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# On Principal Value and Standard Extension of Distributions

Daniel Barlet\*

April 1, 2022

*To the memory of Jan Erik Björk who explains to me the theory of D-modules*

**ABSTRACT.** For a holomorphic function  $f$  on a complex manifold  $\mathcal{M}$  we explain in this article that the distribution associated to  $|f|^{2\alpha}(\text{Log}|f|^2)^q f^{-N}$  by taking the corresponding limit on the sets  $\{|f| \geq \varepsilon\}$  when  $\varepsilon$  goes to 0, coincides for  $\Re(\alpha)$  non negative and  $q, N \in \mathbb{N}$ , with the value at  $\lambda = \alpha$  of the meromorphic extension of the distribution  $|f|^{2\lambda}(\text{Log}|f|^2)^q f^{-N}$ . This implies that any distribution in the  $\mathcal{D}_{\mathcal{M}}$ -module generated by such a distribution has the Standard Extension Property. This implies a non torsion result for the  $\mathcal{D}_{\mathcal{M}}$ -module generated by such a distribution. As an application of this result we determine generators for the conjugate modules of the regular holonomic  $\mathcal{D}$ -modules associated to  $z(\sigma)^\lambda$ , the power  $\lambda$ , where  $\lambda$  is any complex number, of the (multivalued) root of the universal equation of degree  $k$ ,  $z^k + \sum_{j=1}^k (-1)^h \sigma_h z^{k-h} = 0$  whose structure is studied in [4].

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**KEY WORDS.** Principal Value- Meromorphic Extension- Regular Holonomic Distribution- Kashiwara Conjugation Functor- Standard Extension Property for Distributions.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Existence of the Principal Value</b>	<b>3</b>
2.1	The Standard Extension Property . . . . .	3
2.2	Principal Value . . . . .	4
<b>3</b>	<b>Bernstein identity and meromorphic extension</b>	<b>8</b>

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<b>4</b>	<b>Equality of the Principal Value with the value of the Meromorphic Extension</b>	<b>9</b>
4.1	The equality theorem . . . . .	9
4.2	Non torsion of the corresponding $\mathcal{D}$ -modules . . . . .	11
4.3	The case $\alpha < 0$ . . . . .	12
4.4	An easy generalization and an example . . . . .	13
<b>5</b>	<b>Some conjugate <math>\mathcal{D}</math>-modules</b>	<b>15</b>

# 1 Introduction

Let  $f : \mathcal{M} \rightarrow \mathbb{C}$  be a holomorphic function on a domain  $\mathcal{M}$  in  $\mathbb{C}^{n+1}$  and assume that  $f$  has no critical value different from 0. Let  $\alpha$  be a complex number with a non negative real part and  $N$  a positive integer. There are two natural ways to define a distribution on  $\mathcal{M}$  whose restriction to  $\mathcal{M} \setminus \{f = 0\}$  is equal to  $|f|^{2\alpha} f^{-N}$ . The first one is given by the principal value method (see for instance [11]):

Let  $\xi$  be a  $\mathcal{C}_c^\infty(\mathcal{M})$  differential form on  $\mathcal{M}$  of type  $(n+1, n+1)$  and define for  $\varepsilon > 0$  the distribution  $T_{\alpha, N}^\varepsilon$  by

$$\langle T_{\alpha, N}^\varepsilon, \xi \rangle := \int_{|f| \geq \varepsilon} |f|^{2\alpha} f^{-N} \xi.$$

We give in section 2 a rather short proof of the fact that the limit  $T_{\alpha, N}$  when  $\varepsilon$  goes to 0 of  $\langle T_{\alpha, N}^\varepsilon, \xi \rangle$  exists and defines a distribution  $T_{\alpha, N}$  on  $\mathcal{M}$  whose restriction to  $\mathcal{M} \setminus \{f = 0\}$  is equal to  $|f|^{2\alpha} f^{-N}$ .

The second method is to show that for  $\Re(\lambda)$  large enough and for any test differential form  $\xi \in \mathcal{C}_c^\infty(\mathcal{M})^{(n+1, n+1)}$  the function

$$\lambda \mapsto \int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \xi$$

is holomorphic and defines a holomorphic family of distributions on  $\mathcal{M}$ . Moreover this holomorphic family of distributions admits a meromorphic extension to the complex  $\lambda$ -plane with no pole when  $\Re(\lambda) \geq 0$ . Then,  $\alpha$  is not a pole and we define the distribution  $S_{\alpha, N}$  on  $\mathcal{M}$  as the value at  $\lambda = \alpha$  of this meromorphic extension.

We give a proof of the existence of  $T_{\alpha, N}$  and  $S_{\alpha, N}$  and we prove the equality in  $Db_{\mathcal{M}}$ , the sheaf of distributions on  $\mathcal{M}$ ,  $T_{\alpha, N} = S_{\alpha, N}$ .

Our main tools for these proofs are

1. The Asymptotic Expansion Theorem of fiber-integrals given in [2] (which uses Hironaka's Desingularization Theorem); see also [6].

2. The existence of a local Bernstein identity for  $f$ , due to J.E. Bjork in the analytic case (see [8] and [9]).

As an application we first deduce of this result the absence of torsion for some  $\mathcal{D}$ -modules generated by the distributions constructed in the first part. In fact we prove more : any distribution in such a  $\mathcal{D}$ -module has the Standard Extension Property (compare with the result of [5]). We recall the definition of the Standard Extension Property in the beginning of the section 2.

Then we apply this Theorem to the determination of generators of the conjugate  $\mathcal{D}$ -modules of the  $\mathcal{D}$ -modules  $\mathcal{N}_\lambda$  associated to  $z(\sigma)^\lambda$ , the power  $\lambda$ , where  $\lambda$  is any complex number, of the (multivalued) root of the universal equation of degree  $k$ ,  $z^k + \sum_{j=1}^k (-1)^h \sigma_h z^{k-h} = 0$  whose structure is studied in [4].

## 2 Existence of the Principal Value

### 2.1 The Standard Extension Property

First, we recall the definition of this property (see [3] in Appendix and [5] in Paragraph 3.1 or [10])

**Definition 2.1.1** *Let  $\mathcal{M}$  be a complex manifold of pure dimension  $n + 1$  and let  $T$  be a distribution on  $\mathcal{M}$ . We shall say that  $T$  has the **Standard Extension Property** if the following condition are fulfilled:*

1. *Outside a hypersurface  $H$  in  $\mathcal{M}$  the distribution  $T$  is a  $\mathcal{C}^\infty$  function.*
2. *For each point  $x$  in  $H$  there exists an open neighborhood  $U$  of  $x$ , a local holomorphic equation  $\{f = 0\} = H \cap U$  of  $H$  in  $U$  such that for any test differential form  $\xi$  in  $\mathcal{C}_c^\infty(U)^{(n+1, n+1)}$  we have*

$$\langle T, \xi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|f| \geq \varepsilon} T\xi.$$

It is easy to see that if the condition 2 of the previous definition is satisfied for some choice of local equation of  $H$ , it is satisfied for any other choice.

Note also that this property is clearly stable by multiplication by a  $\mathcal{C}^\infty$  function but is not stable, in general, by the action of  $\mathcal{D}_{\mathcal{M}}$ . For instance the locally integrable function  $1/\bar{z}$  defines a distribution on  $\mathbb{C}$  which has the standard extension property, but  $\partial_z(1/\bar{z}) = i\pi\delta_0$  is a non zero torsion element in  $Db_{\mathbb{C}}$ .

The following result is proved in [5]

**Theorem 2.1.2** *Let  $\mathcal{M}$  be a regular holonomic  $\mathcal{D}_{\mathcal{M}}$ -module and let  $H$  be a hypersurface in  $\mathcal{M}$  such that  $\mathcal{M}_{\mathcal{M} \setminus H}$  is  $\mathcal{O}_{\mathcal{M} \setminus H}$ -coherent. Let  $i : \mathcal{M} \rightarrow Db_{\mathcal{M}}$  be a  $\mathcal{D}_{\mathcal{M}}$ -linear*

morphism such that  $i$  belongs to  $L^2(c(\mathcal{M}))$ , the  $L^2$  lattice (see [5]) of the conjugate module of  $\mathcal{M}$  (see [13]). Then any distribution in  $i(\mathcal{M})$  has the Standard Extension Property (relatively to  $H$ ).

The main difficulty to use this theorem is the identification of the  $L^2$ -lattice. When the  $\mathcal{D}_{\mathcal{M}}$ -module has no  $\mathcal{O}_{\mathcal{M}}$ -torsion, the  $L^2$ -lattice is given by a local square integrability condition along  $H$  which is often rather easy to verify. But we must know "a priori" that the  $\mathcal{D}_{\mathcal{M}}$ -module we are considering has no  $\mathcal{O}_{\mathcal{M}}$ -torsion to use this simple characterization of the  $L^2$ -lattice. We shall see in section 5 that the main point is to prove that there is no torsion in some of the regular holonomic  $\mathcal{D}_{\mathcal{M}}$ -modules involved in our computations.

## 2.2 Principal Value

Let us begin by giving the precise Asymptotic Expansion Theorem for fiber-integrals proved in [1] which we shall need below. We keep the situation and the notations introduced in the beginning of section 1.

