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Édouard Daviaud

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EXTRACTION OF OPTIMAL SUBSEQUENCES OF SEQUENCE OF BALLS, AND APPLICATION TO OPTIMALITY ESTIMATES OF MASS TRANSFERENCE PRINCIPLES

EDOUARD DAVIAUD

Abstract. In this article, we prove that from any sequence of balls whose associated limsup set has full μ-measure, one can extract a well-distributed subsequence of balls. From this, we deduce the optimality of various lower bounds for the Hausdorff dimension of limsup sets of balls obtained by mass transference principles. One also gives a version of Borel-Cantelli lemma suitable for limsup sets of balls of full measure.

1. Introduction

Investigating Hausdorff dimensions of sets of points approximable at certain "speed rate" by a given sequence of points $(x_n)_{n\in\mathbb{N}}$ is an important topic in Diophantine approximation (see [6] and [4] among other references), in dynamical systems [13, 19, 21] and in multifractal analysis [16, 1, 3]. These studies consists in general, knowing that $\mu(\limsup_{n\to+\infty} B_n := B(x_n, r_n)) = 1$ for a certain measure μ and a sequence of radius $(r_n)_{n\in\mathbb{N}}$, in investigating the Hausdorff dimension of $\limsup_{n\to+\infty} U_n$ where $U_n \subset B_n$. Typically U_n is a contracted ball inside B_n, but recently, general sets U_n have been considered [17, 23, 8].The so-called ubiquity theorems or mass transference principles mainly focus on finding a lower bound, using an adequate measure μ, for the Hausdorff dimension (or Hausdorff measure) of those sets. However it is key in many situations to understand whether this lower bound is optimal or not. This article is dedicated to this problem.

In order to do so, we introduce, given a probability measure μ on $\mathbb{R}^d$, the concept of μ-asymptotically covering sequence of balls. This notion is a generalization of a covering property used in the KGB Lemma stated in [6]. For a sequence $(B_n)_{n\in\mathbb{N}}$, verifying this condition will be proved to be almost equivalent to verify $\mu(\limsup_{n\to+\infty} B_n) = 1$ (it is equivalent if the measure is doubling for instance, so that working under this settings is very reasonable).

As said above, given a sequence of balls $(B_n)_{n\in\mathbb{N}}$ and another $(U_n \subset B_n)_{n\in\mathbb{N}}$, ubiquity theorems or mass transference principles give lower bounds for the Hausdorff dimension of $\limsup_{n\to+\infty} U_n$ when, roughly speaking, some information is known about the geometry of $\limsup_{n\to+\infty} B_n$. Of course, there is no reason in general for a lower-bound for $\dim_H(\limsup_{n\to+\infty} U_n)$ obtained only knowing that $\mu(\limsup_{n\to+\infty} B_n) = 1$, to be sharp (i.e $= \dim_H(\limsup_{n\to+\infty} U_n)$). If one hopes such a lower-bound to be accurate, the measure μ has to be particularly adapted, in some sense, to the pair of sequences $((B_n)_{n\in\mathbb{N}}, (U_n)_{n\in\mathbb{N}})$. The approach adopted in this article is to extract some sub-sequences from $(B_n)_{n\in\mathbb{N}}$ which are still μ-a.c (or still verifies $\mu(\limsup_{n\to+\infty} B_n) = 1$) but are adapted to the measure μ. Applying mass transference principles (which are proved only for measures presenting enough self-similarity) to those sub-sequences, it will be proved that Hausdorff

dimension of the limsup set associated with the corresponding U_n's is given by the lower-bounds found in [9] (so that it is also the case for the lower-bounds given in [8, 4, 23]).

This shows that those lower-bounds are sharp in a strong sens: for any self-similar measure μ, any μ-a.c sequence $(B_n)_{n\in\mathbb{N}}$, if one only considers the balls that are relevant for the measure μ, the limsup set obtained by considering the sub-sequence of the corresponding U_n's ($(U_n)_{n\in\mathbb{N}}$ being also the sequence of sets involved in the articles mentioned) has the expected dimension.

