'\tau$ ,  $u$  and  $-u_c$  are non-negative, the expression in each row implies  $\wp^{\text{Lag}}(\lambda_*) \leq 0$ .  $\square$

This fresh perspective is not only informative. It also allows more sufficient conditions to be generated, provided that some “educated guess” can be made for the velocity  $\lambda_*$ . As a matter of fact, this approach is inspired from traffic flow modeling [2, 21, 34] where it is customary to prescribe some physically reasonable value  $\lambda_* \in \mathbb{R}$  as characteristic speed of the model, namely,  $\varphi^{\text{Lag}}(\lambda_*) = 0$ . In our problem, most values of  $\zeta_*$  in the last column are taken to be the eigenvalues of (2.1) in special cases such as (2.14) or (2.17). For the moment, the prescribed velocities  $\lambda_*$  for [SC2] and [SC3] seems unmotivated. This will be elaborated on in the upcoming section §3.4.

### 3.4 From the case without adsorption to the case with adsorption

The influence of the adsorption  $a$  on some sufficient conditions of Table 1 can be better understood by means of a change of variables that turns system (3.7) into a system that formally does not contain any adsorption. Let us introduce the new variables

$$\tilde{\tau} = \tau, \quad \tilde{c} = c + a(c)\tau. \quad (3.28)$$

Since the function  $c \mapsto c + a(c)\tau$  is increasing at fixed  $\tau > 0$  and has range in  $[0, +\infty)$ , it can be inverted so as to recover  $c = c(\tilde{\tau}, \tilde{c})$ . Upon defining the new IPV deviation

$$\tilde{q}(\tilde{\tau}, \tilde{c}) = q(\tau, c) - a(c)u(\tau, c), \quad \tilde{g}(\tilde{\tau}, \tilde{c}) = g(\tau, c) - a(c), \quad (3.29)$$

we can rewrite the Lagrangian model (3.7) as

$$\partial_t \tilde{\tau} - \partial_m \tilde{u} = 0, \quad (3.30a)$$

$$\partial_t \tilde{c} + \partial_m \tilde{q} = 0. \quad (3.30b)$$

In (3.30), the velocity

$$\tilde{u}(\tilde{\tau}, \tilde{c}) = u(\tau, c)$$

is the same as earlier but is now seen as a function of the new variables. System (3.30) has the same structure as the old one (3.7) in which the adsorption  $a$  has vanished. This tremendously alleviates the expression of some sufficient conditions for weak hyperbolicity of (3.30).

**Theorem 3.3.** *The sufficient conditions [SC0], [SC2] and [SC3] of Table 1 are equivalent to their counterparts in Table 2 in the new variables  $(\tilde{\tau}, \tilde{c})$ .*

#	practical expression	$\lambda_*$	$\zeta_*$
[SC0]	$\tilde{g}_{\tilde{c}} \tilde{u}_{\tilde{\tau}} - \tilde{g}_{\tilde{\tau}} \tilde{u}_{\tilde{c}} \geq 0$	0	$\tilde{u}$
[SC2]	$\tilde{g}_{\tilde{\tau}} - \tilde{g}\tilde{g}_{\tilde{c}} \geq 0$	$\tilde{g}_{\tilde{c}} \tilde{u}$	$(\gamma c)_{\tilde{c}} \tilde{u}$
[SC3]	$\tilde{q}_{\tilde{\tau}} \geq 0$	$-\tilde{u}_{\tilde{\tau}}$	$f_{\tilde{s}}$

Table 2: Sufficient conditions [SC2] and [SC3] in the new variables.

*Proof.* For each row at issue in Table 1, we apply the chain rule

$$\partial_\tau = \partial_{\tilde{\tau}} + a \partial_{\tilde{c}}, \quad \partial_c = (1 + a'\tau) \partial_{\tilde{c}} \quad (3.31)$$

and use (3.29) to get the corresponding new expression in Table 2. Let us do the calculations for [SC2], the other conditions being similar. Starting from Table 1, we have

$$\begin{aligned} 0 \leq (1 + a'\tau)g_\tau - g(g_c - a') &= (1 + a'\tau)\tilde{g}_\tau - (\tilde{g} + a)\tilde{g}_c \\ &= (1 + a'\tau)[\tilde{g}_\tau + a\tilde{g}_c - (\tilde{g} + a)\tilde{g}_c] \\ &= (1 + a'\tau)(\tilde{g}_\tau - \tilde{g}\tilde{g}_c). \end{aligned}$$

From this, it follows that  $\tilde{g}_\tau - \tilde{g}\tilde{g}_c \geq 0$ .  $\square$

In Table 2, the partial derivatives  $-\tilde{u}_{\tilde{\tau}}$  and  $f_{\tilde{s}}$  are taken at fixed  $\tilde{c}$ . Everything happens as if the effect of adsorption can be hidden behind the modified composition  $\tilde{c}$  and the modified IPV deviation  $\tilde{q}$ . The connection between the case with adsorption and the case without adsorption can be further exploited for the design of IPV laws. Indeed, assume that we have an IPV deviation  $q_0(s, c)$  that ensures weak hyperbolicity for

$$\partial_t(s) + \partial_x(su) = 0, \quad (3.32a)$$

$$\partial_t(sc) + \partial_x(scu + q) = 0, \quad (3.32b)$$

which embodies (3.1) with  $a = 0$ . Then,

$$q_1(s, c) = a(c)u(s, c) + q_0(s, c + \tau a(c)) \quad (3.33)$$

is an IPV deviation that ensures weak hyperbolicity for system (2.1). If  $q_0$  is derived from an IPV factor, namely,

$$q_0(\tilde{s}, \tilde{c}) = (\gamma_0(\tilde{s}, \tilde{c}) - 1)\tilde{s}\tilde{c}\tilde{u}(\tilde{s}, \tilde{c}),$$

then  $q_1$  can be put under the form

$$q_1(s, c) = (\gamma_1(s, c) - 1)scu(s, c)$$

with

$$\gamma_1(s, c) = \left[1 + \frac{a(c)}{c}\tau\right]\gamma_0(s, c + \tau a(c)). \quad (3.34)$$

Unfortunately,  $\gamma_1$  may become unbounded for  $s \downarrow 0$ . Moreover, the requirement  $(\gamma_1)_s \leq 0$  also seems difficult to comply with. This is why we shall not venture farther in this direction.

## 4 Special solutions to some sufficient conditions and their importance

We now illustrate the power of the above sufficient conditions by linking them to some well-known IPV laws. The constant IPV (2.33) appears to be a special solution of [SC0] on a region of the state space (§4.1) for the model without adsorption. The concentration-dependent IPV deviation  $q = q(c)$  emerges (§4.2) as a special solution [SC1] and will play a role in the global construction §5. The percolation law (2.34) reveals itself (§4.3) to be a special solution of [SC2] on a region of the state space.

### 4.1 [SC0] and the constant IPV factor

Let us start with a trivial statement for the full model (2.1). This result will be helpful later.

**Proposition 4.1.** *Over the strip  $[0, s_b] \times \mathbb{R}_+$ , system (2.1) equipped with the constant IPV law  $\gamma(s, c) = 1 + \eta$  is weakly hyperbolic for all  $\eta > 0$ .*

*Proof.* In this region,  $u = 0$  identically so that the equality  $(g_c - a')u_\tau - g_\tau u_c = 0$  trivially holds provided that  $g_c$  and  $g_\tau$  remain bounded, which is the case here.  $\square$

Much more can be said about the combination of a constant IPV factor with the model without adsorption (3.32).

**Proposition 4.2.** *Over the region*

$$\mathcal{U} = \{(s, c) \in \Omega \mid su_s \leq cu_c\}, \quad (4.1)$$

*system (3.32) equipped with the constant IPV law  $\gamma(s, c) = 1 + \eta$  is weakly hyperbolic for all  $\eta > 0$ .*

*Proof.* When  $a = 0$ , [SC0] boils down to

$$g_c u_\tau - g_\tau u_c \geq 0. \quad (4.2)$$

Inserting  $g = \eta c / \tau$  into (4.2) yields

$$\eta \frac{\tau u_\tau + cu_c}{\tau^2} \geq 0. \quad (4.3)$$

This is equivalent to

$$\tau u_\tau + cu_c = -su_s + cu_c \geq 0, \quad (4.4)$$

hence the definition of  $\mathcal{U}$  in (4.1).  $\square$

The region  $\mathcal{U}$  always contains the strip  $[0, s_b] \times \mathbb{R}_+$ , where  $u_s = u_c = 0$ . It also includes the strip  $[s_\sharp, 1] \times \mathbb{R}_+$ , where  $u = 1/s$  and  $su_s = s(-1/s^2) = -1/s < 0 = cu_c$ . A state  $(s, c)$  for which  $s_b < s < s_*(c)$  cannot possibly belong to  $\mathcal{U}$ , since  $u_s > 0 > u_c$  for such a state. The following Proposition delivers a more accurate picture of  $\mathcal{U}$  for  $s \in (s_b, s_\sharp)$ .