**Theorem 2.2.1** *Let  $\varphi \in \mathcal{C}_c^\infty(\mathcal{M})$  be a differential form of type  $(n, n)$  on  $\mathcal{M}$ . Define the function  $\Theta(s) := \int_{f=s} \varphi$  for each  $s \in \mathbb{C}$ . This function, which is  $\mathcal{C}^\infty$  on  $\mathbb{C} \setminus \{0\}$ , admits when  $s \rightarrow 0$  a asymptotic expansion of the form*

$$\Theta(s) \simeq \sum_{m, m'}^{r, j} a_{m, m'}^{r, j} |s|^{2r} (\text{Log } |s|)^j s^m \bar{s}^{m'} \quad (1)$$

where  $m, m'$  are non negative integers,  $r$  describes a finite set  $R \subset [0, 1[ \cap \mathbb{Q}$  and where  $j$  is an integer in  $[0, n + 1]$ . The finite set  $R$  is independent of the choice of  $\varphi$ . Moreover, this asymptotic expansion is term-wise differentiable at any order. ■

### REMARKS.

1. The continuity of the function  $\Theta$  at the point  $s = 0$  implies that for  $r = 0$  and  $j \geq 1$  we have  $m + m' \geq 1$ . In fact it is proved in Proposition 6 of *loc. cit.* that for  $j \geq 1$  we have  $a_{m, 0}^{0, j} = 0$  (resp.  $a_{0, m'}^{0, j} = 0$ ) which implies that for  $r = 0$  and  $j \geq 1$  we have  $m \geq 1$  and  $m' \geq 1$  when  $a_{m, m'}^{0, j} \neq 0$ . This shows that no new type of term appears in such an expansion when we apply  $s\partial_s$  or  $\bar{s}\partial_{\bar{s}}$ . And, of course, neither a constant term nor a term like  $s^m (\text{Log } |s|)^j$  or  $\bar{s}^{m'} (\text{Log } |s|)^j$  with  $j \geq 1$  may appear in these expansions. So the functions  $s\partial_s\Theta$ ,  $\bar{s}\partial_{\bar{s}}\Theta$  and  $s\bar{s}\partial_s\partial_{\bar{s}}\Theta$  are bounded by  $O(|s|^\gamma)$  for some  $\gamma > 0$  when  $s$  goes to 0.
2. It is also proved, in *loc. cit.* that the linear map  $\varphi \mapsto a_{m, m'}^{r, j}(\varphi)$  is a  $(1, 1)$ -current on  $\mathcal{M}$ . This current is supported by the singular set of the hypersurface  $\{f = 0\}$  when  $(r, j) \neq (0, 0)$ , that is to say for terms which do not correspond to an usual term of a Taylor expansion at the origin of a  $\mathcal{C}^\infty$  function.

It will be important in the sequel to use the following corollary of this theorem.

**Corollary 2.2.2** *In the situation of the previous theorem, for any differential  $\mathcal{C}^\infty$  form  $\psi$  and  $\xi$  respectively of type  $(n, n+1)$  and  $(n+1, n+1)$  with  $f$ -proper supports in  $\mathcal{M}$  the fiber-integrals*

$$\eta(s) := \int_{f=s} \bar{f} \frac{\psi}{d\bar{f}} = \bar{s} \int_{f=s} \frac{\psi}{d\bar{f}} \quad \text{and} \quad \zeta(s) := \int_{f=s} f \bar{f} \frac{\xi}{df \wedge d\bar{f}} = s\bar{s} \int_{f=s} \frac{\xi}{df \wedge d\bar{f}}$$

*admit asymptotic expansions of the same type as above when  $s$  goes to 0 (but see the previous remark 1).*

REMARK. For any compact set  $K$  in  $\{f = 0\}$  there exists an integer  $N$  such that we have the inclusion of sheaves in an open neighborhood of  $K$  in  $\mathcal{M}$

$$f^N \Omega_{\mathcal{M}}^{n+1} \subset df \wedge \Omega_{\mathcal{M}}^n \quad \text{and} \quad f^N \bar{f}^N \Omega_{\mathcal{M}}^{n+1} \wedge \bar{\Omega}_{\mathcal{M}}^{n+1} \subset \Omega_{\mathcal{M}}^n \wedge \bar{\Omega}_{\mathcal{M}}^n \wedge df \wedge d\bar{f}.$$

So the only new information in the previous corollary is that we may keep  $m$  and  $m'$  non negative in the expansions.

PROOF. Remark first that the previous theorem is in fact local around the hypersurface  $\{f = 0\}$  in  $\mathcal{M}$  and so the asymptotic expansion is valid for any  $\mathcal{C}^\infty$  differential form of type  $(n, n)$  with  $f$ -proper support in  $\mathcal{M}$ .

Then using the fact that for such a form  $\varphi$  we have the same type of asymptotic expansion for

$$s\partial_s\Theta(s) = \int_{f=s} \frac{fd'\varphi}{df}, \quad \bar{s}\partial_s\Theta(s) = \int_{f=s} \frac{\bar{f}d''\varphi}{d\bar{f}} \quad \text{and for} \quad s\bar{s}\partial_s\partial_s\Theta(s) = \int_{f=s} \frac{f\bar{f}d'd''\varphi}{df \wedge d\bar{f}}$$

we see that it is enough to show that we have the local surjectivity of  $d', d'', d'd''$  with  $f$ -proper supports for the type  $(n+1, n)$ ,  $(n, n+1)$  and  $(n+1, n+1)$  respectively. Using a partition of unity, this is consequence of the following lemma and the local parametrization theorem for the hypersurface  $\{f = 0\}$ <sup>1</sup>.

**Lemma 2.2.3** *Let  $U$  be an open polydisc in  $\mathbb{C}^n$  and  $D$  a disc in  $\mathbb{C}$ . Let  $\psi$  and  $\xi$  be  $\mathcal{C}^\infty$  forms respectively of type  $(n, n+1)$  and  $(n+1, n+1)$  with support in  $K \times D$  where  $K \subset U$  is a compact set. Fix a relatively compact open disc  $D'$  in  $D$ . Then there exists  $\varphi_1$  and  $\varphi_2$  which are  $\mathcal{C}^\infty$  forms of type  $(n, n)$  and with support in  $K \times D$  such that*

$$d''\varphi_1 = \psi \quad \text{and} \quad d'd''\varphi_2 = \xi$$

*on  $U \times D'$ .* ■

---

<sup>1</sup>In order that we may choose, near each point in  $\{f = 0\}$ ,  $U$  and  $D$  such that  $\bar{U} \times \partial D$  does not meet  $\{f = 0\}$  in our next lemma.

PROOF. This is an easy consequence of the fact that  $\partial_z$  and  $\partial_z\partial_{\bar{z}}$  are surjective on  $\mathcal{C}^\infty(D)$  and that we may solve the corresponding equations with  $\mathcal{C}^\infty$  dependence of a parameter on a relatively compact open disc  $D' \subset\subset D$  using a fundamental solution of the corresponding operator (see for instance [7] chapter IV Proposition 5.2.4 for details).  $\blacksquare$

REMARK. Note that  $d'\bar{\varphi}_1 = \bar{\psi}$  gives also the  $d'$ -case.

**Theorem 2.2.4** *In the situation  $f : \mathcal{M} \rightarrow \mathbb{C}$  introduced in section 1, let  $\alpha$  be a complex number such that its real part  $\Re(\alpha)$  is non negative. Then, for any positive integer  $N$  and for any  $\mathcal{C}^\infty$  differential form  $\xi$  of type  $(n+1, n+1)$  with compact support in  $\mathcal{M}$  the limit when  $\varepsilon > 0$  goes to 0 of*

$$\langle T_{\alpha, N}^\varepsilon, \xi \rangle := \int_{|f| \geq \varepsilon} |f|^{2\alpha} f^{-N} \xi$$

*exists and defines a distribution (that is to say a  $(0, 0)$  current)  $T_{\alpha, N}$  on  $\mathcal{M}$ .*

The main argument to prove this result uses the following consequence of the previous corollary of the Asymptotic Expansion Theorem (see also [11] for a proof of the proposition below using a direct computation in a desingularization of the hypersurface  $\{f = 0\}$ ).

**Proposition 2.2.5** *Let  $\psi$  be a  $\mathcal{C}^\infty$  differential form of type  $(n, n+1)$  with  $f$ -proper support in  $\mathcal{M}$ . Then we have for any positive integer  $N$*

$$\lim_{\varepsilon \rightarrow 0} \int_{|f|=\varepsilon} f^{-N} \psi = 0.$$

PROOF. Note first that the result is local and is obvious near a point where  $f$  does not vanish. Around a point where  $f$  vanishes, Milnor's Fibration Theorem allows to use Fubini's Theorem to compute this integral for  $\varepsilon$  small enough as follows:

$$\int_{|f|=\varepsilon} f^{-N} \psi = \int_0^{2\pi} d\theta \int_{f=s} f^{-N} \frac{\psi}{d\theta}$$

where  $\psi/d\theta$  on  $\{f = s = \varepsilon e^{i\theta}\}$  is equal to  $\bar{f}\psi/idf$  because we have  $d\theta = id\bar{f}/\bar{f}$  and taking in account the type of  $\psi$  and the fact that  $\{f = s\}$  is a complex  $n$ -dimensional sub-manifold in  $\mathcal{M}$  for  $s \neq 0$ . This gives with  $s := \varepsilon e^{i\theta}$

$$\int_{|f|=\varepsilon} f^{-N} \psi = \int_0^{2\pi} d\theta \int_{f=s} f^{-N} \frac{\bar{f}\psi}{idf}.$$

When the integral  $\int_{f=s} \bar{f}\psi/df$  is  $\mathcal{O}(\varepsilon^{N+1})$  it is clear that the limit when  $\varepsilon$  goes to 0 exists and vanishes. So it is enough to show the result when we replace the fiber-integral

$$-i s^{-N} \bar{s} \partial_{\bar{s}} \Theta = \int_{f=s} f^{-N} \bar{f} \frac{\psi}{idf}$$

by its asymptotic expansion at a high enough order, and then, by linearity, to prove the assertion when we replace the fiber-integral by  $|s|^{2r}(\text{Log}|s|)^j s^{m-N} \bar{s}^{m'}$ . In this case the exponent of  $e^{i\theta}$  is given, for  $s = \varepsilon e^{i\theta}$ , by :

$$-N + m - m'$$

and the corresponding exponent of  $\varepsilon$  is given by

$$2r + m + m' - N.$$

In order to find a non zero integral between 0 and  $2\pi$  we need that  $m = m' + N$  and then the exponent of  $\varepsilon$  is then equal to  $2r + 2m'$ . So the only terms where the limit is not clearly equal to 0 appear when  $r = m' = 0$ . But in this case the limit is again zero thanks to Remark 1 following Theorem 2.2.1.  $\blacksquare$

**PROOF OF THEOREM 2.2.4.** The assertion is local near each point of  $\{f = 0\}$  so, using Lemma 2.2.3 (in fact the remark following it) we may assume that  $\xi = d'\psi$  where  $\psi$  is a  $\mathcal{C}^\infty$  differential form of type  $(n, n+1)$  with f-proper support. Then Stokes' Formula gives

$$(\alpha - N) \int_{|f| \geq \varepsilon} |f|^{2\alpha} f^{-N} \frac{df \wedge \psi}{f} + \int_{|f| \geq \varepsilon} |f|^{2\alpha} f^{-N} \xi = \int_{|f| = \varepsilon} |f|^{2\alpha} f^{-N} \psi.$$

We know that the limit of the right hand-side is 0 when  $\varepsilon$  goes to 0 thanks to Proposition 2.2.5. Now write the first integral in the left hand-side as

$$\int_{|s| \geq \varepsilon} |s|^{2\alpha} s^{-N} \frac{ds \wedge d\bar{s}}{s\bar{s}} \int_{f=s} \frac{\bar{f}\psi}{d\bar{f}}$$

and using polar coordinates  $s = \rho e^{i\theta}$  so  $ds \wedge d\bar{s} = -2i\rho d\rho d\theta$  and the asymptotic expansion of the fiber-integral of  $\bar{f}\psi/d\bar{f}$  we may replace this fiber-integral by its asymptotic expansion at the origin at a sufficient large order. Then, using linearity, it is enough to consider the integrability at 0 of each non zero term which only depends on the real part of  $\alpha$ . So we have only to consider the terms for which

$$-N + m - m' = 0.$$

The corresponding real part of the power of  $\rho$  is given by

$$-N - 1 + 2r + m + m' + 2\Re(\alpha) = -1 + 2r + 2m' + 2\Re(\alpha).$$

This is at least equal to  $-1$  for  $\Re(\alpha) \geq 0$  because  $m' + r \geq 0$ . Then either we have  $r + m' + \Re(\alpha) > 0$  and this implies the integrability at 0, or  $r = m' = \Re(\alpha) = 0$ . And in this case, thanks to Remark 1 following Theorem 2.2.1, the term to integrate is  $O(|s|^{\gamma-1})$  for some  $\gamma > 0$ , giving again the local integrability at 0. So the limit of  $\langle T_{\alpha, N}^\varepsilon, \xi \rangle$  exists for each test form  $\xi$  on  $\mathcal{M}$  when  $\varepsilon$  goes to 0.

