Study of the perimeter of a shot noise random field by an elementary approach

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STUDY OF THE PERIMETER OF A SHOT NOISE RANDOM FIELD BY AN ELEMENTARY APPROACH

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Abstract. The study of the geometry of excursion sets of 2D random fields, especially the perimeter or length of level lines, has a growing interest from both the theoretical developments and the statistical applications. In this paper we are interested in the relationship between the perimeter of the excursion sets of a shot noise random field compared to the well known Gaussian framework. Our approach follows the weak framework of special bounded variation functions in which we consider the functions that map the level of the excursion set to the perimeter of the excursion set. In this unified framework we exhibit two different regimes with respect to the intensity of the shot noise random field. The first one is the classical Gaussian regime in high intensity, while the second new one, in low intensity, is related to an elementary approximation. In the explicit case of Gaussian correlation functions, we show the pertinence of such approximation for statistical evaluation. At least, this enables us to propose a classification procedure to discriminate Gaussian or shot noise fields.

1. Introduction and notation

Random fields are a popular branch of probabilities that allow to model a lot of domains, including texture synthesis [36, 23], image analysis with applications in the medical field [16, 21] and extensive spatial modelling. The most popular random field models are certainly the Gaussian fields, but in this paper we will consider shot noise models defined on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2, \quad X(x) = \sum_i m_i g(x - x_i), \tag{1}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is the kernel, the x_i are the points of a homogeneous Poisson point process of intensity λ in $\mathbb{R}^2$ and the m_i are "marks", independent of the Poisson point process such that $\{(x_i, m_i)\}_{i \in I}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}$ of intensity $\lambda \mathcal{L} \otimes F$ with F a probability measure on $\mathbb{R}$, defined on $(\Omega, \mathcal{A}, \mathbb{P})$ a complete probability space. This model can be seen as an extension of the Germ-Grain models [18], and thus of the Boolean model, where the x_i are the germs while the kernel plays the role of the grains, but with the difference that we are interested in the random fields formed by the sum of the grains and not just the random set formed by their unions. These fields were first introduced by Daley [20], although they are based on the one-dimensional process [15, 14, 34], and have been used in many fields [7, 28, 35, 23].

On one side, there is a strong relation between the shot noise framework and the Gaussian framework. The main result is due to Heinrich and Schmidt [25], showing asymptotic normality in high intensity, in the sense of finite dimensional distribution, and even in the topology of uniform convergence on compact sets. Since then, this strong result has aroused a lot of interest in application [28, 17, 4] and in research [10, 32]. On the other side, one purpose of this paper is to show that the low intensity behaviour is really distinct from the Gaussian framework but through its visual definition, based on simple random tools, it allows a certain malleability in the calculations [11, 29, 31, 33] that is rarely found outside the Gaussian case.

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All random field information is contained in the excursion sets, defined for any $u \in \mathbb{R}$ and open subset (the viewing window) $T \subset \mathbb{R}^2$ by

$$E_X(u,T) := \{x \in T \mid X(x) \geq u\}$$

Statistically, it is interesting to study geometric characteristics of these random sets, mainly their area, perimeter and Euler characteristic. It is very rare to know the law of these random functions of the level but it happens that we can calculate their mean values. As it is often the case, under certain assumptions of stationnarity and regularity, the Gaussian framework allows the explicit computation of these 3 geometric quantities [3, 6]. With this exception there are only a few results on the geometry of excursion sets [2, 30], and they usually require assumptions of regularity and bounded density existence.

Especially for shot noise random fields, by adopting a functional point of view, several results for the computation of mean geometry were obtained in [11]. More precisely, without bounded density assumption, they obtained results in two distinct frameworks : the one where the kernel is smooth and the one where the kernel is elementary, that may be unified in the framework of special bounded variation function [5]. The quantity of interest for studying the regularity of excursion sets is the perimeter $u \mapsto \mathrm{Per}(E_X(u,T))$ view as a $\mathrm{L}^1(\mathbb{R})$ function when X is a.s. a locally special bounded variation function. We emphasize that under stronger assumptions, $\mathrm{Per}(E_X(u,T))$ corresponds to the length of the level set $\{x \in T; X(x) = u\}$ [6], also related to the Lipschitz-Killing curvature of order 1 of the excursion set [13]. Since we consider stationary random sets, by ergodicity, one should have $\frac{\mathrm{Per}(E_X(u,T))}{\mathcal{L}(T)} \to \overline{\mathrm{Per}(E_X(u))}$ a.s. as T tends to $\mathbb{R}^2$. Note that asymptotic normality have also been investigated in a Gaussian framework in [8] for a fixed level under regularity assumptions and for shot noise random fields in a weak framework in [31]. Hence our function of interest is the mean density perimeter function

$$u \to \overline{\mathrm{Per}(E_X(u))} := \frac{\mathbb{E}[\mathrm{Per}(E_X(u,T))]}{\mathcal{L}(T)}.$$

However, even for a simple smooth kernel as a Gaussian, the formulas of [11] are not fully explicit and in order to find the mean density perimeter, a Fourier inversion is needed. Through a kernel discretization approach, another aim of this paper will be to obtain formulas that are more easily usable in practice. Finally, we will use the area and perimeter of the excursion sets as a classifier to distinguish a Gaussian and shot noise field with the same first and second order statistics, especially for visually indistinguishable pattern.

The paper is organized as follows. After reviewing the well-known formulas for smooth Gaussian and shot noise fields, we give in Section 2 the two distinct behaviors of the mean densities of the level perimeter integral for shot noise smooth random fields : the high regime for large λ values and the low regime for small λ values. Section 3 is devoted to the case where the kernel is a Gaussian function and its own properties, with in particular its scale invariances. We will study in Section 4 the discretization of this model and the limit of the mean perimeter when the discretization step tends to 0. We will make the link between the identified limit and the low regime. Under normalisation by its total variation, this will lead to more explicit formulas for perimeter than those already known. Finally, using these formulas, in Section 5 we will give a classification method to distinguish a shot noise field from a Gaussian field with the same moments of order 1 and 2.

2. Previous results

2.1. **General formula.** For T an open bounded subset of $\mathbb{R}^2$ and X a "nice" stationary random field on $\mathbb{R}^2$. By stationarity we easily get that the mean density area of excursion sets is given by the tail function of the common distribution. More precisely, for all $u \in \mathbb{R}$,

$$\overline{\mathrm{Area}(E_X(u))} = \frac{\mathbb{E}(\mathcal{L}(E_X(u,T))}{\mathcal{L}(T)} = \mathbb{P}(X(0) \geq u).$$

The computation of the perimeter is more involved and we use the weak functional approach developed in [10]. Following the notation of [11] we define, when it exists, the level perimeter integral by the linear form

$$\begin{array}{lrcl} (2) \quad & \mathrm{LP}_X(\cdot,T)\ :\ \ \mathcal{C}_b(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & h & \longrightarrow & \displaystyle\int_{\mathbb{R}} h(u)\mathrm{Per}(E_X(u),T)\mathrm{d}u \end{array}$$

The knowledge of $\mathbb{E}[\mathrm{LP}_X(h,T)]$ for all $h \in \mathcal{C}_b(\mathbb{R})$ allows us to get information of $\mathbb{E}[\mathrm{Per}(E_X(u),T)]$ for almost every level $u \in \mathbb{R}$. In particular computing (2) for $\nu \in \mathbb{R}$ and h_ν given by

$$h_\nu(u) = e^{i\nu u}, \tag{3}$$

$\mathbb{E}[\mathrm{LP}_X(h_\nu,T)]$ corresponds to the Fourier transform of $u \mapsto \mathbb{E}[\mathrm{Per}(E_X(u),T)]$ at frequency ν. Moreover, the total variation of X may be computed using $h=1$ or $\nu=0$ and its expectation is given by

$$\mathbb{E}[\mathrm{TV}(X,T)] = \mathbb{E}[\mathrm{LP}_X(1,T)].$$

By stationarity, the quantities of interest are related to the mean densities

$$\overline{\mathrm{LP}_X}(h) = \frac{\mathbb{E}[\mathrm{LP}_X(h,T)]}{\mathcal{L}(T)} = \int_{\mathbb{R}} h(u)\overline{\mathrm{Per}(E_X(u))}\,\mathrm{d}u \ \text{ and } \ \overline{\mathrm{TV}(X)} = \overline{\mathrm{LP}_X}(1).$$

Note that we might expect that for nice ergodic random fields and nice growing sequences of T

$$\frac{\mathrm{LP}_X(h,T)}{\mathcal{L}(T)} \longrightarrow \overline{\mathrm{LP}_X}(h) \text{ and } \frac{\mathrm{Per}(E_X(u),T)}{\mathcal{L}(T)} \longrightarrow \overline{\mathrm{Per}(E_X(u))} \text{ a.s.}$$

Assuming that X is a C^1 random field with $X(0)$ and $\partial_j X(0)$ having finite expectations for all $j=1,2$, by Theorem 2 of [11] for a.e. $u \in \mathbb{R}$, the random variables $\mathrm{Per}(E_X(u),T)$ have finite expectation such that for all h bounded continuous function on $\mathbb{R}$, one has

$$\begin{aligned} \mathbb{E}(\mathrm{LP}_X(h,T)) &= \int_{\mathbb{R}} h(u)\mathbb{E}(\mathrm{Per}(E_X(u),T))\,\mathrm{d}t \\ &= \mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right)\mathcal{L}(T). \end{aligned}$$

In particular,

$$\overline{\mathrm{LP}_X}(h) = \mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right) \ \text{ and } \ \overline{\mathrm{TV}(X)} = \mathbb{E}[\|\nabla X(0)\|].$$

It follows that $h \mapsto \overline{\mathrm{LP}_X}(h)$ is a continuous linear form on $\mathcal{C}_b(\mathbb{R})$ with subordinated norm given by $\overline{\mathrm{TV}(X)}$. Moreover, when X is also isotropic, the computation may be simplified as

$$\overline{\mathrm{LP}_X}(h) = \frac{\pi}{2}\mathbb{E}\left(h(X(0))|\partial_1 X(0)|\right) \ \text{ and } \ \overline{\mathrm{TV}(X)} = \frac{\pi}{2}\mathbb{E}[|\partial_1 X(0)|]. \tag{4}$$

Following the case of shot noise fields developed in [10] that are infinitely divisible random fields for which the use of characteristic functions is more tractable, we state the following result.

Proposition 1. *Let X be a stationary isotropic C^1 random field on $\mathbb{R}^2$, such that $\mathbb{E}(|X(0)|) < +\infty$ and $\mathbb{E}(|X_1(0)|^{1+\varepsilon}) < +\infty$ for some $\varepsilon > 0$, and let us denote for $(\nu,v) \in \mathbb{R}^2$,*

$$\varphi(\nu,v) = \mathbb{E}[e^{i\nu X(0) + iv\partial_1 X(0)}]. \tag{5}$$

Then

$$\overline{\mathrm{LP}_X(h_\nu)} = -\int_0^\infty \frac{1}{v}\frac{\partial\varphi}{\partial v}(\nu,v)\mathrm{d}v, \ \textit{ and } \ \overline{\mathrm{TV}(X)} = -\int_0^\infty \frac{1}{v}\frac{\partial\varphi}{\partial v}(0,v)\mathrm{d}v, \tag{6}$$

where h_ν is given by (3) and the improper integrals are defined as $\lim_{V\to+\infty}\int_0^V$.

Proof. Let $\nu \in \mathbb{R}$ since X be a stationary isotropic C^2 random field on $\mathbb{R}^2$, by (4) we have

$$\overline{\mathrm{LP}_X}(h_\nu) = \frac{\pi}{2}\mathbb{E}\left(h_\nu(X(0))|\partial_1 X(0)|\right).$$

According to Proposition 2 of [11], since $|h_\nu(X(0))| \leq 1$ and $\mathbb{E}(|X_1(0)|^{1+\varepsilon}) < +\infty$ for some $\varepsilon > 0$ we get

$$\mathbb{E}\left(h_\nu(X(0))|\partial_1 X(0)|\right) = \frac{2}{\pi}\int_0^{+\infty}\frac{1}{v}\mathbb{E}\left(h_\nu(X(0))\partial_1 X(0)\sin(v\partial_1 X(0))\right)dv.$$

But $\varphi(\nu, v) = \varphi(\nu, -v)$ so that $\varphi(\nu, v) = \mathbb{E}\left(h_\nu(X(0))\cos(v\partial_1 X(0))\right)$ and therefore

$$\frac{\partial \varphi}{\partial v}(\nu, v) = -\mathbb{E}\left(h_\nu(X(0))\partial_1 X(0)\sin(v\partial_1 X(0))\right),$$

from which we deduce the result. $\square$

2.2. **Isotropic smooth Gaussian random fields.** When the random field X is an isotropic smooth (at least C^1) Gaussian random field, we write φ_X, respectively $\varphi_{\partial_1 X}$, the characteristic function of $X(0)$, respectively of $\partial_1 X(0)$. Since $X(0)$ and $\partial_1 X(0)$ are non correlated by stationarity and thus independent by Gaussian distribution, the characteristic function in (5) is $\varphi(\nu, v) = \varphi_X(\nu)\varphi_{\partial_1 X}(u)$. Hence

$$-\int_0^\infty \frac{1}{v}\frac{\partial \varphi}{\partial v}(\nu, v)\mathrm{d}v = \varphi_X(\nu)\left(-\int_0^\infty \frac{1}{v}\frac{\partial \varphi_{\partial_1 X}(v)}{\partial v}\mathrm{d}v\right),$$

with $\varphi_{\partial_1 X}(v) = \exp(-\mathrm{Var}[\partial_1 X(0)]v^2/2)$. It follows that for all $\nu \in \mathbb{R}$,

$$\overline{\mathrm{LP}_X(h_\nu)} = \sqrt{\frac{\pi}{2}}\sqrt{\mathrm{Var}[\partial_1 X(0)]}\varphi_X(\nu). \tag{7}$$

Corollary 1. *Let X be a stationary isotropic C^1 Gaussian random field then for almost all $u \in \mathbb{R}$*

$$\overline{\mathrm{Per}(E_X(u))} = \sqrt{\frac{\pi}{2}}\sqrt{Var[\partial_1 X(0)]}\frac{1}{\sqrt{2\pi\, Var[X(0)]}}\exp\left(-\frac{(u - \mathbb{E}(X(0)))^2}{2\,Var[X(0)]}\right), \text{ and}$$

$$\overline{\mathrm{TV}(X)} = \sqrt{\frac{\pi}{2}}\sqrt{Var[\partial_1 X(0)]}.$$

Note that under the stronger assumption that X is a.s. C^2 the previous stated equality will hold for any level $u \in \mathbb{R}$. We refer to Theorem 6.8 of [6] for instance.

2.3. **Isotropic smooth shot noise fields.** Considering now a shot noise field X given by (1), with a radial kernel $g \in \mathcal{C}^2(\mathbb{R}^2)$, we can also go ahead in computations. We recall the notation $g_m = mg$ and introduce a random variable M of distribution F. For $p > 0$, we denote

$$\mu_p(g_M) := \int_{\mathbb{R}^2}\int_{\mathbb{R}} |g_m(x)|^p F(\mathrm{d}m)\mathrm{d}x = \mathbb{E}(|M|^p)\int_{\mathbb{R}^2}|g(x)|^p\mathrm{d}x. \tag{8}$$

For $g \in \mathcal{C}^2(\mathbb{R}^2)$ and $i = 1, 2$ we note the partial derivative $\partial_i g := \frac{\partial g}{\partial x_i}$ and for $\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2$, $|\mathbf{j}| \le 2$, we note also $D^{\mathbf{j}} g := \frac{\partial^{|\mathbf{j}|} g}{\partial^{j_1} x_1 \partial^{j_2} x_2}$ such that $D^{(1,0)}g = \partial_1 g$.