**Proposition 4.3.** *Under the same assumptions for function  $\Upsilon(s) = s\Lambda_o(1-s)/\Lambda_w(s)$  as in Proposition 2.1 and the additional assumption that  $R$  is a  $\mathcal{C}^2$ -function such that*

$$R'(c) + cR''(c) \geq 0, \quad \text{for all } c \geq 0, \quad (4.5)$$

*the set of states  $(s, c) \in (s_b, s_\sharp) \times \mathbb{R}_+$  for which  $su_s = cu_c$  is the frontier curve  $s = s_{**}(c)$  where, for each  $c \geq 0$ ,  $s_{**}(c)$  is the unique solution in  $(s_b, s_\sharp)$  of the equation*

$$R(c)\Upsilon'(s) + cR'(c)\frac{\Upsilon(s)}{s} = 1. \quad (4.6)$$

*The curve  $s_{**}$  lies entirely in the region  $(s_b, s_\sharp) \times \mathbb{R}_+$ , on the right of  $s_*(c)$ , that is,  $s_{**}(c) > s_*(c)$  except for  $s_{**}(0) = s_*(0)$ . It is increasing with respect to  $s \geq (\Upsilon')^{-1}(1) = s_{**}(0)$  and admits the vertical asymptote  $s = s_\sharp$  when  $c \rightarrow +\infty$ . Then,*

$$\mathcal{U} = \{(s, c) \in \Omega \mid 0 \leq s \leq s_b \text{ or } s_{**}(c) \leq s \leq 1\}. \quad (4.7)$$

*Proof.* The equality  $su_s = cu_c$  is tantamount to

$$\frac{sf_s}{f} = \frac{cf_c}{f} + 1. \quad (4.8)$$

In view of expression (2.25) for  $sf_s/f$  and of

$$\frac{cf_c}{f}(s, c) = -\frac{\Lambda_w(s)\Lambda_o(1-s)}{\Lambda_w(s)\Lambda_o(1-s) + \Lambda_w^2(s)/R(c)} \frac{cR'(c)}{R(c)}, \quad (4.9)$$

which is derived from (2.3), equation (4.8) becomes

$$s[\Lambda_w'(s)\Lambda_o(1-s) + \Lambda_o'(1-s)\Lambda_w(s)] = \Lambda_w(s)\Lambda_o(1-s) \left[ 1 - \frac{cR'(c)}{R(c)} \right] + \frac{\Lambda_w^2(s)}{R(c)} \quad (4.10)$$



after multiplication by  $\Lambda_w(s)\Lambda_o(1-s) + \Lambda_w^2/R(c)$ . Multiplying again by  $R(c)/\Lambda_w^2(s)$  and after some rearrangement, we end up with

$$R(c)\Upsilon'(s) + cR'(c)\frac{\Lambda_o(1-s)}{\Lambda_w(s)} - 1 = 0. \quad (4.11)$$

This is none other than (4.6), since  $\Lambda_o(1-s)/\Lambda_w(s) = \Upsilon(s)/s$ . Let

$$\Phi(s, c) = R(c)\Upsilon'(s) + cR'(c)\frac{\Lambda_o(1-s)}{\Lambda_w(s)} - 1 \quad (4.12)$$

be the left-hand side of (4.11). At a fixed  $c \geq 0$ , the function  $\Phi(\cdot, c)$  is continuous and monotonically decreasing with

$$\lim_{s \downarrow s_b} \Phi(s) = +\infty, \quad \lim_{s \uparrow s_{\#}} \Phi(s) = -1, \quad (4.13)$$

thanks to the assumptions on  $\Upsilon$ ,  $\Lambda_w$  and  $\Lambda_o$ . Therefore, there exists a unique  $s_{**}(c) \in (s_b, s_{\#})$  such that  $\Phi(s_{**}(c), c) = 0$ . Since each summand on the left-hand side of the equality

$$R(c)\Upsilon'(s_{**}(c)) + cR'(c)\frac{\Lambda_o(1-s_{**}(c))}{\Lambda_w(s_{**}(c))} = 1$$

is non-negative, we must have  $R(c)\Upsilon'(s_{**}(c)) \leq 1$ . When  $c \rightarrow +\infty$ , we have  $R(c) \rightarrow +\infty$ . For the product  $R(c)\Upsilon'(s_{**}(c))$  to remain bounded, it is necessary that  $\Upsilon'(s_{**}(c)) \downarrow 0$ , which implies  $s_{**}(c) \uparrow s_{\#}$ . In other words, the line  $s = s_{\#}$  is an asymptote for the curve  $s_{**}$ .

To establish that  $s_{**}$  is increasing with respect to  $c$ , we assume differentiability for  $cR'(c)$  and invoke the implicit function theorem, by virtue of which

$$\frac{ds_{**}}{dc}(c) = -\frac{\Phi_c(s_{**}(c), c)}{\Phi_s(s_{**}(c), c)} = -\frac{R'(c)\Upsilon'(s_{**}(c)) + (R'(c) + cR''(c))(\Lambda_o(1-\cdot)/\Lambda_w)(s_{**}(c))}{R(c)\Upsilon''(s_{**}(c)) + cR'(c)(\Lambda_o(1-\cdot)/\Lambda_w)'(s_{**}(c))}. \quad (4.14)$$

The assumptions  $R' > 0$  and  $R' + cR'' \geq 0$  enforce  $\Phi_c > 0$ . On the other hand,  $\Phi_s < 0$  as argued above. Thus  $ds_{**}/dc > 0$ . By monotonicity, it can be checked that for  $s_{**}(c) < s < s_{\#}$ , we have  $su_s < cu_c$ . Because  $R' > 0$ , it is readily proved that  $s_{**}(c) > s_*(c)$  for all  $c > 0$ . Equality occurs only at  $c = 0$ , where  $s_{**}(0) = s_*(0) = (\Upsilon')^{-1}(1)$ .  $\square$

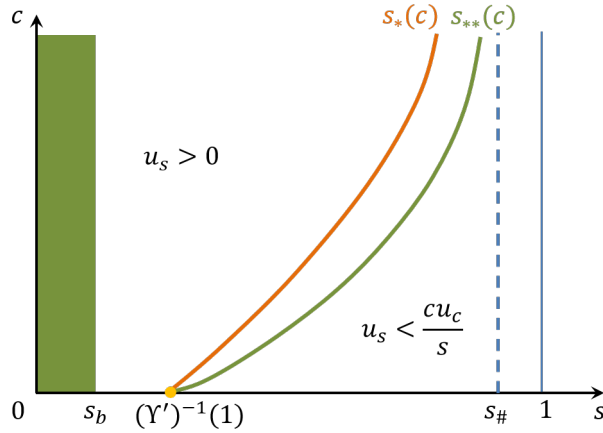


Figure 2: States for which  $su_s = cu_c$  (in green) the  $(s, c)$ -plane.

The behavior of  $s_{**}(\cdot)$  is depicted in Figure 2. Note that the additional requirement (4.5) on  $R$  is quite reasonable. In practice, most mobility reduction functions are of the type [11]

$$R(c) = 1 + \sum_{i \in \mathcal{I}} \varsigma_i c^{m_i}, \quad \varsigma_i > 0, \quad (4.15)$$

where  $\mathcal{I}$  is a finite subset of  $\mathbb{N}$  and  $1 \leq m_1 < m_2 < \dots$  is an increasing sequence of exponents. Any  $R$  of this form fulfills (2.4) and (4.5). The family (4.15) encapsulates other classical mobility reduction laws such as Huggins' [16] and de Gennes' [8].

So far, we have shown that any state in  $\mathcal{U}$  is weakly hyperbolic for the no-adsorption system (3.32) equipped with the constant IPV law (2.33), uniformly in the strength  $\eta > 0$ . We now switch attention to the converse. The following Theorem asserts that the only states in  $\Omega$  that are weakly hyperbolic for model (3.32) uniformly in the strength of the constant IPV are those defined by  $\mathcal{U}$ . It also describes the change in behavior of system (3.32) with respect to this strength  $\eta$  for the states in  $\Omega \setminus \mathcal{U}$ . To this end, [SC0] is not enough and we will have to study the exact discriminant.

**Theorem 4.1.** *If a state  $(s, c) \in \Omega$  is weakly hyperbolic for model (3.32) uniformly in the strength  $\eta > 0$  of the constant IPV law, then it necessarily belongs to region  $\mathcal{U}$ . For any other state  $(s, c) \in \Omega \setminus \mathcal{U}$ , there are two amplitudes  $\eta_m(\omega) \geq 0$  and  $\eta_M(\omega) > 0$  that depend on the ratio  $\omega = -su_s/u$  such that the state is*

- weakly hyperbolic if  $\eta \in [0, \eta_m(\omega)] \cup [\eta_M(\omega), +\infty)$ ;
- elliptic if  $\eta \in (\eta_m(\omega), \eta_M(\omega))$ .

*Proof.* For the system at hand, the discriminant (3.21) of the Lagrangian characteristic polynomial can be successively transformed as

$$\begin{aligned} \Delta &= (u_\tau + q_c)^2 - 4u_c q_\tau \\ &= (-s^2 u_s + q_c)^2 + 4u_c s^2 q_s \\ &= [-s^2 u_s + \eta s (cu)_c]^2 + 4u_c s^2 \eta c (su)_s \\ &= s^2 \{ [-su_s + \eta(u + cu_c)]^2 + 4\eta cu_c (u + su_s) \} \end{aligned}$$

Factorizing by  $u^2$ , we obtain

$$\Delta = (su)^2 \left\{ \left[ -\frac{su_s}{u} + \eta \left( 1 + \frac{cu_c}{u} \right) \right]^2 + 4\eta \frac{cu_c}{u} \left( 1 + \frac{su_s}{u} \right) \right\}. \quad (4.16)$$

The presumes that  $u \neq 0$ , which holds true for  $s \notin [0, s_b]$ . We assume this is the case, since the behavior of the system is trivially known in the strip  $[0, s_b] \times \mathbb{R}_+$ . We introduce the reduced quantities

$$\omega = -\frac{su_s}{u} \in \mathbb{R}, \quad \vartheta = -\frac{cu_c}{u} > 0, \quad (4.17)$$

associated with a state  $(s, c)$ . The first ratio  $\omega$  can be thought of as an algebraic distance from the curve  $u_s = 0$ . If  $\omega < 0$ , we are on the left; if  $\omega > 0$ , we are on the right. Then,

$$\begin{aligned} \Delta/(su)^2 &= [\omega + \eta(1 - \vartheta)]^2 - 4\eta\vartheta(1 - \omega) \\ &= \omega^2 + 2(1 + \vartheta)\omega\eta + (1 - \vartheta)^2\eta^2 - 4\vartheta\eta \\ &=: \mathcal{Q}_\vartheta(\omega, \eta) \end{aligned} \quad (4.18)$$

is a quadratic form in the variables  $(\omega, \eta)$  at a fixed  $\vartheta$ . In the  $(\omega, \eta)$ -plane, the conic  $\mathcal{Q}_\vartheta(\omega, \eta) = 0$  can be shown to be a hyperbola of center

$$(\omega_c, \eta_c) = \left( \frac{1 + \vartheta}{2}, -\frac{1}{2} \right) \quad (4.19)$$

and to have asymptotic directions

$$\frac{\eta}{\omega} = -\frac{1}{(1 \pm \sqrt{\vartheta})^2}. \quad (4.20)$$

Only one of its branches lies in the half-plane  $\eta \geq 0$ . This branch, sketched out in orange in Figure 3, is tangent to the horizontal line  $\eta = 0$  at  $(\omega_h, \eta_h) = (0, 0)$ . On the other hand, it is tangent to the vertical line  $\omega = \vartheta$  at  $(\omega_v, \eta_v) = (\vartheta, \vartheta/(1 - \vartheta))$ .