The fact that the so obtained linear form on test differential forms is a distribution is an easy exercise left to the reader.  $\blacksquare$

Note that in the previous proof, we may conclude with the weaker hypothesis asking that  $\Re(\alpha) + r > 0$  for any  $r$  in  $R \cup \{1\} \setminus \{0\}$ . Remark that with this hypothesis  $\Re(\alpha)$  is not a pole for the meromorphic extension of  $F_{N, \xi}(\lambda)$ ,  $\forall N \in \mathbb{N}$ , defined below.



REMARK R1. If we replace  $|f|^{2\alpha}$  by  $|f|^{2\alpha}(\text{Log}|f|^2)^\beta$  where  $\beta$  is any positive real number, the same result holds true with the same proof.

### 3 Bernstein identity and meromorphic extension

We first recall the fundamental theorem of Bernstein [8], generalized to the local analytic case by Bjork [9].

**Theorem 3.0.1** *Let  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$  be a non zero holomorphic germ. Then there exists a monic polynomial  $b \in \mathbb{C}[\lambda]$  and, near 0 in  $\mathbb{C}^{n+1}$ , a holomorphic partial differential operator  $P$  with polynomial coefficients in  $\lambda$  such that the identity*

$$P(z, \partial_z, \lambda) f^{\lambda+1} = b(\lambda) f^\lambda \quad (2)$$

*holds locally outside the hypersurface  $\{f = 0\}$  on some open neighborhood of the origin.*

The minimal monic polynomial  $b \in \mathbb{C}[\lambda]$  such that an identity of this kind holds near 0 is called the **Bernstein polynomial of  $f$  at the origin**.

Recall also the fundamental result of Kashiwara [12].

**Theorem 3.0.2** *For any non zero holomorphic germ  $f$  at the origin, the roots of the Bernstein polynomial of  $f$  at the origin are rational and negative.* ■

**Corollary 3.0.3** *Let  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$  be a non zero holomorphic germ and let  $\mathcal{M}$  be an open neighborhood of the origin on which the Bernstein identity of  $f$  is valid<sup>2</sup>. For any  $N \in \mathbb{N}$  and any test differential form  $\xi \in \mathcal{C}_c^\infty(\mathcal{M})^{(n+1, n+1)}$  the function defined for  $\lambda \in \mathbb{C}$  such that  $\Re(\lambda) \gg N$  by*

$$F_{N, \xi}(\lambda) := \int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \xi$$

*is holomorphic and admits a meromorphic extension to the complex  $\lambda$ -plane with poles of order at most  $n + 1$  at points in  $\cup_r \{r - \mathbb{N}\}$  where  $r$  is a root of the Bernstein polynomial of  $f$  at the origin.*

*Moreover, for any  $\alpha \in \mathbb{C}$ , the linear forms on  $\mathcal{C}_c^\infty(\mathcal{M})^{(n+1, n+1)}$  given by the coefficient  $P_k(\lambda = \alpha, F_{N, \xi}(\lambda))$  of  $(\lambda - \alpha)^{-k}$ ,  $k \in \mathbb{Z}$ ,  $k \leq n + 1$ , in the Laurent expansion at  $\lambda = \alpha$  of the meromorphic extension of  $F_{N, \xi}$  is a distribution on  $\mathcal{M}$ .*

*For  $k \in [1, n + 1]$  this distribution has support in  $\{f = 0\}$  and, for  $k \in [2, n + 1]$  or for  $k \in [1, n + 1]$  and  $\alpha \notin \mathbb{Z}$ , the support of this distribution is contained in the singular set of  $\{f = 0\}$ .*

---

<sup>2</sup>This means precisely that the Bernstein identity for  $f$  is valid in the universal cover of  $\mathcal{M} \setminus \{f = 0\}$ , on which  $f^\lambda$  is defined as  $\exp(\lambda \text{Log } f)$  for any given determination of  $\text{Log } f$ .

PROOF. The equation (2) implies, as  $b$  has rational coefficients, that we may find for any positive integer  $M$  a anti-holomorphic differential operator  $P_M$  depending polynomially of  $\lambda$  such that we have

$$P_M(\bar{f}^{\lambda+M}) = b(\lambda) \dots b(\lambda + M - 1) \bar{f}^\lambda. \quad (3)$$

Note that the roots of  $B_M(\lambda) := b(\lambda) \dots b(\lambda + M - 1)$  are negative rational numbers. So for  $M \gg N$  we obtain the equality of continuous functions for  $\Re(\lambda) \gg N$

$$|f|^{2\lambda} f^{-N} = \frac{1}{B_M(\lambda)} P_M(|f|^{2\lambda} f^{-N} \bar{f}^M)$$

If  $P_M^*$  is the adjoint of  $P_M$  this implies for  $\Re(\lambda) \gg N$  and  $M$  large enough, that for any test differential form  $\xi \in \mathcal{C}_c^\infty(\mathcal{M})^{(n+1, n+1)}$  we obtain

$$\int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \xi = \frac{1}{B_M(\lambda)} \int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \bar{f}^M P_M^*(\xi). \quad (4)$$

This gives the meromorphic extension to the complex  $\lambda$ -plane of the holomorphic distribution  $\xi \mapsto \int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \xi$  defined for  $\Re(\lambda) \gg N$ , because  $P_M^*(\xi)$  is  $\mathcal{C}_c^\infty$  in  $\mathcal{M}$  and depends polynomially on  $\lambda$  and because the right hand-side of the above formula is holomorphic on the open set  $\Re(\lambda) > -m$  for any given positive integer  $m$  as soon as  $M$  is large enough compare to  $N + m$ .

Moreover this meromorphic extension has no pole at points which are not inside the union of the sets  $r - \mathbb{N}$  where  $r$  is a root of  $b$ .

It is easy to see that near points where  $f$  does not vanish, this meromorphic extension has no pole and that near points where  $f = 0$  but where  $df$  does not vanish the poles of this meromorphic extension are at most simple poles at negative integers. This complete the proof.  $\blacksquare$

REMARK R2. Let  $q$  be a positive integer. The  $q$ -th derivative in  $\lambda$  of the holomorphic function  $F_{N,\xi}(\lambda)$  is given, for  $\Re(\lambda)$  large enough by the absolutely converging integral

$$\int_{\mathcal{M}} |f|^{2\lambda} (\text{Log}|f|^2)^q f^{-N} \xi$$

and the meromorphic extension of these functions allows to define, for each integer  $q$ , a meromorphic distributions on  $\mathcal{M}$  which has no pole for  $\Re(\lambda) \geq 0$ .

Now analog arguments as above give easily a generalization of the previous theorem to these cases.

## 4 Equality of the Principal Value with the value of the Meromorphic Extension

### 4.1 The equality theorem

We keep the notations of the introduction.

**Definition 4.1.1** *In the situation above we define the distribution  $S_{\alpha,N}$  on  $\mathcal{M}$  by the formula*

$$\langle S_{\alpha,N}, \xi \rangle := P_0(\lambda = \alpha, \int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \xi).$$

The aim of this paragraph is to prove the following result:

**Theorem 4.1.2** *Assume that  $\Re(\alpha) \geq 0$ . Then for any positive integer  $N$  we have for any test differential form  $\xi$ :*

$$\langle S_{\alpha,N}, \xi \rangle = \langle T_{\alpha,N}, \xi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|f| \geq \varepsilon} |f|^{2\alpha} f^{-N} \xi.$$

PROOF. We want to show that the analog of the equality (4) for  $\lambda = \alpha$  holds if we perform the integration only on the subset  $\{|f| \geq \varepsilon\}$  with an error which goes to zero when  $\varepsilon$  goes to 0. This would be enough to complete the proof. But, of course, the error comes from the boundary terms which are integrals on  $\{|f| = \varepsilon\}$  appearing in the various Stokes Formulas necessary to pass from  $P_M$  to its adjoint  $P_M^*$ . It is easy to see that such "error" terms have the following shape: a polynomial in  $\lambda$  with coefficient like

$$\int_{|f|=\varepsilon} |f|^{2\alpha} f^{-N} \bar{f}^{M'} \psi$$

where  $\psi$  is in  $\mathcal{C}_c^\infty(\mathcal{M})^{(n+1,n)}$  and  $M'$  is an integer in  $[0, M]$ . Now using the same arguments than in the proof of Theorem 2.2.4 we see that the only non zero term in such an integral comes from the coefficient of  $s^m \bar{s}^{m'}$  in the asymptotic expansion at  $s = 0$  of the function  $s \mapsto \int_{f=s} \bar{f} \psi / d\bar{f}$  such that

$$m - m' - N - M' = 0.$$

And this non zero term comes with a power of  $\varepsilon$  which is at least equal to

$$2\Re(\alpha) + m + m' - N + M' + 2r = 2\Re(\alpha) + 2m' + 2M' + 2r \geq 0$$

and may be some  $(\text{Log } \varepsilon)^q$  factor. So such term goes to 0 when  $\varepsilon$  goes to 0 when  $m' + r + M' > 0$  for  $\Re(\alpha) \geq 0$  but also in the case where  $\Re(\alpha) + m' + M' + r = 0$ , thanks again to Remark 1 following Theorem 2.2.1. This concludes the proof.  $\blacksquare$

Again in the previous proof, we may conclude with the weaker hypothesis asking that  $\Re(\alpha) + r > 0$  for any  $r$  in  $R \cup \{1\} \setminus \{0\}$ .

