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Two-player Boundedness Counter Games

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Abstract

We consider two-player zero-sum games with winning objectives beyond regular languages, expressed as a parity condition in conjunction with a Boolean combination of boundedness conditions on a finite set of counters which can be incremented, reset to 0, but not tested. A boundedness condition requires that a given counter is bounded along the play. Such games are decidable, though with non-optimal complexity, by an encoding into the logic WMSO with the unbounded and path quantifiers, which is known to be decidable over infinite trees. Our objective is to give tight or tighter complexity results for particular classes of counter games with boundedness conditions, and study their strategy complexity. In particular, counter games with conjunction of boundedness conditions are easily seen to be equivalent to Streett games, so, they are CoNP-c . Moreover, finite-memory strategies suffice for Eve and memoryless strategies suffice for Adam. For counter games with a disjunction of boundedness conditions, we prove that they are in solvable in $\text{NP} \cap \text{CoNP}$, and in PTime if the parity condition is fixed. In that case memoryless strategies suffice for Eve while infinite memory strategies might be necessary for Adam. Finally, we consider an extension of those games with a max operation. In that case, the complexity increases: for conjunctions of boundedness conditions, counter games are EXPTIME-c .

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1 Introduction

Games on graphs are a popular mathematical framework to reason on reactive synthesis problems [1, 8]: the system to synthesize is seen as a protagonist which must enforce a given specification (its winning objective) against any adversarial behaviour of its environment. In this framework, executions of reactive systems are modelled as infinite sequences alternating between actions of the systems and actions of its environment. In the ω -regular setting, the set of correct executions of reactive systems is modelled as an automaton, for example, a non-deterministic Büchi automaton, then determinized into a parity automaton. The synthesis problem then boils down to solving a game played on the graph of the parity automaton, where the goal of the protagonist (Eve) is to satisfy, in the long run, the parity condition whatever her opponent (Adam) does. Motivated by the synthesis of more complex systems, the literature is rich in extensions of this basic two-player zero-sum ω -regular setting: multiple players, imperfect information, quantitative objectives, infinite graphs ... (see [1, 8] for some references). In this paper, we follow this line of work and consider an extension of two-player games beyond ω -regularity: counter games with boundedness conditions.

Counter games In this paper, a *two-player counter game with boundedness objectives*, only called counter game hereafter, is given by a finite arena, called counter arena, whose vertices are labelled by counter operations over a finite set of counters C . Those operations



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46 can: increment a counter, reset it, or skip it (i.e. leave its value unchanged). We consider
 47 objectives given as Boolean combinations of *counter boundedness conditions*. For $c \in C$, the
 48 condition $\mathbb{B}(c)$ is satisfied by all infinite paths $\pi = v_0v_1\dots$, called plays, such that for some
 49 $N \in \mathbb{N}$, the value of c along π is bounded by N . Note that the bound N is not uniform, in
 50 the sense that it depends on π , and as a consequence, the set of plays satisfying $\mathbb{B}(c)$ is not
 51 ω -regular in general. In this paper, we consider particular classes of Boolean combinations of
 52 boundedness conditions. Since they do not necessarily capture all ω -regular objectives, we
 53 also, by default, equip counter games with a parity condition.

54 Given an objective W as a Boolean formula Φ over atoms $\mathbb{B}(c)$ for all $c \in C$, the goal of
 55 the protagonist, Eve, is to enforce plays which satisfy W and the parity condition, whatever
 56 the adversary, Adam, does. If she has a strategy to meet this objective, she is said to win
 57 the game. Counter games are zero-sum, meaning that the goal of Adam is to enforce the
 58 complementary objective. The goal of this paper is to study the complexity of deciding,
 59 given a counter game G , if Eve wins G .

60 **Motivations** On infinite words, classes of counter automata with boundedness conditions
 61 have appeared in various papers, e.g. in [5, 14, 2, 7]. The most relevant models in the
 62 context of counter games are the ω BS-automata of [5] and the max-automata of [7]. They
 63 are equipped with the same counter operations as the counter games of this paper, plus
 64 a max operation in the case of max-automata, and some boundedness conditions. As a
 65 consequence, winning objectives in counter games can naturally be expressed with these
 66 automata. However, while they are known to have decidable emptiness problem, not much is
 67 known when they are used to define objectives in two-player games. A motivation for this
 68 paper is to investigate this question, for games where the winning conditions is not given by
 69 such an automaton but where counter operations are explicitly given in the arena.

70 In the same line of works, max-automata, which are deterministic, are known by [2] to
 71 correspond to the logic WMSO+U, which extends weak MSO on infinite words with the
 72 *unbounded quantifier* $\mathbb{U}X$. A formula $\mathbb{U}X.\phi(X)$ holds if there are arbitrarily large sets X
 73 satisfying ϕ . An important result by Bojańczyk states that the extension of WMSO+U to
 74 infinite binary trees and with a path quantifier which allows to quantify over infinite paths,
 75 has decidable satisfiability problem [6]. Since strategies are definable, modulo a tree encoding,
 76 in this latter logic, a direct consequence of this result is that two-player games with objectives
 77 given by max-automata are decidable (see also Example 2 of [6]). As a consequence, counter
 78 games with boundedness conditions are decidable, though with non-elementary complexity.
 79 Another motivation for our work is to obtain tight complexity results for particular classes of
 80 counter games with boundedness conditions, with the goal of providing conceptually simpler
 81 decidability proofs and insights for these particular instances, instead of using the general
 82 result of [6].

83 Finally, counter games with boundedness objectives are closely related to synthesis
 84 problems over infinite alphabets of data. In particular, the problem of synthesising Mealy
 85 machines with *registers* satisfying specifications given as deterministic register automata
 86 over $(\mathbb{N}, <, 0)$, has recently been studied in [18]. It is shown that this problem is decidable,
 87 and, even though the decidability proof is not based on counter games, it is proved that
 88 the synthesis problem reduces to a game with winning conditions given as a (deterministic)
 89 max-automaton whose acceptance is a disjunction of a parity condition and a condition of
 90 the form "counter c is unbounded". Here, we also study the complexity of counter games
 91 with max operation, giving an alternative procedure to decide the former synthesis problem.



92 **Contributions** Our contributions are summarized in Fig. 1. We consider objectives given
 93 as a conjunction of a parity condition and a formula over atoms $\mathbb{B}(c)$ in the following classes:
 94 conjunctions, disjunctions, disjunctions of conjunctions, and negation-free formulas. We also
 95 consider the extension of counter games with a max operator which can assign a counter with
 96 the maximal value of several counters. The table also mentions the strategy complexity. For
 97 conditions in $\bigwedge \mathbb{B}$, counter games are easily proved to be interreducible in polynomial time
 98 to Streett games, yielding CoNP-completeness [20]. More interestingly, we prove that when
 99 the number of counters is fixed, then, they are interreducible to parity games in polynomial
 100 time, using another reduction (Thm 6).

101 We then prove, in it is our main contribution, that for conditions in $\bigvee \mathbb{B}$, counter games
 102 are solvable in $\text{NP} \cap \text{CoNP}$ and in polynomial time when the index of the parity function
 103 is fixed. To prove this result, we introduce the notion of *finitely switching strategies* which
 104 are, to the best of our knowledge new, and we believe, interesting on their own. This notion
 105 is specifically designed for disjunctions of prefix-independent objectives (which is the case
 106 of counter boundedness conditions): in a finitely switching strategy, Eve announces which
 107 objective from the disjunction she aims to satisfy, and she can change her mind along the
 108 play, but only a finite number of times. Eventually, she is bound to satisfy one the objectives.
 109 We give general conditions to decide whether Eve has a finitely switching strategy in a
 110 two-player game with a disjunction of prefix-independent objectives, and prove that such
 111 strategies are sufficient for Eve to win objectives in $\bigvee \mathbb{B}$ and more generally in $\bigvee \bigwedge \mathbb{B}$.

112 **Related works** Two-player games with boundedness conditions have been studied in the
 113 literature, first as finitary parity and Streett games [11], then generalized to cost-parity and
 114 cost-Streett games [19]. Finitary parity- and Streett-games are request-response games [12],
 115 with the additional constraint that the delay (number of edges) between a request and its
 116 response is bounded (by a bound which depends on the play). For cost-parity and cost-Streett,
 117 instead of the number of edges, costs (including 0) label edges and the delay is defined as
 118 the sum of the costs. Cost-parity and cost-Streett games can be encoded as counter games
 119 with conditions in $\bigwedge \mathbb{B}$, though with an exponential blowup. The difference between those
 120 counter games and finitary- and cost-games can be seen in their complexity: counter games
 121 with conditions in $\bigwedge \mathbb{B}$ are CoNP-c, finitary parity games are in PTIME, cost-parity in
 122 $\text{NP} \cap \text{CoNP}$, and finitary Streett and cost-Streett are EXPTIME-c.

123 Delay games with objectives given by a max-automaton have been proved to be decidable
 124 in [24]. This result is orthogonal to ours: first, those games allow for some delay, here in
 125 the sense that Eve has some look-ahead on Adam's future actions. Second, the decision
 126 procedure is non-elementary and rely on an encoding into WMSO+UP on infinite trees, some
 127 argument we avoid here, but for less expressive boundedness objectives.

128 Finally, infinite-state games with boundedness conditions have been considered in [10],
 129 over pushdown arenas. Finitary games over these arenas are shown to be decidable, as well as
 130 (pushdown) counter games with conditions in $\bigwedge \mathbb{B}$, without complexity results. Interestingly,
 131 it is shown that those games are equivalent to games where the objective of Eve is to uniformly
 132 bound all counters, for a bound which only depends on her strategy, and not on the plays.
 133 For counter games in $\bigwedge \mathbb{B}$ over a *finite* arena, this result can easily be seen as a consequence
 134 that finite-memory strategies suffice for Eve.



Winning objective parity \wedge	Complexity	Memory of Eve	Memory of Adam	Theorem
$\bigwedge \mathbb{B}$	coNP-C	Finite	none	Th 4
$\bigvee \mathbb{B}$	NP \cap CoNP PTIME for fixed index	Parity Index	Infinite	Th 13
$\bigvee \bigwedge \mathbb{B}$	coNP-C	Finite	Infinite	Th 14
Bool ⁺ (\mathbb{B})	PSPACE, CoNP-H	Finite	Infinite	Th 15
$\bigwedge \mathbb{B} + \max$	EXPTIME-C	Finite	Finite	Th 16
Bool(\mathbb{B}) + \max	Decidable	Infinite	Infinite	from [4]

■ **Figure 1** Complexity of deciding whether Eve has a winning strategy in a counter game for various winning objectives, always taken in conjunction with a parity objective. Bool⁺(\mathbb{B}) means any negation-free Boolean combination of objectives of the form $\mathbb{B}(c)$. Hardness results hold for any parity function of fixed constant index. The notation $+\max$ indicates that counter games are also equipped with a max operation. Since counter games with boundedness objectives are determined, this yields the complexity of deciding whether Eve wins for the complementary objectives: for example, it is NP-C for objectives $\text{parity} \vee \bigvee \mathbb{U}$ and memoryless strategies are sufficient for Eve, and in PTIME for $\text{parity} \vee \bigwedge \mathbb{U}$ but infinite memory might be necessary for Eve.

2 Preliminaries

In this paper, for any set Σ , we denote by Σ^* (resp. Σ^ω) the finite (resp. infinite) sequences of elements of Σ .

Two-player arenas A two-player *arena* is a tuple $\mathcal{A} = (V, E, V_\exists, V_\forall, v_0)$, where V is finite set, $E \subseteq V \times V$, and V_\exists and V_\forall are two subsets of V such that $\{V_\exists, V_\forall\}$ is a partition of V , and v_0 is an initial vertex. In this paper, we assume that arenas are deadlock-free, *i.e.* that for any $v \in V$, there exists $v' \in V$ such that $(v, v') \in E$. Given $v \in V$, we denote $\mathcal{A}[v] = (V, E, V_\exists, V_\forall, v)$ the arena \mathcal{A} where v_0 has been substituted by v . A play ρ of \mathcal{A} is a mapping from \mathbb{N} to V such that $(\rho(i), \rho(i+1)) \in E$, for all integer $i \in \mathbb{N}$. The set of plays is denoted by $\text{Plays}(\mathcal{A})$. Any play can also be seen as an element of V^ω , and we call a *history* any finite prefix of a play, and denote by $\text{Hist}(\mathcal{A})$ the set of histories of \mathcal{A} .

Strategies and finite-memory A *strategy* for Eve (resp. Adam) is a function σ from $\text{Hist}(\mathcal{A})$ to V defined for all histories $h = h_0 \cdots h_n$ with $h_n \in V_\exists$ (resp. $h_n \in V_\forall$), and such that $(h_n, \sigma(h)) \in E$. A play ρ is *consistent* with a strategy for Eve (resp. Adam) if, for any integer n such that $\rho(n) \in V_\exists$ (resp. $\rho(n) \in V_\forall$), σ is defined on $\rho(0) \cdots \rho(n)$, and $\rho(n+1) = \sigma(\rho(0) \cdots \rho(n))$. We let $\text{Plays}(\mathcal{A}, \sigma)$ (or just $\text{Plays}(\sigma)$ when \mathcal{A} is clear from the context) the set of plays consistent with σ .

A strategy σ of Eve (resp. Adam) is said to be *finite-memory* if there exists a finite set M , an element $m_I \in M$, a mapping δ from $V \times M$ to V , and a mapping g from $V \times M$ to M such that the following is true. When $h = v_0 v_1 \cdots v_l$ is a prefix of a play consistent with σ such that $v_l \in V_\exists$ (resp. $v_l \in V_\forall$), and the sequence m_0, m_1, \dots, m_l is determined by $m_0 = m_I$ and $m_{i+1} = g(v_i, m_i)$, then $\sigma(w) = \delta(v_l, m_l)$. In that case, we say that (δ, g) is a *memory mapping pair* of σ , and that m_l is the memory state of g at move l . We also say that σ is of memory $|M|$, and *memoryless* if it is of memory 1. Note that a memoryless



159 strategy can just be identified with a mapping from V to V .

160 **Two-player games** A *winning condition* for \mathcal{A} is a subset $W \subseteq V^\omega$. A *strategy* σ of Eve or
161 Adam is said to be winning for objective W if $\text{Plays}(\sigma) \subseteq W$. A *two-player game* is a pair
162 $G = (\mathcal{A}, W)$ where \mathcal{A} is an arena and W is a winning condition. We say that a strategy (of
163 Eve or Adam) is winning in G if it is winning for W . A game $G = (\mathcal{A}, W)$ is *determined* if
164 either Eve wins G or Adam wins $(\mathcal{A}, V^\omega \setminus W)$.

165 In this paper, we consider the problem of deciding, given a game G with a finitely
166 represented winning condition, whether Eve wins G . For a complexity class \mathcal{C} and a class of
167 games \mathcal{G} , we say that games in \mathcal{G} are in \mathcal{C} (resp. \mathcal{C} -hard, \mathcal{C} -complete) if the latter problem
168 for games $G \in \mathcal{G}$ is in \mathcal{C} (resp. \mathcal{C} -hard, \mathcal{C} -complete).

169 We also consider the complexity of strategies sufficient or necessary for Eve and Adam to
170 win a game. We say that finite-memory strategies are *sufficient* for Eve (resp. Adam) to win
171 \mathcal{G} if for all $G \in \mathcal{G}$, whenever Eve (resp. Adam) wins G , she has (resp. he has) a finite-memory
172 winning strategy in G . We say that finite-memory is *necessary* for Eve (resp. Adam) to win
173 \mathcal{G} if memoryless strategies do not suffice for Eve (resp. Adam) to win \mathcal{G} . Finally, we say that
174 infinite-memory is *necessary* for Eve (resp. Adam) to win \mathcal{G} if finite-memory strategies do
175 not suffice for Eve (resp. Adam) to win \mathcal{G} .

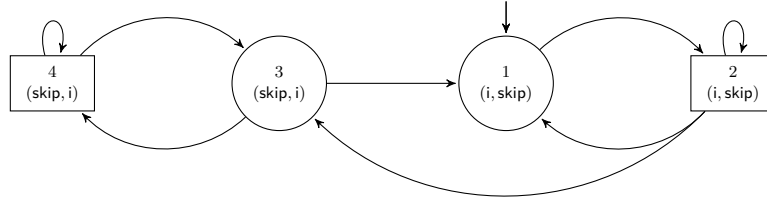
176 **Parity games** Let \mathcal{A} be an arena with set of vertices V . Let $Q \subseteq \mathbb{N}$ be a finite set of
177 elements called *colours* and $\kappa : V \rightarrow Q$ a mapping from vertices to colours called *parity*
178 *function* or *priority function*. The size $|Q|$ of Q is called the *index* of κ . The mapping κ
179 defines a winning condition denoted $\text{Parity}(\kappa)$, called a *parity condition*, as follows: $\text{Parity}(\kappa)$
180 is the set of all infinite words $w = w_0w_1 \dots \in V^\omega$ such that the greatest colour occurring an
181 infinitely often in $\kappa(w_0)\kappa(w_1) \dots$ is even. A parity game is a game whose winning condition is
182 a parity condition. We refer to $\mathcal{A}' = (\mathcal{A}, Q, \kappa)$ as a *coloured arena*, and also denote $\text{Parity}(\kappa)$
183 as $\text{Parity}(\mathcal{A}')$ to avoid an explicit mention of the colouring κ . Note that a coloured arena
184 $\mathcal{A}' = (\mathcal{A}, Q, \kappa)$ uniquely defines a parity game $G = (\mathcal{A}, \text{Parity}(\mathcal{A}'))$. It is well-known that
185 parity games are in $\text{NP} \cap \text{CoNP}$ [16], and even solvable in quasi-polynomial time [9].