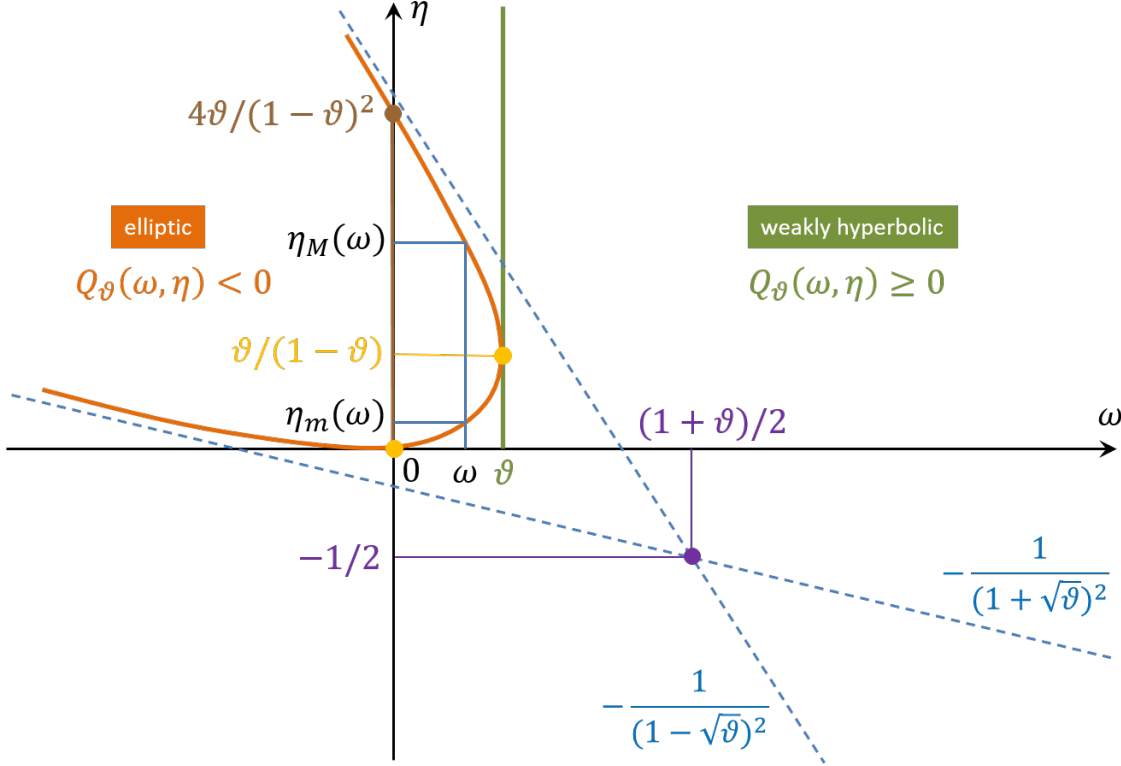


Figure 3: Elliptic and weakly hyperbolic regions for the model without adsorption equipped with the constant IPV law.

The elliptic region  $\mathcal{Q}_\vartheta(\omega, \eta) < 0$  matches with the upper part of the half-plane bordered by this branch of the hyperbola. As illustrated in Figure 3, this elliptic region contains the vertical segments  $(\eta_{\min}(\omega), \eta_{\max}(\omega))$ . Each of these segments stems from the intersection of the vertical line at abscissa  $\omega < \vartheta$  with this branch of the hyperbola. But  $\omega < \vartheta$  is just equivalent to  $-su_s < -cu_c$ , that is,  $su_s > cu_c$ . In other words, we are on the left side of the curve  $s = s_{**}(c)$  in the  $(s, c)$ -plane.

The weakly hyperbolic region  $\mathcal{Q}_\vartheta(\omega, \eta) \geq 0$  is the complementary part of the elliptic region in the half-plane. For  $\omega < \vartheta$ , weak hyperbolicity holds only for  $\eta \leq \eta_m(\omega)$  or  $\eta \geq \eta_M(\omega)$ . In such conditions, we do not have weak hyperbolicity uniformly with respect to  $\eta$ . For weak hyperbolicity to be uniform in  $\eta$ , we must be able to freely move along the vertical direction without crossing the interior region bounded by the branch of hyperbola. This happens only for  $\omega \geq \vartheta$ , in other words, for  $su_s \leq cu_c$ .  $\square$

It is worth mentioning that when  $\omega = 0$ , the interval of ellipticity is given by

$$\eta_m(0) = 0, \quad \eta_M(0) = \frac{4\vartheta}{(1 - \vartheta)^2},$$

which results from solving  $\mathcal{Q}_\vartheta(0, \eta) = 0$  for  $\eta$ . This corroborates Bartelds et al.'s observation [4] that when  $\eta$  is small enough, the states  $(s, c)$  such that  $u_s = 0$  become elliptic. However, this also tells us how small is actually enough, at least for the no-adsorption system. For  $u_s \neq 0$ ,  $\eta_m(\omega) > 0$  and ellipticity can no longer be claimed for  $\eta$  small enough.

## 4.2 [SC1] and the concentration-dependent IPV deviation

As pointed out in Remark 3.2, the sufficient condition  $q_\tau \geq 0$  does not involve the adsorption  $a$  and is therefore highly relevant to the full system (2.1). A special solution of [SC1] corresponds to the equality

$$q_\tau = 0, \quad (4.21)$$

which amounts to saying that the IPV deviation  $q = (\gamma - 1)fc$  depends only on  $c$ .

**Theorem 4.2.** *The IPV factor*

$$\gamma(s, c) = 1 + \frac{\Xi(c)}{f(s, c)}, \quad (4.22)$$

where  $\Xi \geq 0$  is an arbitrary function of  $c$ , solves [SC1] for  $s \in (s_b, 1]$ . The associated IPV deviation reads

$$q(c) = c\Xi(c). \quad (4.23)$$

*Proof.* Imposing  $q$  by (4.23) as a solution of (4.21), we infer (4.22) whenever division by  $f$  is legitimate, i.e., for  $s \in (s_b, 1]$ .  $\square$

The choice  $\Xi = 0$  leads to  $\gamma = 1$  and has no physical interest. When  $\Xi > 0$ , the right-hand side of (4.22) makes sense only for  $s \in (s_b, 1]$ . Due to the blow-up

$$\lim_{s \downarrow s_b} \gamma(s, c) = +\infty, \quad (4.24)$$

the IPV law (4.22) must be restricted to an interval of the form  $[s_\bullet, 1]$  with  $s_\bullet > s_b$ . We shall make use of (4.22) in §5 to build an IPV law that ensures global weak hyperbolicity for (2.1). The model without adsorption (3.32) equipped with an IPV deviation of the type  $q = q(c)$  was studied by Silva and Marchesin [30]. It was shown in particular that the Riemann problem may have structurally stable Riemann solutions without the standard constant state.

Another solution of [SC1] corresponding to the strict inequality  $q_\tau > 0$  deserves our attention, even though it may not be adequate from the physics point of view.

**Proposition 4.4.** *The IPV factor*

$$\gamma(s, c) = 1 + \Xi(s, c) \frac{\Lambda_o(1-s)}{\Lambda_w(s)}, \quad (4.25)$$

where  $\Xi \geq 0$  is an arbitrary function of  $(s, c)$  that is decreasing with respect to  $s$ , solves [SC1] for  $s \in (s_b, 1]$ . Strict inequality  $q_\tau > 0$  holds in  $(s_b, s_\#) \times \mathbb{R}_+^*$ .

The interest of (4.25) lies in taking a positive constant for  $\Xi$ , which yields an IPV factor that does not depend on  $c$ , similarly to the percolation law. The shortcoming of (4.25) is that  $\gamma(s_\#, c) = 1$  for all  $c \geq 0$ . This loss of the IPV effect does not make sense for engineers.