REMARK R3. Using Remarks R1 and R2 we obtain again with the same proof, that the previous theorem is still valid if we replace  $|f|^{2\alpha}$  by  $|f|^{2\alpha} (\text{Log } |f|^2)^q$  for any positive integer  $q$ .

## 4.2 Non torsion of the corresponding $\mathcal{D}$ -modules

We shall deduce from Theorem 4.1.2 an important corollary. To formulate this result we need the following definition, where we keep the situation described in the introduction.

**Definition 4.2.1** *Let  $\alpha$  be a complex number with a non negative real part. Let  $V$  be an holomorphic vector field on  $\mathcal{M}$ . We define the **formal action** of  $V$  on  $|f|^{2\alpha} f^{-N}$  by the formula*

$$\langle V, |f|^{2\alpha} f^{-N} \rangle := (\alpha - N)V(f)|f|^{2\alpha} f^{-N-1}.$$

*Then this defines a "formal action" of  $\mathcal{D}_{\mathcal{M}}$  on the  $\mathcal{O}_{\mathcal{M}}$ -module  $\mathcal{O}_{\mathcal{M}}|f|^{2\alpha}[f^{-1}]$ .*

Remark that, thanks to Theorem 4.1.2, each element in  $\mathcal{O}_{\mathcal{M}}|f|^{2\alpha}[f^{-1}]$  defines an unique distribution on  $\mathcal{M}$  having the standard extension property along the hypersurface  $\{f = 0\}$ . This gives a natural  $\mathcal{O}_{\mathcal{M}}$ -linear embedding of this  $\mathcal{O}_{\mathcal{M}}$ -module in  $Db_{\mathcal{M}}$ .

**Corollary 4.2.2** *The action of any  $P \in \mathcal{D}_{\mathcal{M}}$  on any element in the sub- $\mathcal{O}_{\mathcal{M}}$ -module  $\mathcal{O}_{\mathcal{M}}|f|^{2\alpha}[f^{-1}] \subset Db_{\mathcal{M}}$  coincides with the formal action defined above.*

*In particular, the sub- $\mathcal{D}_{\mathcal{M}}$ -module generated by  $Z_{\alpha}$  in  $Db_{\mathcal{M}}$  has no torsion, where  $Z_{\alpha}$  is the distribution on  $\mathcal{M}$  associated to the locally bounded function  $|f|^{2\alpha}$  on  $\mathcal{M}$ .*

PROOF. Let  $V$  be a holomorphic vector field on  $\mathcal{M}$ . Then let  $T_{\alpha,N}$  be the distribution defined by

$$\langle T_{\alpha,N}, \xi \rangle := P_0(\lambda = \alpha, \int_{\mathcal{M}} |f|^{2\alpha} f^{-N} \xi)$$

and let  $V^*$  be the adjoint of  $V$ ; we obtain:

$$\begin{aligned} \langle V(T_{\alpha,N}), \xi \rangle &= \langle T_{\alpha,N}, V^*(\xi) \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|f| \geq \varepsilon} |f|^{2\alpha} f^{-N} V^*(\xi) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{|f| \geq \varepsilon} V(|f|^{2\alpha} f^{-N}) \xi \end{aligned}$$

because the boundary term on  $\{|f| = \varepsilon\}$  has limit 0 when  $\varepsilon$  goes to 0 using the fact that

$$V(|f|^{2\alpha} f^{-N}) = (\alpha - N)V(f)|f|^{2\alpha} f^{-N-1},$$

the same argument than in the proof of Theorem 4.1.2 and the remark that  $V(f)\xi$  is in  $\mathcal{C}_c^{\infty}(\mathcal{M})^{(n+1, n+1)}$ .

This gives, using again Theorem 4.1.2, that  $V(T_{\alpha,N}) = V(f)T_{\alpha, N+1}$  in  $Db_{\mathcal{M}}$ .

As holomorphic vector fields generate the  $\mathcal{O}_{\mathcal{M}}$ -algebra  $\mathcal{D}_{\mathcal{M}}$ , this is enough to complete the proof. ■

REMARK R4. The generalization of the previous corollary to the cases where we replace  $|f|^{2\alpha}$  by  $|f|^{2\alpha}(\text{Log}|f|^2)^q$  for any positive integer  $q$  is again an easy exercise.

### 4.3 The case $\alpha < 0$

We shall explain now how to define, when  $\alpha$  is a negative real number<sup>3</sup> a "Finite Part" of the integral

$$\int_{\mathcal{M}} |f|^{2\alpha} f^{-N} \xi$$

using the Asymptotic Expansion Theorem. For a differential form  $\xi \in \mathcal{C}_c^\infty(\mathcal{M})^{(n+1, n+1)}$ , we shall write as follows the asymptotic expansion when  $s$  goes to 0 of the fiber-integral

$$s \mapsto \int_{f=s} f \bar{f} \xi / df \wedge d\bar{f} \simeq \sum_{r, j, m, m'} \tilde{T}_{m, m'}^{r, j}(\xi) |s|^{2r} s^m \bar{s}^{m'} (\text{Log}|s|)^j$$

where  $\xi \mapsto \tilde{T}(\xi)$  is a distribution with support in  $\{f = 0\}$  for each  $r \in R$  a finite subset in  $[0, 1[\cap \mathbb{Q}$ , for  $j \in [0, n+1]$  and for  $m, m' \in \mathbb{N}$  (see the Corollary 2.2.2 and also also Remark 1 following Theorem 2.2.1).

**Theorem 4.3.1** *For  $\alpha$  a negative real number such that  $-\alpha \notin R + \mathbb{N}$  the following limit exists and defines a distribution on  $\mathcal{M}$*

$$\lim_{\varepsilon \rightarrow 0} \left( \int_{\{|f| \geq \varepsilon\}} |f|^{2\alpha} f^{-N} \xi + 2i\pi \sum_{m=m'+N}^{\alpha+r+m' < 0} \tilde{T}_{m, m'}^{r, j}(\xi) \frac{\varepsilon^{2(\alpha+r+m')}}{2(\alpha+r+m')} (-\text{Log}\varepsilon)^j \right)$$

which extends to  $\mathcal{M}$  the function  $|f|^{2\alpha} f^{-N}$  on  $\{f \neq 0\}$ .

If  $-\alpha$  is in  $R + \mathbb{N}$ , write  $-\alpha = r + m'$ ,  $r \in R$  and  $m' \in \mathbb{N}$  and add to the sum inside the limit in the left hand-side above the sum

$$2i\pi \sum_{j=0}^{n+1} \tilde{T}_{m'+N, m'}^{r, j}(\xi) (-\text{Log}\varepsilon)^{j+1} / (j+1).$$

Then the result is analogous.

PROOF. Using the asymptotic expansion for the fiber-integral  $\int_{f=s} f \bar{f} \xi / df \wedge d\bar{f}$  recalled above, the proof that the limit in the left hand-side exists is analogous to the proof of Theorem 2.2.4. ■

Of course, the constant term of the Laurent development at  $\lambda = \alpha$  of the meromorphic extension of the function  $\lambda \mapsto \int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \xi$  gives also such a distribution. But in the case  $\alpha < 0$  the relation between the distribution defined by

$$\xi \mapsto P_0(\lambda = \alpha, \int_{\mathcal{M}} |f|^{2\lambda} f^{-N} \xi)$$

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<sup>3</sup>We leave the case where  $\alpha$  is a complex number with a negative real part as an exercise.

and the distribution defined in the previous theorem is not so clear in general, even if the difference between these two distributions has clearly its support in  $\{f = 0\}$ . Of course, when  $\alpha$  satisfies the condition  $\alpha + r > 0$  for each  $r$  in  $R \cup \{1\} \setminus \{0\}$ , not only there is no term in the sum where  $\alpha + r + m' \leq 0$  (using Remark 1 following Theorem 2.2.1 for the case 0) and the equality of the two distributions follows from the remark following Theorem 4.1.2.

#### 4.4 An easy generalization and an example

Consider now  $p \geq 2$  domains  $\mathcal{M}_i$  in  $\mathbb{C}^{n_i+1}$  for  $i \in [1, p]$  and  $p$  holomorphic functions  $f_i : \mathcal{M}_i \rightarrow \mathbb{C}$ . Then on  $\mathcal{M} := \prod_{i=1}^p \mathcal{M}_i$  it is easy to prove, for  $\alpha := (\alpha_1, \dots, \alpha_p)$  satisfying  $\Re(\alpha_i) \geq 0, \forall i \in [1, p]$ , using Fubini's Theorem, the existence of the distribution

$$\langle T_{\alpha, N}, \xi \rangle := \lim_{\varepsilon \rightarrow 0} \langle T_{\alpha}^{\varepsilon}, \xi \rangle$$

where  $\varepsilon := (\varepsilon_1, \dots, \varepsilon_p)$  is in  $(\mathbb{R}^{+*})^p$ ,  $N := (N_1, \dots, N_p)$  is in  $\mathbb{N}^p$ ,  $\xi$  is in  $\mathcal{C}_c^{\infty}(\mathcal{M})^{(n+1, n+1)}$ ,  $n+1 := \sum_{i=1}^p n_i + 1$  and where

$$\langle T_{\alpha, N}^{\varepsilon}, \xi \rangle := \int_{\prod_{i=1}^p \{|f_i| \geq \varepsilon_i\}} \left( \prod_{i=1}^p |f_i|^{2\alpha_i} f_i^{-N_i} \right) \xi.$$

It is also easy to make the meromorphic extension to  $\lambda := (\lambda_1, \dots, \lambda_p) \in \mathbb{C}^p$  of the holomorphic distributions on  $\mathcal{M}$  defined for  $\prod_{i=1}^p \{\Re(\lambda_i) \gg N_i\}$  and then to prove the following generalization of Theorem 4.1.2 to this "product" case:

**Theorem 4.4.1** *In the product situation described above the meromorphic extension of the distribution*

$$\xi \mapsto \int_{\mathcal{M}} \left( \prod_{i=1}^p |f_i|^{2\lambda_i} f_i^{-N_i} \right) \xi$$