186 **Counter operations** Our goal is now to define counter games. First, we introduce counter
187 operations and their semantics. In the rest of the paper, we fix a countable set \mathcal{C} whose
188 elements are called *counters*. A *counter operation* is a mapping from a finite subset C of
189 \mathcal{C} to $\{\text{i}, \text{r}, \text{skip}\}$. We let $\text{Op}(C)$ denote the set of counter operations over $C \subseteq \mathcal{C}$. A *counter*
190 *valuation* is a mapping ν from C to \mathbb{N} . For any infinite word $w \in \text{Op}(C)^\omega$, we define $\lambda(w)$
191 as the infinite sequence of counter valuations $\nu_0, \nu_1, \nu_2, \dots$ such that for any counter $c \in C$,
192 $\nu_0(c) = 0$ and for any non-negative integer n , $\nu_{n+1}(c) = \nu_n(c) + 1$ if $w_n(c) = \text{i}$, $\nu_{n+1}(c) = 0$
193 if $w_n(c) = \text{r}$ and $\nu_{n+1}(c) = \nu_n(c)$ if $w_n(c) = \text{skip}$. We define $\lambda(w)$ for $w \in \text{Op}(C)^*$. To ease
194 notations, we write $\lambda(w, c)_i$ instead of $\lambda(w)_i(c)$. We say that λ is the *evaluation* of w .

195 **Counter games with boundedness objectives** Let \mathcal{A}' be an arena with set of vertices V ,
196 $C \subseteq \mathcal{C}$ a finite set of counters, and $\zeta : V \rightarrow \text{Op}(C)$ a mapping from vertices to counter
197 operations, called *vertex labeling*. Let Q be a set of colours and $\kappa : V \rightarrow Q$ be a colouring of
198 V . To avoid cumbersome notations, for any vertex $v \in V$ and counter $c \in C$, we let $\zeta_c(v)$
199 denote $(\zeta(v))(c)$. We refer to $\mathcal{A} = (\mathcal{A}', C, \zeta, Q, \kappa)$ as a *counter arena*, to \mathcal{A}' as its *underlying*
200 *arena* and to (\mathcal{A}, Q, κ) as its underlying coloured arena. We let $\text{Parity}(\mathcal{A}) = \text{Parity}(\kappa)$.

We consider a particular type of winning objective for counter games, called boundedness conditions, always together with a parity condition. Let $c \in C$. We let $\mathbb{B}(c)$ be an atomic





■ **Figure 2** Counter arena $\mathcal{A} = (V, E, V_{\exists}, V_{\forall}, v)$ with $V = \{1, 2, 3, 4\}$, $V_{\exists} = \{1, 3\}$, $V_{\forall} = \{2, 4\}$, $v = 1$. There are two counters (c, d) whose updates are represented on the figure as pairs. We assume no parity condition and a counter condition $\mathbb{B}(c) \vee \mathbb{B}(d)$. From vertex 3, Eve has a memoryless winning strategy σ : always move to 4. However, she does not have a strategy from 1 to bound counter c , neither does she have a strategy from 1 to bound d . However, she has a memoryless strategy β winning for $\mathbb{B}(c) \vee \mathbb{B}(d)$: from 1, she moves to 2, and from 3 she moves to 4. If the play stays in $\{1, 2\}$, then d is bounded, and if the play eventually moves to 3, then c is bounded.

formula which intuitively requires that counter c is bounded along a play, by some constant. Formally, $\mathbb{B}(c)$ is interpreted in \mathcal{A} by the set of plays ρ of \mathcal{A} , denoted $Plays(\mathcal{A}, \mathbb{B}(c))$, such that the sequence $\lambda(\zeta(\rho), c)$ is bounded, *i.e.*

$$Plays(\mathcal{A}, \mathbb{B}(c)) = \{\rho \in Plays(\mathcal{A}) \mid \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \lambda(\zeta(\rho), c)_n \leq N\}$$

201 The set $Plays(\mathcal{A}, \mathbb{B}(c))$ is called a *boundedness condition*. We let $\mathbb{U}(c)$ as a shortcut for $\neg\mathbb{B}(c)$.
 202 A *counter condition* for \mathcal{A} is a Boolean formula ϕ over the set of propositions $\{\mathbb{B}(c) \mid c \in C\}$.
 203 Its interpretation $Plays(\mathcal{A}, \phi) \subseteq Plays(\mathcal{A})$ over \mathcal{A} is defined naturally.

204 Given a counter condition ϕ , the pair $G = (\mathcal{A}, \phi)$ is called a *counter game*. The game
 205 *induced* by $G = (\mathcal{A}, \phi)$ is the game $G_{\phi} = (\mathcal{A}', Plays(\mathcal{A}, \phi) \cap Parity(\mathcal{A}))$. Note that in a
 206 counter game, both the counter condition and the parity condition must be satisfied. The
 207 notion of strategies and winning strategies carry over to counter games by considering the
 208 games they induce. In particular, Eve wins G if she wins G_{ϕ} , *i.e.*, she has a strategy winning
 209 for the objective $Plays(\mathcal{A}, \phi) \cap Parity(\mathcal{A})$.

210 In this paper, we consider several classes of counter conditions. The class of counter
 211 conditions of the form $\bigwedge_{c \in C} \mathbb{B}(c)$ for some finite set $C \subseteq \mathcal{C}$ is denoted $\bigwedge \mathbb{B}$. Similarly, we
 212 denote by $\bigvee \mathbb{B}$, $\bigvee \bigwedge \mathbb{B}$ and $Bool^+(\mathbb{B})$ the classes of counter conditions which are respectively,
 213 disjunctions of atoms $\mathbb{B}(c)$, disjunction of conjunctions of atoms $\mathbb{B}(c)$ (DNF), any negation-free
 214 Boolean formula.

215 ► **Example 1.** First, Fig. 2 illustrates an example with a disjunction of boundedness objectives.
 216 Our second example is given by an arena with two counters and a single state controlled
 217 by Adam. At each step, Adam can either increment c_1 and leave c_2 unchanged (called
 218 transitions t_1), or increment c_2 and leave c_1 unchanged (called transition t_2). Clearly, Eve
 219 has a strategy to win the objective $\bigvee_{i=1,2} \mathbb{U}(c_i)$. Indeed, in any play, there exists i such that
 220 t_i is taken infinitely many times, and therefore c_i is unbounded. Suppose now that Adam
 221 wants to win objectives $\bigwedge_{i=1,2} \mathbb{U}(c_i)$. He can do so by playing longer and longer sequences of
 222 transition t_1 in alternation with longer and longer sequences of transition t_2 , which requires
 223 infinite memory.

224 ► **Lemma 2.** *Counter games (with Boolean combinations of boundedness objectives) are*
 225 *determined and decidable.*

Proof. Given a counter arena \mathcal{A} and a counter c of \mathcal{A} , the set $Plays(\mathcal{A}, \mathbb{B}(c))$ is a Borel set.



Indeed, it is equal to the countable union for all $N \geq 0$ of the sets

$$\text{Plays}_N(\mathcal{A}, \mathbb{B}(c)) = \{\rho \in \text{Plays}(\mathcal{A}) \mid \forall n \in \mathbb{N}, \lambda(\zeta(\rho), c)_n \leq N\}$$

226 which are ω -regular. Indeed, a Büchi automaton needs $|V| \times N \times |C|$ states to recognize
 227 $\text{Plays}_N(\mathcal{A}, \mathbb{B}(c))$. Since ω -regular sets are Borel, so is $\text{Plays}(\mathcal{A}, \mathbb{B}(c))$, as well as any Boolean
 228 combination of the latter. By Martin's determinacy theorem [22], the result follows.

229 To prove decidability, it suffices to notice that winning strategies in counter games are
 230 infinite trees such that all of their branches are accepted by a deterministic max-automaton as
 231 defined in [2]: such automata have a finite set of counters which can be incremented, reset to 0,
 232 and take the maximal value of several counters and put it in another one. Such automata are
 233 closed under intersection and can recognize any regular language, so, we can encode the parity
 234 condition as well as the counter operations. Deterministic max-automata corresponds exactly
 235 to the logic WMSO+U over infinite words (weak MSO with the unbounding quantifier).
 236 WMSO+U has been extended to WMSO+UP on infinite trees with an additional quantifier
 237 over infinite paths (P). Therefore, winning strategies of two-player games with winning
 238 conditions definable in WMSO+U over infinite words are definable in WMSO+UP (see
 239 Example 2 of [4]). The result follows since WMSO+UP has decidable satisfiability problem,
 240 again by [4]. ◀

241 **3 Counter games with conjunctions of boundedness conditions**

242 In this section, we study games with counter conditions in the class $\bigwedge \mathbb{B}$. Such games
 243 are easily shown to be decidable using known results. Indeed, we prove that they are
 244 equivalent in polynomial time to Streett games, known to be CoNP-complete [17]. Let
 245 us define Streett games. Given an arena \mathcal{A} with set of vertices V , and a set of k pairs
 246 $S = \{(E_i, F_i) \mid 1 \leq i \leq k, E_i, F_i \subseteq V\}$, we let $\text{Streett}(S)$ be the set of words $w \in V^\omega$ such
 247 that for all $i = 1, \dots, k$, if w contains infinitely many occurrences of some $e \in E_i$, then it must
 248 contain infinitely many occurrences of some $f \in F_i$. A Streett game is a pair $G = (\mathcal{A}, W)$
 249 where W is given as set of k Streett pairs S , i.e., $W = \text{Streett}(S)$. We prove that $\bigwedge \mathbb{B}$ -counter
 250 games are interreducible to Streett games in polynomial time.

251 ▶ **Lemma 3.** *There is a bijection Ψ from counter games with condition in $\bigwedge \mathbb{B}$ to Streett*
 252 *games such that Ψ is computable in PTIME, such that for all counter games G , Eve wins G*
 253 *iff she wins the Streett game $\Psi(G)$, and such that for all Streett game G' , $\Psi^{-1}(G')$ has a*
 254 *trivial parity function (with color 0 only).*

255 **Proof.** Let G be a counter game over an underlying arena \mathcal{A} and a set of counters C , with
 256 winning condition of the form $\bigwedge_{c \in C'} \mathbb{B}(c)$ for some $C' \subseteq C$. To simplify our explanations,
 257 we first assume that G does not contain any parity condition. To construct a Streett game
 258 G' , we keep the same arena \mathcal{A} and construct, for all $c \in C'$, a Streett pair (E_c, F_c) which
 259 is satisfied by all sequences of counter operations such that if c is incremented infinitely
 260 often, then c is also reset infinitely often. So, E_c is the set of vertices where c is incremented,
 261 while F_c are those where c is reset. Let σ be a winning strategy for Eve in G and suppose
 262 it is not winning in G' , then it means that there exists some play $\rho \in \text{Plays}(\mathcal{A}, \sigma)$ and a
 263 counter $c \in C'$ which is incremented infinitely often but reset finitely often. So, its value is
 264 not bounded along ρ , contradicting that σ is winning.

265 The converse uses the fact that finite-memory strategies suffice to win Streett games: let
 266 σ be a finite-memory strategy winning for Eve in G' . Suppose it is not winning in G . Then,
 267 there is some play $\rho \in \text{Plays}(\mathcal{A}, \sigma)$ and a counter $c \in C'$ whose value is unbounded along ρ .



268 Therefore, ρ increases c infinitely many times. Since σ is winning for G' , ρ resets c infinitely
269 many times. So, it can be decomposed into infinitely many fragments ending with a reset of
270 c : $\rho = \rho_1 v_1 \rho_2 v_2 \dots$ such that each ρ_i does not contain a reset of c and v_i resets c . Since the
271 value of c is unbounded, the ρ_i contains an arbitrarily large number of increments of c : for
272 all $n \in \mathbb{N}$, there exists i such that ρ_i increments c at least n times. By taking n large enough,
273 there is necessarily some ρ_i which can be decomposed into $\pi_1 \pi_2 \pi_3$ such that π_2 increments c
274 at least once, and the strategy σ cycles on π_2 , i.e., has the same memory state and vertex
275 before and after π_2 . This can be seen using standard pumping arguments. Indeed, if k is
276 the number of memory states of σ , there are at most $k|V|$ positions of ρ_i which are not on a
277 cycle. So, if the number of increments is bigger than $k|V|$, there is a cycle π_2 which contains
278 an increment. Therefore, $\rho_1 v_1 \dots \rho_{i-1} v_{i-1} \pi_1 \pi_2^\omega \in \text{Plays}(\mathcal{A}, \sigma)$, it increments infinitely many
279 times c , and resets c finitely many times, contradicting that σ is winning in G' .

280 If now G also has a parity condition with x colors, we add x Streett pairs to the latter
281 game G' , using a standard parity-to-Streett conversion: those Streett pairs enforce that if an
282 odd color α is seen infinitely often, then some even color $\beta > \alpha$ is seen infinitely often.

283 Conversely, let us explain how to convert any Streett game into a $\bigwedge \mathbb{B}$ -counter game. As
284 a matter of fact, the latter reduction is a bijection, so, it suffices to apply its inverse, which
285 we explicit here. If there are k Streett pairs, then we introduce k counters $C = \{c_1, \dots, c_k\}$.
286 No parity condition is needed in the resulting counter game (formally, we introduce a trivial
287 colouring which colors all vertices by 0). If (E_i, F_i) is a Streett pair, we assume wlog that
288 $E_i \cap F_i = \emptyset$, and for any vertex v , we add the following operation on c_i to v : increments c_i
289 if $v \in E_i$, reset it if $v \in F_i$, and skip otherwise. The counter condition is $\bigwedge_{i=1}^k \mathbb{B}(c_i)$. \blacktriangleleft

290 As a corollary of the latter Lemma, by applying $\Psi^{-1} \circ \Psi$ to a counter game in $\bigwedge \mathbb{B}$, we
291 get in polynomial time an equivalent counter game in $\bigwedge \mathbb{B}$ with a trivial parity condition.
292 Streett games are known to be CONP -complete and in PTIME for a fixed number of Streett
293 pairs [23], and finite-memory strategies suffice for Eve while memoryless strategies suffice for
294 Adam. Therefore, Lemma 3 immediately yields the following result:

295 **► Theorem 4.** *Counter games with winning conditions in $\bigwedge \mathbb{B}$ are CONP -complete, and*
296 *in PTIME if both the index of the priority function and the number of counters are fixed*
297 *constants. Finite memory suffice for Eve and memoryless strategies suffice for Adam. CONP -*
298 *hardness holds even if the index of the parity function is any fixed constant.*

299 The latter theorem does not cover the case where only the number of counters is fixed.
300 We prove that in this case, the complexity is at most $\text{NP} \cap \text{CONP}$. The proof of Theorem 4
301 is based on Lemma 3 which explicits a bijection between counter games in $\bigwedge \mathbb{B}$ and Streett
302 games. In particular, it constructs a game whose winning condition is a conjunction of $|C|$
303 Streett conditions and a parity condition (which is then itself converted as Streett pairs).
304 Each Streett pair can be seen as a parity condition over colors $\{0, 1, 2\}$. Therefore, when the
305 number of counters is fixed, the reduction of Lemma 3 yields a game with a winning condition
306 which is a conjunction of a fixed number ℓ of parity conditions over colors $\{0, 1, 2\}$ and a
307 single arbitrary parity condition. We prove that such games are reducible in polynomial time
308 to parity games for $\ell = 1$ in the following lemma, later on applied recursively to show the
309 result the result for any fixed ℓ (Theorem 6).

310 **► Lemma 5.** *Games of the form $G = (\mathcal{A}, W)$ where $W = \text{Parity}(\kappa) \cap \text{Parity}(\kappa_3)$ for κ an*
311 *arbitrary colouring of index k and κ_3 a colouring in $\{0, 1, 2\}$, reduce in polynomial time to*
312 *parity games of index $2k + 1$. Moreover, finite-memory strategies of memory size equal to k*
313 *are sufficient for Eve to win G .*



314 **Proof.** Let V be the set of vertices of \mathcal{A} , Q the set of colours of κ and m the minimal
315 even number greater than or equal to every element of Q . We construct in polynomial
316 time a parity game G' over a coloured arena $\mathcal{A}' = (V', E', V'_3, V'_\forall, Q', \kappa', v'_0)$ such that Eve
317 wins G iff she wins G' , and such that the index of κ' is equal to $2k + 1$. To prove that
318 finite-memory strategies of memory size equal to k suffice to win G' , we use the known
319 result that memoryless strategies suffice to win parity games, and prove that any memoryless
320 winning strategy in G' can be translated back to a finite-memory winning strategy in G with
321 memory size equal to k .

The construction of G' is as follows. We let $V' = V \times Q$ and $Q' = \{0, \dots, 2m + 2\}$ with
the following parity function κ' : any vertex $(u, q) \in V'$ is coloured by

$$\kappa'(u, q) = \begin{cases} \kappa(u) & \text{if } \kappa_3(u) = 0 \\ m + 1 & \text{if } \kappa_3(u) = 1 \\ m + 2 + \max\{\kappa(u), q\} & \text{if } \kappa_3(u) = 2 \end{cases}$$

322 Before defining the transitions, let us prove some property (called P) about the colouring
323 κ' . Let $\pi = v_0 v_1 \dots \in \text{Plays}(\mathcal{A})$. For all $i \geq 0$, we let q_i be the color by κ seen since the
324 last vertex v_j , $j < i$, such that $\kappa_3(v_j) = 2$. Formally, j is the largest integer such that $j < i$
325 and $\kappa_3(v_j) = 2$, and we let $q_i = \max\{\kappa(v_k) \mid j \leq k < i\}$. If j does not exist, then $q_i = 0$.
326 Let $\pi' = (v_0, q_0)(v_1, q_1) \dots$. We prove that $\pi \in W$ iff $\pi' \in \text{Parity}(\kappa')$. Let x be the maximal
327 priority occurring infinitely often in $\kappa_3(\pi)$ (κ_3 here, is extended morphically to sequences in
328 V^ω). We consider three cases:

- 329 ■ If $x = 0$, then $\kappa'(\pi') = \kappa(\pi)$, so, $\pi \in W$ iff $\pi \in \text{Parity}(\kappa)$ iff $\pi' \in \text{Parity}(\kappa')$.
- 330 ■ If $x = 1$, then $\kappa'(\pi')$ sees $m + 1$ (which is odd) infinitely often, and therefore $\pi' \notin \text{Parity}(\kappa')$
331 and $\pi \notin W$.
- 332 ■ If $x = 2$, this is the most interesting case. In that case, π can be decomposed into
333 fragments $\pi = f_1 f_2 f_3 \dots$ such that each f_i contains exactly one node v , at its end, such
334 that $\kappa_3(v) = 2$. Let α_i be the maximal color of fragment f_i . Then, the maximal colour
335 seen infinitely often in $\kappa'(\pi')$ is the same as in the sequence $(m + 2 + \alpha_1)(m + 2 + \alpha_2) \dots$,
336 which is equal to $m + 2 + \alpha$, where α is the maximal color occurring infinitely often in
337 $\kappa(\pi)$. As m is even, $\pi' \in \text{Parity}(\kappa')$ iff $\pi \in \text{Parity}(\kappa)$ iff $\pi \in W$.