Actually, under the assumption that $\mu_1(D^{\mathbf{j}} g_M) < +\infty$ for all $|\mathbf{j}| \le 2$, the shot noise field X is a.s a stationary isotropic C^1 random field and the characteristic function of $(X(0), \partial_1 X(0))$ is explicitly given for $(\nu, v) \in \mathbb{R}^2$ by

$$\varphi(\nu, v) = \exp\left(\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}}[e^{i\nu m g(x) + i v m \partial_1 g(x)} - 1]F(\mathrm{d}m)\mathrm{d}x\right). \tag{9}$$

Hence we get $\frac{\partial \varphi}{\partial v}(\nu, v) = -S_0(\nu, v)\varphi(\nu, v)$, with

$$\begin{aligned} S_0(\nu, v) &= -i\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}} m\partial_1 g(x)e^{i\nu m g(x) + i v m\partial_1 g(x)}F(\mathrm{d}m)\mathrm{d}x \\ &= \lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}} m\partial_1 g(x)e^{i\nu m g(x)}\sin\left(v m\partial_1 g(x)\right)F(\mathrm{d}m)\mathrm{d}x, \end{aligned}$$

and, assuming moreover that $\mu_2(D^{\mathbf{j}} g_M) < +\infty$ for all $|\mathbf{j}| \le 2$, , by Proposition 1 with $\varepsilon = 1$ (see also Theorem 3 of [11]), we obtain

$$\forall \nu \in \mathbb{R},\ \overline{\mathrm{LP}_X}(h_\nu) = \int_0^\infty \frac{1}{v}\varphi(\nu, v)S_0(\nu, v)\mathrm{d}v. \tag{10}$$

Lemma 1. *Under the assumption that g is a C^1 radial kernel and that $\mu_1(\partial_1 g_M) < +\infty$ and $\mu_2(\partial_1 g_M) < +\infty$, we get*

$$\int_0^\infty \frac{1}{v} S_0(\nu, v)\mathrm{d}v = \lambda \frac{\pi}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} |m\partial_1 g(x)| e^{i\nu m g(x)} F(\mathrm{d}m)\mathrm{d}x.$$

Proof. We follow the proof of Proposition 2 in [11]. Let $V > 0$ and recall that for $y \in \mathbb{R}$ we have

$$\int_0^\infty \frac{\sin(vy)}{v}\mathrm{d}v := \lim_{V\to+\infty} \int_0^V \frac{\sin(vy)}{v}\mathrm{d}v = \frac{\pi}{2}\mathrm{sgn}(y),$$

and $C := \sup_{V>0} \left|\int_0^V \frac{\sin(v)}{v}\mathrm{d}v\right| < +\infty$. Hence we obtain that for a.e $x \in \mathbb{R}^2$ and F a.e. $m \in \mathbb{R}$,

$$m\partial_1 g(x) e^{i\nu m g(x)} \int_0^V \frac{\sin(vm\partial_1 g(x))}{v}\mathrm{d}v \underset{V\to+\infty}{\longrightarrow} \frac{\pi}{2} |m\partial_1 g(x)| e^{i\nu m g(x)}.$$

But for all $V > 0$

$$\left| m\partial_1 g(x) e^{i\nu m g(x)} \int_0^V \frac{\sin(vm\partial_1 g(x))}{v}\mathrm{d}v \right| \le C\, |m\partial_1 g(x)|\,,$$

with

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}} |m\partial_1 g(x)|\, F(\mathrm{d}m)\mathrm{d}x < +\infty.$$

By Lebesgue's theorem we can conclude that

$$\lim_{V\to+\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}} m\partial_1 g(x) e^{i\nu m g(x)} \int_0^V \frac{\sin(vm\partial_1 g(x))}{v} dv F(\mathrm{d}m)\mathrm{d}x = \frac{\pi}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} |m\partial_1 g(x)| e^{i\nu m g(x)} F(\mathrm{d}m)\mathrm{d}x.$$

Now it remains to justify that for all $V > 0$,

$$\int_0^V \frac{1}{v} S_0(\nu, v) dv = \int_{\mathbb{R}^2} \int_{\mathbb{R}} m\partial_1 g(x) e^{i\nu m g(x)} \int_0^V \frac{\sin(vm\partial_1 g(x))}{v}\mathrm{d}v F(\mathrm{d}m)\mathrm{d}x.$$

But it simply follows from Fubini's theorem once remarked that

$$\left| m\partial_1 g(x) e^{i\nu m g(x)} \frac{\sin(vm\partial_1 g(x))}{v} \right| \le |m\partial_1 g(x)|^2\,.$$

□

2.4. **Asymptotics for high and small λ.** In this section we investigate asymptotics of the level perimeter function of shot noise random fields with respect to λ. In the sequel we denote X_λ the shot noise field X given by (1) for an homogeneous Poisson point process of intensity $\lambda > 0$. We assume that g is a C^2 radial kernel such that $\mu_1(D^{\mathbf{j}} g_M) < +\infty$ and $\mu_2(D^{\mathbf{j}} g_M) < +\infty$ for all $|\mathbf{j}| \le 2$.

Since $\mu_2(g_M) < +\infty$, we let B_λ be a stationary Gaussian field with the same second order statistics than X_λ meaning that for all $y, y' \in \mathbb{R}^2$

$$\mathbb{E}(B_\lambda(y)) = \mathbb{E}(X_\lambda(y)) = \lambda \int_{\mathbb{R}^2} \int_{\mathbb{R}} g_m(-x)\mathrm{d}x F(\mathrm{d}m) = \lambda \mathbb{E}(M) \int_{\mathbb{R}^2} g(x)\mathrm{d}x,$$

and

$$\begin{aligned} \mathrm{Cov}(B_\lambda(y), B_\lambda(y')) = \mathrm{Cov}(X_\lambda(y), X_\lambda(y')) &= \lambda \int_{\mathbb{R}^2} \int_{\mathbb{R}} g_m(y-x) g_m(y'-x) F(\mathrm{d}m)\mathrm{d}x \\ &= \lambda \mathbb{E}(M^2) \int_{\mathbb{R}^2} g(x) g(x - (y - y'))\mathrm{d}x, \end{aligned} \tag{11}$$

using Campbell's Theorem [26] or [18].

Under our assumptions, up to choose a modification, we can assume that both B_λ and X_λ are a.s. C^1. It follows that $\partial_1 B_\lambda$ and $\partial_1 X_\lambda$ are centered, stationary and

$$\begin{aligned}\operatorname{Cov}(\partial_1 B_\lambda(y), \partial_1 B_\lambda(y')) &= \operatorname{Cov}(\partial_1 X_\lambda(y), \partial_1 X_\lambda(y')) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \partial_1 g_m(y-x)\partial_1 g_m(y'-x)F(dm)dx \\ &= \lambda\mathbb{E}(M^2)\int_{\mathbb{R}^2}\partial_1 g(x)\partial_1 g(x-(y-y'))dx,\end{aligned}$$

Moreover, writing $B(y) = \frac{B_\lambda(y)-\mathbb{E}(B_\lambda(y))}{\sqrt{\lambda}}$, and $Z_\lambda(y) = \frac{X_\lambda(y)-\mathbb{E}(X_\lambda(y))}{\sqrt{\lambda}}$, as $\lambda \to +\infty$, we obtain as in [25],

$$(Z_\lambda(y), \partial_1 Z_\lambda(y)) \underset{\lambda\to+\infty}{\xrightarrow{d}} (B(y), \partial_1 B(y)).$$

Note that this asymptotic normality for high intensity can be strenghen with an invariance principle and even a coupling between Z_λ and B for a strong invariance principle as in [32]. Hence it is not surprising to observe a Gaussian behavior for geometry of excursion sets in high intensity. This is the object of our first result, which proof is postponed to Appendix 6.1.

Theorem 1 (High regime)**.** *Let us assume that g is a C^2 radial kernel and that $\mu_p(D^{\mathbf{j}}g_M) < +\infty$ for all $|\mathbf{j}| \le 2$ and $1 \le p \le 4$. Then, there exists C_0^+, $C_1{}^+$ and $C_2{}^+$ positive constants such that for $|\nu| \le C_0{}^+$*

$$\begin{aligned}\left|\overline{\mathrm{LP}_{X_\lambda}}(h_\nu) - \overline{\mathrm{LP}_{B_\lambda}}(h_\nu)\right| &= \left|\overline{\mathrm{LP}_{X_\lambda}}(h_\nu) - \sqrt{\lambda}\varphi_{B_\lambda}(\nu)\sqrt{\frac{\pi}{2}\mu_2(\partial_1 g_M)}\right| \\ &\le C_1^+ + (1-\delta_0(\nu))C_2^+,\end{aligned}$$

where

$$\varphi_{B_\lambda}(\nu) = \exp\left[i\nu\lambda\mathbb{E}(M)\int_{\mathbb{R}^2} g(x)dx - \frac{\lambda\mu_2(g_M)\nu^2}{2}\right],$$

$\delta_0(\nu) = 1$ if $\nu = 0$ and $\delta_0(\nu) = 0$ otherwise and

$$\begin{aligned}C_0^+ &\le \frac{1}{6}\min\left(\left(\frac{\mu_4(g_M)}{\mu_2(g_M)}\right)^{1/2}, \frac{5\mu_2(\partial_1 g_M)}{\mu_4(g_M)^{1/2}\mu_4(\partial_1 g_M)^{1/2}}\right) \\ C_1^+ &\le 10\left(\frac{\mu_4(\partial_1 g_M)}{\mu_2(\partial_1 g_M)}\right)^{1/2} \\ C_2^+ &\le \sqrt{\pi}\left(\frac{3\mu_4(g_M)^{1/2}\mu_2(\partial_1 g_M)^{1/2}}{\mu_2(g_M)} + \left(\frac{\mu_4(\partial_1 g_M)}{\mu_2(\partial_1 g_M)}\right)^{1/2}\right).\end{aligned}$$

Observe the interest of the result since $\forall \nu \in \mathbb{R}$ the two quantities $\overline{\mathrm{LP}_{X_\lambda}}(h_\nu)$ and $\overline{\mathrm{LP}_{B_\lambda}}(h_\nu)$ tend to infinity when $\lambda \to \infty$. Note also that, using

$$\overline{\mathrm{LP}_{Z_\lambda}}(h_\nu) = \frac{e^{-i\nu\mathbb{E}(X_\lambda(x))/\sqrt{\lambda}}}{\sqrt{\lambda}}\overline{\mathrm{LP}_{X_\lambda}}(h_{\nu/\sqrt{\lambda}}) \text{ and } \overline{\mathrm{LP}_{B}}(h_\nu) = \frac{e^{-i\nu\mathbb{E}(X_\lambda(x))/\sqrt{\lambda}}}{\sqrt{\lambda}}\overline{\mathrm{LP}_{B_\lambda}}(h_{\nu/\sqrt{\lambda}}),$$

our result extends Theorem 4 of [10] for the normalized shot noise field : for any $\nu \in \mathbb{R}$,

$$\overline{\mathrm{LP}_{Z_\lambda}}(h_\nu) \underset{\lambda\to+\infty}{\longrightarrow} \overline{\mathrm{LP}_B}(h_\nu).$$

This implies the following weak convergence for $u \in \mathbb{R}$,

$$\overline{\mathrm{Per}(E_{X_\lambda}(\mathbb{E}(X_\lambda(x)) + u\sqrt{\lambda}))} \underset{\lambda\to+\infty}{\to} \overline{\mathrm{Per}(E_B(u))},$$

as well as (taking $\nu = 0$)

$$\frac{\overline{\mathrm{TV}(X_\lambda)}}{\sqrt{\lambda}} \underset{\lambda\to+\infty}{\to} \overline{\mathrm{TV}(B)} = \sqrt{\frac{\pi}{2}\mu_2(\partial_1 g_M)}.$$

But Theorem 1 actually gives a rate for convergence, following ideas in [9], since

$$\left|\frac{\overline{\mathrm{TV}(X_\lambda)}}{\sqrt{\lambda}} - \sqrt{\frac{\pi}{2}\mu_2(\partial_1 g_M)}\right| = \left|\frac{\overline{\mathrm{TV}(X_\lambda)}}{\sqrt{\lambda}} - \frac{\overline{\mathrm{TV}(B_\lambda)}}{\sqrt{\lambda}}\right| \leq \frac{C_1^+}{\sqrt{\lambda}}.$$

More surprisingly we observe an other behavior for small values of λ. This is the object of the following theorem which proof is also postponed to Appendix Section 6.2.

Theorem 2 (Low regime)**.** *Let us assume that g is a C^2 radial kernel and that $\mu_p(D^{\mathbf{j}} g_M) < +\infty$ for all $|\mathbf{j}| \leq 2$ and $1 \leq p \leq 2$. Assume moreover that there exists $\beta \in (0,1)$ such that $\mu_\beta(\partial_1 g_M) < +\infty$, then, for $C_0^- > 0$ and $\lambda \leq C_0^-$, there exists $C_1^- > 0$ such that*

$$\left|\overline{\mathrm{LP}_{X_\lambda}}(h_\nu) - \lambda \varphi_{X_\lambda}(\nu) \int_{\mathbb{R}} \mathrm{LP}_{g_m}(h_\nu, \mathbb{R}^2) F(\mathrm{d}m)\right| \leq C_1^- \lambda^{\frac{2}{1+\beta}},$$

with

$$\varphi_{X_\lambda}(\nu) = \exp\left(\lambda \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left[e^{i\nu g_m(x)} - 1\right] F(\mathrm{d}m)\mathrm{d}x\right),$$

$$\mathrm{LP}_{g_m}(h_\nu, \mathbb{R}^2) := \int_{\mathbb{R}} h_\nu(u) \mathrm{Per}(E_{g_m}(u), \mathbb{R}^2)\mathrm{d}u = \frac{\pi}{2} \int_{\mathbb{R}^2} |\partial_1 g_m(x)| e^{i\nu g_m(x)} \mathrm{d}x,$$

and

$$C_1^- \leq \beta^{-1} C_\beta^- \mu_1(\partial_1 g_M) \exp(C_\beta^- C_0^{-\frac{1-\beta}{1+\beta}}) + 2 + 2C_\beta^- C_0^{-\frac{1-\beta}{1+\beta}}, \ \textit{for } C_\beta^- = 2^{1-\beta}\mu_\beta(\partial_1 g_M).$$

A particular consequence of this result is that, in view of Lemma 1, taking $\nu = 0$ we get

$$\left|\frac{\overline{\mathrm{TV}(X_\lambda)}}{\lambda} - \frac{\pi}{2}\mu_1(\partial_1 g_M)\right| \leq C_1^- \lambda^{\frac{1-\beta}{1+\beta}},$$

and therefore

$$\frac{\overline{\mathrm{TV}(X_\lambda)}}{\lambda} \underset{\lambda \to 0}{\to} \frac{\pi}{2}\mu_1(\partial_1 g_M).$$

The next section is devoted to a special kernel g for which we will be able to identify this limit.

3. Gaussian shot noise random field and previous results

From now on, we consider for $\lambda, \mu, \sigma \in (0, +\infty)$, the Gaussian shot noise random field with exponential marks defined by

$$X_{\lambda,\mu,\sigma}(x) = \sum_i g_{m_i,\sigma}(x - x_i), \tag{12}$$

with $(x_i)_i$ an homogeneous Poisson point process of intensity λ in $\mathbb{R}^2$, the $(m_i)_i$ are independent marks of exponential law of parameter μ and $g_{m,\sigma} = m g_\sigma$ with the Gaussian density g_σ

$$g_\sigma(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right). \tag{13}$$

Let us emphasize that, g_σ is of course a radial $\mathcal{C}^\infty$ kernel and moreover, if M denotes a random variable with exponential law of parameter μ whose distribution is denoted by F, then for any $p > 0$

$$\mu_p(g_{M,\sigma}) = \mathbb{E}(M^p) \frac{1}{p(2\pi\sigma^2)^{p-1}} = \frac{\Gamma(p+1)}{\mu^p} \frac{1}{p(2\pi\sigma^2)^{p-1}} = \frac{\Gamma(p+1)}{p(2\pi)^{p-1}} \frac{\sigma^{2-2p}}{\mu^p},$$

while

$$\begin{aligned}
\mu_p(\partial_1 g_{M,\sigma}) &= \mathbb{E}(M^p) \frac{1}{(2\pi\sigma^4)^p} \left(\int_{\mathbb{R}} |x_1|^p e^{-\frac{p x_1^2}{2\sigma^2}} dx_1\right) \left(\int_{\mathbb{R}} e^{-\frac{p x_2^2}{2\sigma^2}} dx_2\right) \\
&= \mathbb{E}(M^p) \frac{1}{(2\pi\sigma^4)^p} \times \Gamma\left(\frac{p+1}{2}\right) \left(\frac{2\sigma^2}{p}\right)^{\frac{p+1}{2}} \times \sqrt{2\pi \frac{\sigma^2}{p}} \\
&= \frac{2^{1-p/2}\Gamma(p+1)\Gamma\left(\frac{p+1}{2}\right)}{p^{1+p/2}\pi^{p-1/2}} \frac{\sigma^{2-3p}}{\mu^p}.
\end{aligned}$$

Hence, the model satisfies all the assumptions of the previous section. Moreover, in [11], the characteristic function of $(X_{\lambda,\mu,\sigma}(x), \partial_1 X_{\lambda,\mu,\sigma}(x))$ is explicitly given for $(\nu, v) \in \mathbb{R}^2$ by

$$\begin{aligned}\varphi(\nu,v) &= \mathbb{E}[e^{i\nu X_{\lambda,\mu,\sigma}(x)+iv\partial_1 X_{\lambda,\mu,\sigma}(x)}] \\ &= \exp\left(\lambda \int_0^\infty \int_0^{2\pi} \frac{ir(\nu\sigma^2 - vr\cos(\theta))e^{-r^2/2\sigma^2}}{2\pi\sigma^4\mu - i(\nu\sigma^2 - vr\cos(\theta))e^{-r^2/2\sigma^2}} \mathrm{d}\theta \mathrm{d}r\right) \end{aligned} \tag{14}$$

and in particular, taking $v = 0$, this shows that $X_{\lambda,\mu,\sigma}(x)$ follows a Gamma law of parameters $2\pi\mu\sigma^2$ and $2\pi\lambda\sigma^2$. As a consequence we simply have $\mathbb{E}(X_{\lambda,\mu,\sigma}(x)) = \frac{\lambda}{\mu}$ and $\mathrm{Var}(X_{\lambda,\mu,\sigma}(x)) = \lambda\mu_2(g_{M,\sigma})$ with $\mu_2(g_{M,\sigma}) = \frac{1}{2\pi\sigma^2\mu^2}$. We can also simply compute the covariance function in (11), that remains a Gaussian function, namely

$$\mathrm{Cov}(X_{\lambda,\mu,\sigma}(x), X_{\lambda,\mu,\sigma}(y)) = \frac{\lambda}{2\pi\sigma^2\mu^2}\exp\left(\frac{-||x-y||^2}{4\sigma^2}\right).$$

Finally, in view of Equation (10), we have also

$$S_0(\nu,v) = i\lambda \int_0^\infty \int_0^{2\pi} \frac{2\pi\sigma^4\mu r^2\cos(\theta)e^{-r^2/2\sigma^2}}{\left(2\pi\sigma^4\mu - i(\nu\sigma^2 - vr\cos(\theta))e^{-r^2/2\sigma^2}\right)^2}\mathrm{d}\theta\mathrm{d}r \tag{15}$$

So we can numerically approach $\nu \to \overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}(h_\nu)$ and a Fourier inversion allows us to obtain the desired function $u \to \overline{\mathrm{Per}(E_X(u))}$. We refer to Appendix Section 6.3 for algorithm and illustration.