*is holomorphic near the point  $\alpha \in \mathbb{C}^p$  satisfying  $\Re(\alpha_i) \geq 0, \forall i \in [1, p]$ , and we have the equality*

$$P_0\left(\lambda = \alpha, \int_{\mathcal{M}} \left( \prod_{i=1}^p |f_i|^{2\lambda_i} f_i^{-N_i} \right) \xi\right) = \langle T_{\alpha, N}, \xi \rangle$$

*where the left hand-side denotes the value at  $\lambda = \alpha$  of the meromorphic extension.*

PROOF. The only point to precise in order to apply the  $p = 1$  case and Fubini's Theorem successively to prove this generalization is the following remark:

- Let  $\mathcal{M}_1 \times \mathcal{M}_2$  be the product of two complex manifolds  $\mathcal{M}_1 \times \mathcal{M}_2$  and let  $\xi$  be a  $\mathcal{C}_c^{\infty}$  test differential form on  $\mathcal{M}_1 \times \mathcal{M}_2$  and  $T_2$  a distribution on  $\mathcal{M}_2$ . Then the test differential form defined on  $\mathcal{M}_1$  by  $\langle T_2, \xi \rangle$  is a  $\mathcal{C}_c^{\infty}$  test differential form on  $\mathcal{M}_1$ . So for any distribution  $T_1$  on  $\mathcal{M}_1$  the distribution  $T_1 \boxtimes T_2$  is well defined on  $\mathcal{M}_1 \times \mathcal{M}_2$  by the rule

$$\langle T_1 \boxtimes T_2, \xi \rangle := \langle T_1, \langle T_2, \xi \rangle \rangle.$$

REMARK. Note that the definition of  $T_{\alpha,N}$  implies that we have

$$\langle T_{\alpha,N}, \xi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|\prod_{i=1}^p f_i| \geq \varepsilon} \left( \prod_{i=1}^p |f_i|^{2\alpha_i} f_i^{-N_i} \right) \xi.$$

Then, using the argument in Corollary 4.2.2 on each  $\mathcal{M}_i, i \in [1, p]$  we obtain that for any  $P \in \mathcal{D}_{\mathcal{M}}$  the distribution  $PT_{\alpha,N}$  has the standard extension property for the hypersurface  $\{\prod_{i=1}^p f_i = 0\}$ . So the sub- $\mathcal{D}_{\mathcal{M}}$ -module of  $Db_{\mathcal{M}}$  generated by  $T_{\alpha,N}$  has no torsion.

Note also that this implies that the sub- $\mathcal{D}_{\mathcal{M}}$ -module generated by  $T_{\alpha,0}$  is contained in the  $\mathcal{O}_{\mathcal{M}}$ -module generated by the  $T_{\alpha,N}$  when  $N$  is in  $\mathbb{N}^p$ .

REMARK R5. Again it is easy to generalize the previous theorem to the cases where we replace  $|f_i|^{2\alpha_i}$  by  $|f_i|^{2\alpha_i} (\text{Log}|f_i|^2)^{q_i}$  for any non negative integers  $q_i, i \in [1, p]$ .

AN EXAMPLE.

NOTATIONS. Let  $\mathcal{M} := \mathbb{C}^k$  with coordinates  $z_1, \dots, z_k$  and let  $\pi : \mathcal{M} \rightarrow \mathcal{N} \simeq \mathbb{C}^k$  be the quotient by the action of the permutation group  $\mathfrak{S}_k$  on  $\mathcal{M}$ . Let  $\sigma_1, \dots, \sigma_k$  be the elementary symmetric polynomials in  $z_1, \dots, z_k$  which give a coordinate system on  $\mathcal{N}$ . Let also  $\Delta := \prod_{1 \leq i < j \leq k} (z_i - z_j)^2$  be the discriminant which is a polynomial in  $\sigma_1, \dots, \sigma_k$ .

**Lemma 4.4.2** Consider on  $\mathcal{M} := \mathbb{C}^k$  with coordinates  $z_1, \dots, z_k$  the holomorphic functions  $f := z_1$  and  $\Delta := \prod_{1 \leq i < j \leq k} (z_i - z_j)^2$ . Then defining new coordinates  $x_1 := z_1$  and  $x_h := z_h - z_1$  for  $h \in [2, k]$  we obtain a decomposition  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$  where  $\mathcal{M}_1 := \mathbb{C}$  with coordinate  $x_1$  and  $\mathcal{M}_2 := \mathbb{C}^{k-1}$  with coordinates  $x_h, h \in [2, k]$ . Moreover  $f$  is in  $\mathbb{C}[x_1]$  and  $\Delta$  is in  $\mathbb{C}[x_2, \dots, x_k]$ .

PROOF. The reader will see easily that  $\Delta = \prod_{h=2}^k x_h^2 \prod_{2 \leq i < j \leq k} (x_i - x_j)^2$ . ■

This easy lemma allows to apply Theorem 4.4.1 to the situation described in the previous lemma.

**Corollary 4.4.3** For any  $\alpha \in \mathbb{C}$  such that  $\Re(\alpha) \geq 0$  and any non negative integers  $q$  and  $N_1, N_2$  the distribution on  $\mathcal{M}$  associated to the locally integrable function

$$X_{\alpha, N_1, N_2, q} := \left( \sum_{j=1}^k |z_j|^{2\alpha} (\text{Log}|z_j|^2)^q z_j^{-N_1} \right) \Delta^{-N_2}$$

generates a sub- $\mathcal{D}_{\mathcal{M}}$ -module of the  $\mathcal{D}_{\mathcal{M}}$ -module  $Db_{\mathcal{M}}$  which has no torsion. Moreover, any distribution in this sub- $\mathcal{D}_{\mathcal{M}}$ -module has the standard extension property along the divisor  $\{\pi^{-1}(\sigma_k \Delta(\sigma)) = 0\}$  in  $\mathcal{M}$  and is  $\mathcal{C}^\infty$  (in fact real analytic) outside

this divisor.

The same properties are true along the divisor  $\{\sigma_k \Delta(\sigma) = 0\}$ , for the sub- $\mathcal{D}_{\mathcal{N}}$ -module of  $Db_{\mathcal{N}}$  generated by the distribution  $\pi_*(X_{\alpha, N_1, N_2, q})$ .

PROOF. For our assertion on  $\mathcal{M}$  it is enough to prove that for any holomorphic vector field  $V$  on  $\mathcal{M}$  the distribution  $V(|z_1|^{2\alpha} z_1^{-N_1} \Delta^{-N_2})$  has the standard extension property and this is clear from the remark following Theorem 4.4.1 and Remark R5. The assertion on the corresponding  $\mathcal{D}_{\mathcal{N}}$ -module is consequence of the fact that if a distribution  $T$  on  $\mathcal{M}$  has the standard extension property along the divisor  $\{\pi^{-1}(\sigma_k \Delta(\sigma)) = 0\}$  then the distribution  $\pi_*(T)$  has the standard extension property along the divisor  $\{(\sigma_k \Delta(\sigma) = 0\}$  because the quotient map  $\pi$  is a finite étale map outside  $\{\Delta(\sigma) = 0\}$ .  $\blacksquare$

REMARK. If  $S$  and  $T$  are two distributions on a complex manifold  $\mathcal{N}$  such that  $\mathcal{D}_{\mathcal{N}}S \subset Db_{\mathcal{N}}$  and  $\mathcal{D}_{\mathcal{N}}T \subset Db_{\mathcal{N}}$  have no  $\mathcal{O}_{\mathcal{N}}$ -torsion, it is not clear that there is no  $\mathcal{O}_{\mathcal{N}}$ -torsion in  $\mathcal{D}_{\mathcal{N}}(S+T) \subset Db_{\mathcal{N}}$ . But if we know that  $\mathcal{D}_{\mathcal{N}}S$  and  $\mathcal{D}_{\mathcal{N}}T$  contain only distributions having the standard extension property for a given hypersurface  $H$ , then this also the case for any distribution in  $\mathcal{D}_{\mathcal{N}}S + \mathcal{D}_{\mathcal{N}}T \subset Db_{\mathcal{N}}$  which contains  $\mathcal{D}_{\mathcal{N}}(S+T)$ .

## 5 Some conjugate D-modules

The aim of this section is to give some examples of explicit computations of the conjugate module (in Kashiwara sense, see [13]) of some regular holonomic  $\mathcal{D}$ -modules. We consider here the case of the  $\mathcal{D}$ -modules associated to the multivalued functions  $z(\sigma)^\lambda$  for  $\lambda \in \mathbb{C}$  where  $z(\sigma)$  is the root of the universal degree  $k$  equation

$$z^k + \sum_{h=1}^k (-1)^h \sigma_h z^{k-h}$$

where  $\sigma := (\sigma_1, \dots, \sigma_k)$  is in  $N := \mathbb{C}^k$ . The structure of these regular holonomic  $\mathcal{D}_{\mathcal{N}}$ -modules has been described in [4] and we describe here for each such  $\mathcal{D}$ -module and for each simple factor which appears in its decomposition a distribution  $T$  on  $\mathcal{N}$  which generates the sub- $\mathcal{D}_{\mathcal{N}}$ -module of  $Db_{\mathcal{N}}$  which is the conjugate of the module we consider.

The reader will notice that even if in each case the corresponding distribution  $T$  is rather easily constructed from the horizontal (multivalued) basis of the corresponding vector bundle with a simple pole connection associated to the  $\mathcal{D}_{\mathcal{N}}$ -module under consideration, the proof that this distribution generates the conjugate module uses in a crucial way the non trivial argument of "non torsion" which is proved in the previous section (see Corollary 4.4.3).