338 The transitions of \mathcal{A}' are constructed so that any play $\pi \in \text{Plays}(\mathcal{A})$ bijectively corresponds
339 to the play π' defined above. In particular, when a vertex (v, q) such that $\kappa_3(v) = 2$ is
340 visited and there is a transition (v, v') in \mathcal{A} , q is reset to 0 (we add a transition to $(v', 0)$ in
341 \mathcal{A}'), and if $\kappa_3(v) \neq 2$, we add a transition to $(v', \max(\kappa(v), q))$. Such a construction ensures
342 that there is a bijection Ψ between $\text{Plays}(\mathcal{A})$ and $\text{Plays}(\mathcal{A}')$ such that, by property P above,
343 $\pi \in \text{Parity}(\kappa) \cap \text{Parity}(\kappa_3)$ iff $\Psi(\pi) \in \text{Parity}(\kappa')$, so, correctness follows. Moreover, any
344 memoryless strategy in G' is translated into a finite-memory strategy in G with a memory
345 size equal to the index of κ , concluding the proof. ◀

346 Note that Lemma 5 entails that games with a conjunction of a parity condition of index
347 k and a fixed number N of parity conditions over colors $\{0, 1, 2\}$ are solvable in $\text{NP} \cap \text{CoNP}$.
348 Indeed, by iterating Lemma 5 N times, the latter games reduce to parity games of index
349 $2^N(k + 1) - 1$. Games with Boolean combinations of parity objectives have been studied
350 in [13]. However, the former complexity result is not covered by [13]. As explained before,
351 Lemma 5 together with the bijection of Lemma 3 imply the following theorem:

352 ► **Theorem 6.** *For any fixed positive integer N , counter games of parity index k with winning*
353 *conditions in $\bigwedge \mathbb{B}$ and at most N counters, are in $\text{NP} \cap \text{CoNP}$ (and parity-hard). Finite*
354 *memory strategies with memory size $2^{N-1}(k + 1) - 1$ suffice for Eve and Adam.*



355 **4 Finitely switching strategies for games with disjunction of**
 356 **prefix-independent objectives**

357 Let \mathcal{A} be an arena, let V be its set of vertices, and let \mathcal{W} be a finite set of prefix-independent¹
 358 winning conditions for \mathcal{A} , i.e., $\mathcal{W} \subseteq 2^V$. We let $\bigvee \mathcal{W} = \bigcup\{W \mid W \in \mathcal{W}\}$. In this section, we
 359 consider a class of strategies for Eve, called *finitely switching*, whose existence entail that
 360 she wins $(\mathcal{A}, \bigvee \mathcal{W})$. We characterize the existence of finitely switching strategies via a least
 361 fixpoint and, for some particular classes of winning objectives $\bigvee \mathcal{W}$ of interest in this paper,
 362 prove that such strategies suffice for Eve to win $(\mathcal{A}, \bigvee \mathcal{W})$. The complexity of computing the
 363 fixpoint for those particular classes of objectives is deferred to Section 5.

Let us first give intuition on the notion of finitely switching strategies. In such a strategy, Eve announces an initial goal $W \in \mathcal{W}$ she wants to satisfy, but she may switch her mind during the play, i.e., announce another goal $W' \in \mathcal{W}$, depending on what Adam does. She can do this only a finite number of times and eventually keep the same goal forever and satisfy it. Formally, for $k \geq 0$, a k -switching strategy for Eve is a strategy σ such that there exists a mapping goal from finite histories of σ to \mathcal{W} such that for all $\pi = v_1 v_2 \dots \in \text{Plays}(\sigma)$, there exists $W_1, \dots, W_{k+1} \in \mathcal{W}$ such that $\pi \in W_{k+1}$ and

$$\text{goal}(v_0)\text{goal}(v_0 v_1)\text{goal}(v_0 v_1 v_2) \dots \in W_1^* W_2^* \dots W_k^* W_{k+1}^\omega$$

364 The goal W_{k+1} is called the *ultimate* goal of π . We say that σ is finitely switching if it is
 365 k -switching for some $k \geq 0$.

366 ► **Example 7.** Consider the example of Fig. 2. The described strategy β is 1-switching for
 367 $\mathcal{W} = \{\mathbb{B}(c), \mathbb{B}(d)\}$: initially, her goal is $\mathbb{B}(d)$. If Adam ever tries to make it so that counter d
 368 becomes unbounded, by going to vertex 3 from vertex 2, Eve can now set her new goal to
 369 $\mathbb{B}(c)$.

370 Consider now the single-state arena of Example 1 in which Eve wants to satisfy $\bigvee_{c=1,2} \mathbb{U}(c)$.
 371 She has no finitely switching strategy: whenever she announces she wants to satisfy $\mathbb{U}(c_i)$
 372 for some i , Adam loops on transition t_{3-i} until Eve switches her mind. If her ultimate goal
 373 is $\mathbb{U}(c_i)$ for some i , then Adam will loop forever on t_{3-i} and c_i will be bounded, so that
 374 Eve does not meet the ultimate goal she announced. By seeing operations on c_1 and c_2 as
 375 priority functions, this example also shows that finitely switching strategies are not sufficient
 376 to win disjunctions of parity objectives in general. More precisely, for $i = 1, 2$, we can define
 377 the priority functions p_i , here on transitions, which colors transition t_i by 0 and transition
 378 t_{3-i} by 1. If she ultimately announces her goal is to satisfy priority p_i , then Adam takes
 379 transition t_{3-i} forever and p_i sees infinitely many times color 1.

380 Since in a finitely switching strategy, any play consistent with that strategy must satisfy
 381 its ultimate goal, the following result is immediate:

382 ► **Lemma 8 (Soundness).** *Any finitely switching strategy for Eve in \mathcal{A} is winning for $(\mathcal{A}, \bigvee \mathcal{W})$.*

383 We will see later on that the converse holds for some particular classes of boundedness
 384 objectives, but for now, let us characterize the existence of finitely switching strategies
 385 via some least fixpoint. For a set $X \subseteq V$, we denote the objective of reaching X by
 386 $\text{Reach}(X) = V^* X V^\omega$. We let f be the function which associates any $X \subseteq V$ to the set of
 387 vertices u from which Eve can win the objective $W \cup \text{Reach}(X)$ for some $W \in \mathcal{W}$. Formally,

¹ A winning condition W is prefix-independent if, for all $(w, u) \in (V^\omega, V^*)$, $w \in W$ iff $uw \in W$.



388 $f(X) = \{u \in V \mid \exists W \in \mathcal{W}, \text{ Eve wins } (\mathcal{A}[u], W \cup \text{Reach}(X))\}$. Note that $X \subseteq f(X)$ for
 389 all $X \subseteq V$. Indeed, if $u \in X$, then Eve has a trivial strategy from u to reach X , and so
 390 $u \in f(X)$. Since $(2^V, \subseteq)$ is a complete lattice, by Knaster–Tarski theorem, f has a unique
 391 least fixpoint denoted $S^{\mathcal{W}}$. To compute $S^{\mathcal{W}}$, it suffices to compute the following sequence of
 392 sets until it stabilizes:

- 393 ■ $S_0^{\mathcal{W}} = \emptyset$,
- 394 ■ for $i \geq 0$, $S_{i+1}^{\mathcal{W}} = \{u \in V \mid \exists W \in \mathcal{W}, \text{ Eve wins } (\mathcal{A}[u], W \cup \text{Reach}(S_i^{\mathcal{W}}))\}$.

395 For all $i \geq 1$ and $u \in S_i^{\mathcal{W}}$ (if it exists), we denote by $\sigma_{u,i}$ a strategy for Eve winning in the
 396 game $(\mathcal{A}[u], W \cup \text{Reach}(S_{i-1}^{\mathcal{W}}))$ for some $W \in \mathcal{W}$. It exists by definition of $S_i^{\mathcal{W}}$.

397 We now prove the following characterization.

398 ► **Lemma 9** (Fixpoint characterization of finitely switching strategies). *Let \mathcal{A} be an arena with*
 399 *set of vertices V and \mathcal{W} a finite set of prefix-independent winning conditions for \mathcal{A} . For all*
 400 *$u \in V$, the following are equivalent:*

- 401 1. *Eve has a finitely switching strategy from u*
- 402 2. *Eve has a $|V|$ -switching strategy from u*
- 403 3. *$u \in S^{\mathcal{W}}$*

404 **Proof.** Clearly $2 \Rightarrow 1$. We first prove $1 \Rightarrow 3$ and then $3 \Rightarrow 2$.

405 Let σ be a k -switching strategy for some $k \geq 0$. By induction on k , we prove that
 406 $u \in S_{k+1}^{\mathcal{W}}$. This implies the claim as $S_{k+1}^{\mathcal{W}} \subseteq S^{\mathcal{W}}$.

407 If $k = 0$, then Eve never changes her mind and therefore all plays of (σ) are in $\text{goal}(u)$
 408 (the history with only the vertex u), so, $u \in S_1^{\mathcal{W}}$. Suppose that $k > 0$. We take $W = \text{goal}(u)$.
 409 Let $\pi \in \text{Plays}\sigma$. We prove that $\pi \in W \cup \text{Reach}(S_k^{\mathcal{W}})$. If Eve never switches her mind during
 410 π , then $\pi \in W$. Otherwise, let h the smallest prefix of π such that $\text{goal}(h) \neq W$. Let v be
 411 the last vertex of h . Note that the strategy² $\sigma|_h$ is a $(k-1)$ -switching strategy from v . By
 412 IH, $v \in \text{Reach}(S_k^{\mathcal{W}})$, which means that $\pi \in \text{Reach}(S_k^{\mathcal{W}})$ and we are done.

413 We now prove $3 \Rightarrow 2$. Let $u \in S^{\mathcal{W}}$. Let i be smallest index such that $u \in S_i^{\mathcal{W}}$. Note that
 414 $i \leq |V|$. We prove by induction on i that Eve has an $(i-1)$ -switching strategy $\beta_{u,i}$ witnessed
 415 by a goal function $\text{goal}_{u,i}$. If $u \in S_1^{\mathcal{W}}$, then $\sigma_{u,1}$ wins $(\mathcal{A}[u], W)$ for some $W \in \mathcal{W}$ and so we
 416 let $\text{goal}_{u,1}(h) = W$ for any history h of $\sigma_{u,1}$.

417 Suppose that $i > 1$ and $u \in S_i^{\mathcal{W}}$. Remind that the strategy $\sigma_{u,i}$ wins $(\mathcal{A}[u], W \cup$
 418 $\text{Reach}(S_{i-1}^{\mathcal{W}}))$. We modify $\sigma_{u,i}$ into a strategy $\beta_{u,i}$ as follows: $\beta_{u,i}$ is the same as $\sigma_{u,i}$ as long
 419 as $S_{i-1}^{\mathcal{W}}$ has not been reached. If eventually $S_{i-1}^{\mathcal{W}}$ is reached, say at a vertex v , then $\beta_{u,i}$ plays
 420 according to $\beta_{v,i-1}$ (which exists by IH).

421 We prove that $\beta_{u,i}$ is $(i-1)$ -switching. We let $\text{goal}_{u,i}(h) = W$ for any history h which
 422 does not visit $S_{i-1}^{\mathcal{W}}$. For any history $h = h_1 v h_2$ such that $|h_1|$ is minimal and $v \in S_{i-1}^{\mathcal{W}}$, we
 423 let $\text{goal}_{u,i}(h) = \text{goal}_{v,i-1}(v h_2)$. Let $\pi \in \text{Plays}(\beta_{u,i})$. If $\pi = v_0 v_1 \dots$ never visits $S_{i-1}^{\mathcal{W}}$, then
 424 $\text{goal}(v_0) \text{goal}(v_0 v_1) \dots \in W^\omega$, and $\pi \in W^\omega$. If there exists j minimal such that $v_j \in S_{i-1}^{\mathcal{W}}$,
 425 then, by IH, there exists $W_1, \dots, W_i \in \mathcal{W}$ such that $\text{goal}_{v_j,i-1}(v_j) \text{goal}_{v_j,i-1}(v_j v_{j+1}) \dots \in$
 426 $W_1^* \dots W_{i-1}^* W_i^\omega$. By definition of $\text{goal}_{u,i}$, we obtain that $\text{goal}_{u,i}(v_0) \text{goal}_{u,i}(v_0 v_1) \dots \in$
 427 $W^* W_1^* \dots W_{i-1}^* W_i^\omega$. Finally, it remains to prove that $\pi \in W_i$: by IH, its suffix $v_j v_{j+1} \dots$ is
 428 in W_i , and since W_i is prefix-independent, so is π , concluding the proof. ◀

² The restriction $\sigma|_h$ is defined by $\sigma|_h(h') = \sigma(hh')$ for all h' .



429 According to Lemma 9, when Eve has a finitely switching strategy, then she has a
 430 $|V|$ -switching strategy. Interestingly, observe that the number of times she possibly needs to
 431 switch her mind does not depend on the number of winning objectives in \mathcal{W} .

432 The proof of Lemma 9 constructs, for all $1 \leq i \leq |V|$ and $u \in \mathbf{S}_i^{\mathcal{W}}$, a finitely switching
 433 strategy $\beta_{u,i}$, which either mimics $\sigma_{u,i}$ or switch to a strategy $\beta_{v,i-1}$. So, Eve needs to
 434 remember the current vertex u and index i , in order to know whether she must play according
 435 to $\sigma_{u,i}$ or to switch to a strategy $\beta_{v,i-1}$. So, even if for some N , all the strategies $\sigma_{u,i}$ are
 436 finite-memory of size at most N , $\beta_{u,i}$ needs memory $O(N \cdot |V|^2)$ in general. We now prove
 437 that Eve can do better.

438 **► Lemma 10 (Memory transfer).** *Let \mathcal{A} be a counter arena, V be its set of vertices, and \mathcal{W}
 439 a finite set of prefix-independent winning conditions for \mathcal{A} . Let $N \in \mathbb{N}$ and suppose that for
 440 all $X \subseteq V$, $u \in V$ and $W \in \mathcal{W}$, strategies of memory size at most N suffice for Eve to win
 441 $(\mathcal{A}[u], W \cup \text{Reach}(X))$. Then for all $u \in \mathbf{S}^{\mathcal{W}}$, Eve wins $(\mathcal{A}[u], \bigvee \mathcal{W})$ with memory at most N .*

442 **Proof.** For all $1 \leq i \leq |V|$ and $u \in \mathbf{S}_i^{\mathcal{W}}$, we assume the existence of a strategy $\sigma_{u,i}$ winning
 443 in $(\mathcal{A}[u], W \cup \text{Reach}(\mathbf{S}_i^{\mathcal{W}}))$ for some $W \in \mathcal{W}$, of memory states $\{1, \dots, N\}$. Note that the $\sigma_{u,i}$
 444 share the same memory states. The main idea of the proof is to provide a way for Eve, given
 445 a current vertex $v \in \mathbf{S}^{\mathcal{W}}$ and a memory state m , to uniquely identify the (finite memory)
 446 strategy $\sigma_{u,i}$ according to she must play in vertex v and state m . Given v , we take i to
 447 be the smallest integer such that $v \in \mathbf{S}_i^{\mathcal{W}} \setminus \mathbf{S}_{i-1}^{\mathcal{W}}$. To identify u , we take the smallest vertex
 448 u according to an arbitrary linear order on V such that $u \in \mathbf{S}_i^{\mathcal{W}} \setminus \mathbf{S}_{i-1}^{\mathcal{W}}$ and v is reachable
 449 from u by a finite path p consistent with $\sigma_{u,i}$, such that if $\sigma_{u,i}$ is in its initial memory state
 450 m_0 at u , then after p , it is in memory state m . Then, Eve moves to vertex $g(v, m)$, for
 451 (δ, g) the memory mapping of $\sigma_{u,i}$. Intuitively, if Eve stays forever in $\mathbf{S}_i^{\mathcal{W}} \setminus \mathbf{S}_{i-1}^{\mathcal{W}}$, she will by
 452 monotonicity eventually always play according to the same strategy $\sigma_{u,i}$, and so satisfy some
 453 W , as conditions in \mathcal{W} are prefix-independent. Otherwise, she will reach another $\mathbf{S}_j^{\mathcal{W}} \setminus \mathbf{S}_{j-1}^{\mathcal{W}}$
 454 for some $j < i$ and we make the same reasoning inductively. See Appendix for details. ◀

455 The converse of Lemma 8 does not hold in general, as illustrated in Example 1 for
 456 disjunction of unboundedness objectives. However, we show here that it holds for disjunctions
 457 of conjunctions of boundedness objectives.