*Proof.* Since  $q_s = c[\gamma_s f + (\gamma - 1)f_s]$  and  $c \geq 0$ , the inequality  $q_s \leq 0$  can be enforced by

$$\gamma_s f + (\gamma - 1)f_s \leq 0. \quad (4.26)$$

Assuming  $\gamma > 1$  and  $f > 0$ , this is equivalent to

$$\frac{\gamma_s}{\gamma - 1} + \frac{f_s}{f} \leq 0. \quad (4.27)$$

The second summand

$$\frac{f_s}{f} = \left[ \frac{\Lambda'_w(s)}{\Lambda_w(s)} + \frac{\Lambda'_o(1-s)}{\Lambda_o(1-s)} \right] \cdot \left[ 1 + \frac{\Lambda_w(s)}{\Lambda_o(1-s)R(c)} \right]^{-1} \quad (4.28)$$

can be majorized—with strict inequality for  $(s, c) \in (s_b, s_\#) \times \mathbb{R}_+^*$ —by the  $c$ -independent upper-bound

$$\frac{f_s}{f} \leq \frac{\Lambda'_w(s)}{\Lambda_w(s)} + \frac{\Lambda'_o(1-s)}{\Lambda_o(1-s)} \quad (4.29)$$

thanks to  $R(c) \geq 1$ . As a consequence, we can enforce (4.27) by demanding

$$\frac{\gamma_s(s, c)}{\gamma(s, c) - 1} + \frac{\Lambda'_w(s)}{\Lambda_w(s)} + \frac{\Lambda'_o(1-s)}{\Lambda_o(1-s)} \leq 0. \quad (4.30)$$

The left-hand side of the previous inequality is the logarithmic derivative with respect to  $s$  of the function

$$(s, c) \mapsto [\gamma(s, c) - 1] \frac{\Lambda_w(s)}{\Lambda_o(1-s)} =: \Xi(s, c).$$

Let us call  $\Xi(s, c)$  this function. Then,  $\Xi$  must be positive and decreasing in  $s$  at fixed  $c$ . Equation (4.25) follows from inverting the definition of  $\Xi$ .  $\square$

### 4.3 [SC2] and the percolation law

As indicated in Remark 3.2, the sufficient condition  $g_\tau - gg_c + a'(\tau g)_\tau \geq 0$  does not involve the velocity  $u$  and is physically interesting. The stronger version (3.27) even makes it transparent to the adsorption  $a$ . Let us look for a solution of [SC2'] such that

$$g_\tau - gg_c = 0, \quad \gamma_s \leq 0. \quad (4.31)$$

Leaving aside the inequality  $\gamma_s \leq 0$  momentarily, we first take care of the equality  $g_\tau - gg_c = 0$ .

**Lemma 4.1.** *All smooth solutions of  $g_\tau - gg_c = 0$  are given by*

$$g(\tau, c) = g_1(c_1(\tau, c)), \quad (4.32)$$

where  $g_1$  is an arbitrary  $\mathcal{C}^1$ -function over  $\mathbb{R}_+$  representing the initial condition at  $\tau = 1$ , i.e.,

$$g_1(c) = g(1, c), \quad c \geq 0,$$

and  $c_1(\tau, c)$  solves the implicit equation

$$c = c_1(\tau, c) - g_1(c_1(\tau, c))(\tau - 1). \quad (4.33)$$

This solution (4.32)–(4.33) is well-defined for

$$1 \leq \tau < 1 + \min_{c \geq 0} \frac{1}{\max(g'_1(c), 0)} =: \tau_\bullet. \quad (4.34)$$

*Proof.* Up to a sign, equation  $g_\tau - gg_c = g_\tau - (\frac{1}{2}g^2)_c = 0$  can be recognized as a Burgers equation, in which  $\tau$  plays the role of time (which must start at 1, so that  $s = 1/\tau$  remains below 1) and  $c$  the role of space. Smooth solutions can then be found by the characteristic method [22].

In the  $(c, \tau)$ -plane, we introduce the characteristic curves  $\tau \mapsto C(\tau; c_1)$  defined by

$$\frac{dC}{d\tau}(\tau; c_1) = -g(\tau, C(\tau; c_1)), \quad (4.35a)$$

$$C(1; c_1) = c_1, \quad (4.35b)$$

for  $c_1 \geq 0$ . Along such a characteristic curve,  $g$  is constant. Indeed,

$$\frac{d}{d\tau}g(\tau, C(\tau; c_1)) = [g_\tau - gg_c](\tau, C(\tau; c_1)) = 0. \quad (4.36)$$

Therefore,  $g(\tau, C(\tau; c_1)) = g(1, C(1; c_1)) = g_1(c_1)$  and the characteristic curve (4.35) is a straight line, the Cartesian representation of which is

$$c = c_1 - g_1(c_1)(\tau - 1). \quad (4.37)$$

Equation (4.33) thus gives the foot  $c_1(\tau, c)$  of the characteristic line going through  $(\tau, c)$ . Inversion can be carried out as long as the right-hand side is a monotone function of  $c_1$ , namely,  $\tau < \tau_\bullet$  as supplied by (4.34).  $\square$

Next, by selecting an appropriate function  $g_1 = g(1, \cdot)$ , we can make sure that  $\gamma_s \leq 0$ .

**Theorem 4.3.** *The percolation law*

$$\gamma(s, c) = \frac{s}{s - s_\bullet}, \quad (4.38)$$

where  $s_\bullet \in (0, 1)$  is a parameter, solves (4.31) on  $s \in (s_\bullet, 1]$ . It corresponds to the linear initial function

$$g_1(c) = \frac{s_\bullet c}{1 - s_\bullet} \quad (4.39)$$

and the maximal time  $\tau_\bullet = 1/s_\bullet$ .

*Proof.* By the choice (4.39), equation (4.33) becomes linear in  $c_1$ , that is,

$$c = c_1 - \frac{s_\bullet c_1}{1 - s_\bullet}(\tau - 1) = \frac{1 - s_\bullet \tau}{1 - s_\bullet} c_1. \quad (4.40)$$

The solution

$$c_1 = \frac{1 - s_\bullet}{1 - s_\bullet \tau} c \quad (4.41)$$

makes sense only for  $1 - s_\bullet \tau > 0$ , namely,  $\tau < 1/s_\bullet$ . It turns out that  $\tau_\bullet = 1/s_\bullet$  also coincides with the value dictated by (4.34). In this case,

$$g(\tau, c) = g_1(c_1) = \frac{s_\bullet c}{1 - s_\bullet} \frac{1 - s_\bullet}{1 - s_\bullet \tau} c = \frac{s_\bullet c}{1 - s_\bullet \tau} = \frac{s_\bullet sc}{s - s_\bullet}. \quad (4.42)$$

Comparing this with the definition  $g = (\gamma - 1)sc$ , we deduce the value (4.38) for  $\gamma$  and notice that  $\gamma_s = -s_\bullet/(s - s_\bullet)^2 \leq 0$ .  $\square$

At this stage, it is natural to wonder about those solutions of (3.27) with (possibly strict) inequality  $g_\tau - gg_c \geq 0$  instead of equality  $g_\tau - gg_c = 0$ . Although we do not have a full characterization (as in Lemma 4.1 for equality), the following Proposition singles out a class of such solutions.

**Proposition 4.5.** *Whenever it is well-defined, the generalized percolation law*

$$\gamma(s, c) = \frac{s}{s - \sigma(s, c)}, \quad (4.43)$$

where the  $\mathcal{C}^1$ -function  $(s, c) \mapsto \sigma(s, c) \in (0, s)$  is non-increasing with respect to  $s$  and  $c$ , is a solution of [SC2'] given by (3.27).

*Proof.* Plugging  $g = (\gamma - 1)c/\tau$  into  $g_\tau - gg_c \geq 0$  and using

$$g_\tau = \gamma_\tau c/\tau - (\gamma - 1)c/\tau^2, \quad g_c = \gamma_c c/\tau + (\gamma - 1)/\tau$$

results in

$$\tau\gamma_\tau - \gamma(\gamma - 1) - c(\gamma - 1)\gamma_c \geq 0. \quad (4.44)$$

This is equivalent to

$$s\gamma_s + \gamma(\gamma - 1) + c(\gamma - 1)\gamma_c \leq 0. \quad (4.45)$$

A special class of solutions can be obtained by separately

$$s\gamma_s + \gamma(\gamma - 1), \quad (\gamma - 1)\gamma_c \leq 0. \quad (4.46)$$

To solve the first inequality, let us consider the function  $\psi = s(1 - 1/\gamma)$ . Its derivative

$$\psi_s = 1 - \frac{1}{\gamma} + s \frac{\gamma_s}{\gamma^2} = \frac{s\gamma_s + \gamma(\gamma - 1)}{\gamma^2} \quad (4.47)$$

must then be non-positive, i.e.,  $\psi_s \leq 0$ . Let  $\sigma(s, c)$  be arbitrary function non-increasing in  $s$ . Setting  $\psi = \sigma$  and solving for  $\gamma$ , we get (4.43). The condition  $0 < \sigma(s, c) < s$  is required for  $\gamma$  to be well-defined and greater than 1. Since  $\gamma > 1$ , the second inequality  $(\gamma - 1)\gamma_c \leq 0$  reduces to

$$\gamma_c = \frac{s\sigma_c}{[s - \sigma(s, c)]^2} \leq 0. \quad (4.48)$$

Therefore,  $\sigma_c \leq 0$  and  $\sigma$  must also be non-increasing with respect to  $c$ . Finally, it is obvious that  $\gamma_s \leq 0$ , since  $\gamma_s \leq -\tau\gamma(\gamma - 1)$ .  $\square$

To conclude this section, we report a most remarkable property of the percolation law (4.38) that seems to have been little emphasized in the literature. The authors of this paper briefly mentioned it in [10]. Below is a more comprehensive statement.