The model satisfies several invariances as stated in the next proposition.

Proposition 2. *Let $\kappa > 0$ and $k > 0$. Then we have*

$$\left\{(X_{\lambda,\mu,\sigma}(x), \partial_1 X_{\lambda,\mu,\sigma}(x)) \mid x \in \mathbb{R}^2\right\} \overset{fdd}{=} \left\{\left(X_{\lambda k^2,\mu k^2,\frac{\sigma}{k}}\left(\frac{x}{k}\right), \frac{1}{k}\partial_1 X_{\lambda k^2,\mu k^2,\frac{\sigma}{k}}\left(\frac{x}{k}\right)\right) \mid x \in \mathbb{R}^2\right\},$$

and

$$\{(X_{\lambda,\mu,\sigma}(x), \partial_1 X_{\lambda,\mu,\sigma}(x)) \mid x \in \mathbb{R}^2\} \overset{fdd}{=} \left\{\frac{1}{\kappa}\left(X_{\lambda,\frac{\mu}{\kappa},\sigma}(x), \partial_1 X_{\lambda,\frac{\mu}{\kappa},\sigma}(x)\right) \mid x \in \mathbb{R}^2\right\}.$$

Proof. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}^2$ and $u_{1,1}, u_{1,2}, \ldots, u_{n,1}, u_{n,2} \in \mathbb{R}$. For $x \in \mathbb{R}^2$ we note $x^{(1)}$ and $x^{(2)}$ its spatial coordinates. By Campbell's Theorem [26] we have that

$$\begin{aligned} & -\log\left(\mathbb{E}\left[\exp\left(i\sum_{l=1}^n u_{l,1}X_{\lambda k^2,\mu k^2,\frac{\sigma}{k}}(\tfrac{x_l}{k}) + \tfrac{u_{l,2}}{k}\partial_1 X_{\lambda k^2,\mu k^2,\frac{\sigma}{k}}(\tfrac{x_l}{k})\right)\right]\right) \\ = & \int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\left[1-\exp\left(i\sum_{l=1}^n \tfrac{u_{l,1}mk^2}{2\pi\sigma^2}e^{\frac{-k^2||\frac{x_l}{k}-c||^2}{2\sigma^2}} + \tfrac{u_{l,2}mk^4x_l^{(1)}}{2\pi k^2\sigma^4}e^{\frac{-k^2||\frac{x_l}{k}-c||^2}{2\sigma^2}}\right)\right]\lambda k^2\mu k^2 e^{-\mu k^2 m}\mathrm{d}m\mathrm{d}c \\ = & \int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\left[1-\exp\left(i\sum_{l=1}^n \tfrac{u_{l,1}m'}{2\pi\sigma^2}e^{\frac{-||x_l-c'||^2}{2\sigma^2}} + \tfrac{u_{l,2}m'x_l^{(1)}}{2\pi\sigma^2}e^{\frac{-||x_l-c'||^2}{2\sigma^2}}\right)\right]\lambda\mu e^{-\mu m'}\mathrm{d}m'\mathrm{d}c' \\ = & -\log\left(\mathbb{E}\left[\exp\left(i\sum_{l=1}^n u_{l,1}X_{\lambda,\mu,\sigma}(x_l) + u_{l,2}\partial_1 X_{\lambda,\mu,\sigma}(x_l)\right)\right]\right) \end{aligned}$$

using the change of variables $c' = kc$ et $m' = k^2 m$, which proves the first equality. The same arguments allow us to check the second equality, observing that

$$\begin{aligned} & \int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\left[1-\exp\left(i\sum_{l=1}^n \tfrac{u_{l,1}m}{2\pi\kappa\sigma^2}e^{\frac{-||x_l-c||^2}{2\sigma^2}} + \tfrac{u_{l,2}mx_l^{(1)}}{2\pi\kappa\sigma^4}e^{\frac{-||x_l-c||^2}{2\sigma^2}}\right)\right]\lambda\tfrac{\mu}{\kappa}e^{-\frac{\mu m}{\kappa}}\mathrm{d}m\mathrm{d}c \\ = & \int_{\mathbb{R}^2}\int_{\mathbb{R}_+}\left[1-\exp\left(i\sum_{l=1}^n \tfrac{u_{l,1}m'}{2\pi\sigma^2}e^{\frac{-||x_l-c||^2}{2\sigma^2}} + \tfrac{u_{l,2}m'x_l^{(1)}}{2\pi\sigma^4}e^{\frac{-||x_l-c||^2}{2\sigma^2}}\right)\right]\lambda\mu e^{-\mu m'}\mathrm{d}m'\mathrm{d}c \end{aligned}$$

using $m' = \frac{m}{\kappa}$. $\square$

Note that introducing the Gaussian random field $B_{\lambda,\mu,\sigma}$ sharing the same second order statistics than $X_{\lambda,\mu,\sigma}$, it will also verify the previous invariances. Moreover, since $\mu_1(g_{M,\sigma}) = \frac{1}{\mu}$, $\mu_2(g_{M,\sigma}) = \frac{1}{2\pi\sigma^2\mu^2}$ and $\mu_2(\partial_1 g_{M,\sigma}) = \frac{1}{4\pi\sigma^4\mu^2}$, we can compute by (7)

$$\overline{\mathrm{LP}_{B_{\lambda,\mu,\sigma}}}(h_\nu) = \frac{1}{2\sigma^2\mu}\sqrt{\frac{\lambda}{2}}e^{i\frac{\lambda}{\mu}\nu}\exp\left(-\frac{\nu^2}{2}\frac{\lambda}{2\pi\sigma^2\mu^2}\right).$$

and we can restate Theorem 1 in terms of resolution σ.

Proposition 3. *There exist $c_0(\mu), c_1(\mu) > 0$ such that for $|\nu| \le \sigma c_0(\mu)$, one has*

$$\left|\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}(h_\nu) - \overline{\mathrm{LP}_{B_{\lambda,\mu,\sigma}}}(h_\nu)\right| \le \frac{c_1(\mu)}{\sigma^3}.$$

Proof. Using Proposition 2 and (4), it follows that

$$\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}(h_\nu) = \frac{1}{\sigma}\overline{\mathrm{LP}_{X_{\lambda\sigma^2,\mu\sigma^2,1}}}(h_\nu) = \frac{1}{\sigma^3}\overline{\mathrm{LP}_{X_{\lambda\sigma^2,\mu,1}}}(h_{\nu/\sigma}),$$

and similarly for the Gaussian field. We can take $c_0(\mu) = C_0^+$ and $c_1(\mu) = C_1^+ + C_2^+$ for $M \sim \mathcal{E}(\mu)$ and $\sigma = 1$, according to Theorem 1, to get for $|\nu/\sigma| \le c_0(\mu)$

$$\left|\overline{\mathrm{LP}_{X_{\lambda\sigma^2,\mu,1}}}(h_{\nu/\sigma}) - \overline{\mathrm{LP}_{B_{\lambda\sigma^2,\mu,1}}}(h_{\nu/\sigma})\right| \le c_1(\mu),$$

that gives the result. □

Theorem 1 tells us that for $\mu > 0$ and $\sigma > 0$ fixed, we have that $\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}(h_\nu)$ is close to $\overline{\mathrm{LP}_{B_{\lambda,\mu,\sigma}}}(h_\nu)$ and grows in high intensity. The previous proposition allows us to interpret this result in high resolution. For $\lambda > 0$ and $\mu > 0$ fixed, we have that $\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}(h_\nu)$ is close to $\overline{\mathrm{LP}_{B_{\lambda,\mu,\sigma}}}(h_\nu)$, decreases in high resolution ($\sigma \to \infty$) and the difference between it tends to 0 faster (in $1/\sigma^3$) than their decreases (in $1/\sigma^2$).

4. Elementary approximation

4.1. The elementary Gaussian function.

The idea of this paper is to discretize the Gaussian kernel to use the other framework of elementary functions in the sense of [11]. To take into account that the range of g_σ depends of σ, we write $k_\sigma(x) = e^{-\|x\|^2/2\sigma^2}$. The range of k_σ is now $(0,1]$, independently from parameters.

For $n \ge 1$ we denote the elementary Gaussian function by

$$k_\sigma^{(n-1)}(x) = \frac{1}{n}\sum_{i=1}^{n-1}\mathbb{1}_{B(0,r_i^{(n)})}(x),$$

for $r_i^{(n)} = \sigma\sqrt{-2\log(i/n)}$ in such a way that $(r_i^{(n)})_{1\le i\le n-1}$ is strictly decreasing with $r_n^{(n)} = 0$ and that when $x \in B(0,r_i^{(n)}) \smallsetminus B(0,r_{i+1}^{(n)})$ one has $\frac{i}{n} \le k_\sigma(x) < \frac{i+1}{n}$ and $k_\sigma^{(n)}(x) = i/n$ such that

$$\|k_\sigma - k_\sigma^{(n)}\|_\infty = 1/n.$$

The function $k_\sigma^{(n)}$ is an elementary function with discontinuity points given by its regular points

$$\mathcal{R}_{k_\sigma^{(n)}} = \bigcup_{i=1}^{n-1}\partial B(0,r_i^{(n)}),$$

meaning that it is piecewize constant with a regular discontinuity set (see Definition 5 in [11]). It also implies that $k_\sigma^{(n)}$ is a special bounded variation function [5]. An illustration is given in Figure 1.

As a discretization of the Gaussian kernel we have the following asymptotics in the Lebesgue $L^p(\mathbb{R}^2)$ spaces for > 0.

Lemma 2. *Let $\sigma > 0$. Then for all $p > 0$, we have*

$$||k_\sigma - k_\sigma^{(n)}||_p \underset{n\to\infty}{\longrightarrow} 0,$$

where $||\cdot||_p$ is the classical norm ($p > 1$) or quasi-norm ($0 < p < 1$) in $L^p(\mathbb{R}^2)$.

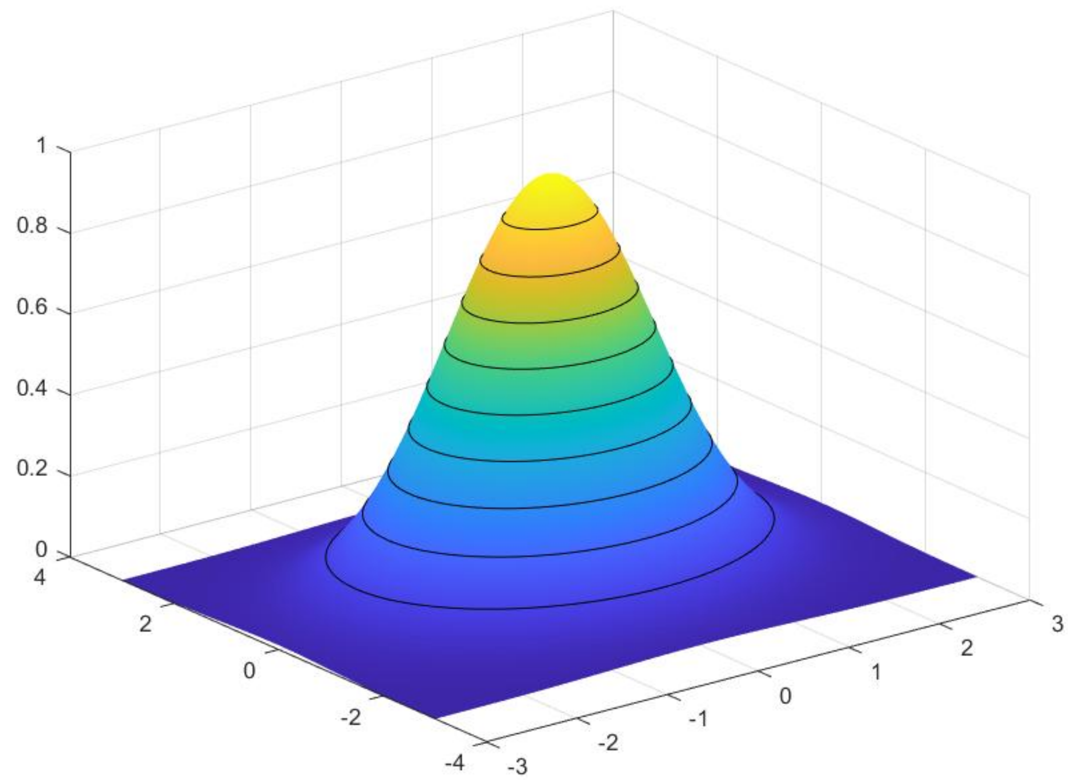

FIGURE 1. Elementary approximation of a Gaussian function

Proof. Let $\sigma > 0$ and $n \in \mathbb{N}$. Since $k_\sigma \geq k_\sigma^{(n)}$ we have

$$\begin{aligned}
||k_\sigma - k_\sigma^{(n)}||_p^p &= \int_{\mathbb{R}^2} \left(k_\sigma(x) - k_\sigma^{(n)}(x)\right)^p \mathrm{d}x \\
&= \sum_{i=1}^{n-1} \int_{B(0,r_i^{(n)}) \setminus B(0,r_{i+1}^{(n)})} \left(k_\sigma(x) - k_\sigma^{(n)}(x)\right)^p \mathrm{d}x + \int_{B(0,r_1^{(n)})^c} k_\sigma(x)^p \mathrm{d}x \\
&\leq \sum_{i=1}^{n-1} \frac{1}{n^p} \mathcal{L}\left(B(0,r_i^{(n)}) \setminus B(0,r_{i+1}^{(n)})\right) + \int_{B(0,r_1^{(n)})^c} k_\sigma(x)^p \mathrm{d}x.
\end{aligned}$$

But for all $1 \leq i \leq n-1$ we have

$$\begin{aligned}
\mathcal{L}\left(B(0,r_i^{(n)}) \setminus B(0,r_{i+1}^{(n)})\right) &= \left(\pi(r_i^{(n)})^2 - \pi(r_{i+1}^{(n)})^2\right) \\
&= \pi\sigma^2 \left(-2\ln\left(\frac{i}{n}\right) + 2\ln\left(\frac{i+1}{n}\right)\right) \\
&= 2\pi\sigma^2 \left(\ln(i+1) - \ln(i)\right).
\end{aligned}$$

Therefore

$$||k_\sigma - k_\sigma^{(n)}||_p^p \leq \frac{2\pi\sigma^2}{n^p} \ln(n) + \int_{B(0,r_1^{(n)})^c} k_\sigma(x)^p dx.$$

Since $p > 0$ and $r_1^{(n)} \underset{n\to\infty}{\longrightarrow} +\infty$ we get $||k_\sigma - k_\sigma^{(n)}||_p^p \underset{n\to\infty}{\longrightarrow} 0$. □

Now let us remark that both k_σ and $k_\sigma^{(n)}$ are functions of $\mathrm{SBV}(\mathbb{R}^2)$, the space of special functions of bounded variation as defined in [5] with total variation given by

$$\mathrm{TV}(k_\sigma, \mathbb{R}^2) = \int_{\mathbb{R}^2} \|\nabla k_\sigma(x)\| dx$$

and

$$\mathrm{TV}(k_\sigma^{(n)}, \mathbb{R}^2) = \sum_{i=1}^{n-1} \frac{1}{n} \mathcal{H}^1(B(0, r_i^{(n)})) = \frac{2\pi\sigma}{n} \sum_{i=1}^{n-1} \sqrt{-2\ln\left(\frac{i}{n}\right)}.$$

Then we can introduce the level perimeter integrals $\mathrm{LP}_{k_\sigma}(\cdot, \mathbb{R}^2)$ and $\mathrm{LP}_{k_\sigma^{(n)}}(\cdot, \mathbb{R}^2)$ that define continuous linear form on $(\mathcal{C}_b(\mathbb{R}), \|\cdot\|_\infty)$.