We begin by recalling the basic results on the conjugation functor of M. Kashiwara. The following theorem is proved in [13]:

**Theorem 5.0.1 (Kashiwara conjugation functor)** *Let  $\mathcal{N}$  be a complex manifold,  $\mathcal{D}_{\mathcal{N}}$  the sheaf of holomorphic partial differential operators on  $\mathcal{N}$  and  $\mathcal{D}_{\bar{\mathcal{N}}}$  the sheaf of anti-holomorphic partial differential operator on  $\mathcal{N}$ . Note  $Db_{\mathcal{N}}$  the sheaf of distributions on  $\mathcal{N}$ . It is a left- $\mathcal{D}_{\mathcal{N}}$ -module but also a left- $\mathcal{D}_{\bar{\mathcal{N}}}$ -module and these two actions commute.*

*For each regular holonomic  $\mathcal{D}_{\mathcal{N}}$ -module  $\mathcal{N}$  the sub- $\mathcal{D}_{\bar{\mathcal{N}}}$ -module*

$$c_{\mathcal{N}}(\mathcal{N}) := \text{Hom}_{\mathcal{D}_{\mathcal{N}}}(\mathcal{N}, Db_{\mathcal{N}})$$

*is regular holonomic (as a  $\mathcal{D}_{\bar{\mathcal{N}}}$ -module) and the contra-variant functor  $c$  is an anti-equivalence of categories which satisfies  $c_{\bar{\mathcal{N}}} \circ c_{\mathcal{N}} = \text{Id}$ .* ■

Moreover, M. Kashiwara also obtains the following proposition which will be useful to describe the  $\mathcal{D}_{\bar{\mathcal{N}}}$ -modules  $c(\mathcal{N}_{\lambda})$ :

**Proposition 5.0.2 ( see [13] Prop. 5)** *If  $\mathcal{N}$  is a regular holonomic  $\mathcal{D}_{\mathcal{N}}$ -module on a complex manifold  $\mathcal{N}$  and if  $T$  is in  $c(\mathcal{N})$ , the following conditions are equivalent:*

a)  *$T$  is an injective sheaf homomorphism of  $\mathcal{N}$  to  $Db_{\mathcal{N}}$ .*

b)  *$T$  generates  $c(\mathcal{N})$  as a  $\mathcal{D}_{\bar{\mathcal{N}}}$ -module.* ■

NOTATIONS. We consider the quotient map  $\pi : \mathcal{M} := \mathbb{C}^k \rightarrow \mathcal{N} := \mathbb{C}^k / \mathfrak{S}_k \simeq \mathbb{C}^k$  with respective coordinates  $z_1, \dots, z_k$  and  $\sigma_1, \dots, \sigma_k$ , where  $\sigma_h$  is the  $h$ -th elementary symmetric function of  $(z_1, \dots, z_k)$ .

The vector fields on  $\mathcal{M}$  associated to partial derivatives in  $z_1, \dots, z_k$  are denoted by  $\partial_{z_1}, \dots, \partial_{z_k}$  and the vector fields on  $\mathcal{N}$  associated to partial derivatives in  $\sigma_1, \dots, \sigma_k$  are denoted by  $\partial_1, \dots, \partial_k$ .

We note  $\mathcal{I}$  the left ideal in  $\mathcal{D}_{\mathcal{N}}$  generated by the following global sections:

$$A_{p,q} = \partial_p \partial_q - \partial_{p+1} \partial_{q-1} \quad \text{where } (p, q) \in [1, k-1] \times [2, k]$$

$$\mathcal{T}^m := \partial_1 \partial_{m-1} + \partial_m E \quad \text{where } E := \sum_{h=1}^k \sigma_h \partial_h \quad \text{and } m \in [2, k]$$

Recall that a **trace function** on  $\mathcal{N}$  is a holomorphic function  $F$  such that there exists a holomorphic function  $f$  in one variable  $z$  such that

$$F(\sigma) = \sum_{j=1}^k f(z_j)$$

where  $\sigma := (\sigma_1, \dots, \sigma_k)$  are the elementary symmetric functions of  $z_1, \dots, z_k$ .

It is proved in [1] that the trace functions are annihilated by the left ideal  $\mathcal{I}$  in  $\mathcal{D}_{\mathcal{N}}$

and this characterizes the trace functions.

We note  $U_{-1}$  the vector field on  $\mathcal{N}$  given by  $U_{-1} := k\partial_1 + \sum_{h=1}^{k-1} (k-h)\sigma_h\partial_{h+1}$  which is the image by the tangent map  $T_\pi$  to  $\pi$  of the vector field  $V_{-1} := \sum_{j=1}^k \partial_{z_j}$  on  $\mathcal{M}$ .

We note  $U_0$  the vector field on  $\mathcal{N}$  given by  $U_0 := \sum_{h=1}^k h\sigma_h\partial_h$  which is the image by  $T_\pi$  of the vector field  $V_0 := \sum_{j=1}^k z_j\partial_{z_j}$  on  $\mathcal{M}$ .

We note  $U_1$  the vector field on  $\mathcal{N}$  given by  $U_1 := \sum_{h=1}^k (\sigma_1\sigma_h - (h+1)\sigma_{h+1})\partial_h$  which is the image by  $T_\pi$  of the vector field  $V_1 := \sum_{j=1}^k z_j^2\partial_{z_j}$  on  $\mathcal{M}$ .

The left ideal  $\mathcal{J}_\lambda$  in  $\mathcal{D}_{\mathcal{N}}$  is, by definition, the sum  $\mathcal{I} + \mathcal{D}_{\mathcal{N}}(U_0 - \lambda)$  and we define  $\mathcal{N}_\lambda := \mathcal{D}/\mathcal{J}_\lambda$ .

The following results which give the structure of the  $\mathcal{D}_{\mathcal{N}}$ -module  $\mathcal{N}_\lambda$  for each  $\lambda \in \mathbb{C}$  are proved in [4]:

1. For each complex number  $\lambda$ , the  $\mathcal{D}_{\mathcal{N}}$ -module  $\mathcal{N}_\lambda$  is holonomic and regular.
2. The right multiplication by  $U_{-1}$  induces a  $\mathcal{D}_{\mathcal{N}}$ -linear map

$$\square U_{-1} : \mathcal{N}_\lambda \rightarrow \mathcal{N}_{\lambda+1}$$

which is an isomorphism for each  $\lambda \neq -1, 0$ . Moreover the right multiplication by  $U_1$  induces an isomorphism  $\square U_1 : \mathcal{N}_{\lambda+1} \rightarrow \mathcal{N}_\lambda$  for any  $\lambda \neq 0, -1$  and we have  $\square U_1 \circ \square U_{-1} = \lambda(\lambda + 1)$  on  $\mathcal{N}_\lambda$ .

3. For  $\lambda \notin \mathbb{Z}$  the  $\mathcal{D}_{\mathcal{N}}$ -module  $\mathcal{N}_\lambda$  is simple.
4. The kernel  $\mathcal{N}_{-1}^\square$  of the  $\mathcal{D}_{\mathcal{N}}$ -linear map  $\varphi_{-1} : \mathcal{N}_{-1} \rightarrow \mathcal{O}_N(\star\sigma_k)$  defined by  $\varphi_{-1}(1) = \sigma_{k-1}/\sigma_k$  is simple.
5. The sub-module  $\mathcal{N}_0^\square$  generated by  $U_1$  in  $\mathcal{N}_0$  is simple and the quotient  $\mathcal{N}_0/\mathcal{N}_0^\square$  is isomorphic to  $\mathcal{O}_N(\star\sigma_k)$
6. The torsion sub-module  $\mathfrak{T}$  in  $\mathcal{N}_1$  is isomorphic to  $H_{[\sigma_k=0]}^1(\mathcal{O}_N)$ ; it is generated by the class of  $\partial_k U_{-1}$  in  $\mathcal{N}_1$ .
7. The sub-module  $Im(\square U_{-1})$  in  $\mathcal{N}_1$  (which is generated by  $U_{-1}$ ) contains  $\mathfrak{T}$  and the quotient  $Im(\square U_{-1})/\mathfrak{T}$  is isomorphic to  $\mathcal{O}_{\mathcal{N}}$  via the map  $\varphi_1 : \mathcal{N}_1 \rightarrow \mathcal{O}_N$  defined by  $\varphi_1(1) = \sigma_1$  (and then  $[U_{-1}] \mapsto k$ ).
8. The quotient  $\mathcal{N}_1^\square := \mathcal{N}_1/Im(\square U_{-1})$  is simple and isomorphic to the quotient

$$\mathcal{D}_{\mathcal{N}}/\mathcal{I} + \mathcal{D}_{\mathcal{N}}(U_0 - 1) + \mathcal{D}_{\mathcal{N}}U_{-1}.$$

9. The right multiplication by  $U_1$  which sends  $\mathcal{N}_1$  in  $\mathcal{N}_0^\square$  vanishes<sup>4</sup> on  $Im(\square U_{-1})$  and induces an isomorphism of  $\mathcal{N}_1^\square$  onto  $\mathcal{N}_0^\square$ .

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<sup>4</sup>It is proved in [4] Formula (19) page 20, that we have  $U_{-1}U_1 = (U_0 + 1)U_0$  modulo  $\mathcal{I}$ .

10. The right multiplication by  $U_{-1}$  which sends  $\mathcal{N}_{-1}$  to  $\mathcal{N}_0$  induces an isomorphism of  $\mathcal{N}_{-1}^\square$  onto  $\mathcal{N}_0^\square$ .

IMPORTANT REMARK. The point 2 recalled above shows that to study the  $\mathcal{D}_{\mathcal{N}}$ -modules  $\mathcal{N}_\lambda$  it is enough to consider the following cases:

- The cases  $\Re(\lambda) \in [0, 1[$  and  $\lambda \neq 0$ . We call it the case  $G$ .
- The cases  $\lambda = -1, 0, 1$ . We call them the case  $\lambda = -1, 0, 1$  respectively.

Then for any  $\lambda \in \mathbb{C}$  we reach one of these previous cases using an isomorphism given either by  $U_1^N$  or by  $U_{-1}^N$  for a suitable  $N \in \mathbb{N}$ .

Now define the following distributions on  $\mathcal{M}$ :

1. For  $\Re(\lambda) \in [0, 1[, \lambda \neq 0$  define  $X_\lambda := \sum_{j=1}^k |z_j|^{2\lambda}$ .
2.  $X_1 := \sum_{j=1}^k |z_j - \sigma_1/k|^2 = \sum_{j=1}^k |z_j|^2 - |\sigma_1|^2/k$ .
3.  $X_0 := \sum_{j=1}^k (\bar{z}_j - \bar{\sigma}_1/k) \text{Log}|z_j|^2 = \sum_{j=1}^k |z_j|^2 z_j^{-1} \text{Log}|z_j|^2 - (\bar{\sigma}_1/k) \text{Log}|\sigma_k|^2$ .
4.  $X_{-1} = \sum_{j=1}^k (\bar{z}_j - \bar{\sigma}_1/k) z_j^{-1} = \sum_{j=1}^k |z_j|^2 z_j^{-2} - (\bar{\sigma}_1/k) \sigma_{k-1}/\sigma_k$ .

For each case we define the distribution  $\mathcal{X}_\lambda := \pi_*(X_\lambda)$  on  $\mathcal{N}$ .