**Theorem 4.4.** *The change of variables*

$$\mathbf{S} = s/\gamma, \quad \mathbf{C} = \gamma c, \quad \mathbf{F}(\mathbf{S}, \mathbf{C}) = f(s, c) \quad (4.49)$$

turns the model without adsorption (3.32), considered for  $(s, c) \in (s_\bullet, 1]$  and equipped with the percolation law (4.38), into the Keyfitz-Kranzer no-IPV model

$$\partial_t(\mathbf{S}) + \partial_x(\mathbf{F}) = 0, \quad (4.50a)$$

$$\partial_t(\mathbf{S}\mathbf{C}) + \partial_x(\mathbf{F}\mathbf{C}) = 0, \quad (4.50b)$$

considered for  $(\mathbf{S}, \mathbf{C}) \in (0, 1 - s_\bullet] \times \mathbb{R}_+$ , and conversely. The equivalence between the two systems holds for smooth as well as discontinuous solutions.

*Proof.* From the identity

$$\frac{s}{\gamma} = s - s_\bullet, \quad (4.51)$$

we have for smooth solutions

$$\partial_t(\mathbf{S}) = \partial_t\left(\frac{s}{\gamma}\right) = \partial_t(s - s_\bullet) = \partial_t(s) = -\partial_x(f) = -\partial_x(\mathbf{F}), \quad (4.52)$$

which proves (4.50a). We also have

$$\partial_t(\mathbf{S}\mathbf{C}) = \partial_t\left(\frac{s}{\gamma}\gamma c\right) = \partial_t(sc) = -\partial_x(\gamma fc) = -\partial_x(\mathbf{F}\mathbf{C}), \quad (4.53)$$

which proves (4.50b). Calculations in the other direction are similar. Thus, the two systems are equivalent.

For discontinuous solutions, let us consider a shock between two states  $(s_L, c_L)$  and  $(s_R, c_R)$  for the system with the percolation law. These states satisfy the Rankine-Hugoniot relations

$$[[f]] = \sigma [[s]], \quad (4.54a)$$

$$[[\gamma fc]] = \sigma [[sc]], \quad (4.54b)$$

where  $[[\cdot]] = \cdot_R - \cdot_L$  denotes the value of the jump. Invoking (4.51) again,

$$\left[ \left[ \frac{s}{\gamma} \right] \right] = [[s - s_\bullet]] = [[s]]. \quad (4.55)$$

Hence,

$$[[F]] = [[f]] = \sigma [[s]] = \sigma \left[ \left[ \frac{s}{\gamma} \right] \right] = [[S]] \quad (4.56a)$$

and

$$[[FC]] = [[f\gamma c]] = \sigma [[sc]] = \sigma \left[ \left[ \frac{s}{\gamma} \gamma c \right] \right] = \sigma [[SC]]. \quad (4.56b)$$

Equalities (4.56) are the Rankine-Hugoniot relations for system (4.50).  $\square$

Owing to this previous equivalence, the model (3.32) equipped with the percolation law (4.38) is weakly hyperbolic on  $(s_\bullet, 1] \times \mathbb{R}_+$ . Its characteristic speeds are deduced from (2.17) as

$$\begin{aligned} F/S &= f/(s/\gamma) = \gamma u, \\ F_S &= f_s s_S + f_c c_S = (su)_s + (\gamma - 1)cu_c, \end{aligned}$$

the first one  $\gamma u$  being linearly degenerate.

## 5 Construction of IPV laws ensuring global weak hyperbolicity

We now proceed with the more ambitious task of building IPV laws that make (2.1) weakly hyperbolic on  $\Omega$ . This is achieved by patching up various sufficient conditions on different strips of  $\Omega$ . For the easier case  $s_\bullet < s_b$ , we expose in §5.1 a rather straightforward extension of the percolation law. For the more delicate case  $s_\bullet > s_b$ , we offer in §5.2 a mathematically consistent construction that appears to us as a significant advance despite some minor physical limitations.

### 5.1 For $s_\bullet < s_b$

The difficulty with the percolation law (4.38) is that  $\gamma$  blows up as  $s \downarrow s_\bullet$ . But if  $s_b > s_\bullet$ , it is advisable to stop the percolation law at  $s_b$  and to extend  $\gamma$  continuously by a constant to the left. The idea is sketched out in Figure 4.

**Theorem 5.1.** *When  $s_\bullet < s_b$ , the IPV law*

$$\gamma(s, c) = \begin{cases} \frac{s}{s - s_\bullet} & \text{if } s_b \leq s \leq 1, \\ \frac{s_b}{s_b - s_\bullet} & \text{if } 0 \leq s \leq s_b, \end{cases} \quad (5.1)$$

*gives rise to a  $\mathcal{C}^1$  flux  $f c \gamma$  and guarantees weak hyperbolicity for system (2.1) over  $\Omega$ .*



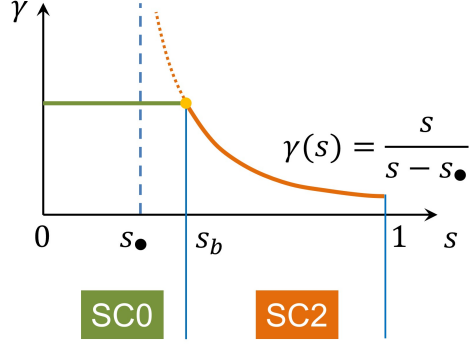


Figure 4: Construction principle for  $s_{\bullet} < s_b$ .

*Proof.* For  $s \in [s_b, 1]$ , weak hyperbolicity is due to [SC2'], as explained in §4.3. For  $s \in [0, s_b]$ , weak hyperbolicity is due to [SC0], as shown in Proposition 4.1. The only thing that remains to be verified is that the flux component  $fc\gamma = scu + q$  is continuously differentiable with respect to  $s$ .

Since  $q = 0$  for  $s \in [0, s_b]$ , we have  $\lim_{s \uparrow s_b} q_s(s, c) = 0$ . For  $s > s_b$ ,

$$q_s = \gamma_s fc + (\gamma - 1)f_s c.$$

When  $s \downarrow s_b$ , the limit of  $\gamma_s fc$  is 0, since  $f(s_b, c) = 0$  and  $\gamma_s = -s/(s - s_{\bullet})^2$  remains bounded. The limit of  $(\gamma - 1)f_s c$  is also 0, thanks to

$$f_s(s, c) = \frac{\Lambda'_w(s)\Lambda_o(1-s) + \Lambda_w(s)\Lambda'_o(1-s)}{R(c)[\Lambda_w(s)/R(c) + \Lambda_o(1-s)]^2} \quad (5.2)$$

and more specifically to  $\Lambda'_w(s_b) = 0$ . Therefore,  $\lim_{s \downarrow s_b} q_s(s, c) = 0$  and finally  $q_s(s_b, c) = 0$ .  $\square$

In the previous proof, what really matters for differentiability of  $fc\gamma$  at  $s_b$  is not the values of  $\gamma$  and  $\gamma_s$ , but the equality  $\Lambda'_w(s_b) = 0$  which comes from the differentiability assumption on  $\Lambda_w$  and from (2.7)–(2.8). The same argument will be reiterated in §5.2.

## 5.2 For $s_{\bullet} > s_b$

### 5.2.1 Theoretical derivation

Because of the blow-up at  $s_{\bullet}$ , the percolation law is no longer applicable. Instead, we deploy the [SC1]-type IPV law (4.25) on  $[s_{\bullet}, 1]$ , that we extrapolate on  $[s_b, s_{\bullet}]$  by an IPV law subject to [SC2'] with strict inequality. On  $[0, s_b]$ , the IPV factor  $\gamma$  is again continuously extended by a constant. The strategy is summarized in Figure 5.

The form imposed to  $\gamma$  on  $[s_b, s_{\bullet}]$  is meant to make

$$h = (\gamma - 1)s$$

affine with respect to  $\tau$ , namely,  $h(\tau, c) = A(c)\tau + B(c)$ . The justification of this ansatz, along with that of the maximal amplitude  $\Xi(c) = \eta/R(c)$  for  $\gamma - 1$  on  $[s_{\sharp}, 1]$ , is postponed until the proof of Theorem 5.2 is completed.

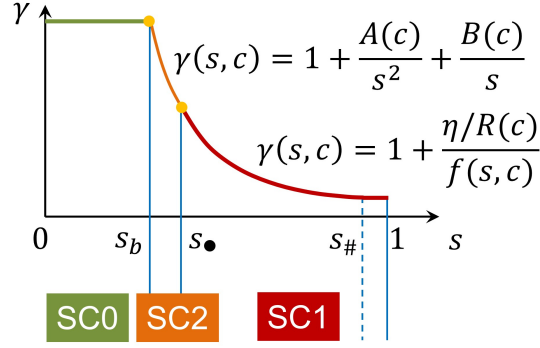


Figure 5: Construction principle for  $s_{\bullet} > s_b$ .

**Theorem 5.2.** *When  $s_{\bullet} > s_b$ , the IPV law*

$$\gamma(s, c) = \begin{cases} 1 + \frac{\eta}{R(c)f(s, c)} & \text{if } s \in [s_{\bullet}, 1], \\ 1 + \frac{A(c)}{s^2} + \frac{B(c)}{s} & \text{if } s \in [s_b, s_{\bullet}], \\ 1 + \frac{A(c)}{s_b^2} + \frac{B(c)}{s_b} & \text{if } s \in [0, s_b], \end{cases} \quad (5.3)$$

where  $\eta > 0$  is a given constant and

$$A(c) = -\frac{\eta}{R(c)} \frac{u_{\tau}(\tau_{\bullet}, c)}{u^2(\tau_{\bullet}, c)}, \quad B(c) = \frac{\eta}{R(c)} \frac{u(\tau_{\bullet}, c) + \tau_{\bullet} u_{\tau}(\tau_{\bullet}, c)}{u^2(\tau_{\bullet}, c)}, \quad (5.4)$$

gives rise to a  $\mathcal{C}^1$  flux  $fc\gamma$  and guarantees weak hyperbolicity for system (2.1) over  $\Omega$ , provided that the assumptions of Proposition 2.1 on  $\Upsilon(s) = -s\Lambda_o(1-s)/\Lambda_w(s)$  are met and that

$$s_{\bullet} < \min\{2s_b, s_*(0)\}, \quad \eta < \frac{-u_{\tau}(\tau_{\bullet}, 0)}{\left[1 - \frac{u_{\tau}(\tau_{\bullet}, 0)}{u}(\tau_{\bullet}, 0)(\tau_b - \tau_{\bullet})\right]^2}, \quad (5.5)$$

with  $\tau_{\bullet} = 1/s_{\bullet}$ ,  $\tau_b = 1/s_b$ .