Proposition 4. *Let $\sigma > 0$. Then $\mathrm{TV}(k_\sigma, \mathbb{R}^2) = \pi\sqrt{2\pi}\sigma$ and for all $n \in \mathbb{N}$,*

$$\mathrm{TV}(k_\sigma^{(n)}, \mathbb{R}^2) \leq \mathrm{TV}(k_\sigma, \mathbb{R}^2).$$

Moreover, for any $h \in \mathcal{C}_b(\mathbb{R})$, one has

$$\mathrm{LP}_{k_\sigma^{(n)}}(h, \mathbb{R}^2) \underset{n\to\infty}{\longrightarrow} \mathrm{LP}_{k_\sigma}(h, \mathbb{R}^2),$$

where

$$\mathrm{LP}_{k_\sigma}(h, \mathbb{R}^2) = \int_0^1 h(s) 2\pi\sigma\sqrt{-2\log(s)} ds.$$

Proof. We use the general co-area formula obtained in Theorem 1 of [10]. Let $h \in \mathcal{C}_b(\mathbb{R})$. It follows that on the one hand,

$$\mathrm{LP}_{k_\sigma}(h, \mathbb{R}^2) := \int_{\mathbb{R}} h(u) \mathrm{Per}(E_{k_\sigma}(u), \mathbb{R}^2) du = \int_{\mathbb{R}^2} h(k_\sigma(x)) \|\nabla k_\sigma(x)\| dx,$$

from which we can compute by a change of variables in polar coordinates that

$$\mathrm{LP}_{k_\sigma}(h, \mathbb{R}^2) = 2\pi \int_0^{+\infty} h\left(e^{-\frac{r^2}{2\sigma^2}}\right) \frac{r^2}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr. \tag{16}$$

Therefore, with the change of variables $s = e^{-\frac{r^2}{2\sigma^2}}$ we get

$$\mathrm{LP}_{k_\sigma}(h, \mathbb{R}^2) = \int_0^1 h(s) 2\pi\sigma\sqrt{-2\log(s)} ds.$$

Note that we could also just remark that $E_{k_\sigma}(u) = B(0, \sigma\sqrt{-2\ln(u)})$ and $\mathrm{Per}(E_{k_\sigma}(u), \mathbb{R}^2) = 2\pi\sigma\sqrt{-2\ln(u)}$, for any $u \in (0, 1]$. From (16) we obtain for $h = 1$ that $\mathrm{TV}(k_\sigma, \mathbb{R}^2) = \pi\sqrt{2\pi}\sigma$.

On the other hand for any $n \geq 1$,

$$\mathrm{LP}_{k_\sigma^{(n)}}(h, \mathbb{R}^2) := \int_{\mathbb{R}} h(u) \mathrm{Per}(E_{k_\sigma^{(n)}}(u), \mathbb{R}^2) \mathrm{d}u = \sum_{i=1}^{n-1} \int_{\partial B(0, r_i^{(n)})} \int_{(i-1)/n}^{i/n} h(s) \mathrm{d}s \mathcal{H}^1(\mathrm{d}x).$$

Therefore,

$$\mathrm{LP}_{k_\sigma^{(n)}}(h, \mathbb{R}^2) = \sum_{i=1}^{n-1} 2\pi r_i^{(n)} \int_{(i-1)/n}^{i/n} h(s) \mathrm{d}s.$$

Let us note $f(s) = \sqrt{-2\log(s)}$. This function is well defined on $]0, 1]$, non-negative and strictly decreasing. Moreover we have for all $n \in \mathbb{N}^*$

$$\int_0^{1/n} f(s) \mathrm{d}s = \frac{\sqrt{2\log(n)}}{n} + \int_{\sqrt{2\log(n)}}^{+\infty} e^{-u^2/2} \mathrm{d}u \leq \frac{\sqrt{2\log(n)}}{n} + \frac{1}{n\sqrt{2\log(n)}},$$

using the inequalities on the tail distribution of the Gaussian (see p175 of [22]). So we have

$$\left|\mathrm{LP}_{k_\sigma}(h,\mathbb{R}^2)-\mathrm{LP}_{k_\sigma^{(n)}}(h,\mathbb{R}^2)\right| = \left|\int_0^1 h(s)2\pi\sigma\sqrt{-2\log(s)}\mathrm{d}s-\sum_{i=1}^{n-1}2\pi\sigma\sqrt{-2\log(i/n)}\int_{(i-1)/n}^{i/n}h(s)\mathrm{d}s\right|$$
$$\leq 2\pi\sigma||h||_\infty\left(A_n+B_n+C_n\right)$$

with

$$A_n=\int_0^{1/n}\left|\sqrt{-2\log(s)}-\sqrt{-2\log(1/n)}\right|\mathrm{d}s\leq\int_0^{1/n}\sqrt{-2\log(s)}\mathrm{d}s\leq 2\frac{\sqrt{2\log(n)}}{n},$$

$$\begin{aligned}B_n&=\sum_{i=2}^{n-1}\int_{\frac{i-1}{n}}^{i/n}\left|\sqrt{-2\log(s)}-\sqrt{-2\log(i/n)}\right|\mathrm{d}s\\&\leq\sum_{i=2}^{n-1}\int_{\frac{i-1}{n}}^{i/n}\left|\sqrt{-2\log\left(\frac{i-1}{n}\right)}-\sqrt{-2\log(i/n)}\right|\mathrm{d}s\\&=\frac{1}{n}\sum_{i=2}^{n-1}\sqrt{-2\log\left(\frac{i-1}{n}\right)}-\sqrt{-2\log(i/n)}\\&=\frac{1}{n}\left(\sqrt{-2\log(1/n)}-\sqrt{-2\log\left(\frac{n-1}{n}\right)}\right),\end{aligned}$$

and

$$C_n=\int_{\frac{n-1}{n}}^1\left|\sqrt{-2\log(s)}\right|\mathrm{d}s\leq\frac{1}{n}\sqrt{-2\log\left(\frac{n-1}{n}\right)}.$$

Hence,

$$\left|\mathrm{LP}_{k_\sigma}(h,\mathbb{R}^2)-\mathrm{LP}_{k_\sigma^{(n)}}(h,\mathbb{R}^2)\right|\leq 2\pi\sigma||h||_\infty\left(\frac{3\sqrt{2\log(n)}}{n}+\right)\underset{n\to\infty}{\longrightarrow}0$$

Finally, the uniform upper bound for $\mathrm{TV}(k_\sigma^{(n)},\mathbb{R}^2)$ can be deduced by taking $h=1$ or by remarking that, for all $u\in\mathbb{R}$,

$$\mathrm{Per}(E_{k_\sigma^{(n)}}(u),\mathbb{R}^2)\leq\mathrm{Per}(E_{k_\sigma}(u),\mathbb{R}^2).$$

□

Let us conclude this part by remarking that for $k_{m,\sigma}=mk_\sigma$ we have for any level $u\in\mathbb{R}$,

$$\mathrm{Per}(E_{k_{m,\sigma}}(u),\mathbb{R}^2)=\mathrm{Per}\left(E_{k_\sigma}\left(\frac{u}{m}\right),\mathbb{R}^2\right).$$

Defining similarly for $n\in\mathbb{N}$ and $m>0$,

$$k_{m,\sigma}^{(n)}=mk_\sigma^{(n)}, \tag{17}$$

we can state the following corollary.

Corollary 2. *Let $\sigma>0$ and $m>0$. Then $\mathrm{TV}(k_{m,\sigma},\mathbb{R}^2)=m\mathrm{TV}(k_\sigma,\mathbb{R}^2)=m\pi\sqrt{2\pi}\sigma$ and for all $n\geq 1$,*

$$\mathrm{TV}(k_{m,\sigma}^{(n)},\mathbb{R}^2)\leq\mathrm{TV}(k_{m,\sigma},\mathbb{R}^2).$$

Moreover, for any $h\in\mathcal{C}_b(\mathbb{R})$, one has

$$\mathrm{LP}_{k_{m,\sigma}^{(n)}}(h,\mathbb{R}^2)\underset{n\to\infty}{\longrightarrow}\mathrm{LP}_{k_{m,\sigma}}(h,\mathbb{R}^2),$$

where

$$\mathrm{LP}_{k_{m,\sigma}}(h,\mathbb{R}^2)=\int_0^1 mh(ms)2\pi\sigma\sqrt{-2\log(s)}ds.$$

Proof. It simply follows from the fact that for $h \in \mathcal{C}_b(\mathbb{R})$,

$$\begin{aligned}
\mathrm{LP}_{k_{m,\sigma}}(h, \mathbb{R}^2) &= \int_{\mathbb{R}} h(u)\mathrm{Per}(E_{k_{m,\sigma}}(u), \mathbb{R}^2)du \\
&= \int_{\mathbb{R}} h(u)\mathrm{Per}(E_{k_\sigma}(u/m), \mathbb{R}^2)du \\
&= \int_{\mathbb{R}} mh(mv)\mathrm{Per}(E_{k_\sigma}(v), \mathbb{R}^2)dv,
\end{aligned}$$

by a change of variable. Hence, denoting $[h]_m := mh(m\cdot)$ we get $\mathrm{LP}_{k_{m,\sigma}}(h, \mathbb{R}^2) = \mathrm{LP}_{k_\sigma}([h]_m, \mathbb{R}^2)$ and similarly $\mathrm{LP}^{(n)}_{k_{m,\sigma}}(h, \mathbb{R}^2) = \mathrm{LP}^{(n)}_{k_\sigma}([h]_m, \mathbb{R}^2)$. □

4.2. **The elementary Gaussian shot noise random field.** Let $\lambda, \mu, \sigma \in (0, +\infty)$ and $X_{\lambda,\mu,\sigma}$ be a Gaussian shot noise random field defined by (12) and note that

$$X_{\lambda,\mu,\sigma}(x) = \sum_i \frac{m_i}{2\pi\sigma^2} k_\sigma(x - x_i),$$

with marks $\{\frac{m_i}{2\pi\sigma^2}\}$ with distribution $\mathcal{E}(2\pi\sigma^2\mu)$ denoted as F_σ.

Proposition 5. *Let $n \geq 1$ and $\sigma > 0$. For $\lambda, \mu > 0$ and $\{x_i\}_{i\in I}$ an homogeneous Poisson point process of intensity λ on $\mathbb{R}^2$ independently marked with $\{m_i\}_{i\in I}$ of exponential distribution F_σ of parameter $2\pi\sigma^2\mu$, the random field*

$$X^{(n)}_{\lambda,\mu,\sigma}(x) := \sum_i m_i k^{(n)}_\sigma(x - x_i), \tag{18}$$

is an elementary field on any open bounded $T \subset \mathbb{R}^2$, in the sense of [11]. *Moreover for all $h \in \mathcal{C}_b(\mathbb{R})$ one has*

$$\overline{\mathrm{LP}_{X^{(n)}_{\lambda,\mu,\sigma}}}(h) = \lambda \int_{\mathbb{R}} \mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2) F_\sigma(dm),$$

where $\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}(s) = \mathbb{E}[h(X^{(n)}_{\lambda,\mu,\sigma}(0) + s)]$ for $s \in \mathbb{R}$.

Proof. Let us check assumptions of Theorem 5 of [11]. Note that $k^{(n)}_{m,\sigma} = mk^{(n)}_\sigma$ is an elementary function with compact support in $B(0, r^{(n)}_1)$ satisfying

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |k^{(n)}_{m,\sigma}(x)| dx F_\sigma(dm) < +\infty,$$

since $\int_{\mathbb{R}} |m| F_\sigma(dm) < +\infty$. Moreover, by Corollary 2

$$\int_{\mathbb{R}} \mathrm{TV}(k^{(n)}_{m,\sigma}) F_\sigma(dm) \leq \int_{\mathbb{R}} \mathrm{TV}(k_{m,\sigma}) F_\sigma(dm) = \int_{\mathbb{R}^2} \|\nabla k_\sigma(x)\| dx \times \int_{\mathbb{R}} |m| F_\sigma(dm) < +\infty,$$

and, using (16) in Proposition 4,

$$\int_{\mathbb{R}} \mathrm{LTaC}(k^{(n)}_{m,\sigma}) F_\sigma(dm) \leq 2\pi \times \int_{\mathbb{R}} |m| F_\sigma(dm) < +\infty,$$

that finishes to prove (21). Equation (22) simply follows from the fact that the supports do not depend on m and again that $\int_{\mathbb{R}} |m| F_\sigma(dm)$. Now, let us note that all discontinuity points are regular points, ensuring (24) while (25) comes from the fact that

$$\mathcal{H}^0\left(\mathcal{R}_{k^{(n)}_{m',\sigma}} \cap \tau_x \mathcal{R}_{k^{(n)}_{m,\sigma}}\right) \leq 2(n-1)^2 \mathbb{1}_{\|x\| \leq 2r^{(n)}_1}. \tag{19}$$

The last point (26) is obtained remarking that for almost all $x \in \mathbb{R}^2$ the regular sets can not intersect in a tangency position. This allows to obtain the fact that $X^{(n)}_{\lambda,\mu,\sigma}$ is an elementary field in T according to Theorem 4 of [11]. To conclude it only remains to check (31) of Theorem 5, that follows from (19). □

We couple this elementary field with our smooth shot noise field by setting

$$X_{\lambda,\mu,\sigma}(x) := \sum_i m_i k_\sigma(x - x_i), \tag{20}$$

with the same marked Poisson point process with marks of distribution $\mathcal{E}(2\pi\sigma^2\mu)$.

Proposition 6. *Let $\lambda, \mu, \sigma \in (0, +\infty)$, $n \in \mathbb{N}$, $X^{(n)}_{\lambda,\mu,\sigma}$ given by* (18) *and $X_{\lambda,\mu,\sigma}$ defined by* (20)*. Then, $X^{(n)}_{\lambda,\mu,\sigma}(x)$ converges towards $X_{\lambda,\mu,\sigma}(x)$ in $L^2(\Omega)$ (and therefore also in $L^1(\Omega)$), for all $x \in \mathbb{R}^2$.*

Proof. By stationarity we can consider $x = 0$. It follows that

$$\begin{aligned}\mathbb{E}[|X_{\lambda,\mu,\sigma}(0) - X^{(n)}_{\lambda,\mu,\sigma}(0)|^2] &= \operatorname{Var}\left(X_{\lambda,\mu,\sigma}(0) - X^{(n)}_{\lambda,\mu,\sigma}(0)\right) + \mathbb{E}\left(X_{\lambda,\mu,\sigma}(0) - X^{(n)}_{\lambda,\mu,\sigma}(0)\right)^2 \\ &= \lambda\mathbb{E}[M_\sigma^2]||k_\sigma - k_\sigma^{(n)}||_2^2 + \lambda^2\mathbb{E}[M_\sigma]^2||k_\sigma - k_\sigma^{(n)}||_1^2,\end{aligned}$$

which tends to 0, according to Lemma 2. Here we introduce M_σ a random variable of distribution $\mathcal{E}(2\pi\sigma^2\mu)$. □

Theorem 3. *Let $\lambda, \mu, \sigma \in (0, +\infty)$, $n \in \mathbb{N}$, $X^{(n)}_{\lambda,\mu,\sigma}$ given by* (18) *and $X_{\lambda,\mu,\sigma}$ defined by* (20)*. For $h \in \mathcal{C}_b(\mathbb{R})$ and Lipschitz on $\mathbb{R}$, we have*

$$\overline{\mathrm{LP}_{X^{(n)}_{\lambda,\mu,\sigma}}}(h) \underset{n\to\infty}{\longrightarrow} \overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h) := \lambda \int_{\mathbb{R}} \mathrm{LP}_{k_{m,\sigma}}(\bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2) F_\sigma(dm) \tag{21}$$

where $\bar{h}_{X_{\lambda,\mu,\sigma}(0)}(s) = \mathbb{E}[h(X_{\lambda,\mu,\sigma}(0) + s)]$ for $s \in \mathbb{R}$ and F_σ is the exponential distribution of parameter $2\pi\sigma^2\mu$.

Proof. According to Proposition 5, we know that

$$\overline{\mathrm{LP}_{X^{(n)}_{\lambda,\mu,\sigma}}}(h) = \lambda \int_{\mathbb{R}} \mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2) F_\sigma(dm) \tag{22}$$

where $\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}(s) = \mathbb{E}[h(X^{(n)}_{\lambda,\mu,\sigma}(0) + s)]$ for $s \in \mathbb{R}$.

Lemma 3. *Let $h \in \mathcal{C}_b(\mathbb{R})$ and Lipschitz with Lipschitz constant $\mathrm{Lip}(h)$. Then, for all $n \geq 1$, we also have $\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)} \in \mathcal{C}_b(\mathbb{R})$ and Lipschitz with $\left\|\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}\right\|_\infty \leq \|h\|_\infty$ and $\mathrm{Lip}\left(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}\right) \leq \mathrm{Lip}(h)$.*

Proof. Let $n \geq 1$. It is clear that for $h \in \mathcal{C}_b(\mathbb{R})$, the function $\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}$ remains bounded and moreover, for all $s \in \mathbb{R}$,

$$|\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}(s)| \leq \mathbb{E}\left[\left|h(X^{(n)}_{\lambda,\mu,\sigma}(0) + s)\right|\right] \leq \|h\|_\infty.$$

Moreover, for $s, t \in \mathbb{R}$ we have

$$\begin{aligned}\left|\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}(s) - \bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}(t)\right| &\leq \mathbb{E}\left[\left|h(X^{(n)}_{\lambda,\mu,\sigma}(0) + s) - h(X^{(n)}_{\lambda,\mu,\sigma}(0) + t)\right|\right] \\ &\leq \mathbb{E}\left[\mathrm{Lip}(h)|s - t|\right] \\ &= \mathrm{Lip}(h)|s - t|,\end{aligned}$$

that concludes the proof. □

Now we write for $h \in \mathcal{C}_b(\mathbb{R})$ and Lipschitz,

$$\begin{aligned}&\left|\mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2) - \mathrm{LP}_{k_{m,\sigma}}(\bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2)\right| \\ &\leq \left|\mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)} - \bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2)\right| + \left|\mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2) - \mathrm{LP}_{k_{m,\sigma}}(\bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2)\right|\end{aligned} \tag{23}$$

Let $\epsilon > 0$. By Proposition 6, there exists $N_\epsilon \in \mathbb{N}$ such that $\forall n \geq N_\epsilon$ we have

$$\mathbb{E}\left[\left|X_{\lambda,\mu,\sigma}(0) - X^{(n)}_{\lambda,\mu,\sigma}(0)\right|\right] \leq \frac{\epsilon}{1+\mathrm{Lip}(h)}.$$

Let $n \geq N_\epsilon$, then for all $u \in \mathbb{R}$,

$$\begin{aligned}|\bar{h}_{X_{\lambda,\mu,\sigma}(0)}(u) - \bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}(u)| &\leq \mathbb{E}\left[\left|h(X_{\lambda,\mu,\sigma}(0)+u) - h(X^{(n)}_{\lambda,\mu,\sigma}(0)+u)\right|\right] \\ &\leq \mathrm{Lip}(h)\mathbb{E}\left[\left|X_{\lambda,\mu,\sigma}(0) - X^{(n)}_{\lambda,\mu,\sigma}(0)\right|\right] \\ &\leq \epsilon.\end{aligned}$$

It follows that $||\bar{h}_{X_{\lambda,\mu,\sigma}(0)} - \bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}||_\infty \underset{n\to\infty}{\longrightarrow} 0$.