458 **► Lemma 11 (Completeness for boundedness conditions in DNF).** *Let \mathcal{A} be a counter arena and
 459 C its set of counters. Let \mathcal{W} be a finite subset of counter conditions for \mathcal{A} in $\text{Parity}(\mathcal{A}) \wedge \bigwedge \mathcal{B}$.
 460 If Eve wins $(\mathcal{A}, \bigvee \mathcal{W})$, then she has a finitely switching strategy.*

461 **Proof.** Let C_1, \dots, C_p be subsets of C such that \mathcal{W} is the set of all counter conditions
 462 $\bigwedge_{c \in C_i} \mathbb{B}(c)$, for $i \in \{1, \dots, p\}$. Suppose that Eve does not have a finitely switching strategy
 463 from the initial vertex v_0 . This means, by Lemma 9, that $v_0 \notin \mathbf{S}^{\mathcal{W}}$. We construct a
 464 winning strategy for Adam for the complementary objective $\text{Comp} = \bigcap_{i \in \{1, \dots, p\}} \bigcup_{c \in C_i} \mathbb{U}(c) \cup$
 465 $\overline{\text{Parity}(\mathcal{A})}$. By definition of $\mathbf{S}^{\mathcal{W}}$, $f(\mathbf{S}^{\mathcal{W}}) = \mathbf{S}^{\mathcal{W}}$. Therefore, for any $v \in V \setminus \mathbf{S}^{\mathcal{W}}$ and
 466 $i \in \{1, \dots, p\}$, Eve does not win the counter game $(\mathcal{A}[v], (\text{Parity}(\mathcal{A}[v]) \cap \bigcap_{c \in C_i} \mathbb{B}(c)) \cup$
 467 $\text{Reach}(\mathbf{S}^{\mathcal{W}}))$. Thus, by determinacy (Lemma 2), Adam has a winning strategy $\sigma_{v,i}$ in $\mathcal{A}[v]$
 468 for the complementary objective $(\bigcup_{c \in C_i} \mathbb{U}(c) \cup \overline{\text{Parity}(\mathcal{A}[v])}) \cap \text{Reach}(\mathbf{S}^{\mathcal{W}})$.

469 Intuitively, a strategy for Adam winning for Comp could be defined by breaking it down
 470 into the following steps:

- 471 ■ Adam begins by step (1, 1): he follows strategy $\sigma_{v_0,1}$ until the play of the game reaches
 472 a vertex where the value of a counter of C_1 is 1. If that is never the case, then Adam



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► **Theorem 13.** *Counter games with counter conditions in $\bigvee \mathbb{B}$ are in $NP \cap coNP$, and are in PTIME if the index of the colouring is fixed. A memory of size equal to the index of the colouring suffices for Eve, and infinite memory is required for Adam.*

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Proof. Let G be a game over counter arena \mathcal{A} with set of counters C , initial vertex v and objective $\bigvee \mathcal{W}$ where $\mathcal{W} = \{\text{Parity}(\mathcal{A}) \cap \mathbb{B}(c) \mid c \in C'\}$ for some $C' \subseteq C$. It should be clear that those conditions are prefix-independent, therefore, by Lemma 9 and Lemma 11, Eve wins G iff she has a finitely switching strategy iff $v \in S^{\mathcal{W}}$. So, to check whether Eve wins G , it suffices to compute the fixpoint $S^{\mathcal{W}}$. We prove that each step of the fixpoint computation

(line 7 in algorithm SOLVE) is done in $NP \cap coNP$, and in PTIME if the index of the colouring is fixed. By Lemma 12, the complexity statement of the theorem follows. It remains to show that for all subset $X \subseteq V$, any vertex $u \in V$ and any counter $c \in C'$, it is decidable in $NP \cap coNP$ (and in ptime for fixed parity) whether Eve wins the game $(\mathcal{A}[u], (\text{Parity}(\mathcal{A}) \cap \mathbb{B}(c)) \cup \text{Reach}(X))$. First, we evacuate the reachability condition, i.e., reduce in ptime the latter problem to solving a game $(\mathcal{A}', \text{Parity}(\mathcal{A}') \cap \mathbb{B}(c))$. This is easily done by adding a sink state to \mathcal{A} reached whenever X is visited, with operation skip on c and priority 0. This reduction works for more general boundedness conditions (see Lemma 17 in Appendix). Finally, the game $(\mathcal{A}', \text{Parity}(\mathcal{A}') \cap \mathbb{B}(c))$ is solvable in $NP \cap coNP$ by Theorem 6, and in ptime for fixed parity, which is the case of \mathcal{A}' when the index of \mathcal{A} is fixed, because they have the same colours.

For Adam, infinite memory might be necessary to enforce the complementary objective, as illustrated by Example 1. For Eve, Theorem 6 states that a memory of size the index of the parity function is sufficient to solve the "local" games $(\mathcal{A}', \text{Parity}(\mathcal{A}') \cap \mathbb{B}(c))$, which can be translated back to strategies of same size in $(\mathcal{A}[u], (\text{Parity}(\mathcal{A}) \cap \mathbb{B}(c)) \cup \text{Reach}(X))$. Therefore, the memory transfer lemma (Lemma 10) yields the result. ◀

We now turn to games on arenas \mathcal{A} with conditions in $\bigvee \bigwedge \mathbb{B}$, i.e., where $\mathcal{W} = \{\text{Parity}(\mathcal{A}) \wedge \bigwedge_{c \in C_i} \mathbb{B}(c) \mid i = 1, \dots, n\}$ for C_1, \dots, C_n finite subsets of counters. The same reasoning as in the proof of Theorem 13 applies. The only difference here is that, to solve the "local" games of the fixpoint computation (line 7 of algorithm SOLVE), we rely on Theorem 4.

► **Theorem 14.** *Counter games with winning conditions in $\bigvee \bigwedge \mathbb{B}$ are $coNP$ -complete. Finite memory suffices for Eve, and infinite memory is required for Adam.*

We conclude this section by the case of Boolean combination of boundedness objectives.

► **Theorem 15.** *Counter games with winning conditions in $Bool^+(\mathbb{B})$ are in PSPACE and $coNP$ -hard. Finite memory suffices for Eve, and infinite memory is required for Adam.*

Proof. Any counter condition which is a positive boolean combination $\phi \in Bool^+(\mathbb{B})$ can be written in disjunctive normal form $\psi = \bigvee_{i \in \{1, \dots, p\}} \bigwedge_{c \in C_i} \mathbb{B}(c)$, where the C_i are subsets of C . Let $\mathcal{W} = \{\text{Parity}(\mathcal{A}) \wedge \bigwedge_{c \in C_i} \mathbb{B}(c) \mid i = 1, \dots, n\}$. A direct application of Theorem 14 yields a $coNExpTime$, because p might be exponential. Instead, we do not construct ψ explicitly. Recall that, from Theorem 4, counter games with counter conditions in $\bigwedge \mathbb{B}$ are in $coNP$, and thus in PSPACE. Thus, since it is well-known that, even if p may be exponential in the size of ϕ , we can enumerate \mathcal{W} in polynomial space, we can use this enumeration algorithm



560 at line 6 of algorithm SOLVE in Fig. 3 to compute the fixpoint S^W in polynomial space. As
 561 a consequence, the problem of deciding whether Eve has a winning finitely switching strategy
 562 for counter games with winning conditions in $Bool^+(\mathbb{B})$ is in PSPACE. Hence, the result
 563 follows because, as for Theorem 14, these strategies suffice for Eve. ◀

564 6 Extensions of counter games with max operation

565 In this section, we consider counter games where the players can, in addition, put into a
 566 counter the maximum value of a subset of counters. In other words, max-counter games are
 567 defined in the same exact way as counter games, the only difference being *counter operations*
 568 are now mappings from a finite subset C of \mathcal{C} to $\{i, r, \text{skip}\} \cup \{\max(c) \mid S \subseteq C\}$.

569 ▶ **Theorem 16.** *Let \mathcal{G} be the class of counter games G with counter condition $\bigwedge_{c \in C} \mathbb{B}(c)$,
 570 where C is the set of counters of G . Given a game G in \mathcal{G} , the problem of deciding whether
 571 Eve wins G is EXPTIME- c . Finite memory is sufficient for Eve and Adam.*

572 **Proof.** For hardness, we reduce the emptiness problem of the intersection of n deterministic
 573 top-down tree automata, which is known to be EXPTIME-hard [15]. We first show PSPACE-
 574 hardness if the case of arenas where Adam plays no role, i.e., $V_\forall = \emptyset$. The proof is by
 575 reduction from the emptiness problem of the intersection of n DFA. The latter reduction
 576 is inspired from the proof that deterministic min-automata have PSPACE- c emptiness
 577 problem [7]. Using the fact that strategies are trees, we lift the latter reduction to tree
 578 automata. The detailed proof is in Appendix, in Lemma 18.

579 It remains to show that solving a game in \mathcal{G} can be done in exponential time. The
 580 difficulty for solving a game G of \mathcal{G} comes from the fact that counters interact with each
 581 other, since the value of counters can "flow" from one to another via the max operation. That
 582 was not case for $\bigwedge \mathbb{B}$ -counter games without max, which are CoNP- c , and we could track
 583 each counter separately, replacing each boundedness condition by a condition of the form
 584 "if c is incremented infinitely often, then it is reset infinitely often". Here, we need to track
 585 sequences of counters that flow one into another, called traces. We rather solve games with
 586 the complementary objective, which is correct since max-counter games are determined (see
 587 Lemma 19 in Appendix). We define a (non-deterministic) automaton \mathcal{B} with a single counter
 588 d that guesses either a new trace, or a valid continuation to the current trace, at every move
 589 of a play of \mathcal{G} . Every operation on the counters of the trace are mimicked on d , and it accepts
 590 a play iff there exists a run such that d is unbounded. That same idea is already used in
 591 the proof of Theorem 1 of [3], from which this proof is inspired. So, solving G boils down
 592 to solving a game on the same arena but with objective given by the language $L(\mathcal{B})$. To
 593 solve the latter, we convert \mathcal{B} into a non-deterministic parity automaton \mathcal{T} , which does not
 594 preserve the language, but preserves the existence of winning strategy for Eve: when playing
 595 on the arena of G , Eve wins the objective $\mathcal{L}(\mathcal{B})$ if and only if she wins the objective $\mathcal{L}(\mathcal{T})$.
 596 Correctness is ensured by a pumping-like argument based on the fact that finite-memory
 597 strategies are sufficient to win ω -regular games, an argument very similar to the one used in
 598 the proof of Lemma 3. The automata \mathcal{B} and \mathcal{T} are constructed in ptime from G . Then we
 599 determinize \mathcal{T} in exponential time, take its product with G , and obtain a classical parity
 600 game of exponential size and linear index. We can conclude since parity games with m edges,
 601 n vertices and index k can be solved in $O(mn^k)$ (see e.g. [13]). The detailed proof is in the
 602 Appendix, in Lemma 16. ◀



603 **7** Future work

604 In this paper, we have proved new complexity results for counter games and important classes
605 of boundedness conditions. Beyond the objective of having an exhaustive complexity table, we
606 believe that considering those different classes advance our understanding of those games, as
607 the techniques required to solve conjunctions and disjunctions are different. By determinacy,
608 those results also yield complexity bounds for the complementary classes of *unboundedness*
609 objectives. For example, we get that games with conjunctions of objectives of the form
610 $\mathbb{U}(c)$ can be solved in $\text{NP} \cap \text{CoNP}$ and that infinite memory is required. However, note that
611 our counter games are always taken in conjunction with a parity condition. Therefore, in
612 the complementary objectives, this parity condition is now taken in disjunction. We leave
613 conjunction of parity and unboundedness objectives as future work. Another important
614 direction is to consider classes of conditions that mix boundedness and unboundedness
615 objectives. Since the techniques used to solve them individually are different, this would
616 require new techniques. More generally, the only known upper bound for any Boolean
617 combination (not necessarily negation-free) of boundedness objective is non-elementary. We
618 believe there is space for improvement.

619 — References —

- 620 1 Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann. Graph games and
621 reactive synthesis. In Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and
622 Roderick Bloem, editors, *Handbook of Model Checking*, pages 921–962. Springer, 2018.
623 doi:10.1007/978-3-319-10575-8_27.
- 624 2 Mikolaj Bojanczyk. Weak MSO with the unbounding quantifier. *Theory Comput. Syst.*,
625 48(3):554–576, 2011. doi:10.1007/s00224-010-9279-2.
- 626 3 Mikolaj Bojanczyk. Weak MSO with the unbounding quantifier. *Theory Comput. Syst.*,
627 48(3):554–576, 2011.
- 628 4 Mikolaj Bojanczyk. Weak MSO+U with path quantifiers over infinite trees. In Javier Esparza,
629 Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and*
630 *Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11,*
631 *2014, Proceedings, Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 38–49.
632 Springer, 2014. doi:10.1007/978-3-662-43951-7_4.
- 633 5 Mikolaj Bojanczyk and Thomas Colcombet. Bounds in w-regularity. In *21th IEEE Symposium*
634 *on Logic in Computer Science (LICS 2006), 12-15 August 2006, Seattle, WA, USA, Proceedings*,
635 pages 285–296. IEEE Computer Society, 2006. doi:10.1109/LICS.2006.17.
- 636 6 Mikolaj Bojanczyk, Tomasz Gogacz, Henryk Michalewski, and Michal Skrzypczak. On
637 the decidability of MSO+U on infinite trees. In Javier Esparza, Pierre Fraigniaud, Thore
638 Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and Programming - 41st*
639 *International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings,*
640 *Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 50–61. Springer, 2014.
641 doi:10.1007/978-3-662-43951-7_5.
- 642 7 Mikolaj Bojanczyk and Szymon Torunczyk. Deterministic automata and extensions of
643 weak MSO. In Ravi Kannan and K. Narayan Kumar, editors, *IARCS Annual Conference*
644 *on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2009,*
645 *December 15-17, 2009, IIT Kanpur, India*, volume 4 of *LIPICs*, pages 73–84. Schloss Dagstuhl
646 - Leibniz-Zentrum für Informatik, 2009. doi:10.4230/LIPICs.FSTTCS.2009.2308.
- 647 8 Véronique Bruyère. Computer aided synthesis: A game-theoretic approach. In Émilie
648 Charlier, Julien Leroy, and Michel Rigo, editors, *Developments in Language Theory - 21st*
649 *International Conference, DLT 2017, Liège, Belgium, August 7-11, 2017, Proceedings*, volume
650 10396 of *Lecture Notes in Computer Science*, pages 3–35. Springer, 2017. doi:10.1007/
651 978-3-319-62809-7_1.



- 652 9 Cristian S. Calude, Sanjay Jain, Bakhadyr Khoussainov, Wei Li, and Frank Stephan. Deciding
653 parity games in quasipolynomial time. In *STOC*, pages 252–263, 2017.
- 654 10 Krishnendu Chatterjee and Nathanaël Fijalkow. Infinite-state games with finitary conditions.
655 In Simona Ronchi Della Rocca, editor, *Computer Science Logic 2013 (CSL 2013)*, *CSL 2013*,
656 *September 2-5, 2013, Torino, Italy*, volume 23 of *LIPICs*, pages 181–196. Schloss Dagstuhl -
657 Leibniz-Zentrum für Informatik, 2013. doi:10.4230/LIPICs.CSL.2013.181.
- 658 11 Krishnendu Chatterjee, Thomas A. Henzinger, and Florian Horn. Finitary winning in omega-
659 regular games. *ACM Trans. Comput. Log.*, 11(1):1:1–1:27, 2009. doi:10.1145/1614431.
660 1614432.
- 661 12 Krishnendu Chatterjee, Thomas A. Henzinger, and Florian Horn. The complexity of request-
662 response games. In *LATA*, volume 6638, pages 227–237, 2011.
- 663 13 Krishnendu Chatterjee, Thomas A. Henzinger, and Nir Piterman. Generalized parity games.
664 In Helmut Seidl, editor, *Foundations of Software Science and Computational Structures, 10th*
665 *International Conference, FOSSACS 2007, Held as Part of the Joint European Conferences*
666 *on Theory and Practice of Software, ETAPS 2007, Braga, Portugal, March 24-April 1, 2007*,
667 *Proceedings*, volume 4423 of *Lecture Notes in Computer Science*, pages 153–167. Springer,
668 2007. URL: https://doi.org/10.1007/978-3-540-71389-0_12.
- 669 14 Thomas Colcombet and Christof Löding. The non-deterministic mostowski hierarchy and
670 distance-parity automata. In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M.
671 Halldórsson, Anna Ingólfssdóttir, and Igor Walukiewicz, editors, *Automata, Languages and*
672 *Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11,*
673 *2008, Proceedings, Part II - Track B: Logic, Semantics, and Theory of Programming & Track*
674 *C: Security and Cryptography Foundations*, volume 5126 of *Lecture Notes in Computer Science*,
675 pages 398–409. Springer, 2008.
- 676 15 Hubert Comon-Lundh, Max Dauchet, Rémi Gilleron, Cristof Löding, Florent Jacquemard,
677 Denis Lugiez, Sophie Tison, and Marc Tommasi. *Tree Automata Techniques and Applications*.
678 November 2007. URL: <http://www.grappa.univ-lille3.fr/tata/>.
- 679 16 E. Allen Emerson and Charanjit S. Jutla. Tree automata, mu-calculus and determinacy
680 (extended abstract). In *32nd Annual Symposium on Foundations of Computer Science, San*
681 *Juan, Puerto Rico, 1-4 October 1991*, pages 368–377. IEEE Computer Society, 1991.
- 682 17 E. Allen Emerson and Charanjit S. Jutla. The complexity of tree automata and logics of
683 programs. *SIAM J. Comput.*, 29(1):132–158, 1999.
- 684 18 Léo Exibard, Emmanuel Filiot, and Ayrat Khalimov. Church synthesis on register automata
685 over linearly ordered data domains. In Markus Bläser and Benjamin Monmege, editors,
686 *38th International Symposium on Theoretical Aspects of Computer Science, STACS 2021,*
687 *March 16-19, 2021, Saarbrücken, Germany (Virtual Conference)*, volume 187 of *LIPICs*, pages
688 28:1–28:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.
- 689 19 Nathanaël Fijalkow and Martin Zimmermann. Cost-parity and cost-streett games. In Deepak
690 D’Souza, Telikepalli Kavitha, and Jaikumar Radhakrishnan, editors, *IARCS Annual Conference*
691 *on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2012,*
692 *December 15-17, 2012, Hyderabad, India*, volume 18 of *LIPICs*, pages 124–135. Schloss Dagstuhl
693 - Leibniz-Zentrum für Informatik, 2012. URL: [https://doi.org/10.4230/LIPICs.FSTTCS.](https://doi.org/10.4230/LIPICs.FSTTCS.2012.124)
694 2012.124.
- 695 20 E. Grädel, W. Thomas, and T. Wilke. Automata, logics, and infinite games. a guide to current
696 research. volume 2500. LNCS, Springer, 2002.
- 697 21 Orna Kupferman, Nir Piterman, and Moshe Y. Vardi. Pushdown specifications. In *LPAR*,
698 volume 2514, pages 262–277, 2002.
- 699 22 Donald A. Martin. Borel determinacy. *Annals of Mathematics*, 102(2):363–371, 1975.
- 700 23 Nir Piterman and Amir Pnueli. Faster solutions of rabin and streett games. In *LICS*, pages
701 275–284, 2006.
- 702 24 Martin Zimmermann. Delay games with WMSO+U winning conditions. *RAIRO Theor.*
703 *Informatics Appl.*, 50(2):145–165, 2016. doi:10.1051/ita/2016018.