To prove this Theorem, we need a few technical Lemmas. The first one clarifies the significance of the coefficients  $A(c)$  and  $B(c)$ .

**Lemma 5.1.** *An alternative expression for  $A(c)$  is*

$$A(c) = \eta s_{\bullet}^2 \left[ \Upsilon'(s_{\bullet}) - \frac{1}{R(c)} \right], \quad (5.6)$$

where  $\Upsilon$  is the function defined in (2.21). The coefficients  $A(c)$  and  $B(c)$  defined in (5.4) are adjusted so that  $h$  (and  $q$ ,  $fc\gamma$ ) is continuously differentiable at  $s_{\bullet}$ , that is,

$$h(\tau_{\bullet}, c) = \frac{\eta}{R(c)} \frac{1}{u(\tau_{\bullet}, c)}, \quad h_{\tau}(\tau_{\bullet}, c) = -\frac{\eta}{R(c)} \frac{u_{\tau}(\tau_{\bullet}, c)}{u^2(\tau_{\bullet}, c)}. \quad (5.7)$$

*Proof.* Taking the limit of  $h = (\gamma - 1)s$  on the right of  $s_{\bullet}$ , i.e., on the left of  $\tau_{\bullet}$ , we have

$$\lim_{\tau \uparrow \tau_{\bullet}} h(\tau, c) = \frac{\eta}{R(c)} \frac{1}{u(\tau_{\bullet}, c)}, \quad \lim_{\tau \uparrow \tau_{\bullet}} h_{\tau}(\tau, c) = -\frac{\eta}{R(c)} \frac{u_{\tau}(\tau_{\bullet}, c)}{u^2(\tau_{\bullet}, c)}. \quad (5.8)$$

Taking the limit of  $h = A\tau + B$  on the left of  $s_\bullet$ , i.e., on the right of  $\tau_\bullet$ , we have

$$\lim_{\tau \uparrow \tau_\bullet} h(\tau, c) = A(c)\tau_\bullet + B(c), \quad \lim_{\tau \uparrow \tau_\bullet} h_\tau(\tau, c) = A(c). \quad (5.9)$$

Requiring that the quantities in (5.9) are equal to their counterparts in (5.8), we obtain a linear system in  $(A(c), B(c))$  whose solution is given by (5.4). The value of  $A(c)$  can still be written as

$$A(c) = \frac{\eta}{R(c)} \left( \frac{1}{u} \right)_\tau (\tau_\bullet, c) = -\frac{\eta s_\bullet^2}{R(c)} \left( \frac{s}{f} \right)_s (s_\bullet, c). \quad (5.10)$$

But from the definition (2.3) of the fractional flux, it is readily checked that

$$\frac{s}{f(s, c)} = s - R(c)\Upsilon(s). \quad (5.11)$$

Hence,

$$\left( \frac{s}{f} \right)_s (s, c) = 1 - R(c)\Upsilon'(s). \quad (5.12)$$

Inserting this into (5.10) yields the expression (5.6).  $\square$

The next Lemma shows that extrapolation of  $h$  by an affine function in  $\tau$  on  $[\tau_\bullet, \tau_b]$  induces favorable behaviors on  $\gamma$ .

**Lemma 5.2.** *Under the assumptions of Proposition 2.1 on  $\Upsilon$  and if  $s_\bullet < s_*(0)$ , we have*

$$A(c) > 0, \quad A'(c) > 0, \quad h > 0, \quad \gamma > 1, \quad \gamma_s < 0, \quad (5.13)$$

for all  $c \geq 0$  and for all  $\tau \in [\tau_\bullet, \tau_b]$ .

*Proof.* According to the study of the curve  $s_*$  in Proposition (2.1), the assumption  $s_\bullet < s_*(0)$  implies  $s_\bullet < s_*(c)$ , which in turn entails  $\Upsilon'(s_\bullet) > 1/R(c)$ , that is,  $A(c) > 0$ . Its derivative

$$A'(c) = \eta s_\bullet^2 \frac{R'(c)}{R^2(c)} \quad (5.14)$$

is positive by virtue of conditions (2.4) on  $R$ .

For  $\tau \in [\tau_\bullet, \tau_b]$ ,

$$h(\tau, c) = h(\tau_\bullet, c) + A(c)(\tau - \tau_\bullet) > 0 \quad (5.15)$$

since  $h(\tau_\bullet, c) > 0$  after (5.7). Consequently,  $\gamma > 1$ . Taking the derivative of  $\gamma = 1 + A(c)\tau^2 + B(c)\tau$  with respect to  $\tau$  gives

$$\gamma_\tau = 2A(c)\tau + B(c) = A(c)\tau + h(\tau, c). \quad (5.16)$$

Thus,  $\gamma_\tau > 0$  for  $\tau \in [\tau_\bullet, \tau_b]$ , which means  $\gamma_s < 0$  for  $s \in [s_b, s_\bullet]$ .  $\square$

The last Lemma will be needed at the very end of the proof of Theorem 5.2.

**Lemma 5.3.** *If  $s_b < s_\bullet < \min\{s_*(0), 2s_b\}$ , then*

$$\min_{c \geq 0} \frac{-R(c)u_\tau(\tau_\bullet, c)}{\left[1 - \frac{u_\tau(\tau_\bullet, c)(\tau_b - \tau_\bullet)}{u}\right]^2} = \frac{-u_\tau(\tau_\bullet, 0)}{\left[1 - \frac{u_\tau(\tau_\bullet, 0)(\tau_b - \tau_\bullet)}{u}\right]^2}. \quad (5.17)$$

*Proof.* The function

$$\Theta(c) = \frac{-R(c)u_\tau(\tau_\bullet, c)}{\left[1 - \frac{u_\tau(\tau_\bullet, c)(\tau_b - \tau_\bullet)}{u}\right]^2} \quad (5.18)$$

is well-defined for all  $c \geq 0$ , since the denominator in the right-hand side is always greater than 1, thanks to  $u(\tau_\bullet, c) > 0$ ,  $-u_\tau(\tau_\bullet, c) > 0$  [due to  $s_\bullet < s_*(0)$ ] and  $\tau_b > \tau_\bullet$ . A more convenient form of (5.18) can be obtained by noting on one hand that from

$$-u_\tau = s^2 u_s = s f_s - f = f \left( \frac{s f_s}{f} - 1 \right)$$

and from the expression (2.25) for  $s f_s / f$ , we have

$$-u_\tau(\tau_\bullet, c) = \frac{\Lambda_w(s_\bullet)/R(c)}{\Lambda_o(1-s_\bullet) + \Lambda_w(s_\bullet)/R(c)} \left[ s_\bullet \frac{\{\Lambda'_w/\Lambda_w\}(s_\bullet) + \{\Lambda'_o/\Lambda_o\}(1-s_\bullet)}{1 + \Lambda_w(s_\bullet)/(R(c)\Lambda_o(1-s_\bullet))} - 1 \right]. \quad (5.19)$$

On the other hand,

$$u(\tau_\bullet, c) = \tau_\bullet \frac{\Lambda_w(s_\bullet)/R(c)}{\Lambda_o(1-s_\bullet) + \Lambda_w(s_\bullet)/R(c)}. \quad (5.20)$$

Plugging (5.19)–(5.20) into (5.18), we end up with

$$\Theta(c) = \frac{\Lambda_w^2(s_\bullet)}{s_\bullet [\Lambda'_w(s_\bullet)\Lambda_o(1-s_\bullet) + \Lambda_w(s_\bullet)\Lambda'_o(1-s_\bullet)]} \frac{z(c)(z(c)+1)}{[1+z(c)(s_\bullet\tau_b-1)]^2}, \quad (5.21)$$

where

$$z(c) = s_\bullet \frac{\{\Lambda'_w/\Lambda_w\}(s_\bullet) + \{\Lambda'_o/\Lambda_o\}(1-s_\bullet)}{1 + \Lambda_w(s_\bullet)/(R(c)\Lambda_o(1-s_\bullet))} - 1 \quad (5.22)$$

is an obviously increasing function of  $c \geq 0$ . When  $c$  ranges from 0 to  $+\infty$ ,  $z(c)$  ranges from

$$z_m = s_\bullet \frac{\{\Lambda'_w/\Lambda_w\}(s_\bullet) + \{\Lambda'_o/\Lambda_o\}(1-s_\bullet)}{1 + \Lambda_w(s_\bullet)/\Lambda_o(1-s_\bullet)} - 1 = \frac{1}{1 + \Lambda_o(1-s_\bullet)/\Lambda_w(s_\bullet)} [\Upsilon'(s_\bullet) - 1]$$

to

$$z_M = s_\bullet \left[ \frac{\Lambda'_w}{\Lambda_w}(s_\bullet) + \frac{\Lambda'_o}{\Lambda_o}(1-s_\bullet) \right] - 1 = \frac{\Lambda_w(s_\bullet)}{\Lambda_o(1-s_\bullet)} \Upsilon'(s_\bullet).$$

Note that  $z_m > 0$  because  $s_\bullet < s_*(0)$ . This leads us to study the function

$$\Psi(z) = \frac{z(z+1)}{[1+z(s_\bullet\tau_b-1)]^2}, \quad (5.23)$$

which coincides with  $\Theta(c)$  up to a positive factor, for  $z \in [z_m, z_M]$ . The derivative of  $\Psi$  reads

$$\Psi'(z) = \frac{(3-s_\bullet\tau_b)z+1}{[1+z(s_\bullet\tau_b-1)]^3}. \quad (5.24)$$

If  $s_\bullet < 2s_b$ , then  $s_\bullet\tau_b < 2$  and  $3-s_\bullet\tau_b > 1 > 0$ . Therefore  $\Psi'(z) > 0$  for all  $z$  of interest and the minimum of  $\Psi$  on  $[z_m, z_M]$  is reached at  $z = z_m$ . Going back  $\Theta$ , we can now state that the minimum of  $\Theta$  on  $\mathbb{R}_+$  is reached at  $c = 0$ , which proves (5.17).  $\square$

At last, we are in a position to prove Theorem 5.2.