More precisely, it will be proven first that, under those very weak condition over a μ-a.c sequence of balls $(B_n)_{n\in\mathbb{N}}$, it is always possible to extract a sub-sequence $(B_{\phi(n)})_{n\in\mathbb{N}}$, still μ-ac, "weakly redundant" (see Definition 2.3) and such that the balls $(B_{\phi(n)})_{n\in\mathbb{N}}$ have prescribed behavior with respect to the measure μ, roughly meaning (see Theorem 2.3) that the balls $(B_{\phi(n)})_{n\in\mathbb{N}}$ satisfies

$$|B_{\phi(n)}|^{\overline{\dim}_P(\mu)} \lessapprox \mu(B_{\phi(n)}) \lessapprox |B_{\phi(n)}|^{\underline{\dim}_H(\mu)}. \tag{1}$$

In a second time, it will be proved that, for weakly redundant sequences satisfying (1), the Hausdorff dimension of lim sup set associated with any sequence of shrunk balls or very thin rectangles $(R_n \subset B_n)_{n\in\mathbb{N}}$ (see Theorem 2.10) can be bounded by above precisely by the lower-bound given in [9], which proves the optimality of those bounds.

2. Notation and definition

Let $d \in \mathbb{N}$.

For $n \in \mathbb{N}$, the set of dyadic cubes of generation n of $\mathbb{R}^d$ is denoted $\mathcal{D}_n(\mathbb{R}^d)$ and defined as $\mathcal{D}_n(\mathbb{R}^d) = \left\{\prod_{i=1}^d [\frac{k_i}{2^n}, \frac{k_i+1}{2^n})\right\}_{(k_i)_{i\in\mathbb{Z}}\in\mathbb{Z}^{\mathbb{Z}}}$.

For $x \in \mathbb{R}^d$, $r > 0$, $B(x,r)$ stands for the closed ball of $(\mathbb{R}^d, ||\ \ ||_\infty)$ of center x and radius r. Given a ball B, $|B|$ is the diameter of B.

For $t \geq 0$, $\delta \in \mathbb{R}$ and $B := B(x,r)$, tB stand for $B(x,tr)$, i.e. the ball with same center as B and radius multiplied by t, and the δ-contracted B^δ is defined by $B^\delta = B(x, r^\delta)$.

Given a set $E \subset \mathbb{R}^d$, $\overset{\circ}{E}$ stands for the interior of the E, $\overline{E}$ its closure and ∂E is the boundary of E, i.e, $\partial E = \overline{E} \setminus \overset{\circ}{E}$.

The σ-algebra of Borel sets of $\mathbb{R}^d$ is denoted by $\mathcal{B}(\mathbb{R}^d)$, $\mathcal{L}^d$ is the Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of Borel probability measure over $\mathbb{R}^d$.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\mathrm{supp}(\mu) := \{x :\ \forall r > 0,\ \mu(B(x,r)) > 0\}$ is the topological support of μ.

Given $E \in \mathcal{B}(\mathbb{R}^d)$, $\dim_H(E)$ and $\dim_P(E)$ denote respectively the Hausdorff and the packing dimension of E.

2.1. **Definition and recalls.**

Definition 2.1. *Let $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be an increasing mapping verifying $\zeta(0) = 0$. The Hausdorff measure at scale $t \in (0, +\infty)$ associated with ζ of a set E is defined*

by

(2)

$$\mathcal{H}^{\zeta}_t(E) = \inf\left\{\sum_{n\in\mathbb{N}} \zeta(|B_n|) :\ (B_n)_{n\in\mathbb{N}} \text{ closed balls, } |B_n|\le t \text{ and } E\subset \bigcup_{n\in\mathbb{N}} B_n\right\}.$$