Thus we have

$$\begin{aligned}\left|\mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)} - \bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2)\right| &\leq \mathrm{TV}(k^{(n)}_{m,\sigma}, \mathbb{R}^2)||\bar{h}_{X_{\lambda,\mu,\sigma}(0)} - \bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}||_\infty \\ &\leq \mathrm{TV}(k_{m,\sigma}, \mathbb{R}^2)||\bar{h}_{X_{\lambda,\mu,\sigma}(0)} - \bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}||_\infty\end{aligned}$$

by Corollary 2. It follows that

$$\left|\mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)} - \bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2)\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

For the second term, we simply use again Corollary 2 with $h := \bar{h}_{X_{\lambda,\mu,\sigma}(0)} \in \mathcal{C}_b(\mathbb{R})$ and combining both we can conclude that

$$\mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2) \underset{n\to\infty}{\longrightarrow} \mathrm{LP}_{k_{m,\sigma}}(\bar{h}_{X_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2).$$

Since we also have

$$|\mathrm{LP}_{k^{(n)}_{m,\sigma}}(\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}, \mathbb{R}^2)| \leq \|h\|_\infty \mathrm{TV}(k^{(n)}_{m,\sigma}, \mathbb{R}^2) \leq |m|\|h\|_\infty \mathrm{TV}(k_\sigma, \mathbb{R}^2),$$

and $\int_{\mathbb{R}} |m| F_\sigma(dm) = \mathbb{E}(|M_\sigma|) < +\infty$, we can use Lebesgue's theorem to obtain the result.

□

At that point it is natural to ask if $\overline{\overline{\mathrm{LP}}}_{X_{\lambda,\mu,\sigma}}(h) = \overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}(h)$. Let us look at this with this next corollary.

Corollary 3. *Let $\lambda, \mu, \sigma \in (0, +\infty)$, $n \in \mathbb{N}$, $X^{(n)}_{\lambda,\mu,\sigma}$ given by (18) and $X_{\lambda,\mu,\sigma}$ defined by (20). Then we have*

$$\overline{\mathrm{TV}(X^{(n)}_{\lambda,\mu,\sigma})} \underset{n\to\infty}{\longrightarrow} \overline{\overline{\mathrm{TV}(X_{\lambda,\mu,\sigma})}} := \overline{\overline{\mathrm{LP}}}_{X_{\lambda,\mu,\sigma}}(1) = \frac{\lambda}{\sigma\mu}\sqrt{\frac{\pi}{2}}. \tag{24}$$

Proof. By taking h constant equal to 1 we also have $\bar{h}_{X^{(n)}_{\lambda,\mu,\sigma}(0)}$ constant equal to 1. Then by Theorem 3 we have

$$\begin{aligned}\overline{\mathrm{TV}(X^{(n)}_{\lambda,\mu,\sigma})} &\underset{n\to\infty}{\longrightarrow} \lambda\int_{\mathbb{R}} \mathrm{LP}_{k_{m,\sigma}}(1, \mathbb{R}^2) F_\sigma(dm) \\ &= \lambda\int_{\mathbb{R}}\int_0^1 m2\pi\sigma\sqrt{-2\log(s)}\,ds F_\sigma(dm) \\ &= \frac{2\pi\sigma\lambda}{2\pi\sigma^2\mu}\int_0^1 \sqrt{-2\ln(s)}\,ds \\ &= \frac{\lambda}{\sigma\mu}\sqrt{\frac{\pi}{2}}.\end{aligned}$$

□

The simplicity of this result has the defect of representing only the low "regime" with small λ of Theorem 2 since we can now identify the limit.

Proposition 7. *Let $\lambda, \mu, \sigma \in (0, +\infty)$ and $\nu \in \mathbb{R}$. Then*

$$\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h_\nu) = \lambda \varphi_{X_{\lambda,\mu,\sigma}}(\nu) \int_{\mathbb{R}} \mathrm{LP}_{g_{m,\sigma}}(h_\nu, \mathbb{R}^2) F(dm).$$

Proof. Actually, by definition of $\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}$ in (21) we have that

$$\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h_\nu) = \lambda \varphi_{X_{\lambda,\mu,\sigma}}(\nu) \int_{\mathbb{R}} \mathrm{LP}_{k_{m,\sigma}}(h_\nu, \mathbb{R}^2) F_\sigma(dm).$$

Now, by the general co-area formula obtained in Theoreme 1 of [10] we have

$$\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h_\nu) = \lambda \varphi_{X_{\lambda,\mu,\sigma}}(\nu) \int_{\mathbb{R}} \int_{\mathbb{R}^2} h_\nu(k_{m,\sigma}(x)) \|\nabla k_{m,\sigma}(x)\| dx F_\sigma(dm),$$

and a change of variable in m allows us to get

$$\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h_\nu) = \lambda \varphi_{X_{\lambda,\mu,\sigma}}(\nu) \int_{\mathbb{R}} \int_{\mathbb{R}^2} h_\nu(g_{m,\sigma}(x)) \|\nabla g_{m,\sigma}(x)\| dx F(dm),$$

where we recognize

$$\mathrm{LP}_{g_{m,\sigma}}(h_\nu, \mathbb{R}^2) = \int_{\mathbb{R}^2} h_\nu(g_{m,\sigma}(x)) \|\nabla g_{m,\sigma}(x)\| dx.$$

□

Then similarly to what we have done for high regime, we can restate Theorem 2 in terms of resolution σ.

Proposition 8. *Assuming $\lambda\sigma^2 \le c_0$, for all $\beta \in (0,1)$, there exists $c_\beta(\mu) > 0$ such that for all $\nu \in \mathbb{R}$*

$$\left| \overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}(h_\nu) - \overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h_\nu) \right| \le c_\beta(\mu) \lambda^{\frac{2}{1+\beta}} \sigma^{\frac{1-3\beta}{1+\beta}}.$$

Proof. Using change of variables we can also check that it follows that

$$\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h_\nu) = \frac{1}{\sigma} \overline{\overline{\mathrm{LP}_{X_{\lambda\sigma^2,\mu\sigma^2,1}}}}(h_\nu) = \frac{1}{\sigma^3} \overline{\overline{\mathrm{LP}_{X_{\lambda\sigma^2,\mu,1}}}}(h_{\nu/\sigma}).$$

For $\beta \in (0,1)$, we can take $c_\beta(\mu) = C_1^-$ for $M \sim \mathcal{E}(\mu)$ and $\sigma = 1$, according to Theorem 2, to get for all $\nu \in \mathbb{R}$,

$$\left| \overline{\mathrm{LP}_{X_{\lambda\sigma^2,\mu,1}}}(h_{\nu/\sigma}) - \overline{\mathrm{LP}_{B_{\lambda\sigma^2,\mu,1}}}(h_{\nu/\sigma}) \right| \le c_\beta(\mu) \left[\lambda\sigma^2\right]^{\frac{2}{1+\beta}},$$

that gives the result. □

Hence we introduce the following quantities that will be used for numerical computations.

Proposition 9. *Let $\lambda, \mu, \sigma \in (0, +\infty)$, and let $f_{\lambda,\mu,\sigma}$ be the density of a gamma law of parameters $2\pi\mu\sigma^2$ and $2\pi\lambda\sigma^2$, then we define the mean elementary perimeter function as*

$$\overline{\overline{\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u))}} := \lambda \int_{\mathbb{R}} \int_0^1 f_{\lambda,\mu,\sigma}(u - ms) m 2\pi\sigma \sqrt{-2\ln(s)} ds F_\sigma(dm), \tag{25}$$

where F_σ denotes the exponential distribution of parameter $2\pi\sigma^2\mu$. Then, for all $h \in \mathcal{C}_b(\mathbb{R})$, one has

$$\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h) = \int_{\mathbb{R}} h(u) \overline{\overline{\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u))}} du.$$

Moreover, its Fourier transform is given for all $\nu \in \mathbb{R}$ by

$$\overline{\overline{\mathrm{LP}_{X_{\lambda,\mu,\sigma}}}}(h_\nu) = \lambda \left(\frac{2\pi\mu\sigma^2}{2\pi\mu\sigma^2 - i\nu} \right)^{2\pi\lambda\sigma^2} 2\pi\sigma \int_0^1 \frac{2\pi\sigma^2\mu}{(i\nu s - 2\pi\sigma^2\mu)^2} \sqrt{-2\ln(s)} ds \tag{26}$$

Proof. Let $h \in \mathcal{C}_b(\mathbb{R})$. By Theorem 3 and Corollary 2 it follows that

$$\begin{aligned}
\overline{\overline{\mathrm{LP}}}_{X_{\lambda,\mu,\sigma}}(h) &= \lambda \int_{\mathbb{R}} \mathrm{LP}_{k_{m,\sigma}}(\bar{h}_{X_{\lambda,\mu,\sigma}}(0), \mathbb{R}^2) F_\sigma(dm) \\
&= \lambda \int_{\mathbb{R}} \int_0^1 \mathbb{E}[h(X_{\lambda,\mu,\sigma}(0) + ms)] m 2\pi\sigma \sqrt{-2\ln(s)} ds F_\sigma(dm) \\
&= \lambda \int_{\mathbb{R}} \int_0^1 \int_{\mathbb{R}} h(t + ms) f_{\lambda,\mu,\sigma}(t) m 2\pi\sigma \sqrt{-2 ln(s)} dt ds F_\sigma(dm) \\
&= \lambda \int_{\mathbb{R}} \int_0^1 \int_{\mathbb{R}} h(u) f_{\lambda,\mu,\sigma}(u - ms) m 2\pi\sigma \sqrt{-2 ln(s)} du ds F_\sigma(dm) \\
&= \int_{\mathbb{R}} h(u) \left(\lambda \int_{\mathbb{R}} \int_0^1 f_{\lambda,\mu,\sigma}(u - ms) m 2\pi\sigma \sqrt{-2 ln(s)} ds F_\sigma(dm) \right) du,
\end{aligned}$$

where we recognize $\overline{\overline{\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u))}}$ under integral.

Let $h_\nu(s) = e^{i\nu s}$ for $s, \nu \in \mathbb{R}$. By Corollary 2 we have that

$$\begin{aligned}
\overline{\overline{\mathrm{LP}}}_{X_{\lambda,\mu,\sigma}}(h_\nu) &= \lambda \varphi_{X_{\lambda,\mu,\sigma}}(\nu) \int_{\mathbb{R}} \int_0^1 e^{i\nu ms} m 2\pi\sigma \sqrt{-2\ln(s)} ds F(dm) \\
&= \lambda \left(\frac{2\pi\mu\sigma^2}{2\pi\mu\sigma^2 - i\nu} \right)^{2\pi\lambda\sigma^2} 2\pi\sigma \int_0^1 \sqrt{-2\ln(s)} \int_{\mathbb{R}} m e^{i\nu ms} F(dm) ds \\
&= \lambda \left(\frac{2\pi\mu\sigma^2}{2\pi\mu\sigma^2 - i\nu} \right)^{2\pi\lambda\sigma^2} 2\pi\sigma \int_0^1 \frac{2\pi\sigma^2\mu}{(i\nu s - 2\pi\sigma^2\mu)^2} \sqrt{-2\ln(s)} ds
\end{aligned}$$

□

Let us remark that the integral giving $u \mapsto \overline{\overline{\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u))}}$ is not easy to compute but can be efficiently numerically approximated as well as its Fourier transform (see Appendix).

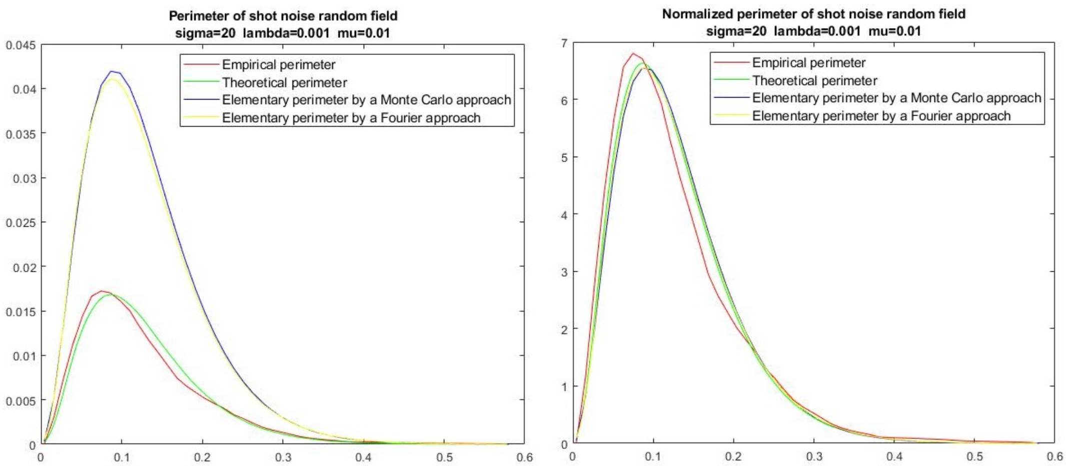

FIGURE 2. Perimeter and normalized perimeter of a Gaussian shot noise random field with exponential marks. The paramaters are in the title. In red, the empirical perimeter from a simulation on $F = [0, 1000]^2$ and in green its theoretical formula $u \mapsto \overline{\mathrm{Per}(E_X(u))}$ approximated by Algorithm 1. In blue and yellow, the elementary asymptotic perimeter $u \mapsto \overline{\overline{\mathrm{Per}(E_X(u))}}$ approximated by Algorithm 3 and 2.

We strongly believe that the difference between the theoretical perimeter and the elementary perimeter comes from the total variation. That is why, for X a random field and $u \in \mathbb{R}$, let us define the normalized perimeter function by

$$\mathrm{Per}^*(E_{X_{\lambda,\mu,\sigma}}(u)) := \frac{\overline{\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u))}}{\overline{\mathrm{TV}(E_{X_{\lambda,\mu,\sigma}(u))}}},$$

and make the following conjecture, motivated by Figure 2.

Conjecture : for almost all $u \in \mathbb{R}$ we have $\mathrm{Per}^*(E_{X_{\lambda,\mu,\sigma}}(u)) = \overline{\overline{\dfrac{\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}(u))}}{\mathrm{TV}(E_{X_{\lambda,\mu,\sigma}(u))}}}}$.

5. Classification between Gaussian and shot noise fields

We have already seen that there is a closed correspondence between Gaussian and shot noise random fields. In practice, this can be used in insurance risk management [28] or in wireless networks [17, 4]. This section has a specific aim : the classification between these two random fields.

For $(\lambda,\mu,\sigma) \in]0,+\infty[^3$, considering a shot noise field $X_{\lambda,\mu,\sigma}$ (*i.e.* a shot noise with a Gaussian kernel and exponential mark), by Campbell's Theorem [27], the mean is $\mathbb{E}[X_{\lambda,\mu,\sigma}(0)] = \frac{\lambda}{\mu}$ and the isotropic covariance function is itself Gaussian, since we have for all $r > 0$

$$C_{X_{\lambda,\mu,\sigma}}(r) = \frac{\lambda}{2\pi\sigma^2\mu^2}\exp(-\frac{r^2}{4\sigma^2}),$$

where $C_{X_{\lambda,\mu,\sigma}}(r) = \mathrm{Cov}(X_{\lambda,\mu,\sigma}(x), X_{\lambda,\mu,\sigma}(y))$ for all $x, y \in \mathbb{R}^2$ with $||x-y|| = r$. That is why, the Gaussian random field of interest $B_{\lambda,\mu,\sigma}$ has the same mean $M_{B_{\lambda,\mu,\sigma}} = \frac{\lambda}{\mu}$ and the same covariance function $C_{B_{\lambda,\mu,\sigma}}(r) = C_{X_{\lambda,\mu,\sigma}}(r)$. When there is no ambiguity, we write $X = X_{\lambda,\mu,\sigma}$ and $B = B_{\lambda,\mu,\sigma}$.