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705 **A Detailed proofs of section 4**

706 ► **Lemma 10** (Memory transfer). *Let \mathcal{A} be a counter arena, V be its set of vertices, and \mathcal{W}*
 707 *a finite set of prefix-independent winning conditions for \mathcal{A} . Let $N \in \mathbb{N}$ and suppose that for*
 708 *all $X \subseteq V$, $u \in V$ and $W \in \mathcal{W}$, strategies of memory size at most N suffice for Eve to win*
 709 *$(\mathcal{A}[u], W \cup \text{Reach}(X))$. Then for all $u \in S^{\mathcal{W}}$, Eve wins $(\mathcal{A}[u], \bigvee \mathcal{W})$ with memory at most N .*

710 **Proof.** Let us recall that the strategies $\beta_{u,i}$ constructed in the proof of Lemma 9 (implication
 711 $3 \Rightarrow 2$) are inductively defined as follows: Eve follows $\sigma_{u,i}$ as long as $S_{i-1}^{\mathcal{W}}$ is not reached. As
 712 soon as she reach a vertex $v \in S_{i-1}^{\mathcal{W}}$, she applies $\beta_{v,i-1}$. We let $\beta = \beta_{u,|V|}$ for some $u \in S^{\mathcal{W}}$,
 713 which is a finitely switching strategy, and winning in $(\mathcal{A}[u], \bigvee \mathcal{W})$ according to Lemma 9.
 714 Note that by construction of the family of strategies $\beta_{u,i}$, in any arbitrary vertex v Eve has
 715 reached by playing β , she is always playing according to some local strategy $\sigma_{w,j}$ for some w
 716 and j . It means that Eve has to remember w and j , but this costs $|V|^2$ states. Instead, we
 717 show a different way of constructing a finitely switching strategy γ in $(\mathcal{A}[u], \bigvee \mathcal{W})$ for any
 718 $u \in S^{\mathcal{W}}$ as follows. Let us fix $u \in S^{\mathcal{W}}$ in the rest of the proof.

719 We take an arbitrary linear order $<_V$ on V . The memory of γ is $\{1, \dots, N\}$, just as the
 720 finite-memory strategies $(\sigma_{v,i})_{v,i}$. For all v, i , we let $(\delta_{v,i}, g_{v,i})$ be the memory mapping of
 721 $\sigma_{v,i}$. Remind that $\delta_{v,i}$ updates the states while $g_{v,i}$ defines the moves. We define γ as follows,
 722 with the invariant that all states γ reaches are in $S^{\mathcal{W}}$. Let i be the smallest integer such that
 723 $u \in S_i^{\mathcal{W}} \setminus S_{i-1}^{\mathcal{W}}$. It exists since $u \in S^{\mathcal{W}}$. The initial state of γ is the initial state m_0 of $\sigma_{u,i}$.

724 Now, let $v \in V_{\exists} \cap S^{\mathcal{W}}$ and m a memory state of γ . We define how γ plays when it is in v
 725 with memory state m . We let i_v be the smallest integer such that $v \in S_{i_v}^{\mathcal{W}} \setminus S_{i_v-1}^{\mathcal{W}}$. We let u_v
 726 the smallest vertex such that

- 727 1. $u_v \in S_{i_v}^{\mathcal{W}} \setminus S_{i_v-1}^{\mathcal{W}}$
- 728 2. there exists a finite path h from u_v to v consistent with σ_{u_v, i_v} such that $\delta_{u_v, i_v}^*(q_0, h) = m$,
 729 where $\delta_{u_v, i_v}^*(h)$ is the memory state of σ_{u_v, i_v} after h , starting in its initial state q_0 at
 730 vertex u_v .

731 Then, γ at vertex v in memory state m plays $g_{u_v, i_v}(v, m)$ and moves to state $\delta_{u_v, i_v}(v, m)$.

732 Note that γ is well-defined: by construction, all the vertices v it reaches have been reached
 733 following some strategy $\sigma_{v,i}$ and hence, there exists always a path h satisfying the conditions
 734 of point 2 above, for such vertices v .

735 We prove that γ wins $(\mathcal{A}[u], \bigvee \mathcal{W})$. Take a play $\pi = v_1 v_2 v_3 \dots$ consistent with γ . Let
 736 $m_0 m_1 m_2 \dots$ the sequence of states of γ on π . First, note that by definition of γ , there exists
 737 n such that all the vertices v visited by π after at least n steps all belong to the same part
 738 $S_{i^*}^{\mathcal{W}} \setminus S_{i^*-1}^{\mathcal{W}}$ for some i^* . It means that for all $k \geq n$, $i_{v_k} = i_{v_{k+1}} = i^*$. To simplify notations,
 739 for all $k \geq n$, let $i_k = i_{v_k}$ and $u_k = u_{v_k}$. Consider the sequence $(u_k)_{k \geq n}$. By definition of
 740 γ , it is decreasing for $<_V$. Indeed, let $k \geq n$. Then, there exists a finite path h_k from u_k
 741 to v_k consistent with σ_{u_k, i_k} . By definition of γ , v_{k+1} is defined by σ_{u_k, i_k} at state v_k and
 742 memory m_k . Therefore, $h_k v_{k+1}$ is consistent with σ_{u_k, i_k} , so, u_k meets the requirements of
 743 point 2 above. Since we want a minimal node, we obtain that $u_{k+1} \leq_V u_k$. Since there are
 744 finitely many vertices, eventually, the sequence $(u_k)_{k \geq n}$ stabilizes on the same vertex u^* , i.e.,
 745 there exists l such that for all $k \geq l$, $u_k = u^*$. From that point on, γ always applies strategy
 746 σ_{u^*, i^*} . Formally, it means that the infinite prefix $v_l v_{l+1} \dots$ is consistent with σ_{u^*, i^*} . We
 747 can conclude since σ_{u^*, i^*} is winning in $(\mathcal{A}[u^*], W \cup \text{Reach}(S_{i^*-1}^{\mathcal{W}}))$ for some $W \in \mathcal{W}$, and since
 748 π never visits $S_{i^*-1}^{\mathcal{W}}$ by definition of i^* , $v_j v_{j+1} \dots \in W$, and since W is prefix-independent,
 749 $\pi \in W$. ◀



750 ► **Lemma 11** (Completeness for boundedness conditions in DNF). *Let \mathcal{A} be a counter arena and*
751 *C its set of counters. Let \mathcal{W} be a finite subset of counter conditions for \mathcal{A} in $\text{Parity}(\mathcal{A}) \wedge \bigwedge \mathbb{B}$.*
752 *If Eve wins $(\mathcal{A}, \bigvee \mathcal{W})$, then she has a finitely switching strategy.*

753 **Proof.** Let C_1, \dots, C_p be subsets of C such that \mathcal{W} is the set of all counter conditions
754 $\bigwedge_{c \in C_i} \mathbb{B}(c)$, for $i \in \{1, \dots, p\}$. Suppose that Eve does not have a finitely switching strategy
755 from the initial vertex v_0 . This means, by Lemma 9, that $v_0 \notin S^{\mathcal{W}}$. We construct a
756 winning strategy for Adam for the complementary objective $\text{Comp} = \bigcap_{i \in \{1, \dots, p\}} \bigcup_{c \in C_i} \mathbb{U}(c) \cup$
757 $\overline{\text{Parity}(\mathcal{A})}$. By definition of $S^{\mathcal{W}}$, $f(S^{\mathcal{W}}) = S^{\mathcal{W}}$. Therefore, for any $v \in V \setminus S^{\mathcal{W}}$ and $i \in$
758 $\{1, \dots, p\}$, Eve does not win the counter game $(\mathcal{A}[v], \text{Parity}(\mathcal{A}) \cap (\bigcap_{c \in C_i} \mathbb{B}(c) \cup \text{Reach}(S^{\mathcal{W}})))$.
759 Thus, by determinacy (Lemma 2), Adam has a winning strategy $\sigma_{v,i}$ in $\mathcal{A}[v]$ for the
760 complementary objective $\bigcup_{c \in C_i} \mathbb{U}(c) \cap \text{Reach}(S^{\mathcal{W}}) \cup \overline{\text{Parity}(\mathcal{A}[v])}$. We now define a well-suited
761 decomposition of every history and play. To that aim, we define a successor function on
762 every element of the set $(\mathbb{N} \setminus \{0\}) \times \{1, \dots, p\}$. For any positive integer i and any integer
763 $j \in \{1, \dots, p\}$, we thus define $s(i, j)$ as $(i, j + 1)$ if $j < p$, and as $(i + 1, 1)$ if $j = p$. We define
764 the *counting decomposition* of a history or a play w as the sequence $(w_i)_{i \in E}$ that satisfies
765 the following properties:

- 766 ■ E is either equal to $\{1, \dots, n\}$ for some $n \in \mathbb{N}$ or equal to $\mathbb{N} \setminus \{0\}$,
- 767 ■ $w = w_1 \cdots w_n$,
- 768 ■ for any integer $i \in E$,
- 769 ■ $w \neq w_1 \cdots w_{i-1}$,
- 770 ■ and, if we let (k, l) denote $s^{i-1}(1, 1)$, h_i is equal to the longest non-empty prefix
771 h' of $(h_1 \cdots h_{i-1})^{-1}h$ such that, for any non-negative m with $m \leq |h'|$, we have
772 $\lambda(\zeta(h'), c)_m < k$ for any $c \in C_l$ (h' can be an infinite word).

773 Notice that the counting decomposition $(h_i)_{i \in \{1, \dots, n\}}$ of any history h is indeed unique,
774 since every h_i is unique by induction on i .

775 Using the above definitions, we formally define a winning strategy σ' of Adam in the
776 following way.

777 For any history h , if we let $(h_i)_{i \in \{1, \dots, n\}}$ denote the counting decomposition of h , v denote
778 the first letter of h_n , and (k, l) denote $s^{n-1}(1, 1)$, we define $\sigma'(h)$ as $\sigma_{v,l}(h_n)$. We show that
779 σ' is a winning strategy for Adam for \overline{G} (game arena of G with objective the complement
780 of G 's objective). Let ρ be a play consistent with σ' , and let $(h_i)_{i \in E}$ denote the counting
781 decomposition of ρ . Let us suppose that there exists $n \in \mathbb{N}$ such that $E = \{1, \dots, n\}$. By
782 definition of σ' , if we let v denote the first letter of h_n and (k, l) denote $s^{n-1}(1, 1)$, h_n is
783 consistent with $\sigma_{v,l}$. However, since $\rho = h_1 \cdots h_n$, by definition of the counting decomposition,
784 for any $m \in \mathbb{N}$, we have $\lambda(\zeta(h_n), c)_m < k$ for any $c \in C_l$. Since $\sigma_{v,l}$ is a winning strategy
785 of Adam for \overline{G}_l , $h_n \notin \text{Parity}(\mathcal{A})$, and as a consequence, $\rho \notin \text{Parity}(\mathcal{A})$ either. Thus, if
786 $\rho \in \text{Parity}(\mathcal{A})$, we have $E = \mathbb{N}$.

787 Furthermore, by definition of the counting decomposition, for any $i \in \mathbb{N}$, if we let
788 (k, l) denote $s^{i-1}(1, 1)$ and v' denote the first letter of h_{i+1} , there exists $c \in C_l$ such that
789 $\lambda(\zeta(h_0 \cdots h_i v'), c)_{|h_i|+1} = k$. Thus, since every C_l is finite, for any $l \in \{1, \dots, p\}$, there
790 exists $c \in C_l$ such that, for any integer k , there exists an integer j with $\lambda(\zeta(\rho), c)_j \geq k$. To
791 conclude, σ' is a winning strategy of Adam for \overline{G} . ◀

792 **B Detailed proofs of section 5**

793 ► **Lemma 12.** *Let $\mathcal{C} \in \{\text{PTIME}, \text{NP}, \text{coNP}, \text{EXPTIME}\}$. Let \mathcal{G} be a class of games with*
794 *prefix-independent objectives, such that deciding whether, given $(\mathcal{A}, W) \in \mathcal{G}$, a vertex v of \mathcal{A} ,*



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795 and a subset X of vertices of \mathcal{A} , Eve wins $(\mathcal{A}[v], W \cup \text{Reach}(X))$, is in \mathcal{C} . Then, deciding, given
 796 an arena \mathcal{A} and a finite subset of winning conditions \mathcal{W} such that $\{(\mathcal{A}, W) \mid W \in \mathcal{W}\} \subseteq \mathcal{G}$,
 797 whether Eve has a winning finitely switching strategy for $(\mathcal{A}, \bigvee \mathcal{W})$, is in \mathcal{C} .

798 **Proof.** Suppose first that $\mathcal{C} = \text{PTIME}$. From Lemma 9, Eve has a winning finitely switching
 799 strategy for $(\mathcal{A}, \bigvee \mathcal{W})$ if and only if the initial vertex v_0 of \mathcal{A} is in $S^{\mathcal{W}}$. Thus, we can decide
 800 whether Eve has a finitely switching strategy by recursively computing the $S_i^{\mathcal{W}}$, one after
 801 the other, until $S_i^{\mathcal{W}} = S_{i+1}^{\mathcal{W}} = S^{\mathcal{W}}$. In order to compute $S_{i+1}^{\mathcal{W}}$ from $S_i^{\mathcal{W}}$, we check for every
 802 vertex v of \mathcal{A} whether Eve wins the game $(\mathcal{A}[v], W \cup \text{Reach}(S_i^{\mathcal{W}}))$. Thus, since $S_{|V|}^{\mathcal{W}} = S^{\mathcal{W}}$,
 803 in order to compute $S^{\mathcal{W}}$, we only need to check whether Eve wins a game of the form
 804 $(\mathcal{A}[v], W \cup \text{Reach}(X))$ at most $|V| \times |V| \times |\mathcal{W}|$ times. As a consequence, the problem of
 805 deciding whether Eve has a winning finitely switching strategy for $(\mathcal{A}, \bigvee \mathcal{W})$ is in PTIME .
 806 In order to better explain the case where $\mathcal{C} = \text{NP}$, we give the full algorithm $\text{SOLVE}(\mathcal{A}, \mathcal{W})$
 807 in Figure 4. In that figure, slv is an algorithm that terminates in polynomial time, and such
 808 that $\text{slv}(\mathcal{A}, v, W, H)$ returns true if and only if Eve wins $(\mathcal{A}[v], W \cup \text{Reach}(H))$. The case
 809 where $\mathcal{C} = \text{EXPTIME}$ is similar.

810 Suppose now that $\mathcal{C} = \text{NP}$. Intuitively, we can use a very similar algorithm to the one
 811 we used in the previous case: the only difference is that we guess the certificates needed
 812 to check if Eve wins the games of the form $(\mathcal{A}[v], W \cup \text{Reach}(X))$. This approach works
 813 because the algorithm returns **True** if and only if the answers to some well-chosen questions
 814 of the type "Does Eve win $(\mathcal{A}[v], W \cup \text{Reach}(X))$?" are true. In order to formalize this idea,
 815 for any arena \mathcal{A} , and any set \mathcal{W} such that $\{(\mathcal{A}, W) \mid W \in \mathcal{W}\} \subseteq \mathcal{G}$, we let $\text{size}(\mathcal{A}, \mathcal{W})$
 816 denote the size of a coding of the arena \mathcal{A} and of the set of winning conditions \mathcal{W} . There
 817 exists an algorithm vrfE such that $\text{vrfE}(\mathcal{A}, v, W, H, w)$ verifies in polynomial time if Eve wins
 818 $(\mathcal{A}[v], W \cup \text{Reach}(S_i^{\mathcal{W}}))$ using the certificate w . More precisely, vrfE terminates in polynomial
 819 time, and there exists a polynomial P , such that for any game $(\mathcal{A}, w) \in \mathcal{G}$, the following is
 820 true: there exists w of length $P(\text{size}(\mathcal{A}, \mathcal{W}))$ such that $\text{vrfE}(H, W, \mathcal{A}, v, w)$ returns **True**
 821 and only if Eve wins $(\mathcal{A}[v], W \cup \text{Reach}(H))$. An algorithm VERIF_EVE that checks whether
 822 Eve has a winning finitely switching strategy for $(\mathcal{A}, \bigvee \mathcal{W})$, given a certificate w of size
 823 $P(\text{size}(\mathcal{A}, \mathcal{W})) \times |V|^2 \times |\mathcal{W}|$ (where V is the set of vertices of \mathcal{A}), $\text{VERIF_EVE}(\mathcal{A}, \mathcal{W}, w)$, is
 824 precisely described in Figure 5. In the algorithm, for any integer k and any sequence of words
 825 $(l_i)_{i \in \{1, \dots, k\}}$, we use the notation $\prod_{i=1, \dots, k} l_i$ for $l_1 \cdots l_n$. It is easy to see that, if the initial
 826 vertex v of \mathcal{A} is in $S^{\mathcal{W}}$, then there exists a certificate w of size $P(\text{size}(\mathcal{A}, \mathcal{W})) \times |V|^2 \times |\mathcal{W}|$
 827 such that $\text{VERIF}(\mathcal{A}, \mathcal{W}, w)$ returns **True**. The converse directly comes the fact that, for any
 828 certificate w , during a run of $\text{VERIF_EVE}(\mathcal{A}, \mathcal{W}, w)$, we always have $H_{\alpha+1} \subseteq f(H_\alpha)$, and
 829 that we thus have $H_n \subseteq S^{\mathcal{W}}$.