The Hausdorff measure associated with ζ of a set E is defined by

$$\mathcal{H}^{\zeta}(E) = \lim_{t\to 0^+} \mathcal{H}^{\zeta}_t(E). \tag{3}$$

For $t\in(0,+\infty)$, $s\ge 0$ and $\zeta : x\to x^s$, one simply uses the usual notation $\mathcal{H}^{\zeta}_t(E)=\mathcal{H}^s_t(E)$ and $\mathcal{H}^{\zeta}(E)=\mathcal{H}^s(E)$. In particular, the s-dimensional Hausdorff outer measure at scale $t\in(0,+\infty]$ of the set E is defined by

$$\mathcal{H}^s_t(E) = \inf\left\{\sum_{n\in\mathbb{N}} |B_n|^s :\ (B_n)_{n\in\mathbb{N}} \text{ closed balls, } |B_n|\le t \text{ and } E\subset \bigcup_{n\in\mathbb{N}} B_n\right\}. \tag{4}$$

Definition 2.2. *Let $\mu\in\mathcal{M}(\mathbb{R}^d)$. For $x\in\operatorname{supp}(\mu)$, the lower and upper local dimensions of μ at x are*

$$\underline{\dim}(\mu,x) = \liminf_{r\to 0^+} \frac{\log\mu(B(x,r))}{\log r}$$

$$\text{and}\qquad \overline{\dim}(\mu,x) = \limsup_{r\to 0^+} \frac{\log\mu(B(x,r))}{\log r}.$$

Then, the lower and upper dimensions of μ are defined by

$$\underline{\dim}_H(\mu) = \mathit{infess}_{\mu}(\underline{\dim}(\mu,x)) \quad \text{and} \quad \overline{\dim}_P(\mu) = \mathit{supess}_{\mu}(\overline{\dim}(\mu,x)). \tag{5}$$

It is known that (for more details see [10])

$$\underline{\dim}_H(\mu) = \inf_{E\in\mathcal{B}(\mathbb{R}^d):\ \mu(E)>0} \dim_H(E) \quad \text{and} \quad \overline{\dim}_P(\mu) = \inf_{E\in\mathcal{B}(\mathbb{R}^d):\ \mu(E)=1} \dim_P(E).$$

A measure verifying $\underline{\dim}_H(\mu)=\overline{\dim}_P(\mu):=\alpha$ will be called an α exact dimensional measure. From Definition 2.2, such measures verify, for μ-almost every $x\in\mathbb{R}^d$, $\lim_{r\to 0^+}\frac{\log\mu(B(x,r))}{\log r}=\alpha$.

2.2. **Main statements.** Before stating the Theorems proved in this article, one starts by recalling the following definition, introduced in [2].

Definition 2.3. *Let $\mathcal{B}=(B_n=:B(x_n,r_n))_{n\in\mathbb{N}}$ be a family of balls in $\mathbb{R}^d$. Denote by $\mathcal{T}_k(\mathcal{B})=\left\{B_n\ :\ 2^{-k-1}<r_n\le 2^{-k}\right\}$. The family $\mathcal{B}$ is said to be weakly redundant when for all k, there exists an integer J_k and $\mathcal{T}_{k,1}(\mathcal{B}),..,\mathcal{T}_{k,J_k}(\mathcal{B})$ a partition of $\mathcal{T}_k(\mathcal{B})$ such that:*

- (C_1) $\mathcal{T}_k(\mathcal{B})=\bigcup_{1\le j\le J_k} T_{k,j}(\mathcal{B})$,
- (C_2) *For every $1\le j\le J_k$ and every pair of balls $B\neq B'\in\mathcal{T}_{k,j}(\mathcal{B})$, $B\cap B'=\emptyset$,*
- (C_3) $\lim_{k\to+\infty}\frac{\log_2(J_k)}{k}=0$.

So, a sequence of balls $(B_n)_{n\in\mathbb{N}}$ is weakly redundant when at each scale 2^{-k}, the balls of the family $\{B_n\}_{n\in\mathbb{N}}$ that have radii $\approx 2^{-k}$ can be sorted in a relatively small number of families of pairwise disjoint balls.