Conversely, let B a Gaussian random field with Gaussian covariance function, then there exist $(\lambda,\mu,\sigma) \in]0,+\infty[^3$ and a shot noise random field $X_{\lambda,\mu,\sigma}$ with the same mean, the same variance and the same covariance function than B. Indeed, if we note $C_B(r) = \mathrm{Var}[B(0)]e^{-\frac{r^2}{4\sigma^2}}$ the covariance function and $M_B = \mathbb{E}[B(0)]$ the mean of the Gaussian random field, the parameters of the corresponding shot noise $X_{\lambda,\mu,\sigma}$ are

$$\begin{cases} \lambda &= \dfrac{M_B^2}{2\pi\sigma^2\mathrm{Var}[B(0)]} \\ \mu &= \dfrac{M_B}{2\pi\sigma^2\mathrm{Var}[B(0)]} \\ \sigma &= \sigma. \end{cases}$$

Some simulations are presented in Figure 3, 4, 5 to compare a Gaussian field with a shot noise field. Each random field is simulated on a window $T = [0, 1000]^2$ with the same mean, same variance and same correlation. We have chosen, arbitrarily, the same intensity of the Poisson process $\lambda = 0.001$ and the same mark $\mu = 0.01$ for all simulations. Although the intensity parameter λ may seem small, it represents on average 1000 points on T. Also, let us not forget that the parameter of an exponential law is the inverse of its mean, hence a mean of 100 for the mark. The difference between Figures 3, 4, 5 is in the σ parameter and note that, similarly to the high intensity convergence [25], we can less and less distinguish the fields with the increase of σ.

We will start by introducing the classification based on the geometries of Gaussian fields. By Corollary 1, since the second spectral moment of B is $\lambda_2 = \mathrm{Var}[\partial B(0)] = \partial_1^2 \mathrm{Cov}_B(0) = \frac{\lambda}{4\pi\mu^2\sigma^4}$, for $u \in \mathbb{R}$ we have that

$$\overline{\mathrm{Per}(E_B(u))} = \frac{1}{\sqrt{8}\sigma}\exp\left(-\pi\mu^2\sigma^2\frac{(u-\frac{\lambda}{\mu})^2}{\lambda}\right) \tag{27}$$

and we have also

$$\overline{\mathrm{Area}(E_B(u))} = \int_u^\infty \frac{1}{\sqrt{2\pi\mathrm{Var}[B(0)]}}\exp\left(-\frac{(x-\lambda/\mu)^2}{2\mathrm{Var}[B(0)]}\right)dx \tag{28}$$

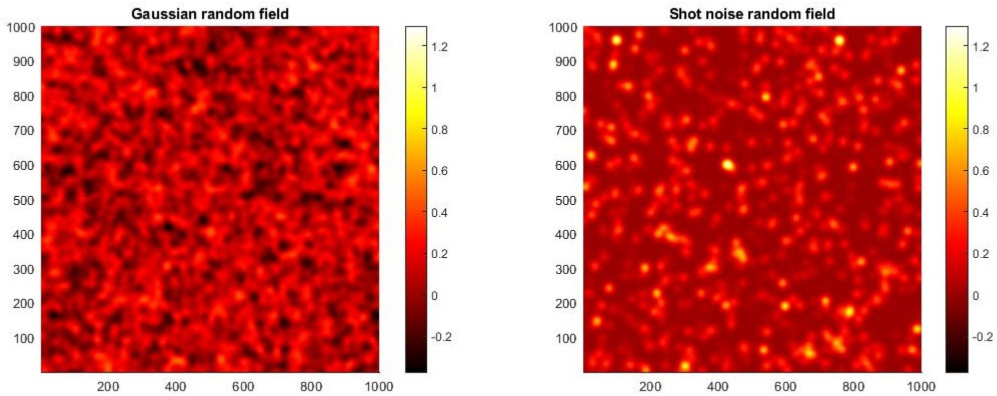

FIGURE 3. Gaussian and shot noise random fields simulated on $F = [0, 1000]^2$ with $\lambda = 0.001$, $\mu = \frac{1}{100}$ and $\sigma = 10$.

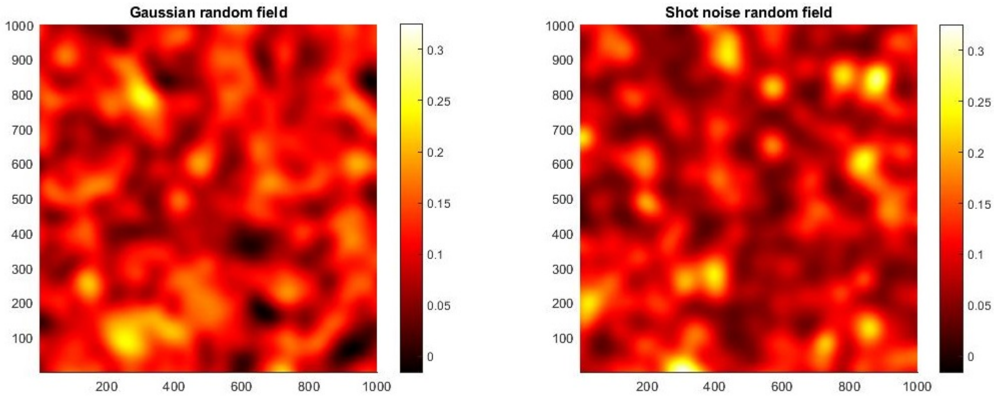

FIGURE 4. Gaussian and shot noise random fields simulated on $F = [0, 1000]^2$ with $\lambda = 0.001$, $\mu = \frac{1}{100}$ and $\sigma = 30$.

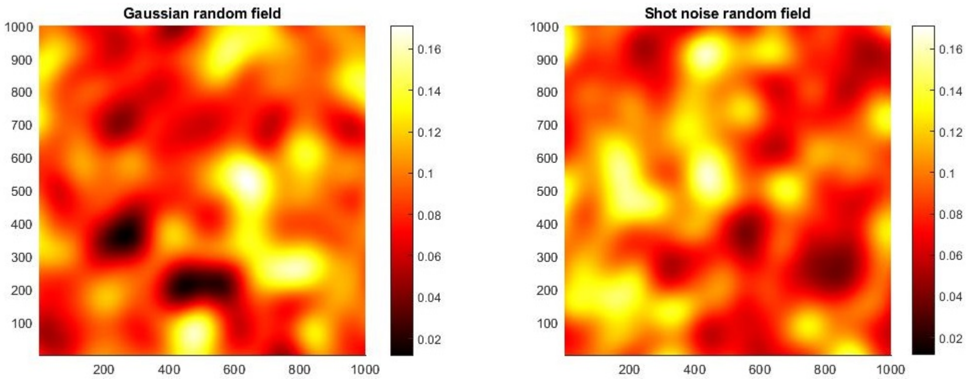

FIGURE 5. Gaussian and shot noise random fields simulated on $F = [0, 1000]^2$ with $\lambda = 0.001$, $\mu = \frac{1}{100}$ and $\sigma = 50$.

To compute (27) and (28), it is necessary to estimate the parameters. To do this, we use the method of moments, similar to what we did for the shot noise random fields in [33]. More precisely, suppose that we observe our field on a rectangle $T \subset \mathbb{Z}^2$ and let us introduce the following moment estimators :

$$\widehat{M_1} = \frac{1}{|T|}\sum_{x\in T} X(x) \tag{29}$$

$$\widehat{M_2} = \frac{1}{|T|}\sum_{x\in T} X(x)^2 \tag{30}$$

$$\widehat{h(w)} = \frac{1}{|T\ominus w|}\sum_{x\in T\ominus w} X(x)X(x+w), \tag{31}$$

where $w \in \mathbb{Z}^2$ and $T \ominus w = \{x \in T \ ; \ x+w \in T\}$.
Since the field is assumed to be isotropic, the estimators of mixed moment of second order $\widehat{h(w)}$ can be grouped according to the norm of w which allows to improve the estimator. That is why we consider, for $r \in \mathbb{Z}_+$,

$$\widehat{\tilde{h}(r)} = \frac{1}{2}(\widehat{h(re_1)} + \widehat{h(re_2)})$$

where (e_1, e_2) denotes the canonical basis of $\mathbb{R}^2$.
From these estimators, let us introduce the estimators of variance $\widehat{V} = \widehat{M_2} - \widehat{M_1}^2$ and of the isotropic correlation function $\widehat{\tilde{\rho}(r)} = \frac{(\widehat{\tilde{h}(r)} - \widehat{M_1}^2)}{\widehat{V}}$. Then, to estimate (λ, μ, σ), we can use for any $r \in \mathbb{Z}^2$ the estimators :

$$\begin{cases} \widehat{\lambda_r} = \dfrac{-2\widehat{M_1}^2 \log(\widehat{\tilde{\rho}(r)})}{\pi r^2(\widehat{M_2} - \widehat{M_1}^2)} \\ \widehat{\mu_r} = \dfrac{-2\widehat{M_1} \log(\widehat{\tilde{\rho}(r)})}{\pi r^2(\widehat{M_2} - \widehat{M_1}^2)} \\ \widehat{\sigma_r} = \sqrt{\dfrac{-r^2}{4\log(\widehat{\tilde{\rho}(r)})}} \end{cases} \tag{32}$$

By property of association of a Gaussian field with Gaussian covariance function, they are consistent estimators and asymptotically normal, similar to what we did for the shot noise fields [33]. Following this work, we can optimize the choice of r by taking $r^* = \underset{r\in\mathbb{N}^*}{\operatorname{argmin}}|\widehat{\tilde{\rho}(r)} - \exp(-\frac{1}{2})|$, the distance where the covariance function decreases fastest.

These good estimates of the parameters allow us to compute the theoretical formulas (28, 27) with $(\widehat{\lambda_{r*}}, \widehat{\mu_{r*}}, \widehat{\sigma_{r*}})$ instead of (λ, μ, σ). To stay in the spirit of the article, we are interested in the normalized perimeter function

$$\mathrm{Per}^*(E_F(u)) = \frac{\overline{\mathrm{Per}(E_F(u))}}{\overline{\mathrm{TV}(E_F(u))}}.$$

We can also estimate these geometries. For f a random field data observed on a rectangle $T \subset \mathbb{Z}^2$, let us introduce u_{min} the minimum of f and u_{max} its maximum. Let val a regular discretization of $[u_{min}, u_{max}]$ of size $lval \geq 2$ and note $sval = \frac{u_{max}-u_{min}}{lval-1}$ the distance between two successive points of val. For all $u \in val$, the area function estimator is

$$\widehat{\mathrm{Area}}(E_f(u)) = \frac{1}{|T|}\sum_{x\in T} \mathbf{1}_{f(x)\geq u}.$$

To estimate the perimeter function we simply use the estimator $\widehat{\mathrm{Per}}(E_f(u), T)$ of [1], also defined as $\hat{P}_f^{(1)}$ in [19], multiplied by $\pi/4$ to correct the bias according to Proposition 4.4 in [12] in view of isotropy of fields under study. Then we normalized according to the level to set $\widehat{\mathrm{Per}^*}(E_f(u))$.

The geometric statistic that will be used as classifier is $d^B(f) = (d^B_{area}(f), d^B_{per}(f)) \in \mathbb{R}^2$ with

$$d^B_{area}(f) = \left(sval \sum_{u \in val} (\overline{\text{Area}(E_B(u))} - \widehat{\text{Area}}(E_f(u)))^2 \right)^{1/2} \tag{33}$$

the discretized $\mathcal{L}^2$ distance between the emperical area and the theoretical Gaussian area, and

$$d^B_{per}(f) = \left(sval \sum_{u \in val} (\text{Per}^*(E_B(u)) - \widehat{\text{Per}^*}(E_f(u)))^2 \right)^{1/2} \tag{34}$$

the discretized $\mathcal{L}^2$ distance between the empirical perimeter and the theoretical normalized Gaussian perimeter.

Our classification approach is that of the supervised learning. To do this, we will use the quadratic discriminant analysis [24]. To use the classical notation of classification theory, let us note $\mathcal{C}_1$ the class of Gaussian random field with Gaussian covariance and $\mathcal{C}_2$ the class of shot noise random field with Gaussian covariance. We use several simulations of Gaussian fields and shot noise fields to understand the behavior of the two distances (33)-(34) under the two different classes. More precisely, we simulate $N \in \mathbb{N}^*$ independent Gaussian random fields $B_1, \ldots, B_N$ in $\mathcal{C}_1$of same distribution than B to compute

$$\left(d^B(B_1), \ldots, d^B(B_N) \right),$$

the distances under the Gaussian assumption. In the same way, we simulation N independent shot noise random fields $X_1, \ldots, X_N$ in $\mathcal{C}_2$ with same second order statistics than B to compute the distances under the shot noise assumption

$$\left(d^B(X_1), \ldots, d^B(X_N) \right).$$

In the classification lexicon, these two samples are called training set. It remains to determine to which sample the observed distance pairs $d^B(f) = (d_B^{area}(f), d_B^{per}(f))$ best belong. To do this, we will use the quadratic discriminant analysis [24]. For f a random field data and $k = 1, 2$, we assume that, conditional on $f \in \mathcal{C}_k$, $d^B(f)$ corresponds to the realisation of a two-dimensional random variable with Gaussian distribution $\mathcal{N}(\mu_k, \Sigma_k)$. Since it is accepted that $\Sigma_1 \neq \Sigma_2$, the boundary of the classification decision rule is quadratic [24]. Noting p^B_k the density of a Gaussian variable $\mathcal{N}(\mu_k, \Sigma_k)$, the classification rule is given by

$$\begin{cases} f \in \mathcal{C}_1 & \text{if } p^B_1(d^B(f)) > p^B_2(d^B(f)) \\ f \in \mathcal{C}_2 & \text{if } p^B_1(d^B(f)) < p^B_2(d^B(f)) \end{cases}$$

When making a classification decision, there is always the possibility of making a mistake. For $d \in \mathbb{R}^2$ and $k = 1, 2$, we introduce the score function $d \to S^B_k(d)$ that represents the probability of choosing class k given that $f \in \mathcal{C}_k$. In the quadratic discriminant analysis, conditionally to the fact that $f \in \mathcal{C}_k$, the score function $S^B_k(d^B(f)) = \frac{p^B_k(d^B(f))}{p^B_1(d^B(f)) + p^B_2(d^B(f))}$. Finally, the error of the classification if $f \in \mathcal{C}_k$ is defined by 1 minus the score $S^B_k(d^B(f))$.

For $k = 1, 2$, to compute the density p^B_k, it only remains to estimate the mean by

$$\widehat{\mu_k} = \frac{1}{N} \sum_{i=1}^{N} \left(d^B_{area}(f_i), d^B_{per}(f_i) \right), \tag{35}$$

and the covariance matrix by

(36)

$$\widehat{\Sigma_k} = \frac{1}{N} \sum_{i=1}^{N} \begin{pmatrix} (d^B_{area}(f_i) - \widehat{\mu_k}(1))^2 & (d^B_{area}(B_i) - \widehat{\mu_k}(1))(d^B_{per}(f_i) - \widehat{\mu_k}(2)) \\ (d^B_{area}(f_i) - \widehat{\mu_k}(1))(d^B_{per}(f_i) - \widehat{\mu_k}(2)) & (d^B_{per}(f_i) - \widehat{\mu_k}(2))^2 \end{pmatrix},$$

where $f_1, \ldots, f_N$ are iid samples in $\mathcal{C}_k$.

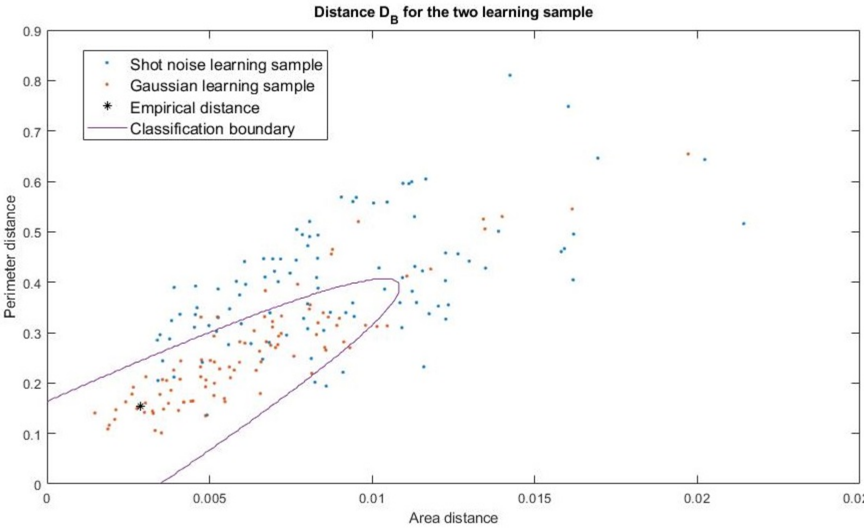

FIGURE 6. Example of a classification, with the two learning samples of size $N = 50$ and the classification boundary. The initial Gaussien field B was simulated using $\lambda = 0.001$, $\mu = 0.01$ and $\sigma = 50$.

Figure 6 is an example of a classification where the initial field is Gaussian, and it can be seen that it is well classified in the class of Gaussian fields. We refer to this classification as the Gaussian classification, because it is based on the geometrical formulas of the Gaussian fields, and performed it on a sample of $N = 50$ Gaussian fields (the class $\mathcal{C}_1$) and $N = 50$ shot noise fields (the class $\mathcal{C}_2$). Figure 7 represents a histogram of the scores of the classification $S_k^B(d^B(f_{k,j}))$ with $1 \leq j \leq M$ and $k = 1, 2$, where $(d^B(f_{1,j}))_j$ are the distances obtained for $M = 50$ Gaussian random fields data $(f_{1,j})_j$ and $(d^B(f_{2,j}))_j$ are the distances obtained for $M = 50$ shot noise random fields $(f_{2,j})_j$. We would like to draw your attention in order not to confuse the role of M and N. For each of the M random fields, we simulate N Gaussian fields and N shot noise fields, which constitute the training set, using the parameters (λ, μ, σ) estimated from the initial field. We observe that the classification of Gaussian fields is very successful with $50/50$ good results and a very high score with a average of about 0.95 and a standard deviation of about 0.07. The classifications of shot noise fields is not as successfull since only $25/50$ gave the results and with a bad score with a average of about 0.49 and a standard deviation of about 0.24.