*Proof of Theorem 5.2.* Weak hyperbolicity on  $[s_\bullet, 1] \times \mathbb{R}_+$  comes from applying Theorem 4.2 to  $\Xi(c) = \eta/R(c)$ . By construction, the flux  $fc\gamma$  is differentiable at  $s_\bullet$ . Weak hyperbolicity on  $[0, s_b] \times \mathbb{R}_+$  results from Proposition 4.1. By an argument similar to Theorem 5.1,  $fc\gamma$  is differentiable at  $s_b$ . The only thing that remains to be established is weak hyperbolicity on  $(s_b, s_\bullet)$ .

To this end, we are going to use (3.27). The second part  $\gamma_s \leq 0$  is already settled in Lemma 5.2. To prove that  $g_\tau - gg_c \geq 0$  on  $(\tau_\bullet, \tau_b)$ , we note that since  $g = hc$ , this is equivalent to

$$h_\tau - h^2 - chh_c \geq 0, \quad (5.25)$$

after simplification by  $c \geq 0$ . Plugging  $h = A(c)\tau + B(c)$  into (5.25), the inequality to be proven becomes

$$A(c) - [A(c)\tau + B(c)]^2 - c[A(c)\tau + B(c)][A'(c)\tau + B'(c)] \geq 0. \quad (5.26)$$

The left-hand side of (5.26) is a quadratic polynomial in  $\tau$ , whose leading term in  $\tau^2$  has negative coefficient, namely,

$$-A^2(c) - cA(c)A'(c) < 0, \quad (5.27)$$

by Lemma 5.2. Put another way, the left-hand side of (5.26) is a concave function. It is known, however, that on a compact interval, a concave function reaches its minimum value on the boundary. Here, this minimum is achieved at  $\tau_\bullet$  or  $\tau_b$ . Therefore, condition (5.26) for all  $\tau \in (\tau_\bullet, \tau_b)$  is equivalent to

$$A(c) - [A(c)\tau_\bullet + B(c)]^2 - c[A(c)\tau_\bullet + B(c)][A'(c)\tau_\bullet + B'(c)] \geq 0, \quad (5.28a)$$

$$A(c) - [A(c)\tau_b + B(c)]^2 - c[A(c)\tau_b + B(c)][A'(c)\tau_b + B'(c)] \geq 0. \quad (5.28b)$$

Let us first try to comply with (5.28a). By taking the derivative of

$$A(c)\tau_\bullet + B(c) = \frac{\eta}{R(c)} \frac{1}{u(\tau_\bullet, c)} = \frac{\eta}{R(c)} \frac{s_\bullet}{f(s_\bullet, c)} = \eta s_\bullet \left[ \frac{1}{R(c)} + \frac{\Lambda_o(1-s_\bullet)}{\Lambda_w(s_\bullet)} \right] \quad (5.29)$$

with respect to  $c$ , we have

$$A'(c)\tau_\bullet + B'(c) = -\eta s_\bullet \frac{R'(c)}{R^2(c)} < 0. \quad (5.30)$$

As a consequence, the third summand of (5.28a) is positive, that is,

$$-c[A(c)\tau_\bullet + B(c)][A'(c)\tau_\bullet + B'(c)] > 0. \quad (5.31)$$

It is thus sufficient to impose  $A(c) - [A(c)\tau_\bullet + B(c)]^2 \geq 0$ , namely,

$$-\frac{\eta}{R(c)} \frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)} - \frac{\eta^2}{R^2(c)} \frac{1}{u^2(\tau_\bullet, c)} \geq 0. \quad (5.32)$$

This boils down to

$$\eta \leq -R(c) u_\tau(\tau_\bullet, c). \quad (5.33)$$

Next, let us investigate the feasibility of (5.28b). We observe that

$$A(c)\tau_b + B(c) = [A(c)\tau_\bullet + B(c)] + A(c)(\tau_b - \tau_\bullet) > 0. \quad (5.34)$$

By taking the derivative of the above equality with respect to  $c$ ,

$$A'(c)\tau_b + B'(c) = -\eta s_\bullet \frac{R'(c)}{R^2(c)} + \eta s_\bullet^2 \frac{R'(c)}{R^2(c)} (\tau_b - \tau_\bullet) = \eta s_\bullet \frac{R'(c)}{R(c)} (s_\bullet \tau_b - 2). \quad (5.35)$$

If  $s_\bullet < 2s_b$ , as supposed by the first condition of (5.5), then  $s_\bullet \tau_b < 2$  and  $A'(c)\tau_b + B'(c) < 0$ . In this case, the third summand of (5.28b) is positive, that is,

$$-c[A(c)\tau_b + B(c)][A'(c)\tau_b + B'(c)] > 0. \quad (5.36)$$

It is thus sufficient to prescribe  $A(c) - [A(c)\tau_b + B(c)]^2 \geq 0$ , namely,

$$-\frac{\eta}{R(c)} \frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)} - \left[ \frac{\eta}{R(c)} \frac{1}{u(\tau_\bullet, c)} - \frac{\eta}{R(c)} \frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)} (\tau_b - \tau_\bullet) \right]^2 \geq 0. \quad (5.37)$$

After some algebra, this boils down to

$$\eta \leq \frac{-R(c) u_\tau(\tau_\bullet, c)}{\left[1 - \frac{u_\tau(\tau_\bullet, c)(\tau_b - \tau_\bullet)}{u}\right]^2}. \quad (5.38)$$

Owing to  $-u_\tau(\tau_\bullet, c) > 0$ ,  $u(\tau_\bullet, c) > 0$  and  $\tau_b > \tau_\bullet$ , the upperbound of (5.38) is more restrictive than that of (5.32). In other words, it is enough to require (5.38) for all  $c \geq 0$ . According to Lemma 5.3,

$$\eta \leq \min_{c \geq 0} \frac{-R(c) u_\tau(\tau_\bullet, c)}{\left[1 - \frac{u_\tau(\tau_\bullet, c)(\tau_b - \tau_\bullet)}{u}\right]^2} = \frac{-u_\tau(\tau_\bullet, 0)}{\left[1 - \frac{u_\tau(\tau_\bullet, 0)(\tau_b - \tau_\bullet)}{u}\right]^2}, \quad (5.39)$$

which is the second condition of (5.5).  $\square$

The existence of a  $(s_b, s_\sharp, s_\bullet)$ -dependent maximal value

$$\eta_M(s_b, s_\sharp, s_\bullet) = \frac{-u_\tau(\tau_\bullet, 0)}{\left[1 - \frac{u_\tau(\tau_\bullet, 0)(\tau_b - \tau_\bullet)}{u}\right]^2} \quad (5.40)$$

allowed for the amplitude of  $\gamma(1, 0) = \eta$  should not come as a surprise. After all, such a limitation was already in place for the percolation law (4.38). Indeed,

$$\eta_M(s_\bullet) := \gamma(1) - 1 = \frac{1}{1 - s_\bullet} - 1 = \frac{s_\bullet}{1 - s_\bullet}. \quad (5.41)$$

## 5.2.2 Numerical illustration

To gain insight into the order of magnitude of the upper-bound (5.40) for the amplitude of  $\eta$ , we undertake a short numerical study by drawing a series of curves. First, in Figure 6, we plot  $s_\bullet \mapsto \eta_M(s_b, s_\sharp, s_\bullet)$  for  $s_\bullet \in (0.2, 0.4)$  corresponding to

$$(s_b, s_\sharp) = (0.2, 0.7), \quad (\alpha_w, \alpha_o) = (2, 3),$$

and

$$(\Lambda_w^m, \Lambda_o^m) = (1.4804 \cdot 10^{-11}, 2.4673 \cdot 10^{-12}) \text{ [m}^2 \cdot \text{Pa}^{-1} \cdot \text{s}^{-1}\text{]},$$

The mobility reduction law is taken to be

$$R(c) = 1 + a_1 c + a_2 c^2 + a_4 c^{15/4}$$

with

$$a_1 = 2 \quad a_2 = 2.8, \quad a_4 = 0.16005.$$

For this configuration, we found the resonant saturation for  $c = 0$  takes the value

$$s_*(0) = 0.42588.$$

Depending on the value of  $s_\bullet$  selected, the maximal amplitude  $\eta_M$  can take values from  $\eta_M^{\min} = 0$  to  $\eta_M^{\max} = 0.20799$ , reached at  $s_\bullet = 0.37255$ .

Next, in Figure 7, the IPV factor  $\gamma$  is displayed as a function of  $s \in [0, 1]$  for different values of concentration  $c \in \{0, 0.5, 1, 2\}$  [ $\text{g} \cdot \text{L}^{-1}$ ], using the same parameters as above and setting

$$s_\bullet = 0.37255$$

so as to benefit from the largest possible amplitude in  $\eta_M$ . We see that the range of  $\gamma$  matches physical observations. As expected by construction,  $\gamma$  decreases when  $c$  increases. The decay is more and more substantial when  $s$  approaches 1.