830 Suppose now that $\mathcal{C} = \text{CONP}$. We show that the problem of deciding whether Eve does
 831 not win $(\mathcal{A}, \bigvee \mathcal{W})$ is in NP , using the same idea as in the previous case, that is by computing
 832 $S^{\mathcal{W}}$ in a non-deterministic way, with an algorithm that returns **True** if and only if the answers
 833 to some well-chosen questions (in fact, to all of the questions in this case) of the type "Does
 834 Eve win $(\mathcal{A}[v], W \cup \text{Reach}(X))$?" are false. However, in this case, we compute $S^{\mathcal{W}}$ by guessing
 835 its complement. There exists an algorithm vrfA such that $\text{vrfA}(\mathcal{A}, v, W, H, w)$ verifies in
 836 polynomial time if Eve does not win $(\mathcal{A}[v], W \cup \text{Reach}(S_i^{\mathcal{W}}))$ using the certificate w . More
 837 precisely, vrfA terminates in polynomial time, and there exists a polynomial P , such that for
 838 any game $(\mathcal{A}, w) \in \mathcal{G}$, the following is true: there exists w of length $P(\text{size}(\mathcal{A}, \mathcal{W}))$ such that
 839 $\text{vrfA}(H, W, \mathcal{A}, v, w)$ returns **True** if and only if Eve does not win $(\mathcal{A}[v], W \cup \text{Reach}(H))$. An
 840 algorithm VERIF_ADAM that checks whether Eve does not have winning finitely switching
 841 strategy for $(\mathcal{A}, \bigvee \mathcal{W})$, given a certificate w of size $|V| + P(\text{size}(\mathcal{A}, \mathcal{W})) \times |V| \times |\mathcal{W}|$ (where



```

SOLVE( $\mathcal{A}, \mathcal{W}$ )
//  $v$  is the initial vertex of  $\mathcal{A}$ 
//  $V = \{v_1, \dots, v_n\}$ 
//  $\mathcal{W} = \{W_1, \dots, W_p\}$ 

```

1. $N \leftarrow n^2 \times p$
2. $\alpha \leftarrow 0$
3. $H_0, H_1, \dots, H_N \leftarrow \emptyset$
4. While $\alpha < n$
5. For $i = 1, \dots, n$
6. For $j = 1, \dots, p$
7. If $\text{slv}(\mathcal{A}, v_i, W_j, H_\alpha)$
8. $H_{\alpha+1} \leftarrow \{v_i\} \cup H_{\alpha+1}$
9. $\alpha \leftarrow \alpha + 1$
10. Return $v \in H_\alpha$

■ **Figure 4** An algorithm for case $\mathcal{C} = \text{PTIME}$

```

VERIF_EVE( $\mathcal{A}, \mathcal{W}, w$ )
//  $v$  is the initial vertex of  $\mathcal{A}$ 
//  $V = \{v_1, \dots, v_n\}$ 
//  $\mathcal{W} = \{W_1, \dots, W_p\}$ 
//  $w = \prod_{\alpha=1, \dots, n} \prod_{i=1, \dots, n} \prod_{j=1, \dots, p} w_{i,j}^\alpha$ ,
// where each  $w_{i,j}^\alpha$  is of length  $P(\text{size}(\mathcal{A}, \mathcal{W}))$ .

```

1. $N \leftarrow n^2 \times p$
2. $\alpha \leftarrow 0$
3. $H_0, H_1, \dots, H_N \leftarrow \emptyset$
4. While $\alpha < n$
5. For $i = 1, \dots, n$
6. For $j = 1, \dots, p$
7. If $\text{vrfE}(\mathcal{A}, v_i, W_j, H_\alpha, w_{i,j}^\alpha)$
8. $H_{\alpha+1} \leftarrow \{v_i\} \cup H_{\alpha+1}$
9. $\alpha \leftarrow \alpha + 1$
10. Return $v \in H_\alpha$

■ **Figure 5** An algorithm in the case $\mathcal{C} = \text{NP}$

842 V is the set of vertices of \mathcal{A}), $\text{VERIF_ADAM}(\mathcal{A}, \mathcal{W}, w)$, is precisely described in Figure 6.
843 For any certificate w , if $\text{VERIF_ADAM}(\mathcal{A}, \mathcal{W}, w)$ returns **True**, if we let S' denote the value
844 of the set S at the beginning of line 4 of a run of $\text{VERIF_ADAM}(\mathcal{A}, \mathcal{W}, w)$, we always have,
845 for any $u \in S'$ and for any $W \in \mathcal{W}$, Eve does not win $(\mathcal{A}[u], W \cup \text{Reach}(V \setminus S'))$. Suppose
846 now that $S' \cap S^{\mathcal{W}} \neq \emptyset$, and let k be the smallest integer such that $S_k^{\mathcal{W}} = S_{k+1}^{\mathcal{W}}$. We show by
847 induction on i that, for any $i \in \{0, \dots, k\}$, $S' \cap S_{k-i}^{\mathcal{W}} \neq \emptyset$. Indeed, the property is obvious for
848 $k = 0$ since $S_k^{\mathcal{W}} = S^{\mathcal{W}}$. Furthermore, if there exists u such that $u \in S' \cap S_{k-i}^{\mathcal{W}}$, then Eve wins
849 $(\mathcal{A}[u], W \cup \text{Reach}(S_{k-i-1}^{\mathcal{W}}))$, but Eve does not win $(\mathcal{A}[u], W \cup \text{Reach}(V \setminus S'))$. As a consequence,
850 $S_{k-i-1}^{\mathcal{W}}$ is not a subset of $V \setminus S'$, and thus $S_{k-i-1}^{\mathcal{W}} \cap S' \neq \emptyset$. Therefore, $S_0^{\mathcal{W}} \cap S' \neq \emptyset$, which
851 is impossible since $S_0^{\mathcal{W}} = \emptyset$. Thus, we have $S' \subseteq V \setminus S^{\mathcal{W}}$. As a consequence, if there exists
852 a certificate w such that $\text{VERIF_ADAM}(\mathcal{A}, \mathcal{W}, w)$ returns **True**, then Eve does not have
853 a winning finitely switching strategy for $(\mathcal{A}, \bigvee \mathcal{W})$. The converse is straightforward. To
854 conclude, the problem of deciding whether Eve does not win $(\mathcal{A}, \bigvee \mathcal{W})$ is in NP.

856 The following lemma assesses the complexity of games of the form $(\mathcal{A}[v], W \cup \text{Reach}(X))$.
857 We show that, when W comes from a counter condition, we can reduce games of the form
858 $(\mathcal{A}[v], W \cup \text{Reach}(X))$ to a game (\mathcal{A}', W) in polynomial time.

859 ► **Lemma 17.** *Let \mathcal{A} be a counter arena, V be its set of vertices, v be a vertex in V ,
860 X be a subset of V , and W be a counter condition for \mathcal{A} . We can construct a counter
861 arena \mathcal{A}' in polynomial time with respect to $|V|$, such that Eve wins $(\mathcal{A}[v], (\text{Plays}(\mathcal{A}, W) \cap$
862 $\text{Parity}(\mathcal{A})) \cup \text{Reach}(X))$ if and only if Eve wins (\mathcal{A}', W) . Furthermore, if strategies with
863 memory size at most N suffice for (\mathcal{A}', W) , then strategies with memory size at most N
864 suffice for $(\mathcal{A}[v], (\text{Plays}(\mathcal{A}, W) \cap \text{Parity}(\mathcal{A})) \cup \text{Reach}(X))$.*

865 **Proof.** The idea is to construct an arena \mathcal{A}' that comes from \mathcal{A} by replacing all the elements of



```

VERIF_ADAM( $\mathcal{A}, \mathcal{W}, w$ )

//  $V = \{v_1, \dots, v_n\}$ 
//  $\mathcal{W} = \{W_1, \dots, W_p\}$ 
//  $v$  is the initial vertex of  $\mathcal{A}$ 
//  $w = w_1 w_2$ , with
//  $w_1 = \overrightarrow{u_1 \# u_2 \# \dots \# u_n}$ , and
//  $w_2 = \prod_{\alpha=1, \dots, n} \prod_{j=1, \dots, p} w_{i,j}$ ,
// where each  $w_{i,j}$  is of length  $P(\text{size}(\mathcal{A}, \mathcal{W}))$ 

1.  $S \leftarrow \emptyset$ 
2. For  $i = 1, \dots, n$ 
3.    $S = S \cup \{u_i\}$ 
4. For  $i = 1, \dots, n$ 
5.   For  $j = 1, \dots, p$ 
6.     If  $\text{vrfA}(\mathcal{A}, v_i, W_j, V \setminus S, w_{i,j})$  returns False
7.     Return False
8. Return True

```

■ **Figure 6** An algorithm in NP for the complement problem, for the case $\mathcal{C} = \text{CoNP}$

866 X by a single one, that we call -1 . When a play would reach X in \mathcal{A} , there is a corresponding
867 play in \mathcal{A}' with the same previous history, that reaches -1 , and that subsequently stays at
868 that vertex *ad. infinitum*. Furthermore, the labeling and colouring of the vertex -1 are
869 defined appropriately so that, if a play stays at the vertex $\{-1\}$, it is always winning for Eve.

870 Let C be the counter the set of counters of \mathcal{A} . Since W is a counter condition, it is a
871 boolean formula ϕ over the set of propositions $\{\mathbb{B}(c) \mid c \in C\}$. If $v \in V$, then any strategy of
872 Eve is winning for $(\mathcal{A}[v], (\text{Plays}(\mathcal{A}, W) \cap \text{Parity}(\mathcal{A})) \cup \text{Reach}(X))$. We thus suppose in the
873 following that $v \notin V$.

874 Suppose now that ϕ is not satisfiable, then $\text{Plays}(\mathcal{A}, W)$ is empty. As a consequence, Eve
875 wins $(\mathcal{A}[v], (\text{Plays}(\mathcal{A}, W) \cap \text{Parity}(\mathcal{A})) \cup \text{Reach}(X))$ if and only if Eve wins $(\mathcal{A}[v], \text{Reach}(X))$.
876 Since deciding reachability in two-player games can done in polynomial time, and since
877 memoryless strategies are sufficient for Eve and for Adam [20], the theorem follows.

878 We can thus suppose that ϕ is satisfiable. Then ϕ is implied by some condition of the form
879 $\bigwedge_{c \in C_1} \mathbb{B}(c) \wedge \bigwedge_{c \in C_2} \mathbb{U}(c)$, with $C_1, C_2 \subseteq C$. Let $(V, E, V_\exists, V_\forall, u)$ be the underlying two-player
880 arena of \mathcal{A} , let ζ be the vertex labeling of \mathcal{A} , and κ be its vertex colouring. We can suppose
881 without loss of generality that -1 is not in V . We let V' denote $X \cup \{-1\}$, and E' denote
882 the union of the following sets:

- 883 ■ the set $(E \cap ((V \setminus X) \times (V \setminus X)))$
- 884 ■ the set of all pair of vertices $(u, -1)$ such that there exists $t \in X$ such that $(u, t) \in E$
- 885 ■ the singleton $\{-1, -1\}$.

Furthermore, we let V'_\exists denote the set $V_\exists \cup \{-1\}$, Q' denote $Q \cup \{0\}$, κ' denote the vertex
colouring equal to κ on X , and such that $\kappa(-1) = 0$, and ζ' denote the vertex labeling equal
to ζ on X and such that

$$\zeta_c(-1) = \begin{cases} r & \text{if } c \in C_1 \\ i & \text{if } c \in C_2. \end{cases}$$



886 Finally, we let \mathcal{A}' denote the counter arena with underlying two-player arena equal to
 887 $(V', E', V'_\exists, V'_\forall \cap X, v)$, set of counters C , vertex labeling ζ' , set of colours Q' and colouring
 888 κ' .

889 The lemma is direct consequence of the fact that, if σ and σ' are two strategies of Eve for
 890 the game $(\mathcal{A}[v], (\text{Plays}(\mathcal{A}, W) \cap \text{Parity}(\mathcal{A})) \cup \text{Reach}(X))$ and for the counter game (\mathcal{A}', W)
 891 respectively that are equal on $(V \setminus X)^\omega$, then σ is winning if and only if σ' is winning.
 892 ◀

893 **C** Detailed proofs of section 6

894 **► Theorem 16.** *Let \mathcal{G} be the class of counter games G with counter condition $\bigwedge_{c \in C} \mathbb{B}(c)$,
 895 where C is the set of counters of G . Given a game G in \mathcal{G} , the problem of deciding whether
 896 Eve wins G is EXPTIME- c . Finite memory is sufficient for Eve and Adam.*

897 The proof of Theorem 16 is split into two parts, each covered by a different lemma.
 898 Lemma 18 gives the EXPTIME-hardness, and Lemma 20 gives the EXPTIME-easiness.

899 **► Lemma 18.** *Max-counter games with a single winning condition $\mathbb{B}(c)$ for some counter c ,
 900 and no parity condition, are EXPTIME-hard.*

901 **Proof of Lemma 18.** We prove EXPTIME-hardness of max-counter games with no parity
 902 condition and a conjunction of boundedness conditions $\bigwedge_{c \in C} \mathbb{B}(c)$. This entails the result
 903 because one can always add a counter c_m which takes the maximal value of all other counters
 904 $c \in C$ at each step, so that $\bigwedge_{c \in C} \mathbb{B}(c)$ is satisfied iff $\mathbb{B}(c_m)$ is satisfied.

905 To prove the theorem for conjunctions of boundedness conditions, we reduce the problem,
 906 called $\bigcap_n \text{DTOP}$, of deciding if the intersection of n languages recognized by deterministic
 907 top-down tree automata (DTOP) is empty, which is known to be EXPTIME- c [21]. Before
 908 giving the EXPTIME-hardness proof, we first prove PSPACE-hardness for the particular
 909 class of counter games where $V_\forall = \emptyset$, i.e., where Adam plays no role. We reduce the problem
 910 of deciding if the intersection of n languages recognized by deterministic finite-automata
 911 (DFA) is empty. We call the latter problem $\bigcap_n \text{DFA}$. The proof is inspired by a PSPACE-
 912 hardness proof of deciding non-emptiness of the language recognized by a deterministic
 913 min-automaton [7]. Then we lift the reduction from $\bigcap_n \text{DFA}$ to the problem $\bigcap_n \text{DTOP}$,
 914 i.e., to trees, by using the branching nature of counter games induced by Adam.

915 Consider an alphabet Σ and n complete DFA $D_i = (\Sigma, Q_i, q_0^i, F_i, \delta_i)$ such that all Q_i are
 916 pairwise disjoint. We construct a counter arena $\mathcal{A}[D_1, \dots, D_n]$ with $V_\forall = \emptyset$ and a set C of
 917 $n + 1$ counters, and no parity condition, such that Eve has a strategy to satisfy objective
 918 $\bigwedge_{c \in C} \mathbb{B}(c)$ iff $\bigcap_i L(D_i) \neq \emptyset$. This construction is similar to that of [7], which is a reduction
 919 from the universality problem for NFA. We assume that Σ contains a symbol $\# \in \Sigma$ and
 920 for all i , $L(D_i) \subseteq (\Sigma - \#)^* \#$. The counter arena $\mathcal{A}[D_1, \dots, D_n]$ is defined by $V_\exists = \Sigma$ and
 921 $V_\forall = \emptyset$, and the set of transitions is $E = V_\exists \times V_\exists$. The vertex $\#$ is initial. The set of counters
 922 is $C = \{c_0\} \cup \{c_q \mid q \in Q_i, i = 1, \dots, n\}$, and they are updated as follows for $i = 1, \dots, n$,
 923 where $\max(\emptyset) = 0$:

- 924 on vertex $f \neq \#$: for all $q \in Q_i$, $c_q := \max\{c_{q'} + 1 \mid \exists q' \in Q_i, \delta(q', f) = q\}$ and $c_0 := c_0 + 1$
- 925 on vertex $\#$: $c_{q_0^i} := \max\{c_q \mid q \in Q_{i'} \text{ for some } i' \text{ and } \delta_{i'}(q, \#) \notin F_{i'}\}$, and the counters
 926 c_q for all $q \in Q_i \setminus \{q_0^i\}$ are reset, as well as c_0 .

927 Note that for $f \neq \#$, two operations are performed at once: increment counters $c_{q'}$ and take
 928 the max. This is done to simplify the presentation and can be simulated by doubling the
 929 number of vertices of the arena.



930 Now, observe that $\text{Plays}(\mathcal{A}[D_1, \dots, D_n]) = \#\Sigma^\omega$ and a strategy for Eve is nothing but
 931 an infinite word w in $\#\Sigma^\omega$. We prove the following claims:

932 **Claim 1** For all non-empty finite set $X \subseteq \bigcap_{i=1}^n L(D_i)$, any play in $\#.X^\omega$ satisfies $\bigwedge_{c \in C} \mathbb{B}(c)$.