Now, thanks to the study of the perimeter of Gaussian shot noise random fields, we can make exactly the same classification method, but this time based on shot noise geometries. Once the model parameters (λ, μ, σ) have been estimated by the method of moments presented in [33], we can compute the theoretical volume easily by

$$\overline{\text{Area}(E_X(u))} = \int_u^\infty \frac{x^{2\pi\lambda\sigma^2 - 1} e^{-x 2\pi\mu\sigma^2} (2\pi\mu\sigma^2)^{2\pi\lambda\sigma^2}}{\Gamma(2\pi\lambda\sigma^2)} dx,$$

given that $X(0)$ is gamma distributed.

For the theoretical perimeter function, we approximate it by Algorithm 2 or 3. Figure 8 is the theoretical perimeter of a Gaussian and shot noise field, by increasing the values of σ and with the same values of λ and μ. Let us remember that, in view of Proposition 3, that it is consistent to observe a convergence of the curves with the increase of σ.

So, for f a random field data, the geometric statistic that will be used as classifier is $d^X(f) = (D^X_{area}(f), D^X_{per}(f))$ defined by (33,34) replacing B by X. With the same N Gaussian and N shot

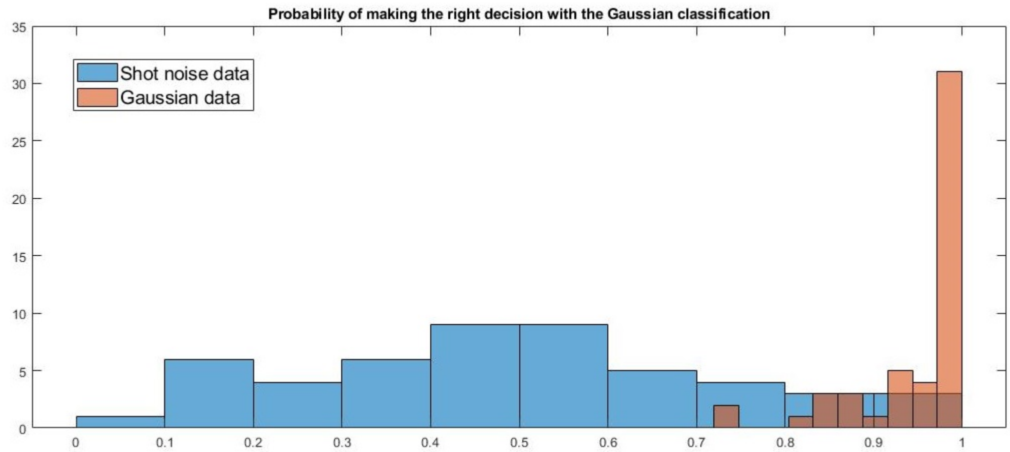

FIGURE 7. Results for the Gaussian classification. In red, a histogram of the classifications score $\left(S_1^B(d^B(f_{1,j}))\right)_j$ obtained from a sample of $M = 50$ Gaussian fields. In blue, the score $\left(S_2^B(d^B(f_{2,j}))\right)_j$ for $M = 50$ shot noise fields.

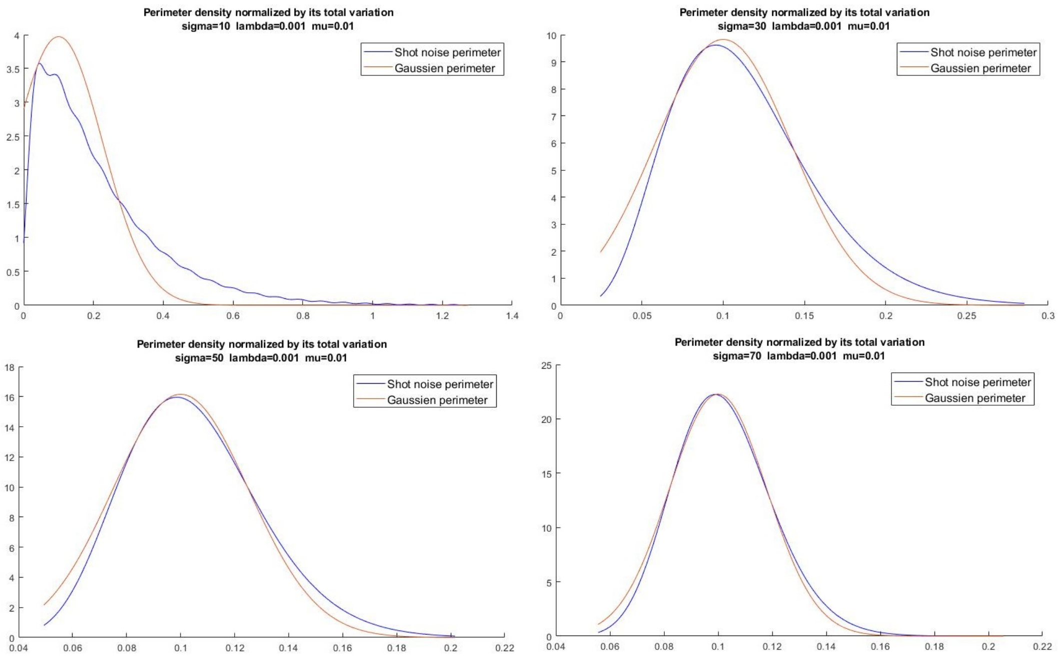

FIGURE 8. Gaussian and shot noise perimeter computed with (27) and Algorithm 2. The parameters are $\lambda = 0.001$, $\mu = \frac{1}{100}$ and from top left to bottom right $\sigma = 10, 30, 50, 70$.

noise random field simulations than before, we have the two new training sets $\left(d^X(B_1), \dots, d^X(B_N)\right)$ and $\left(d^X(X_1), \dots, d^X(X_N)\right)$.

As before, we suppose that for f a random field data and $k = 1, 2$, conditional on the event $f \in \mathcal{C}_k$, $d^X(f)$ correponds to the realisation of a two-dimensional random variable with Gaussian distribution and we note p_k^X its density. That is why the classification rule is

$$\begin{cases} f \in \mathcal{C}_1 & \text{if } p_1^X(d^X(f)) > p_2^X(d^X(f)) \\ f \in \mathcal{C}_2 & \text{if } p_1^X(d^X(f)) < p_2^X(d^X(f)) \end{cases}$$

where for $i = 1, 2$ the density p_i^X is compute with the analogue of (35) and (36) replacing B by X.

This time, we refer to this classification as the shot noise classification, because it is based on the geometrical formulas of the shot noise fields, and performed it on the same sample of $M = 50$ Gaussian fields (the class $\mathcal{C}_1$) and $M = 50$ shot noise fields (the class $\mathcal{C}_2$) than in the Gaussian classification. Figure 9 represents a histogram of the classifications score $S_k^X(d^X(f_{k,j}))$ with $1 \leq j \leq M$ and $k = 1, 2$, where $(d^X(f_{1,j}))_j$ are the distances obtained by the shot noise classification for the same $M = 50$ Gaussian random fields $(f_{1,j})_j$ than in the Gaussien classification, and $(d^X(f_{2,j}))_j$ are the distances obtained for the same $M = 50$ shot noise random fields $(f_{2,j})_j$ than in the Gaussien classification. We observe that the classification of Gaussian fields is bad with 26/50 good results and a bad score with a average of about 0.55 and a standard deviation of about 0.24. The classifications of shot noise fields is, this time, very successfull with 50/50 good results and with a high score with a average of about 0.95 and a standard deviation of about 0.06.

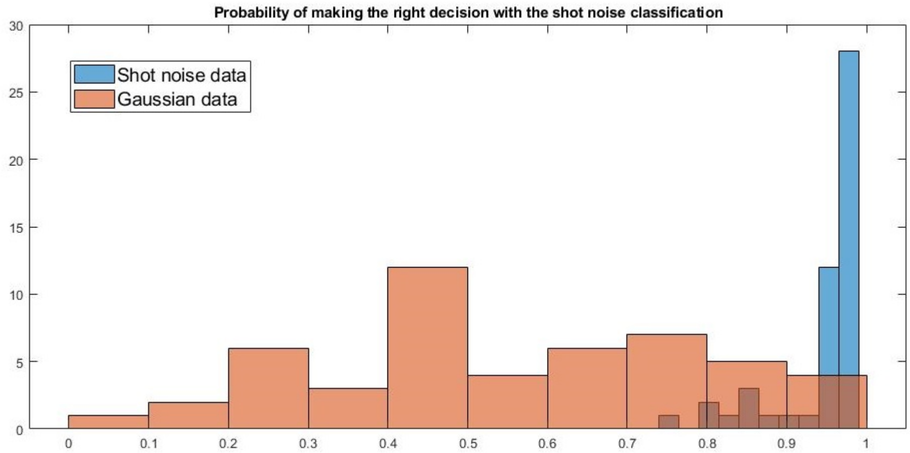

Figure 9. Results for the shot noise classification. In red, a histogram of the classifications score $\left(S_1^X(d^X(f_{1,j}))\right)_j$ obtained from a sample of $M = 50$ Gaussian fields. In blue, the score $\left(S_2^X(d^X(f_{2,j}))\right)_j$ for a sample of $M = 50$ shot noise fields. The samples are the same as in Figure 7.

This highlights the usefulness of the shot noise classification. These two classifications should be used together and when they do not have the same conclusion, the classification with the largest difference between the probability densities is preferred reducing the error as much as possible. In Figure 10 we compare the classification score for the sample of Gaussian fields. For better visibility, we have sorted the scores, so that they are ascending for the shot noise classification. A score below 1/2 means that this is not the choice retained by the classification procedure. We do the same for the 50 shot noise fields in Figure 11, where we have this time sorted the scores to be increasing for the Gaussian classification.

6. Appendix

6.1. **Proof of Theorem 1.** Recall that φ denotes the characteristic function of $(X_\lambda(0), \partial_1 X_\lambda(0))$. According to (10) we have for $\nu \in \mathbb{R}$,

$$\overline{\mathrm{LP}_{X_\lambda}}(h_\nu) = \int_0^\infty \frac{1}{v}\varphi(\nu, v) S_0(\nu, v) dv,$$

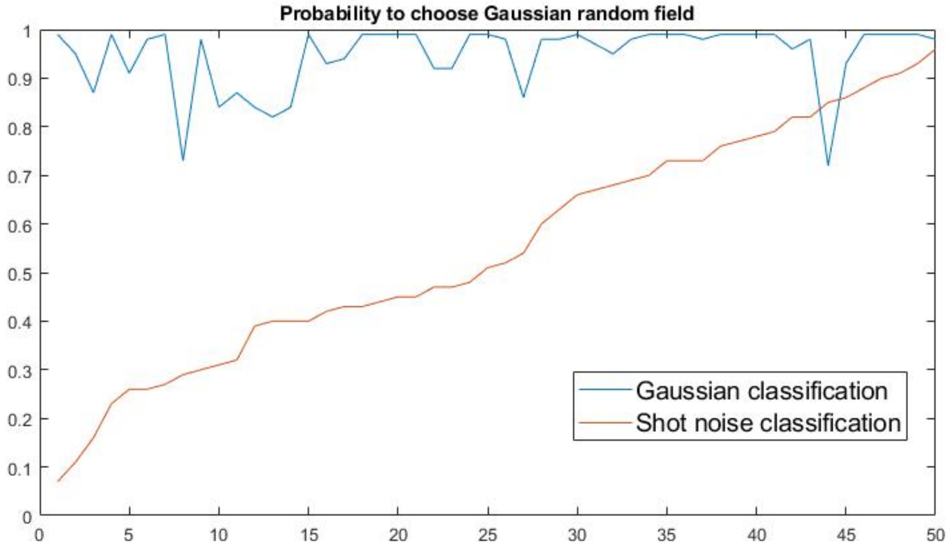

FIGURE 10. The two scores $S_1^B(d^B(f_{1,j}))_j$ and $S_1^X(d^X(f_{1,j}))_j$ for the Gaussian sample.

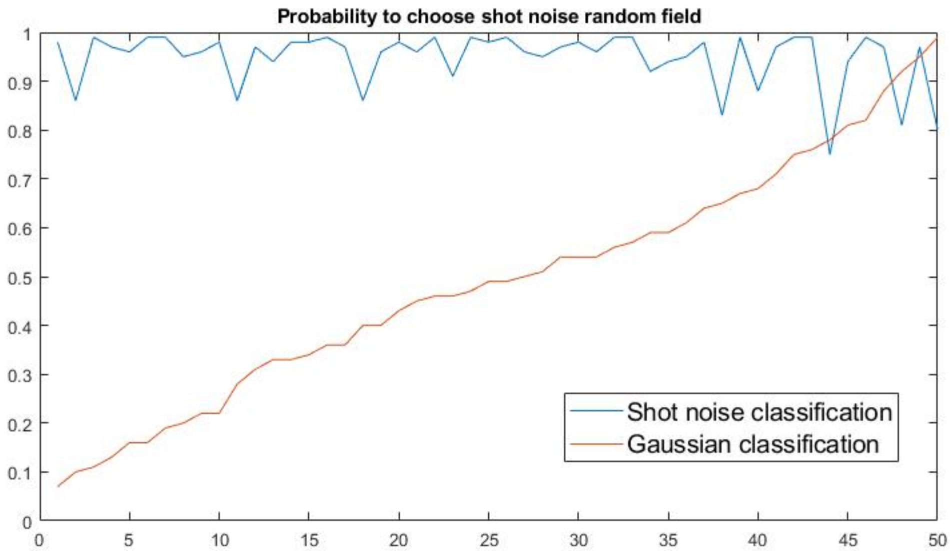

FIGURE 11. The two scores $S_2^B(d^B(f_{2,j}))_j$ and $S_2^X(d^X(f_{2,j}))_j$ for the Gaussian sample.

where the improper integral $\int_0^\infty$ is obtained as $\lim_{V\to+\infty}\int_0^V$. Actually, for $V'>V>0$, integrating by parts,

$$\int_V^{V'} \frac{1}{v}\frac{\partial\varphi}{\partial v}(\nu,v)dv = \frac{\varphi(\nu,V')}{V'} - \frac{\varphi(\nu,V)}{V} + \int_V^{V'} \frac{\varphi(\nu,v)}{v^2}dv.$$

Since φ is a characteristic function it is bounded by 1 and we can let V' tends to ∞ and obtain

$$\left|\int_V^{+\infty} \frac{1}{v}\frac{\partial\varphi}{\partial v}dv\right| \le \frac{2}{V}. \tag{37}$$

We introduce

$$\psi(\nu,v) := \exp\left(i\nu\mathbb{E}(X_\lambda(0))\right)\exp\left(-\frac{\lambda\mu_2(g_M)}{2}\nu^2\right)\exp\left(-\frac{\lambda\mu_2(\partial_1 g_M)}{2}v^2\right),$$

the characteristic function of the Gaussian vector $(B_\lambda(0),\partial_1 B_\lambda(0))$ and note that we have

$$\overline{\mathrm{LP}_{B_\lambda}}(h_\nu) = -\int_0^{+\infty}\frac{1}{v}\frac{\partial\psi}{\partial v}(\nu,v)dv,$$

with similarly as (37)

$$\left|\int_V^{+\infty}\frac{1}{v}\frac{\partial\psi}{\partial v}(\nu,v)dv\right| \le \frac{2}{V}.$$

Moreover introducing $S_1(v)=\lambda\mu_2(\partial_1 g_M)v$ we can write $\frac{\partial\psi}{\partial v}(\nu,v)=-S_1(v)\psi(\nu,v)$. It follows that

$$\begin{aligned}\left|\overline{\mathrm{LP}_{X_\lambda}}(h_\nu)-\overline{\mathrm{LP}_{B_\lambda}}(h_\nu)\right| &= \left|\int_0^{+\infty}\frac{1}{v}S_0(\nu,v)\varphi(\nu,v)dv-\int_0^{+\infty}\frac{1}{v}S_1(v)\psi(\nu,v)dv\right|\\ &\le \int_0^V\frac{1}{v}|S_0(\nu,v)\varphi(\nu,v)-S_1(\nu,v)\psi(\nu,v)|+\frac{4}{V}.\end{aligned}$$

Then we write

$$|S_0(\nu,v)\varphi(\nu,v)-S_1(v)\psi(\nu,v)| \le |S_0(\nu,v)||\varphi(\nu,v)-\psi(\nu,v)|+|S_0(\nu,v)-S_1(v)||\psi(\nu,v)|.$$

For the first term, we can write

$$\varphi(\nu,v)-\psi(\nu,v)=\psi(\nu,v)\left[\exp\left(z(\nu,v)\right)-1\right],$$

where

$$z(\nu,v) \;:=\; \lambda\int e^{i[\nu g_m+v\partial_1 g_m]}-1-i[\nu g_m+v\partial_1 g_m]-\frac{[\nu g_m+v\partial_1 g_m]^2}{2}F(dm)dx,$$

using the fact that, since $\partial_1 g_m$ is an odd function $\int\partial_1 g_m=0$ and $\int g_m\partial_1 g_m=0$. Hence, using finite increments inequality,

$$|\varphi(\nu,v)-\psi(\nu,v)| \le |\psi(\nu,v)||z(\nu,v)|\exp(\Re(z(\nu,v))),$$

where

$$\Re(z(\nu,v)) := \lambda\int\cos[\nu g_m+v\partial_1 g_m]-1-\frac{[\nu g_m+v\partial_1 g_m]^2}{2}F(dm)dx.$$