The main property we introduce for a sequence of balls $\mathcal{B}=(B_n)_{n\in\mathbb{N}}$ is meant to ensure that any set can be covered efficiently by the limsup of the B_n's, with

respect to a measure μ. This property is a general version of the KGB Lemma of Beresnevitch and Velani, stated in [6], using a Borel probability measure μ. Such properties (like the KGB Lemma) are usually key (cf [16, 6, 2] for instance) to prove ubiquity or mass transference results.

Definition 2.4. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. The sequence $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ of balls of $\mathbb{R}^d$ is said to be μ-asymptotically covering (in short, μ-a.c) when there exists a constant $C > 0$ such that for every open set $\Omega \subset \mathbb{R}^d$ and $g \in \mathbb{N}$, there is an integer $N_\Omega \in \mathbb{N}$ as well as $g \leq n_1 \leq ... \leq n_{N_\Omega}$ such that:*

- $\forall\, 1 \leq i \leq N_\Omega$, $B_{n_i} \subset \Omega$,
- $\forall\, 1 \leq i \neq j \leq N_\Omega$, $B_{n_i} \cap B_{n_j} = \emptyset$,
- *one has*

$$\mu\left(\bigcup_{1\leq i\leq N_\Omega} B_{n_i}\right) \geq C\mu(\Omega). \tag{6}$$

In other words, for any open set Ω and any $g > 0$, there exists a finite set of disjoint balls of $\{B_n\}_{n\geq g}$ covering a large part of Ω from the μ-standpoint.

This notion of μ-asymptotically covering is related to the way the balls of $\mathcal{B}$ are distributed according to the measure μ. In particular, given a measure μ, this property is slightly stronger than being of lim sup of full μ-measure, as illustrated by the following Theorem.

Theorem 2.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ be a sequence of balls of $\mathbb{R}^d$ with $\lim_{n\to+\infty} r_n = 0$.*

(1) If $\mathcal{B}$ is μ-a.c, then $\mu(\limsup_{n\to+\infty} B_n) = 1$.

(2) If there exists $v < 1$ such that $\mu\big(\limsup_{n\to+\infty}(vB_n)\big) = 1$, then $\mathcal{B}$ is μ-a.c.

Moreover, it results from the proof of the KGB-Lemma [6] that if the μ is doubling, $\mu\Big(\limsup_{n\to+\infty} B_n\Big) = 1 \Leftrightarrow (B_n)_{n\in\mathbb{N}}$ is μ-a.c.

An interesting consequences of Theorem 2.1 is the following version of Borel-Cantelli lemma which is extends the case where μ is doubling, established [5].

Proposition 2.2. *Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of closed balls satisfying $|B_n| \to 0$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure.*

(A): *Assume that $(B_n)_{n\in\mathbb{N}}$ is μ-a.c, then, there exists $C > 1$ such that for any open ball B, there exists a sub-sequence of $(B_n)_{n\in\mathbb{N}}$, $(L_{n,B})_{n\in\mathbb{N}}$ satisfying, for any $n \in \mathbb{N}$, $L_{B,n} \subset B$ and*

$$\sum_{n\geq 0} \mu(L_{B,n}) = +\infty \tag{7}$$

and for infinitely many Q,

$$\sum_{s,t=1}^{Q} \mu(L_{B,s} \cap L_{B,t}) \leq \frac{C}{\mu(B)} \left(\sum_{n=1}^{Q} \mu(L_{B,n})\right)^2. \tag{8}$$

(B): *Assume that there exists $C > 1$ such that for any open ball B, there exists a sub-sequence of $(B_n)_{n\in\mathbb{N}}$ $(L_{n,B})_{n\in\mathbb{N}}$ with, for any $n \in \mathbb{N}$, $L_{n,B} \subset B$, satisfying (19) and (20), then $\mu(\limsup_{n\to+\infty} B_n) = 1$, so that, for any $\kappa > 1$, $(\kappa B_n)_{n\in\mathbb{N}}$ is μ-a.c.*

One now states the main result about extraction of sub-sequences of balls of this article.