933

934 **Claim 2** No play in $\#.\left(\bigcup_{i=1}^n ((\Sigma - \#)^* \#) \setminus L(D_i)\right)^\omega$ satisfies $\bigwedge_{c \in C} \mathbb{B}(c)$.

935 **Proof of Claim 1.** Let $m = \max\{|u| \mid u \in X\}$. Let $w = \#u_1u_2\dots$ such that for all $j \geq 1$,
 936 $u_j \in X$. We prove that w , which is a play of $\mathcal{A}[D_1, \dots, D_n]$ satisfies that all the counters
 937 are bounded by $2m$. First, note that each u_j is of the form $v_j\#$, because $u_j \in \bigcap_i L(D_i)$ and
 938 the DFA D_i are assumed to accept words where $\#$ is an endmarker. First, consider counter
 939 c_0 : it is reset every time $\#$ is read, so, its maximal value is bounded by m . Now, for all $j \geq 1$
 940 and $q \in \bigcup_i Q_i$, we let $in_{j,q}$ be the value of counter c_q after prefix $\#u_1\dots u_{j-1}$ and $out_{j,q}$ is
 941 value after prefix $\#u_1\dots u_{j-1}v_j$. By definition of the counter updates, we have:

- 942 1. $in_{j,q} = 0$ for all $j \geq 1$ and q not initial
- 943 2. $in_{j,q_0^i} = \max\{out_{j-1,q} \mid q \in Q_{i'}$ for some i' and $\delta_{i'}(q, \#) \notin F_{i'}\}$ for all $j \geq 1$
- 944 3. $out_{j,q} = in_{j,q_0^i} + |v_j|$ if $q \in Q_i$ for some i and there exists a run of D_i from q_0^i to q on v_j
- 945 4. otherwise, $out_{j,q} = |r|$ where r is a run of maximal length on a prefix of v_j , ending in q .

946 Let $q \in Q_i$ for some i such that $\delta_i(q, \#) \notin F_i$. For all $j \geq 1$, there is no run from q_0^i to q
 947 on v_j , since $u_j = v_j\# \in L(D_i)$. So, we are in case 4 above and we have $out_{j,q} \leq |v_j| \leq m$.
 948 From the latter fact and 2, we get that $in_{j,q_0^i} \leq m$ for all i, j . From that and 3, we get that
 949 $out_{j,q} \leq m + |v_j| \leq 2m$ for all j . So, all the counter have value at most $2m$ after each v_j ,
 950 which concludes the proof that they are bounded. ◀

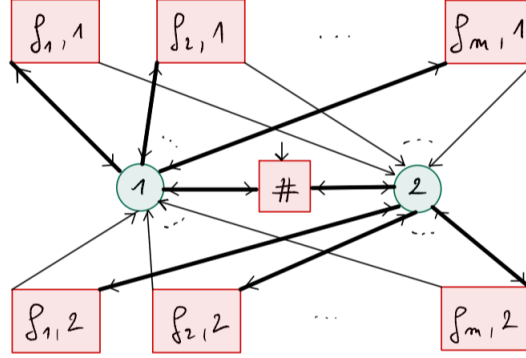
951 **Proof of Claim 2.** Let w be a play of $\mathcal{A}[D_1, \dots, D_n]$ in $\#.\left(\bigcup_{i=1}^n ((\Sigma - \#)^* \#) \setminus L(D_i)\right)^\omega$. Then,
 952 $w = \#w_1\#w_2\#w_3\#\dots$ such that $w_j \in (\Sigma - \#)^*$ for all $j \geq 1$. Moreover, for all $j \geq 1$, there
 953 exists $i_j \in \{1, \dots, n\}$ such that $w_j\# \notin L(D_{i_j})$ and there exists a run of D_{i_j} on w_j from $q_0^{i_j}$
 954 to some non-accepting state q_{i_j} . Denote by $in(i_j)$ the value of counter $c_{q_{i_j}^0}$ before reading
 955 $w_j\#w_{j+1}\dots$ in w , and by $out(i_j)$ the value of counter $c_{q_{i_j}}$ before reading $\#w_{j+1}\#w_{j+1}\dots$ in
 956 w . By definition of the counter updates, we have $out(i_1) \geq in(i_1) + |u|$, $out(i_2) \geq in(i_2) + |u|$,
 957 and so on. Moreover, $in(i_2) \geq out(i_1)$, $in(i_3) \geq out(i_2)$, and so on, since the states q_{i_j}
 958 are non-accepting. This yields that the sequence $(in(i_j))_j$ is unbounded, concluding the
 959 proof. ◀

960 As a side note, observe that the two claims imply the following: $\bigcap_{i=1}^n L(D_i) \neq \emptyset$ iff there
 961 exists a word $w \in \#\Sigma^\omega$ which satisfies $\bigwedge_{c \in C} \mathbb{B}(c)$. Indeed, if there exists $u \in \bigcap_{i=1}^n L(D_i)$,
 962 then it suffices to apply Claim 1 to $X = \{u\}$. Conversely, if $\bigcap_{i=1}^n L(D_i) = \emptyset$, then
 963 $\left(\bigcup_{i=1}^n (\Sigma^* \setminus L(D_i))\right)^\omega = \Sigma^\omega$ and Claim 2 implies that no word of Σ^ω satisfy $\bigwedge_{c \in C} \mathbb{B}(c)$.

964 We now lift the latter reduction to (binary) trees. We let Σ be a finite alphabet containing
 965 a symbol $\#$ called a constant symbol, and all other symbols are called *binary* symbols. We
 966 let $\Sigma_2 = \Sigma - \#$ be the set of binary symbols. A Σ -tree is defined as a term where terms t
 967 are inductively defined by $t, t_1, t_2 ::= \# \mid f(t_1, t_2), f \in \Sigma_2$. The set of branches of a Σ -tree t
 968 is inductively defined as $br(\#) = \{\#\}$, and $br(f(t_1, t_2)) = \{(f, d).b \mid d \in \{1, 2\}, b \in br(t_d)\}$.

A *deterministic top-down tree automaton* is a tuple $\mathcal{T} = (Q, q_0, F, \delta)$ where Q is a finite set
 of states, $q_0 \in Q$ the initial state, $F \subseteq Q$ the final states, and $\delta : Q \times (\{\#\} \cup (\Sigma_2 \times \{1, 2\})) \rightarrow Q$
 is a (total) transition function. We see \mathcal{T} as a DFA $DFA(\mathcal{T})$ recognizing a language
 in $(\Sigma_2 \times \{1, 2\})^* \#$ naturally as follows: $DFA(\mathcal{T}) = (Q, q_0, F, \delta')$ where for all $q \in Q$,
 for all $(f, d) \in \Sigma_2 \times \{1, 2\}$, $\delta'(q, f) = \text{proj}_d(\delta(q, f))$, with proj_d the d th projection, and





■ **Figure 7** Arena for the proof of Theorem 18, where $\Sigma = \{\#, f_1, \dots, f_m\}$. Transitions in bold are in both directions. Square vertices are controlled by Adam, and the initial vertex is $\#$. When Adam picks a direction $d \in \{1, 2\}$, then Eve is forced to pick a vertex in $\Sigma_2 \times \{d\}$, or $\#$.

$\delta'(q, \#) = \delta(q, \#)$, and we denote by $L_{br}(\mathcal{T})$ the language recognized by this DFA. The language of Σ -trees accepted by \mathcal{T} is the set

$$L(\mathcal{T}) = \{t \in Trees_\Sigma \mid br(t) \subseteq L_{br}(\mathcal{T})\}$$

969 Deciding³, given n DTOP $\mathcal{T}_1, \dots, \mathcal{T}_n$, whether $\bigcap_{i=1}^n L(\mathcal{T}_i) = \emptyset$ is EXPTIME-c [15].

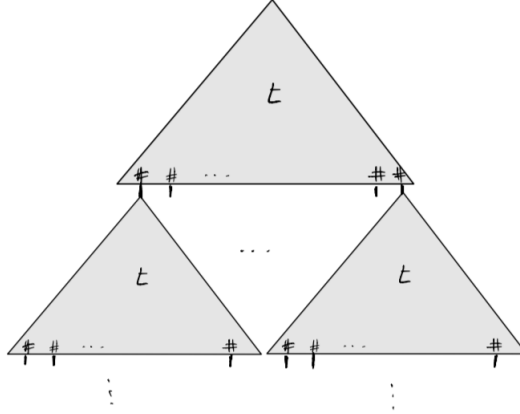
970 Given $\mathcal{T}_1, \dots, \mathcal{T}_n$ such that $\mathcal{T}_i = (Q_i, q_0^i, F_i, \delta_i)$ for all i , we construct a max-counter game
 971 G winnable by Eve iff $\bigcap_{i=1}^n L(\mathcal{T}_i) \neq \emptyset$. The main idea of the proof is construct a game where
 972 Adam picks a direction $d \in \{1, 2\}$ (1 means left and 2 right), while Eve picks the labels in Σ .
 973 The arena $\mathcal{A}[\mathcal{T}_1, \dots, \mathcal{T}_n]$ of G (without the counters) is depicted on Fig. 7.

974 We now define counter conditions which make sure that if Eve has a strategy to keep
 975 all the counters bounded iff there exists $t \in \bigcap_i L(\mathcal{T}_i)$. For all i , let $\mathcal{T}_i = (Q_i, q_0^i, F_i, \delta_i)$.
 976 The set of counters is $C = \{c_q \mid q \in \bigcup_i Q_i\} \cup \{c_0\}$ (we assume wlog that all the sets Q_i
 977 are pairwise disjoint). Let us define counter updates. They are defined as for the arena
 978 $\mathcal{A}[DFA(\mathcal{T}_1), \dots, DFA(\mathcal{T}_n)]$. To simplify the presentation (and in particular the structure
 979 of the arena), we perform several operations at once. Let us define the updates, for all
 980 $1 \leq i \leq n$:

- 981 ■ on vertex $(f, j) \in \Sigma_2 \times \{1, 2\}$: for all $q_j \in Q_i$, $c_{q_j} := \max\{c_q + 1 \mid \exists q, q_{3-j} \in Q_i, \delta(q, f) =$
 982 $(q_1, q_2)\}$ and $c_0 := c_0 + 1$
- 983 ■ on vertex $\#$: $c_{q_0^i} := \max\{c_q \mid q \in Q_{i'} \text{ for some } i' \text{ and } \delta_{i'}(q, \lambda) \notin F_{i'}\}$, and the counters
 984 c_q for all $q \in Q_i \setminus \{q_0^i\}$ are reset, as well as c_0 .
- 985 ■ on vertices $i \in \{1, 2\}$: counters are unchanged.

986 There is no parity condition and the counter condition is that the counters in C must
 987 be bounded. Let G be the constructed max-counter game. Before showing correctness, let
 988 us introduce some useful notation. Note that the histories and plays of G are elements of
 989 $\{\#\} \cup \Sigma_2 \times \{1, 2\}$ alternating with directions in $\{1, 2\}$. The following function removes the

³ In [15], the definition of DTOP is slightly different, but less general: there are no accepting states but the transition function can be partial. A tree is accepted if there is a run on it which traverses the whole tree (it is not in an inner node). Those automata can easily be encoded into (our) DTOP by completing the transition function into a sink state q_s , declaring all states to be final but q_s .



■ **Figure 8** Strategy σ_t constructed from a Σ -tree t .

990 intermediate directions. Given $w = \lambda_1 d_1 \lambda_2 d_2 \dots \lambda_n d_n$ such that for all i , $\lambda_i \in \{\#\} \cup \Sigma_2 \times \{1, 2\}$
 991 and $d_i \in \{1, 2\}$, we let $\text{lab}(w) = \lambda_1 \lambda_2 \dots \lambda_n$.

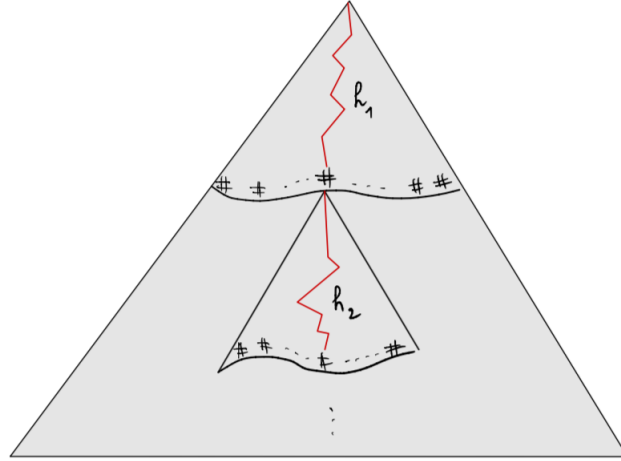
992 We now show correctness of the reduction. Suppose that there exists some $t \in \bigcap_i L(\mathcal{T}_i)$.
 993 We first define a strategy σ_t for Eve and then show it is winning in G . The strategy σ_t just
 994 mimics t : it plays as t dictates when a leaf of t is reached, its behaviour is reset to the root of
 995 t . It is illustrated on Fig.8. Formally, the construction of σ_t satisfies the following invariant:
 996 all histories ending with an Eve vertex are words of the form $h = \#h_1 h_2 \dots h_k p d$ where:

- 997 ■ all h_i are such that $\text{lab}(h_i) \in \text{br}(t)$,
- 998 ■ $\text{lab}(p)$ is a prefix of a branch of t
- 999 ■ $d \in \{1, 2\}$

1000 Given such a history h , we consider two cases: if $\text{lab}(p) \in \text{br}(t)$, then σ_t is reset to the root of t ,
 1001 which means that $\sigma_t(h) = (f, d)$ such that f is the label of the root of t . Otherwise, $\sigma_t(h) = \#$
 1002 if $\text{lab}(p)\# \in \text{br}(t)$, and $\sigma_t(h) = (f, d)$ if $\text{lab}(p)(f, d) \in \text{br}(t)$. Let us show that σ_t is winning.
 1003 Let $\pi \in \text{Plays}(\sigma_t)$. First, we observe that $\text{lab}(\pi)$ is a play of $\mathcal{A}[\text{DFA}(\mathcal{T}_1), \dots, \text{DFA}(\mathcal{T}_n)]$.
 1004 Let C' be the set of counters of the latter arena. By definition of σ_t , $\text{lab}(\pi)$ is of the form
 1005 $\#b_1\#b_2\#\dots$ with infinitely many $\#$ such that for all $j \geq 1$, $b_j\#$ is a branch of t . Since
 1006 $t \in \bigcap_i L(\mathcal{T}_i)$, we also get that $b_j\# \in \bigcap_i L(\text{DFA}(\mathcal{T}_i))$. The set $X = \{b_j\# \mid j \geq 1\}$ is
 1007 finite since its elements correspond to branches of t . Therefore, by Claim 1, π satisfies
 1008 $\bigwedge_{c \in C'} \mathbb{B}(c)$. We conclude by observing that $C = C'$, that $\mathcal{A}[\mathcal{T}_1, \dots, \mathcal{T}_n]$ has the same vertices
 1009 as $\mathcal{A}[\text{DFA}(\mathcal{T}_1), \dots, \text{DFA}(\mathcal{T}_n)]$ plus the two vertices 1 and 2, with the same counter updates
 1010 for their common vertices and no update on 1 and 2. Therefore, π satisfies $\bigwedge_{c \in C} \mathbb{B}(c)$ in
 1011 $\mathcal{A}[\mathcal{T}_1, \dots, \mathcal{T}_n]$.

1012 Conversely, suppose that $\bigcap_i L(\mathcal{T}_i) = \emptyset$. Take an arbitrary strategy σ of Eve. We show
 1013 it is not winning. Intuitively, σ can be seen as an infinite tree. If there is a branch of the
 1014 tree which visits $\#$ finitely many times, then σ is not winning because by following the
 1015 directions corresponding to that branch, Adam can guarantee that counter c_0 is unbounded.
 1016 So, we can assume that σ is such that all plays consistent with it sees infinitely many $\#$. We
 1017 construct a play π of the form $\#h_1\#h_2\#\dots$ such that for all $j \geq 1$, there exists i such that
 1018 $\text{lab}(h_j)\# \notin L(\text{DFA}(\mathcal{T}_i))$, and we conclude by Claim 2. The construction of π is illustrated
 1019 on Fig.9.

1020 Consider the set of histories H_1 of σ which contains a $\#$ symbol only at their end.
 1021 Clearly, H_1 can be identified with a Σ -tree t_1 . Since $t_1 \notin \bigcap_i L(\mathcal{T}_i)$, there exists i such



1022 **Figure 9** Losing play $\pi = \#h_1\#h_2\#\dots$ constructed from a strategy σ , seen as a tree.