Since $0\le\cos(s)-1+\frac{s^2}{2}\le\frac{1}{4!}s^4$ for all $s\in\mathbb{R}$, we have $0\le\Re(z(\nu,v))\le\frac{\lambda}{4!}\int[\nu g_m+v\partial_1 g_m]^4F(dm)dx$.
Now let us choose $V=\left(\frac{\mu_2(\partial_1 g_m)}{\mu_4(\partial_1 g_m)}\right)^{1/2}$ and ν with $|\nu|\le C_0^+$. Then it follows that

- $\nu^2\int g_m^4\le 6\int g_m^2$;
- $\nu^2\int g_m^2\partial_1 g_m^2\le\frac{5}{6}\int(\partial_1 g_m)^2$;
- $V^2\int(\partial_1 g_m)^4\le\int(\partial_1 g_m)^2$,

where we used Cauchy-Schwarz inequality for the second inequality. It follows that for $v\le V$ we get

$$\begin{aligned}\int[\nu g_m+v\partial_1 g_m]^4 &= \nu^4\int g_m^4+6v^2\nu^2\int g_m^2\partial_1 g_m^2+v^4\int\partial_1 g_m^4\\ &\le 6\nu^2\int g_m^2+5v^2\int\partial_1 g_m^2+v^2\int\partial_1 g_m^2\\ &\le 6\int[\nu g_m+v\partial_1 g_m]^2,\end{aligned}$$

and therefore

$$0\le\Re(z(\nu,v))\le\frac{\lambda}{4}\int[\nu g_m+v\partial_1 g_m]^2F(dm)dx.$$

Hence we have

$$|\psi(\nu,v)|\exp\left(\Re(z(\nu,v))\right)\leq \exp\left(-\frac{\lambda}{4}\mu_2(g_m)\nu^2\right)\exp\left(-\frac{\lambda}{4}\mu_2(\partial_1 g_m)v^2\right).$$

Moreover we can simply bound

$$\begin{aligned}|z(\nu,v)| &\leq \frac{\lambda}{3!}\int|\nu g_m+v\partial_1 g_m|^3\\ &\leq \frac{2\lambda}{3}[|\nu|^3\int|g_m|^3+|v|^3\int|\partial_1 g_m|^3]\end{aligned}$$

and, since

$$S_0(\nu,v)=\lambda\int e^{iug_m}\partial_1 g_m\sin(v\partial_1 g_m)F(dm)dx, \tag{38}$$

we simply bound

$$\frac{|S_0(\nu,v)|}{v}\leq\lambda\int(\partial_1 g_m)^2.$$

It follows that

$$\begin{aligned}&\int_0^V\frac{|S_0(\nu,v)|}{v}|\varphi(\nu,v)-\psi(\nu,v)|dv\\
\leq\;& \frac{2}{3}\lambda^2\exp\left(-\frac{\lambda\mu_2(g_M)}{4}\nu^2\right)\mu_2(\partial_1 g_M)\int_0^V[|\nu|^3\mu_3(g_M)+v^3\mu_3(\partial_1 g_M)]\exp\left(-\frac{\lambda\mu_2(\partial_1 g_M)}{4}v^2\right)dv\\
\leq\;& \mu_2(\partial_1 g_M)\left[c_3\lambda^{1/2}\frac{\mu_3(g_M)}{\mu_2(g_M)^{3/2}}\int_0^V\exp\left(-\frac{\lambda\mu_2(\partial_1 g_M)}{4}v^2\right)dv+\frac{2}{3}\lambda^2\mu_3(\partial_1 g_M)\int_0^V v^3\exp\left(-\frac{\lambda\mu_2(\partial_1 g_M)}{4}v^2\right)dv\right],\\
\leq\;& \mu_2(\partial_1 g_M)\left[c_3\frac{\mu_3(g_M)}{\mu_2(g_M)^{3/2}\mu_2(\partial_1 g_M)^{1/2}}\sqrt{\pi}+\frac{2^4}{3}\frac{\mu_3(\partial_1 g_M)}{\mu_2(\partial_1 g_M)^2}\right]\\
\leq\;& \left[c_3\frac{\mu_3(g_M)\mu_2(\partial_1 g_M)^{1/2}}{\mu_2(g_M)^{3/2}}\sqrt{\pi}+\frac{2^4}{3}\frac{\mu_3(\partial_1 g_M)}{\mu_2(\partial_1 g_M)}\right].\end{aligned}$$

where $c_3=\frac{2}{3}\times 2^3\times\sup_{s\in\mathbb{R}}|s|^3e^{-s^2}=4\sqrt{6}e^{-3/2}$ if $\nu\neq 0$ and $c_3=0$ else.
Now let us focus on the second term and recall that $S_0(\nu,v)=\lambda\int\partial_1 g_m e^{i\nu g_m}\sin(v\partial_1 g_m)F(dm)dx$ and $S_1(v)=\lambda v\int(\partial_1 g_m)^2F(dm)dx$ so that we write

$$\begin{aligned}|S_0(\nu,v)-S_1(v)| &\leq |S_0(\nu,v)-S_0(0,v)|+|S_0(0,v)-S_1(v)|\\ &\leq \lambda|v|\left[|\nu|\int(\partial_1 g_m)^2|g_m|+\frac{v^2}{3!}\mu_4(\partial_1 g_M)\right].\end{aligned}$$

It follows that

$$\begin{aligned}&\int_0^V\frac{|S_0(\nu,v)-S_1(v)|}{v}|\psi(\nu,v)|dv\\
\leq\;&\left[c_1\sqrt{\frac{\pi}{2}}\frac{\int(\partial_1 g_m)^2|g_m|}{\mu_2(g_M)^{1/2}\mu_2(\partial_1 g_M)^{1/2}}+\frac{\mu_4(\partial_1 g_M)^{1/2}}{3!\mu_2(\partial_1 g_M)^{1/2}}\min\left(\lambda^{-1/2}\sqrt{\frac{\pi}{2}}\frac{\mu_4(\partial_1 g_M)^{1/2}}{\mu_2(\partial_1 g_M)},1\right)\right],\end{aligned}$$

using the fact that $|\nu|\exp(-\frac{1}{2}\lambda\mu_2(g_M)\nu^2)\leq c_1\lambda^{-1/2}\mu_2(g_M)^{-1/2}$ with $c_1=\sup_{s\in\mathbb{R}}|s|e^{-\frac{1}{2}s^2}=e^{-1/2}$ if $\nu\neq 0$ and $c_1=0$ else, $\sqrt{\frac{\pi}{2}}=\int_0^{+\infty}e^{-s^2/2}ds=\int_0^{+\infty}s^2e^{-s^2/2}ds$ and $v\leq V$ with $\int_0^{+\infty}se^{-s^2/2}ds=1$.

We get the result of upper bounds for C_1^+ and C_2^+, combining previous inequalities with Cauchy-Schwarz inequalities.

6.2. **Proof of Theorem 2.** Let $\nu \in \mathbb{R}$ and write again

$$\overline{\mathrm{LP}}_{X_\lambda}(h_\nu) = \int_0^\infty \frac{1}{v}\varphi(\nu,v)S_0(\nu,v)dv.$$

Let $V > 0$ and remark that after using Fubini's theorem, since $\mu_2(\partial_1 g_m) < +\infty$, we can also integrate by parts to obtain for $V' > V$

$$\int_V^{V'} \frac{1}{v}S_0(\nu,v)dv = \lambda \int e^{i\nu g_m}\left(\frac{1-\cos(V'\partial_1 g_m)}{V'} - \frac{1-\cos(V\partial_1 g_m)}{V} + \int_V^{V'} \frac{1-\cos(v\partial_1 g_m)}{v^2}dv\right)F(dm)dx.$$

It follows choosing $\beta \in (0,1)$ we can use that $1-\cos(s) = 2\sin^2(s/2) \le 2^{1-\beta}|s|^\beta$ for $s \in \mathbb{R}$ to bound

$$\left|\int_V^{V'} \frac{1}{v}S_0(\nu,v)dv\right| \le \lambda 2^{1-\beta}\mu_\beta(\partial_1 g_m)\left(V'^{-(1-\beta)} + 2V^{-(1-\beta)}\right),$$

letting V' tends to infinity we obtain

$$\left|\int_V^{+\infty} \frac{1}{v}S_0(\nu,v)dv\right| \le 2\lambda C_\beta^- V^{-(1-\beta)},$$

with $C_\beta^- = 2^{1-\beta}\mu_\beta(\partial_1 g_M)$. Then we write

$$\overline{\mathrm{LP}}_{X_\lambda}(h_\nu) - \varphi(\nu,0)\int_0^{+\infty} \frac{1}{v}S_0(\nu,v)dv = \int_0^V \frac{1}{v}S_0(\nu,v)(\varphi(\nu,v)-\varphi(\nu,0))dv$$

$$+ \int_V^{+\infty} \frac{1}{v}S_0(\nu,v)\varphi(\nu,v)dv - \varphi(\nu,0)\int_V^{+\infty} \frac{1}{v}S_0(\nu,v)dv.$$

Now we write

$$\varphi(\nu,v) - \varphi(\nu,0) = \varphi(\nu,0)[\exp(\tilde{z}(\nu,v)) - 1],$$

where

$$\tilde{z}(\nu,v) := \lambda \int e^{i\nu g_m}\left(e^{iv\partial_1 g_m} - 1\right)F(dm)dx = -\lambda \int e^{i\nu g_m} 2\sin^2(v\partial_1 g_m/2)F(dm)dx.$$

Hence for $\beta \in (0,2]$

$$|\tilde{z}(\nu,v)| \le \lambda C_\beta^- v^\beta,$$

and for $v \in [0,V]$,

$$\begin{aligned} |\varphi(\nu,v) - \varphi(\nu,0)| &\le |\tilde{z}(\nu,v)|\exp(|\tilde{z}(\nu,v)|) \\ &\le \lambda C_\beta^- v^\beta \exp(C_\beta^- \lambda V^\beta) \end{aligned}$$

Recall (38) but use that $|S_0(\nu,v)| \le \lambda\mu_1(\partial_1 g_M)$ to get

$$\left|\int_0^V \frac{1}{v}S_0(\nu,v)(\varphi(\nu,v)-\varphi(\nu,0))dv\right| \le \lambda^2\mu_1(\partial_1 g_M)C_\beta^- \exp(C_\beta^-\lambda V^\beta)\frac{V^\beta}{\beta}.$$

We choose $V = \lambda^{-\frac{2}{1+\beta}}$ such that $\lambda V^\beta = \lambda^{\frac{1-\beta}{1+\beta}}$ and $\lambda^2 V^\beta = V^{-1} = \lambda^{\frac{2}{1+\beta}}$ to obtain finally

$$\left|\overline{\mathrm{LP}}_{X_\lambda}(h_\nu) - \varphi(\nu,0)\int_0^{+\infty} \frac{1}{v}S_0(\nu,v)dv\right| \le \lambda^{\frac{2}{1+\beta}}\left(\frac{C_\beta^-\mu_1(\partial_1 g_M)\exp(C_\beta^-\lambda^{\frac{1-\beta}{1+\beta}})}{\beta} + 2 + 2C_\beta^-\lambda^{\frac{1-\beta}{1+\beta}}\right).$$

6.3. **Algorithms.**

Algorithm 1: Calculate the mean perimeter function from smooth framework

Choose n_1 the number of item of the vector ν ;
Choose ν_{max} maximum of the vector ν ;
$\nu = -\frac{\nu_{max}}{2} + (0 : n_1 - 1)\frac{\nu_{max}}{n_1-1}$;
Choose n_2 the number of item of the vector v ;
Choose v_{max} the maximum of the vector v ;
$v = (1 : n_2)\frac{v_{max}}{n_2}$;
Choose r_{max} the maximum in the integral of $\varphi(\nu, v)$ and $S_0(\nu, v)$;
Choose n_3 the number of item of the vector u ;
Choose u_{min} and u_{max} the minimum and maximum of the vector u ;
$u = u_{min} + (0 : n_3 - 1)\frac{u_{max}-u_{min}}{n_3}$;

1: **for** $k = 1 : n_1$ **do**
2: **for** $l = 1 : n_2$ **do**
3: Approximate $\varphi(\nu(k), v(l))$ by numerically integration of formula (14)
4: Approximate $S_0(\nu(k), v(l))$ by numerically integration of formula (15)
5: **end for**
6: **end for**
7: Approximate $\nu \to \mathbb{E}[\mathrm{LP}_X(h_\nu, T)]$ from the formula (10)
8: Approximate $u \to \mathbb{E}[\mathrm{Per}(E_X(u), T)]$ by inverse fourier transform

Beyond the choice of precisions n_1, n_2 and n_3; it is the choice of r_{max}, v_{max} andν_{max} that is not easy since they depend on the parameters of the field. The choices of u_{min} and u_{max} can be chosen with the minimum and maximum field.
Let $\sigma = 20$, $\mu = \frac{1}{100}$ and $\lambda = 9.5e^{-4}$ which corresponds to an average of 1000 points over an observation window $[0, 1024]^2$. We choose a good precision $n_1 = n_2 = n_3 = 2^{10}$, and we recommend to choose $r_{max} = 200$ and $v_{max} = 6000$. Using the tic toc command from Matlab, the algorithm lasts 1000 seconds or almost 3 hours. It is annoying for applications such as parameter applications or classification. The green curve in Figure 2 comes from this algorithm.

Recall the formula (25) obtained in Proposition 9 :

$$\overline{\overline{\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u))}} = \lambda 2\pi\sigma\mathbb{E}\left(f_{\lambda,\mu,\sigma}(u - M_\sigma V)M_\sigma\sqrt{-2ln(V)}\right),$$

with V, a random variable uniform on $(0, 1]$ and independent from M_σ with distribution $F(dm)$, that is exponential distribution of parameter $2\pi\sigma^2\mu$. Using Monte-Carlo approximation we can approximate the function $u \mapsto \lambda 2\pi\sigma\mathbb{E}\left(f_{\lambda,\mu,\sigma}(u - M_\sigma V)M_\sigma\sqrt{-2ln(V)}\right)$ by

$$\frac{\lambda 2\pi\sigma}{N}\sum_{j}^{N} f_{\lambda,\mu,\sigma}(u - M_{\sigma,j}V_j)\sqrt{-2\ln(V_j)}M_{\sigma,j},$$

choosing $(V_j, M_{\sigma,j})_{1\le j\le N}$ iid of same law than (V, M_σ) and N sufficiently large.

Algorithm 2: Calculate the mean perimeter function from elementary framework with Monte Carlo

Choose n_3 the number of item of the vector u ;
Choose u_{min} and u_{max} the minimum and maximum of the vector u ;
$u = u_{min} + (0 : n_3 - 1)\frac{u_{max}-u_{min}}{n_3}$;
Choose N the number of repeats in the Monte Carlo method ;

1: Draw $(V_j)_{1\le j\le N}$ iid with uniform law on $[0, 1]$
2: Draw $(M_{\sigma,j})_{1\le j\le N}$ iid with exponential law of parameter $2\pi\sigma^2\mu$
3: Approximate $u \to \mathbb{E}[\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u), T)]$ by $\frac{\lambda 2\pi\sigma}{N}\sum_{j=1}^{N} f_{\lambda,\mu,\sigma}(u - M_{\sigma,j}V_j)\sqrt{-2\ln(V_j)}M_{\sigma,j}$

Here there is no choice except for the precision n_3 and the sample size N of the Monte Carlo method, since u_{min} and u_{max} can be chosen experimentally. Let the same example with $\sigma = 20$, $\mu = \frac{1}{100}$ and $\lambda = 9.5e^{-4}$. We choose a good precision $n_3 = 2^{10}$ and $N = 2000$. Using the tic toc command from Matlab, the algorithm lasts 12 seconds. It is an excellent performance for applications.

It is also possible to use the Fourier framework with the equation (26).

Algorithm 3: Calculate the mean perimeter function from elementary framework with Fourier transform

Choose n_1 the number of item of the vector ν ;
Choose ν_{max} the maximum of the vector ν ;
$\nu = -\frac{\nu_{max}}{2} + (0 : n_1 - 1)\frac{\nu_{max}}{n_1}$;
Choose n_3 the number of item of the vector u ;
Choose u_{min} and u_{max} the minimum and maximum of the vector u ;
$u = u_{min} + (0 : n_3 - 1)\frac{u_{max}-u_{min}}{n_3}$;
1: Approximate $\nu \to \mathbb{E}[\mathrm{LP}_{X_{\lambda,\mu,\sigma}}(h_\nu, T)]$ from the formula (26)
2: Approximate $u \to \mathbb{E}[\mathrm{Per}(E_{X_{\lambda,\mu,\sigma}}(u), T)]$ by inverse Fourier transform

Here there is no choice except for the precision n_1 and n_3, since ν_{max}, u_{min} and u_{max} can be chosen experimentally.
Let the same example with $\sigma = 20$, $\mu = \frac{1}{100}$ and $\lambda = 9.5e^{-4}$. We choose a good precision $n_1 = n_3 = 2^{10}$. Using the tic toc command from Matlab, the algorithm lasts 1 seconds. It is a perfect performance for applications.

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