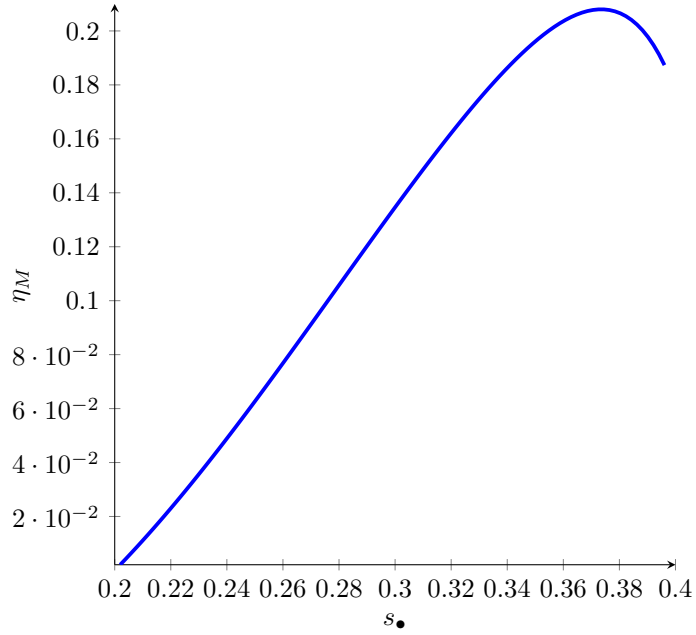


Figure 6: Upper-bound  $\eta_M(s_b, s_{\#}, s_{\bullet})$  as a function of  $s_{\bullet} \in [s_b, \min\{2s_b, s_*(0)\}]$ .

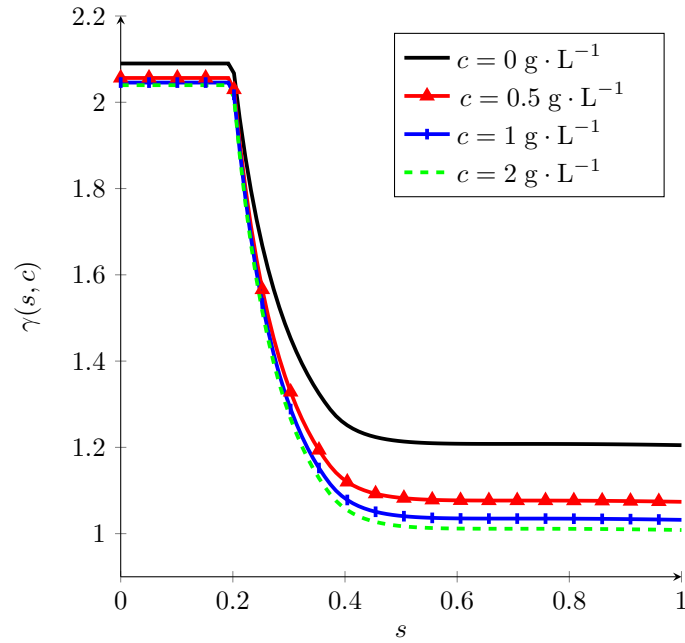


Figure 7: IPV factor  $\gamma$  as a function of  $s$  for different concentration values  $c \in \{0, 0.5, 1, 2\} [g \cdot L^{-1}]$ .

Finally, in Figure 8, we investigate the maximal amplitude with respect to  $s_\bullet$ , that is,

$$\eta_M^{\max}(s_b, s_\#) = \max_{s_\bullet \in [s_b, \min\{2s_b, s_*(0)\}]} \eta_M(s_b, s_\#, s_\bullet) \quad (5.42)$$

as a function of  $s_\#$  for different values of  $s_b$ . It can be seen that this quantity decreases with respect to  $s_\#$ . Moreover, it can range from unrealistically small values (for  $s_\#$  approaching 1) to physically acceptable values (for small  $s_\#$ ).

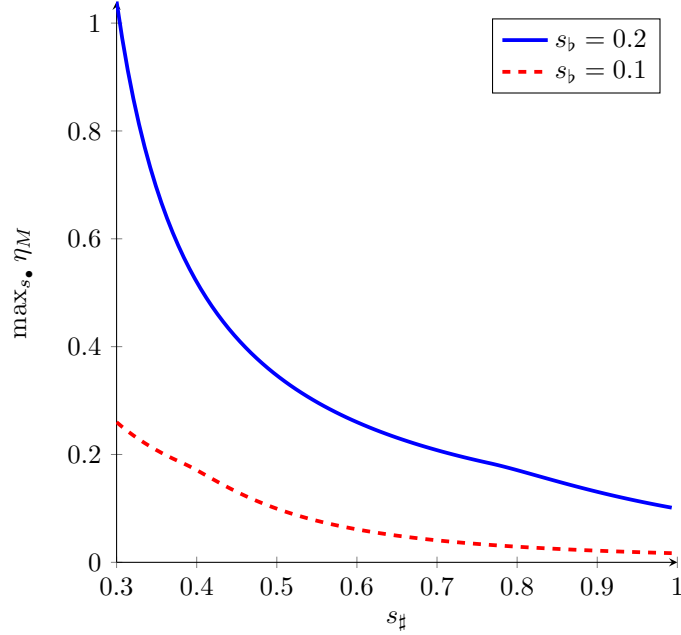


Figure 8: Maximal upper-bound  $\eta_M^{\max}(s_b, s_\#)$  as a function of  $s_\# \in [0.3, 1)$  for  $s_b \in \{0.1, 0.2\}$ .

### 5.2.3 Further discussion

The previous study seems to show that it is delicate to guarantee a physically correct magnitude for  $\eta$ . However, it is capital to underline the fact that the upper-bound (5.40) is very far from being optimal. Indeed, in the proof of Theorem 5.2, we have neglected the favorable positive terms (5.31), (5.36) for the sake of simplicity. The proposed conditions  $A(c) - [A(c)\tau + B(c)]^2 \geq 0$  for  $\tau \in \{\tau_\bullet, \tau_b\}$  are therefore quite abrupt. To put it another way, the actual maximal value for  $\eta$  that is compatible with weak hyperbolicity may be much larger, but the challenge of estimating it in a pragmatic and efficient way seems out of our reach for the time being.

We now comment on two fundamental aspects of the construction (5.3). The first one is concerned with the choice of the form  $h(\tau, c) = A(c)\tau + B(c)$  for the IPV law on  $[s_b, s_\bullet]$ . Broadly speaking, the most natural extrapolation is the affine one. But the question remains as to which quantity should be affinely extrapolated and with respect to which variable. Several pairs of (quantity, variable) have been attempted. The difficulty is that, except for the pair  $(h, \tau)$ , they do not supply us with an acceptable maximal value on the IPV amplitude  $\eta$  for [SC2'] on  $(s_b, s_\bullet)$ . For instance, the pair  $(\gamma - 1, s)$  leads to the condition

$$\eta < \min_{c \geq 0} \min \left\{ -R(c)u_\tau(\tau_\bullet, c), -\frac{R(c)[u_\tau(\tau_\bullet, c) + 2(s_\bullet - s_b)f_s(s_\bullet, c)]}{\left[1 + \frac{f_s}{f}(s_\bullet, c)(s_\bullet - s_b)\right]^2} \right\}. \quad (5.43)$$



The sign of the second argument in the minimum function of the right-hand side cannot be controlled. It is not easy either to study the minimum over  $c \geq 0$  of this upper-bound, which makes it highly impractical.

The second aspect to be discussed is the division of  $\eta$  by  $R(c)$  on  $[s_\bullet, 1]$ , which seems objectionable at first sight. The purpose of this operation is to have a non-zero upper-bound  $\eta_M(s_b, s_\sharp, s_\bullet)$  that guarantees weak hyperbolicity for all  $c \geq 0$ . Had we merely searched for weak hyperbolicity on a finite range  $c \in [0, c_M]$ , it would not have been necessary to divide  $\eta$  by  $R(c)$ . Nevertheless, the corresponding upper-bound would have depended on  $c_M$  in a decreasing manner. This purely mathematical argument can be supplemented by a more physical one: the more polymer molecules there are, the less likely they are accelerated, but the more likely they collide with each other or with the pores of the medium, thus lowering their overall speed. A rough but solid analogy is perhaps to be found in traffic flow. If some exceptional vehicles (police, emergency) are authorized to move faster than the mainstream cars, the speed-up they gain must be necessarily tempered when there are too many of them.

## 6 Conclusion

In this work, we laid down some foundational principles for the design of IPV laws that guarantee weak hyperbocity of a prototype polymer flow model. Derived from a passage to Lagrangian coordinates, the sufficient conditions for local weak hyperbolicity were given a clearer physical meaning and appeared to be easier to deal with. Special solutions to these sufficient conditions were shown to coincide with classical IPV laws and brought more insight into them. A strategy using these sufficient conditions piecewise was proposed for the construction of an IPV law that ensured global weak hyperbolicity.

Further to these encouraging theoretical results, there is still a long way to go in order to expand this idea of solution into a fully satisfactory one for the weak hyperbolicity problem. First, it is important to improve the practical upper-bound  $\eta_M(s_b, s_\sharp, s_\bullet)$  for the maximal IPV amplitude  $\eta = \gamma(1, 0) - 1$ . Regardless of the success of this theoretical effort, it is also essential to perform numerical simulations with the new global IPV law using physically “sensible” values for  $\eta$ ,  $s_b$ ,  $s_\sharp$  and  $s_\bullet$ . The observation of numerical results on typical test cases will ultimately validate the IPV laws. This part requires appropriate numerical schemes to be developed and will be the subject of an upcoming work.

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