Theorem 2.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of balls of $\mathbb{R}^d$.*

(1) If $(B_n)_{n\in\mathbb{N}}$ is μ-a.c, then there exists a μ-a.c sub-sequence $(B_{\phi(n)})_{n\in\mathbb{N}}$ which is weakly redundant.

(2) If there exists $v < 1$ such that $\mu(\limsup_{n\to+\infty} vB_n) = 1$, then there exists a μ-a.c sub-sequence $(B_{\phi(n)})_{n\in\mathbb{N}}$ verifying

$$\underline{\dim}_H(\mu) \leq \liminf_{n\to+\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \limsup_{n\to+\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \overline{\dim}_P(\mu). \tag{9}$$

Remark 2.4. *Theorem 2.3 implies in particular that if the sequence of balls $(B_n)_{n\in\mathbb{N}}$ verifies $\mu(\limsup_{n\to+\infty} vB_n) = 1$, for some $v < 1$, it is possible to extract a μ-a.c sub-sequence verifying items* (1) *and* (2).

Section 3.3 and Section 4 are respectively dedicated to the proof of Theorem 2.1 and Theorem 2.3.

Section 5 provides some explicit applications of Theorem 2.3.

In the last section, Section 6, one proves Theorem 2.6 and some final remark are given about Corollary 2.7.

2.3. **Application to the study of the optimality of lower-bounds in obtained via mass transference principles.**

2.3.1. *An upper-bound for ubiquity Theorem in the self-similar case.* In this section, one shows how the previous extraction theorem can be used to investigate optimal bounds in inhomogeneous mass transference principles. Let us recall first the definition of the following geometric quantity, introduced in [9].

Definition 2.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, and $s \geq 0$. The s-dimensional μ-essential Hausdorff content at scale $t \in (0, +\infty]$ of a set $A \subset \mathcal{B}(\mathbb{R}^d)$ is defined as*

$$\mathcal{H}_t^{\mu,s}(A) = \inf \left\{ \mathcal{H}_t^s(E) : \ E \subset A, \ \mu(E) = \mu(A) \right\}. \tag{10}$$

Let us also recall the definition of a self-similar measure.

Definition 2.6. *A self-similar IFS is a family $S = \{f_i\}_{i=1}^m$ of $m \geq 2$ contracting similarities of $\mathbb{R}^d$.*

Let $(p_i)_{i=1,\dots,m} \in (0,1)^m$ be a positive probability vector, i.e. $p_1 + \cdots + p_m = 1$.

The self-similar measure μ associated with $\{f_i\}_{i=1}^m$ and $(p_i)_{i=1}^m$ is the unique probability measure such that

$$\mu = \sum_{i=1}^m p_i \mu \circ f_i^{-1}. \tag{11}$$

The topological support of μ is the attractor of S, that is the unique non-empty compact set $K \subset X$ such that $K = \bigcup_{i=1}^m f_i(K)$.

The existence and uniqueness of K and μ are standard results [15]. Recall that due to a result by Feng and Hu [12] any self-similar measure is exact dimensional.

Our goal is to investigate whether the lower-bound given by the following theorem, proved in [9], is sharp.

Theorem 2.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a self-similar measure and $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ be a μ-a.c. sequence of closed balls of $\mathbb{R}^d$ centered in $\mathrm{supp}(\mu)$. Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of open sets such that $U_n \subset B_n$ for all $n \in \mathbb{N}$, and $0 \leq s \leq \dim(\mu)$. If, for every $n \in \mathbb{N}$ large enough, $\mathcal{H}^{\mu,s}_\infty(U_n) \geq \mu(B_n)$, then*

$$\dim_H(\limsup_{n\to+\infty} U_n) \geq s.$$

One now states the main result of this section.