Define also the following distributions on  $\mathcal{N}$ :

1. For the case  $G$ :  $\mathcal{Y}_\lambda := 0$ .
2. For the case  $\lambda = 1$ :  $\mathcal{Y}_1 := \sigma_1/\bar{\sigma}_k$
3. For the case  $\lambda = 0$ :  $\mathcal{Y}_0 := 1/\bar{\sigma}_k$ .
4. For the case  $\lambda = -1$ :  $\mathcal{Y}_{-1} := \sigma_{k-1}/\sigma_k$ .

Then we have the following results:

**Theorem 5.0.3** [conjugate modules]

In all cases  $G, -1, 0, 1$  the distribution  $\mathcal{X}_\lambda$  defines an element in  $c(\mathcal{N}_\lambda)$  via the  $\mathcal{D}_{\mathcal{N}}$ -linear map sending 1 to  $\mathcal{X}_\lambda$ .

In case  $G$  the distribution  $\mathcal{X}_\lambda$  defines a generator of the (simple)  $\mathcal{D}_{\mathcal{N}}$ -module  $c(\mathcal{N}_\lambda)$ .

In cases  $\lambda = 1$  the distribution  $\mathcal{X}_1$  defines an element in  $c(\mathcal{N}_1^\square) \subset c(\mathcal{N}_1)$  and gives a generator of this (simple)  $\mathcal{D}_{\mathcal{N}}$ -sub-module.

In cases  $\lambda = -1, 0$  the simple module  $c(\mathcal{N}_\lambda^\square)$  is a quotient of  $c(\mathcal{N}_\lambda)$  and the image of  $\mathcal{X}_\lambda$  in  $c(\mathcal{N}_\lambda^\square)$  gives a generator of this (simple)  $\mathcal{D}_{\mathcal{N}}$ -module.

In all cases  $\lambda = -1, 0, 1$  the map sending 1 to  $\mathcal{Y}_\lambda$  is in  $c(\mathcal{N}_\lambda)$  and the  $\mathcal{D}_{\mathcal{N}}$ -module  $c(\mathcal{N}_\lambda)$  is generated by  $\mathcal{X}_\lambda$  and  $\mathcal{Y}_\lambda$ .

PROOF. Consider the first point. We have to prove that the left ideal  $\mathcal{J}_\lambda$  annihilates the distribution  $\mathcal{X}_\lambda$  for each case.

Corollary 4.4.3 and Remark R5 give that for  $\Re(\lambda) \geq 0$  any distribution in the  $\mathcal{D}_{\mathcal{M}}$ -module generated by the distribution

$$|z_j|^{2\lambda} (\text{Log}|z_j|^2)^q z_j^{-N}$$

has the standard extension property. So the same property holds for any distribution in the  $\mathcal{D}_{\mathcal{N}}$ -module generated by the distribution

$$\sum_{j=1}^k |z_j|^{2\lambda} (\text{Log}|z_j|^2)^q z_j^{-N}$$

along the hypersurface  $\{\sigma_k \Delta(\sigma) = 0\}$  thanks to Corollary 4.4.3 .

Remark also that the generator of the left ideal  $\mathcal{I}$  and the vector field  $U_0$  are in the left ideal of  $\mathcal{D}_{\mathcal{N}}$  generated by  $\partial_h, h \in [1, k-1]$  and  $\sigma_k \partial_k$  and that these vector fields annihilate the distribution  $1/\bar{\sigma}_k$  (see Lemma 5.0.4 below).

We have also  $\sigma_k \partial_k (\text{Log}|\sigma_k|^2) = 1$  in  $Db_{\mathcal{N}}$ .

Now the verification that in each case the distribution  $\mathcal{X}_\lambda$  is annihilated by the generator of the left ideal  $\mathcal{J}_\lambda$  is "formal" (see Corollary 4.4.3 ) because  $\mathcal{X}_\lambda$  is locally on the open set  $\{\sigma_k \Delta(\sigma) \neq 0\}$  a linear combination with anti-holomorphic coefficients of the holomorphic trace functions  $z_j(\sigma)^\lambda, j \in [1, k]$ .

Moreover it is clear that  $z_j(\sigma)^\lambda$  is homogeneous of degree  $\lambda$  on  $\mathcal{M}$  and the vector field  $U_0$  is equal to  $\pi_*(V_0)$  where  $V_0$  is the Euler vector field on  $\mathcal{M}$ .

In the case  $G$  we know that  $c(\mathcal{N}_\lambda)$  is a simple  $\mathcal{D}_{\mathcal{N}}$ -module. So the second point of the theorem is proved as  $\mathcal{X}_\lambda$  is clearly not 0 in  $Db_{\mathcal{N}}$ .

For  $\lambda = 1$ , as we know that ideal  $\mathcal{J}_1 = \mathcal{I} + \mathcal{D}_{\mathcal{N}}(U_0 - 1)$  already annihilates  $\mathcal{X}_1$  it is enough, thanks to the point 8 recalled above, to check that  $U_{-1}(\mathcal{X}_1) = 0$  in  $Db_{\mathcal{N}}$  to conclude because we know that  $\mathcal{N}_1^\square$  is simple and  $\mathcal{X}_1$  is not 0 in  $Db_{\mathcal{N}}$ . Again the fact that it is enough to check this on the open set  $\{\Delta(\sigma) \neq 0\}$  makes the computation easy :

$$V_{-1}(|z_j|^2) = \bar{z}_j \quad \text{and} \quad V_{-1}(|\sigma_1|^2) = k\bar{\sigma}_1 \quad \text{so} \quad U_{-1}(\mathcal{X}_1) = 0.$$

**Lemma 5.0.4** *Let  $\mathcal{M}$  be the  $\mathcal{D}_{\mathbb{C}}$ -module quotient of  $\mathcal{D}_{\mathbb{C}}$  by the left ideal generated by  $z\partial_z$ . Then  $c(\mathcal{M})$  is isomorphic to  $\mathcal{O}_{\bar{\mathbb{C}}}(*\bar{z})$ .*

PROOF. First recall that the distribution on  $\mathbb{C}$  associated to the locally integrable function  $1/\bar{z}$  satisfies  $\partial_z(1/\bar{z}) = i\pi\delta_0$ , where  $\delta_0$  is the Dirac mass at the origin. So the  $\mathcal{D}_{\bar{\mathbb{C}}}$ -sub-module of  $Db_{\mathbb{C}}$  generated by  $1/\bar{z}$  is contained in  $c(\mathcal{M})$ . To prove that it is equal to  $c(\mathcal{M})$ , thanks to Proposition 5.0.2, it is enough to prove that the kernel of the  $\mathcal{D}_{\mathbb{C}}$ -linear map  $\mathcal{M} \rightarrow Db_{\mathbb{C}}$  defined by  $1 \mapsto 1/\bar{z}$  is injective. But modulo the left ideal generated by  $z\partial_z$  any  $P \in \mathcal{D}_{\mathbb{C}}$  may be written as  $a_0 + \sum_{p=1}^m a_p \partial_z^p$  where  $a_0, \dots, a_p$  are complex numbers. Then  $a_0$  has to vanish and the  $\mathbb{C}$ -linear independence of the distributions  $\partial_z^{p-1}\delta_0, p \geq 1$  allows to conclude.  $\blacksquare$

**Corollary 5.0.5** *The conjugate of the sub- $\mathcal{D}_{\mathcal{N}}$ -module  $Im(\square U_{-1}) := \mathcal{D}_{\mathcal{N}}U_{-1} \subset \mathcal{N}_1$  is equal to  $\mathcal{D}_{\mathcal{N}}\sigma_1/\bar{\sigma}_k \subset Db_{\mathcal{N}}$  which is isomorphic to the  $\mathcal{D}_{\mathcal{N}}$ -module  $\mathcal{O}_{\mathcal{N}}(*\bar{\sigma}_k)$ . It is generated by the  $\mathcal{D}_{\mathcal{N}}$ -linear map sending the class of  $U_{-1}$  in  $Im(\square U_{-1}) \subset \mathcal{N}_1$  to the distribution  $\mathcal{Y}_1 = \sigma_1/\bar{\sigma}_k$ . So the distribution  $\mathcal{X}_1$  and  $\mathcal{Y}_1$  in  $Db_{\mathcal{N}}$  generate  $c(\mathcal{N}_1)$  as a  $\mathcal{D}_{\mathcal{N}}$ -sub-module of  $Db_{\mathcal{N}}$ .*

PROOF. It is proved in Proposition 4.1.7 and Lemma 4.1.8 of [4] that the annihilator of the class of  $U_{-1}$  in  $\mathcal{N}_1$  is generated by  $\partial_h, h \in [1, k-1]$  and  $\sigma_k\partial_k$ . Note that this is also the annihilator in  $\mathcal{D}_{\mathcal{N}}$  of the distribution  $1/\bar{\sigma}_k$  in  $Db_{\mathcal{N}}$  thanks to the previous lemma. This shows that the  $\mathcal{D}_{\mathcal{N}}$ -linear map defined by  $[U_{-1}] \mapsto 1/\bar{\sigma}_k \in Db_{\mathcal{N}}$  is injective, proving the isomorphism of  $\mathcal{D}_{\mathcal{N}}$ -modules  $c(Im(\square U_{-1})) \simeq \mathcal{D}_{\mathcal{N}}1/\bar{\sigma}_k \simeq \mathcal{O}_{\mathcal{N}}(*\bar{\sigma}_k)$  thanks to Proposition 5.0.2.

Then the exact sequence of  $\mathcal{D}_{\mathcal{N}}$ -modules

$$0 \rightarrow Im(\square U_{-1}) \rightarrow \mathcal{N}_1 \rightarrow \mathcal{N}_1^{\square} \rightarrow 0$$

gives the exact sequence of  $\mathcal{D}_{\mathcal{N}}$ -modules

$$0 \rightarrow c(\mathcal{N}_1^{\square}) \rightarrow c(\mathcal{N}_1) \xrightarrow{\alpha} c(Im(\square U_{-1})) \rightarrow 0$$

where the map  $\alpha$  is the quotient map. Now, let us prove that the distribution  $\mathcal{Y}_1 = \sigma_1/\bar{\sigma}_k$  is a generator of  $c(Im(\square U_{-1}))$  via the  $\mathcal{D}_{\mathcal{N}}$ -linear map  $U_{-1} \mapsto \mathcal{Y}_1$ . As  $\sigma_1$  commutes with the action of  $\mathcal{D}_{\mathcal{N}}$ , the annihilator of  $\sigma_1/\bar{\sigma}_k$  in  $\mathcal{D}_{\mathcal{N}}$  is the same than the annihilator of  $1/\bar{\sigma}_k$  proving our claim.