1022 that $t_1 \notin L(\mathcal{T}_i)$ and therefore, a history $h_1\# \in H_1$ such that $\text{lab}(h_1)\# \notin L(\text{DFA}(\mathcal{T}_i))$. To
 1023 construct h_2, h_3, \dots , we proceed similarly. Let us explain how to construct h_2 . We let H_2 be
 1024 the set of histories of the form $h_1\#g_2\#$ such that $h_1\#g_2\#$ is a history of σ such that g_2 does
 1025 not contain $\#$. The set $(h_1\#)^{-1}H_2$ can be identified with a Σ -tree t_2 . Now, it suffices to
 1026 take $h_2\# \in (h_1\#)^{-1}H_2$ such that $\text{lab}(h_2)\# \notin L(\text{DFA}(\mathcal{T}_i))$ for some $i = 1, \dots, n$. It exists
 1027 since $t_2 \notin \bigcap_i L(\mathcal{T}_i)$. This concludes the proof. \blacktriangleleft

1028 In order to prove Lemma 20, we first prove the following, in a very similar way to the
 1029 proof of Lemma 2.

1030 **► Lemma 19.** *Max-counter games (with Boolean combinations of boundedness objectives)*
 1031 *are determined.*

Proof. Given a counter arena \mathcal{A} and a counter c of \mathcal{A} , the set $\text{Plays}(\mathcal{A}, \mathbb{B}(c))$ is a Borel set. Indeed, it is equal to the countable union for all $N \geq 0$ of the sets

$$\text{Plays}_N(\mathcal{A}, \mathbb{B}(c)) = \{\rho \in \text{Plays}(\mathcal{A}) \mid \forall n \in \mathbb{N}, \lambda(\zeta(\rho), c)_n \leq N\}$$

1032 which are ω -regular. Indeed, a Büchi automaton that stores, in every state, the maximums
 1033 between N and the value of each counter of C needs $|V| \times N^{|C|}$ states to recognize
 1034 $\text{Plays}_N(\mathcal{A}, \mathbb{B}(c))$. Since ω -regular sets are Borel, so is $\text{Plays}(\mathcal{A}, \mathbb{B}(c))$, as well as any Boolean
 1035 combination of the latter. By Martin's determinacy theorem [22], the result follows. \blacktriangleleft

1036 **► Lemma 20.** *Given a game in \mathcal{G} , the problem of deciding whether Eve wins \mathcal{G} is in*
 1037 *EXPTIME. Finite memory is sufficient for Eve and Adam.*

1038 **Proof.** We show that counter games G with condition of the form $\text{Plays}\left(\mathcal{A}, \bigvee_{c \in C} \mathbb{U}(c)\right) \cup$
 1039 $\text{Parity}(\mathcal{A})$, where C is the set of counters of G , and \mathcal{A} its underlying two-player arena, can
 1040 be solved in EXPTIME, which implies the theorem by Lemma 19.

1041 We construct, from a max-counter game G , a parity game G' , of exponential size and
 1042 linear index, such that Eve wins G' if and only if Eve wins G . To simplify the presentation,
 1043 we introduce an intermediate automata model, called \mathbb{UB} -automata. A (non-deterministic)
 1044 \mathbb{UB} -automaton \mathcal{B} is a pair (\mathcal{A}, W) , where

- 1045 ■ \mathcal{A} is a counter arena without max where Adam plays no role ($V_{\forall} = \emptyset$), where each edge is
1046 labeled by a letter of an alphabet Σ , *i.e.* the set of edges of the counter game is replaced
1047 by a subset E of $V \times V \times \Sigma$, and where the domain of the colouring κ is Σ rather than
1048 V (we colour the edges rather than colouring the vertices)
1049 ■ W is a winning condition for \mathcal{A} , *i.e.* a subset of V^{ω} .

1050 A *run* in \mathcal{B} is an infinite word $\pi = y_0 y_1 \cdots \in Y^{\omega}$ such that the first element of y_0 is the
1051 initial vertex of \mathcal{B} , and such that the second element of each y_i is the first element of y_{i+1}
1052 for any non-negative integer i . We let $\text{Play}(\pi)$ denote the word $v_0 v_1 \cdots$, where each v_i is
1053 the first element of y_i , and we let $\text{Input}(\pi)$ denote the word $z_0 z_1 \cdots$, where each z_i is the
1054 third element of y_i (*i.e.* the label of the edge y_i). A word w is *accepted* by \mathcal{B} if $\text{Input}(\pi)$ is
1055 in $\text{Parity}(\kappa)$, or if there exists a run π of \mathcal{B} such that $\text{Input}(\pi) = w$, and such that $\text{Play}(\pi)$
1056 is winning in \mathcal{B} (*i.e.* satisfies the winning condition of \mathcal{B}). The language accepted by \mathcal{B} is
1057 the set of accepted words.

1058 Let G be a counter game with underlying two-player arena $\mathcal{A} = (V, E, V_{\exists}, V_{\forall}, v)$, vertex
1059 labeling ζ , set of colors Q , colouring κ , and winning condition $\text{Plays}(\mathcal{A}, \bigvee_{c \in C} \mathbb{U}(c)) \cup$
1060 $\text{Parity}(\mathcal{A})$. We construct a \mathbb{UB} -automaton \mathcal{B} , of size polynomial in $|C|$, with only one counter
1061 $d \notin C$, that recognizes the language of all words $w \in V^{\omega}$ such that either $w \in \text{Parity}(\kappa)$, or
1062 $\zeta(w)$ satisfies the condition $\bigvee_{c \in C} \mathbb{U}(c)$. To make the construction more easily understood, we
1063 first introduce the notion of trace. A *trace* of a word $w = z_0 z_1 \cdots \in \mathbf{Op}(C)^{\omega}$ is a mapping θ
1064 from $\{i, \dots, j\}$ to C , where $i \leq j$ are two integers, such that, for any $l \in \{i, \dots, j - 1\}$,

- 1065 ■ either $\theta(l) = \theta(l + 1)$ and $z_l(\theta(l)) \in \{i, r, \text{skip}\}$,
1066 ■ or $\theta(l + 1) \neq \theta(l)$ and $z_l(\theta(l + 1)) = \max_{c \in S}(c)$ with $S \subseteq C$ and $\theta(l) \in S$.

1067 The *value* of θ at move $t \in \{i, \dots, j\}$ is defined inductively as 0 if $t = i$, one plus the value at
1068 move $t - 1$ if $z_{t-1}(\theta(t - 1)) = i$, 0 if $z_{t-1}(\theta(t - 1)) = r$, and the value at move $t - 1$ otherwise.
1069 If a counter c reaches a value $N \geq 1$ at some point in w , then it is always possible to "track
1070 back", with a trace of w , the sequence of counter operations which led to c having that value,
1071 by choosing, every time we go back to a previous counter operation of the type $c' = \max_{d \in S}(d)$
1072 with $S \subseteq C$, the good counter d of S (the one with the maximum value), until reaching a
1073 counter whose value is 0. Thus, there exists a counter $c \in C$ and two integers t and N such
1074 that $\lambda(w, c)_t = N$ if and only if there exists a trace θ of w , such $\theta(t) = c$, and such that
1075 the value of θ at move t is N . As a consequence, there exists counter c such that $\lambda(w, c)$
1076 is unbounded if and only if the values of the traces of w are unbounded.

1077 This result allows us to define \mathcal{B} in the following way. The \mathbb{UB} -automaton \mathcal{B} works, on
1078 input w , by guessing all the possible traces of $\zeta(w)$, by using non-determinism. The value of
1079 a trace is stored inside the counter d . More precisely, every time \mathcal{B} reads a letter, it either
1080 guesses a new trace, or guesses the next counter c' of C of the trace it is following, while
1081 applying, if c' is equal to the current counter c of the trace, the operation over c induced by
1082 the letter read, to counter d . Thus, the \mathbb{UB} -automaton \mathcal{B} is constructed so that the value of d
1083 is unbounded if and only if there are traces of its input of arbitrarily large values. Moreover,
1084 we set the colouring of \mathcal{B} as κ . Thus, \mathcal{B} recognizes the language of all words $w \in V^{\omega}$ such that
1085 either $w \in \text{Parity}(\kappa)$, or $\zeta(w)$ satisfies the condition $\bigvee_{c \in C} \mathbb{U}(c)$, *i.e.* the language recognized
1086 by \mathcal{B} is the winning condition of the game G . The precise definition of \mathcal{B} is given below.

1087 We let $V_1 = C \times \{i, r, \text{skip}\}$, and $v_1 = (c, r)$ where c is any counter in C . Furthermore, we
1088 let ζ^1 denote the mapping from V_1 to $\mathbf{Op}(\{d\})$ such that $(\zeta^1(c, \alpha))(d) = \alpha$, and E_1 denote



1089 the the smallest subset of $V_1 \times V_1 \times V$ such that, for any $\alpha \in \{i, r, \text{skip}\}$ and any $v \in V$, we
 1090 have

- 1091 ■ for any $c, c' \in C$, $((c, \alpha), (c', r), v) \in E_1$ (this comes from the fact that \mathcal{B} should be able
 1092 to guess a new trace at any time),
- 1093 ■ for any $c \in C$, if $\zeta_c(v) \in \{i, \text{skip}\}$, $((c, \alpha), (c, \zeta_c(v)), v) \in E_1$ (the trace follows the increment
 1094 or skip operation of a counter while updating d),
- 1095 ■ for any $c \in C$, if $\zeta_c(v) = \max_{c' \in S}(c)$, then $((c', \alpha), (c, \text{skip}), v) \in E_1$, for any $c' \in S$ (the trace
 1096 changes counters on a max operation while leaving d unchanged).

1097 The \mathbb{UB} -automaton \mathcal{B} is the \mathbb{UB} -automaton with set of vertexes V_1 , set of edges E_1 , initial
 1098 vertex v_1 , set of counters $\{d\}$, vertex labeling ζ_1 , set of colors Q , colouring κ , and its winning
 1099 condition is defined as the set of all words $w \in V_1^\omega$ that satisfy $\mathbb{U}(d)$.

1100 We now transform \mathcal{B} into a parity automaton. To achieve that goal, we first transform \mathcal{B}
 1101 into an automaton with two colourings and without counters. To simplify the presentation of
 1102 that construction, we introduce the notion of dual parity automaton. A (non-deterministic)
 1103 *dual parity automaton* is a two-player game \mathcal{T} where Adam plays no role, with an alphabet
 1104 Σ , a set of colours P' in addition to the original set of colours P , a colouring η' over Σ in
 1105 addition to the original colouring η over the set X of vertices of G , and where every edge is
 1106 labeled by Σ (*i.e.* the set of edges of \mathcal{T} is a subset of $X \times X \times \Sigma$). Furthermore, a word
 1107 $w = b_0 b_1 \dots$ is *accepted* by \mathcal{T} if and only if either $b_0 b_1 \dots$ is in $\text{Parity}(\eta')$, or there exists a
 1108 run $(v_0, v_1, b_0)(v_1, v_2, b_1) \dots$ of \mathcal{T} such that $v_0 v_1 \dots$ is in $\text{Parity}(\eta)$. The language recognized
 1109 by \mathcal{T} , denoted $\mathcal{L}(\mathcal{T})$, is the set of words accepted by \mathcal{T} . Notice that \mathcal{T} can be converted into
 1110 a non-deterministic automaton \mathcal{T}_1 , with two colours whose domains are the set of vertexes, by
 1111 copying each vertex for every edge that goes to it. The acceptance condition of \mathcal{T}_1 is expressed
 1112 by the union of the parity conditions induced by its two colourings. That automaton \mathcal{T}_1
 1113 can be further converted into a non-deterministic parity automaton \mathcal{T}_2 with a single colour
 1114 and two initial states, by duplicating it, colouring the first copy with the first colouring and
 1115 the second copy with the second colouring. Thus, there exists a parity automaton \mathcal{T}_2 that
 1116 recognizes the same language as \mathcal{T} , and $\mathcal{L}(\mathcal{T})$ is thus an ω -regular language.

We now define \mathcal{T} as the dual parity automaton with set of vertices $V_1 = V \times \{i, r, \text{skip}\}$,
 alphabet V , edges E_1 , initial vertex v_1 , colouring κ from V to Q , and colouring κ' from V_1
 to $\{1, 2, 3\}$ such that

$$\kappa'(v, \alpha) = \begin{cases} 1 & \text{if } \alpha = \text{skip} \\ 2 & \text{if } \alpha = i \\ 3 & \text{if } \alpha = r \end{cases}$$

1117 Notice that \mathcal{T} and \mathcal{B} have the same edges with the same labels, and the same initial state
 1118 We let $\mathcal{L}(\mathcal{B})$ denote the language accepted by \mathcal{B} . Since $\mathcal{L}(\mathcal{T}) \subseteq \mathcal{L}(\mathcal{B})$, Eve wins $(\mathcal{A}, \mathcal{L}(\mathcal{B}))$ if
 1119 Eve wins $(\mathcal{A}, \mathcal{L}(\mathcal{T}))$. We now show the converse.

1120 Suppose that Eve has a winning strategy σ for $(\mathcal{A}, \mathcal{L}(\mathcal{B}))$, and that Eve does not win
 1121 $(\mathcal{A}, \mathcal{L}(\mathcal{T}))$. Since $\mathcal{L}(\mathcal{T})$ is an ω -regular objective, $(\mathcal{A}, \mathcal{L}(\mathcal{T}))$ is determined, and Adam has a
 1122 finite memory winning strategy τ with memory mapping pair (δ, g) and memory size N , for
 1123 $(\mathcal{A}, V^\omega \setminus \mathcal{L}(\mathcal{T}))$. Let ρ be a play of \mathcal{A} consistent with σ and τ . Then ρ satisfies both of the
 1124 following properties:

- 1125 (1) either $\rho \in \text{Parity}(\kappa)$, or there exists a run π in \mathcal{B} such that $\text{Input}(\pi) = \rho$ and $\text{Play}(\pi)$
 1126 satisfies $\mathbb{U}(d)$
- 1127 (2) $\rho \notin \text{Parity}(\kappa)$, and for any run π in \mathcal{T} such that $\text{Input}(\pi) = \rho$, the greatest colour
 1128 appearing infinitely often in $\text{Play}(\pi)$ is odd.



1129 If $\rho \in \text{Parity}(\kappa)$, then ρ cannot satisfy property (2). Thus, from property (1), there exists a
1130 counter $c \in C$, and a run π in \mathcal{B} such that $\text{input}(\pi) = \rho$ and $\text{play}(\pi)$ satisfies $\mathbb{U}(d)$. Therefore,
1131 if we let $\mu_0\mu_1 \cdots = \text{play}(\pi)$, there exists two integers i and j such that there is no occurrences
1132 of r in $\zeta_d(\mu_i \cdots \mu_j)$ but at least $N \times \#V \times \#V_1 + 1$ occurrences of i . As a consequence, there
1133 exists two integers i' and j' such that there is at least one occurrence of i in $\zeta_d(\mu_{i'} \cdots \mu_{j'})$ and
1134 no occurrences of r , such that $\mu_{i'} = \mu_{j'}$, such that the g -memory state of $\text{play}(\pi)$ at move i'
1135 equal to the g -memory state of $\text{play}(\pi)$ at move j' , and such that $\rho(i') = \rho(j')$. Thus, if we
1136 let π' denote the run $\pi(0) \cdots \pi(i' - 1)(\pi(i') \cdots \pi(j' - 1))^\omega$ (where $\pi(0)\pi(1) \cdots = \pi$), the play
1137 $\rho' = \rho(0) \cdots \rho(i' - 1)(\rho(i') \cdots \rho(j' - 1))^\omega$ is consistent with σ and τ , $\text{input}(\pi') = \rho'$, and the
1138 greatest colour occurring infinitely often in $\text{play}(\pi')$ is equal to 2, which is even. This directly
1139 contradicts property (2). Therefore, Eve wins $(\mathcal{A}, \mathcal{L}(\mathcal{B}))$ if and only if Eve wins $(\mathcal{A}, \mathcal{L}(\mathcal{T}))$.

1140 Eve thus wins G , if and only if Eve wins $(\mathcal{A}, \mathcal{L}(\mathcal{B}))$, if and only if Eve wins $(\mathcal{A}, \mathcal{L}(\mathcal{T}_2))$.
1141 We let k be the index of G , and n be the number of its vertices. Notice that \mathcal{T}_2 is a parity
1142 automaton with a polynomial size and index, with respect to nk . It is well-known that there
1143 is a deterministic parity automaton \mathcal{D} with an exponential size and polynomial index with
1144 respect to the product of the size and index of \mathcal{T}_2 , that recognizes the same language as \mathcal{T}_2 .
1145 Thus, \mathcal{D} has exponential size and polynomial index with respect to the size of G , and Eve
1146 wins G if and only if Eve wins $(\mathcal{A}, \mathcal{L}(\mathcal{D}))$.

1147 We now construct a game G' as the product of G and \mathcal{D} . More precisely, if $V_{\mathcal{D}}$ be the
1148 set of vertices of \mathcal{D} , $E_{\mathcal{D}}$ is its set of edges, $v_{\mathcal{D}}$ is its initial state, $Q_{\mathcal{D}}$ is its set of colours,
1149 and $\kappa_{\mathcal{D}}$ is its colouring, we let Z denote the set of all pairs $((u_1, v_1), (u_2, v_2))$ of elements of
1150 $V \times V_{\mathcal{D}}$ such that (u_1, u_2) is in E and (v_1, v_2, u_1) is in $E_{\mathcal{D}}$, and we let χ denote the mapping
1151 from $V \times V_{\mathcal{D}}$ to $Q_{\mathcal{D}}$ that maps (u, u') to $\kappa_{\mathcal{D}}(u')$. We define G' as the parity game with
1152 underlying two-player arena $(V \times V_{\mathcal{D}}, Y, V_{\exists} \times V_{\mathcal{D}}, V_{\forall} \times V_{\mathcal{D}}, (v, v_{\mathcal{D}}))$, set of colours $Q_{\mathcal{D}}$, and
1153 colouring χ . A play $(v_0, u_0)(v_1, u_1) \cdots$ is winning in G' if and only if $v_0v_1 \cdots$ is winning in
1154 G . Thus, Eve wins G if and only if wins G' . Furthermore, since finite memory strategies are
1155 sufficient for Eve and Adam for G' , finite memory strategies are also sufficient for Eve and
1156 Adam for G : the winning player of G only needs to simulate a winning play in G' with the
1157 help of a finite memory winning strategy for G' , by remembering at every move the exact
1158 vertex the corresponding play of G' would be in. In addition, it is well-known that solving a
1159 parity game with m edges, n vertices, and index $2k$ can be done in $O(m(n^k))$. Therefore,
1160 G' , can be solved in time $O((2^{P(n)})^{Q(n)})$, where P and Q are two polynomials. To conclude,
1161 max-counter games in \mathcal{G} can be solved in EXPTIME. \blacktriangleleft