Theorem 2.6. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a self-similar measure, K its support and $(B_n)_{n\to+\infty}$ be a weakly redundant sequence of balls of $\mathbb{R}^d$ verifying $|B_n| \to 0$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open sets satisfying $U_n \subset B_n$. For any $0 \leq s < \dim(\mu)$ such that, for all large enough $n \in \mathbb{N}$, $\mathcal{H}^{\mu,s}_\infty(U_n) \leq \mu(B_n)$,*

$$\dim_H(\limsup_{n\to+\infty} U_n \cap K) \leq s. \tag{12}$$

Note that if, for any $n \in \mathbb{N}$, the ball B_n intersects K, $\limsup_{n\to+\infty} U_n \cap K = \limsup_{n\to+\infty} U_n$.

Theorem 2.5 and Theorem 2.6 yields the following useful corollary.

Corollary 2.7. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a self-similar measure. Let $(B_n)_{n\in\mathbb{N}}$ be a weakly redundant μ-a.c sequence of balls satisfying $|B_n| \to 0$ and $B_n \cap K \neq \emptyset$ for any $n \in \mathbb{N}$.*

Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open sets satisfying that, for any $n \in \mathbb{N}$, $U_n \subset B_n$. Assume that there exists s_0 such that

- *for any $s < s_0$, for n large enough, $\mathcal{H}^{\mu,s}_\infty(U_n) \geq \mu(B_n)$,*
- *for any $s > s_0$, for n large enough, $\mathcal{H}^{\mu,s}_\infty(U_n) \leq \mu(B_n)$.*

Then by Theorem 2.5 and Theorem 2.6,

$$\dim_H(\limsup_{n\to+\infty} U_n) = s_0.$$

Remark 2.8. *It is easily seen from the proof that the condition $\mathcal{H}^{\mu,s}_\infty(U_n) \leq \mu(B_n)$ in Theorem 2.6 can be weakened into $\liminf_{n\to+\infty} \frac{\log \mathcal{H}^{\mu,s}_\infty(U_n)}{\log \mu(B_n)} \geq 1$.*

2.3.2. *Application in the case of balls and rectangles.* We can now show in which sense, in view of Theorem 2.3, Theorem 2.5 is sharp by applying Corollary 2.7 to the specific cases where the sets U_n are balls or rectangles.

Corollary 2.9. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a self-similar measure of support K and $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ be a sequence of balls centered in K satisfying $|B_n| \to 0$ and $\mu\Big(\limsup_{n\to+\infty} B_n\Big) = 1$. Then* [9]*,*

$$\dim_H(\limsup_{n\to+\infty} B_n^\delta) \geq \frac{\dim(\mu)}{\delta}.$$

Assume furthermore that $\mathcal{B}$ is weakly redundant and $\limsup_{n\to+\infty} \frac{\log \mu(B_n)}{\log(|B_n|)} = \dim(\mu)$, then for every $\delta \geq 1$,

$$\dim_H(\limsup_{n\to+\infty} B_n^\delta) = \frac{\dim(\mu)}{\delta}.$$

Corollary 2.10. *Let μ be a self-similar measure verifying that its support, K, is the closure of its interior. Let $1 \leq \tau_1 \leq ... \leq \tau_d$, $\tau = (\tau_1, ..., \tau_d)$ and $(B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ be a sequence of balls of $\mathbb{R}^d$ satisfying $r_n \to 0$, $\mu(\limsup_{n\to+\infty} B_n) = 1$. Define $R_n = \mathring{R}_\tau(x_n, r_n)$, where $R_\tau(x_n, r_n) = x_n + \prod_{i=1}^d [-\frac{1}{2}r_n^{\tau_i}, \frac{1}{2}r_n^{\tau_i}]$. Then* [9]

$$\dim_H(\limsup_{n\to+\infty} R_n) \geq \min_{1\leq i\leq d} \left\{ \frac{\dim(\mu) + \sum_{1\leq j\leq i} \tau_i - \tau_j}{\tau_i} \right\}. \tag{13}$$

Assume furthermore that $(B_n)_{n\in\mathbb{N}}$ is weakly redundant and $\lim_{n\to+\infty} \frac{\log\mu(B_n)}{\log|B_n|} = \dim(\mu)$, then

$$\dim_H(\limsup_{n\to+\infty} R_n) = \min_{1\leq i\leq d} \left\{ \frac{\dim(\mu) + \sum_{1\leq j\leq i} \tau_i - \tau_j}{\tau_i} \right\}. \tag{14}$$

Remark 2.11.