But we know that  $\mathcal{X}_1$  is a generator of  $c(\mathcal{N}_1^{\square})$ . So, it is enough to prove that  $\mathcal{Y}_1$  is in  $c(\mathcal{N}_1)$  to see that with  $\mathcal{X}_1$  they generate this  $\mathcal{D}_{\mathcal{N}}$ -module. This is proved in our next lemma and completes the proof of this corollary.  $\blacksquare$

**Lemma 5.0.6** *For  $\lambda = 1, 0, -1$  the distribution  $\mathcal{Y}_{\lambda}$  defines an element in  $c(\mathcal{N}_{\lambda})$  via the  $\mathcal{D}_{\mathcal{N}}$ -linear map  $1 \mapsto \mathcal{Y}_{\lambda}$ .*

PROOF. The fact that  $\mathcal{Y}_1$  is in  $c(\mathcal{N}_1)$  is easy because  $\sigma_1$  is a trace function which satisfies  $U_0(\sigma_1) = \sigma_1$ , because  $\partial_h, h \in [1, k-1]$  and  $\sigma_k\partial_k$  annihilate the distribution  $1/\bar{\sigma}_k$  and it is easy to check that the generators of  $\mathcal{I}$  and  $U_0$  are in this left ideal of  $\mathcal{D}_{\mathcal{N}}$  generated by these vector fields.

Now we have also, using the same argument, that  $\mathcal{Y}_0$  is in  $c(\mathcal{N}_0)$ .

To see that  $\mathcal{Y}_{-1}$  is annihilated by  $\mathcal{J}_{-1}$ , remark first that  $\mathcal{D}_{\mathcal{N}}\mathcal{Y}_{-1} \subset Db_{\mathcal{N}}$  has no torsion (see Corollary 4.4.3 for  $\alpha = 0, N = -1, m = 0$ ). Then note that

$$\sigma_{k-1}/\sigma_k = \pi_*\left(\sum_{j=1}^k 1/z_j\right)$$

is a trace function on the open set  $\{\sigma_k\Delta(\sigma) \neq 0\}$  and that  $U_0 + 1$  kills also this holomorphic function on this open set. This is enough to complete the proof of the

lemma using again the absence of torsion. ■

**Lemma 5.0.7** *The  $\mathcal{D}_{\mathcal{N}}$ -linear map defined by  $[1] \mapsto \mathcal{X}_0 \in Db_{\mathcal{N}}$  is a generator of  $c(\mathcal{N}_0^{\square})$  and the  $\mathcal{D}_{\mathcal{N}}$ -linear map defined by  $[1] \mapsto \mathcal{X}_{-1} \in Db_{\mathcal{N}}$  is a generator of  $c(\mathcal{N}_{-1}^{\square})$ .*

PROOF. As we already know that the distribution  $\mathcal{X}_1$  is a generator of  $c(\mathcal{N}_1^{\square})$  the point 9 above implies that a distribution  $Z$  in  $c(\mathcal{N}_0)$  which satisfies  $U_1(Z) = \mathcal{X}_1$  must induce a generator in  $c(\mathcal{N}_0^{\square})$ . But  $\mathcal{X}_0$  is such a distribution.

Thanks to point 10 above and the previous result, the distribution  $U_{-1}(\mathcal{X}_0)$  gives a generator in  $c(\mathcal{N}_{-1}^{\square})$  and we have, using as above the absence of torsion the equality  $U_{-1}(\mathcal{X}_0) = \mathcal{X}_{-1}$  in  $Db_{\mathcal{N}}$ . ■

END OF PROOF OF THEOREM 5.0.3. For  $\lambda = 0$  we have the exact sequence of  $\mathcal{D}_{\mathcal{N}}$ -modules (see point 5 above)

$$0 \rightarrow \mathcal{N}_0^{\square} \rightarrow \mathcal{N}_0 \rightarrow \mathcal{O}_N(*\sigma_k) \rightarrow 0$$

where the simple sub-module  $\mathcal{N}_0^{\square}$  is generated by  $U_1$ . So we have an exact sequence of  $\mathcal{D}_{\bar{\mathcal{N}}}$ -modules

$$0 \rightarrow c(\mathcal{O}_N(*\sigma_k)) \rightarrow c(\mathcal{N}_0) \rightarrow c(\mathcal{N}_0^{\square}) \rightarrow 0$$

and if we find a distribution  $Z$  in  $c(\mathcal{N}_0)$  such that its image in  $c(\mathcal{N}_0^{\square})$  is equal to  $\mathcal{X}_{-1}$ , and a distribution  $T$  which generates  $c(\mathcal{O}_N(*\sigma_k))$ , then  $Z$  and  $T$  will generate  $c(\mathcal{N}_0)$ .

But  $U_1(\mathcal{Y}_0) = 0$  in  $Db_{\mathcal{N}}$  because the coefficient of  $\partial_k$  in  $U_1$  is  $\sigma_1\sigma_k$  and  $\sigma_k\partial_k$  kills the distribution  $1/\bar{\sigma}_k$ . So  $\mathcal{X}_0$  and  $\mathcal{Y}_0$  generates  $c(\mathcal{N}_0)$ .

The situation is analogous for  $\lambda = -1$  because  $\mathcal{X}_{-1}$  is in  $c(\mathcal{N}_{-1})$  and its image generates  $c(\mathcal{N}_{-1}^{\square})$  thanks to the previous lemma.

Now the distribution  $\mathcal{Y}_{-1}$  is in  $c(\mathcal{N}_{-1})$  and generates a sub- $\mathcal{D}_{\bar{\mathcal{N}}}$ -module isomorphic to  $\mathcal{O}_{\bar{\mathcal{N}}}(*\bar{\sigma}_k)$  because its annihilator is generated by  $\bar{\partial}_h, h \in [1, k-1]$  and  $\bar{\sigma}_k\bar{\partial}_k$ . Its image in  $c(\mathcal{N}_{-1}^{\square})$  is either a generator or 0. But it has to be non 0 because there is no non zero morphism between  $\mathcal{O}_{\bar{\mathcal{N}}}(*\bar{\sigma}_k)$  and  $c(\mathcal{N}_{-1}^{\square})$ . So  $\mathcal{X}_{-1}$  and  $\mathcal{Y}_{-1}$  generate  $c(\mathcal{N}_{-1})$ . ■

COMPLEMENT. We give in the following lemma, for each case  $\lambda = -1, 0, 1$ , a distribution which generates the  $\mathcal{D}_{\bar{\mathcal{N}}}$ -module  $c(\mathcal{N}_{\lambda})$ .

**Lemma 5.0.8** *Consider the distribution  $Z_{-1} := \sum_{j=1}^k \bar{z}_j/z_j$  on  $\mathcal{M}$  and define  $\mathcal{Z}_{-1} := \pi_*(Z_{-1})$ . Then  $\mathcal{Z}_{-1}$  is a generator of  $c(\mathcal{N}_{-1})$ .*

PROOF. First write  $Z_{-1} = \sum_{j=1}^k |z_j|^2 / z_j^2$  in order to show that any distribution in the  $\mathcal{D}_{\mathcal{M}}$ -module generated by  $Z_{-1}$  has the standard extension property, by applying Corollary 4.4.3. Then the same is true for any distribution in the  $\mathcal{D}_{\mathcal{N}}$ -module generated by  $\mathcal{Z}_{-1}$ . Then it is easy to see that  $\mathcal{Z}_{-1}$  is in  $c(\mathcal{N}_{-1})$ . Then we have  $\bar{U}_{-1}(\mathcal{Z}_{-1}) = \pi_*(\sum_{j=1}^k 1/z_j) = \mathcal{Y}_{-1}$  and

$$\mathcal{X}_{-1} = \mathcal{Z}_{-1} - (\bar{\sigma}_1/k)\sigma_{k-1}/\sigma_k = (1 - (\bar{\sigma}_1/k)\bar{U}_{-1})(\mathcal{Z}_{-1}) \in \mathcal{D}_{\mathcal{N}}\mathcal{Z}_{-1}.$$

So  $\mathcal{D}_{\mathcal{N}}\mathcal{Z}_{-1}$  is equal to  $c(\mathcal{N}_{-1})$  thanks to Theorem 5.0.3. ■

**Lemma 5.0.9** *Let  $Z_1 := \sum_{j=1}^k |z_j|^2 - \sigma_1/\sigma_k$  in  $Db_{\mathcal{M}}$  and  $\mathcal{Z}_1 := \pi_*(Z_1) \in Db_{\mathcal{N}}$ . Then  $\mathcal{Z}_1$  is a generator of  $c(\mathcal{N}_1)$ .*

PROOF. Any distribution in  $\mathcal{D}_{\mathcal{N}}\mathcal{Z}_1$  has the standard extension property because this is true for the  $\mathcal{D}_{\mathcal{M}}$ -module generated by  $\pi_*(\sum_{j=1}^k |z_j|^2)$  and by  $1/\sigma_k$ . So  $\mathcal{Z}_1$  is in  $c(\mathcal{N}_1)$ . Moreover we have

$$(\bar{U}_0 + k)(\mathcal{Z}_1) = (k + 1)\pi_*(\sum_{j=1}^k |z_j|^2) = (k + 1)\mathcal{X}_1 + \frac{k + 1}{k}\bar{\sigma}_1\bar{\sigma}_k\mathcal{Y}_1$$

and

$$(\bar{U}_0 - 1)(\mathcal{Z}_1) = (k - 1)\mathcal{Y}_1.$$

So  $\mathcal{D}_{\mathcal{N}}\mathcal{Z}_1$  is equal to  $c(\mathcal{N}_1)$  thanks to Theorem 5.0.3. ■

**Lemma 5.0.10**  *$\mathcal{X}_0 + \mathcal{Y}_0$  is a generator of  $c(\mathcal{N}_0)$ .*

PROOF. We already know that  $\mathcal{X}_0$  and  $\mathcal{Y}_0$  are in  $c(\mathcal{N}_0)$ . Moreover we have

$$(\bar{U}_0 - 1)(\mathcal{X}_0 + \mathcal{Y}_0) = -(k + 1)/\bar{\sigma}_k = -(k + 1)\mathcal{Y}_0$$

and

$$(\bar{U}_0 + k)(\mathcal{X}_0 + 1/\bar{\sigma}_k) = \mathcal{X}_0.$$

This is enough to conclude, thanks to Theorem 5.0.3. ■

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