- *Corollaries 2.9 and 2.10 are direct consequences of second item of Remark 5.1 and Remark 5.3 in* [9]*, together with Corollary 2.7 (applied to, respectively, $s_0 = \frac{\dim(\mu)}{\delta}$ and $s_0 = s(\mu, \tau)$).*
- *Note that, by Theorem 2.3 combined with Corollary 2.9 and Corollary 2.10, for any sequence of balls $(B_n)_{n\in\mathbb{N}}$ satisfying $\mu\Big(\limsup_{n\to+\infty} \frac{1}{2}B_n\Big) = 1$ (μ a self-similar measure satisfying the of hypothesis of Corollaries 2.9 or 2.10 for μ), it is always possible to extract a μ-a.c sub-sequence of balls so that the Hausdorff dimension of the limsup set associated with corresponding U_n's is the bound stated in* [9] *and recalled in those corollaries. This in particular proves that those bounds are sharp.*
- *In the case of the Lebesgue measure, it is always verified that $\lim_{n\to+\infty} \frac{\log\mu(B_n)}{\log|B_n|} = \dim(\mu)$. As a consequence, the lower-bound provided by Theorem 2.5 (which is established in* [17]*) in the case of balls or rectangles is precisely the dimension of $\limsup_{n\to+\infty} U_n$ as soon as the sequence (B_n) is weakly redundant. More explicitly, given a weakly redundant sequence of balls $(B_n)_{n\in\mathbb{N}}$ of $[0,1]^d$ satisfying $|B_n| \to 0$ and $\mathcal{L}^d(\limsup_{n\to+\infty} B_n) = 1$, for any sequence rectangles associated with a vector τ as in Theorem 2.10, one has*

$$\dim_H(\limsup_{n\to+\infty} R_n) = \min_{1\leq i\leq d} \left\{ \frac{d + \sum_{1\leq j\leq i} \tau_i - \tau_j}{\tau_i} \right\}.$$

Section 3 is dedicated to the proof of Theorem 2.1. In the next section, Section 4, Theorem 2.3 is established. Then some explicit examples of application of Theorem 2.3 are given in Section 5.

In the penultimate section, Section 6, Theorem 2.6 is proved.

The last section, Section 7, draws some conclusions and gives some perspectives about the results established in this article.

3. Proof of Theorem 2.1

3.1. A useful modified version of Besicovitch covering Lemma.

One focuses on a modified version of Besicovitch's covering Lemma.

Proposition 3.1. *For any $0 < v \leq 1$, there exists $Q_{d,v} \in \mathbb{N}^\star$, a constant depending only on the dimension d and v, such that for every $E \subset [0,1]^d$, for every set*

$\mathcal{F} = \big\{B(x, r_{(x)}) : x \in E, r_{(x)} > 0\big\}$, there exists $\mathcal{F}_1, ..., \mathcal{F}_{Q_{d,v}}$ finite or countable sub-families of $\mathcal{F}$ such that:

- *$\forall 1 \leq i \leq Q_{d,v}$, $\forall L \neq L' \in \mathcal{F}_i$, one has $\frac{1}{v}L \cap \frac{1}{v}L' = \emptyset$.*
- *E is covered by the families $\mathcal{F}_i$, i.e.*

$$E \subset \bigcup_{1 \leq i \leq Q_{d,v}} \bigcup_{L \in \mathcal{F}_i} L. \tag{15}$$

The case $v = 1$ corresponds to the standard Besicovich's covering lemma (see [20], Chapter 2, pp. 28-34 for instance).

A first step toward Proposition 3.1 is the next lemma, that allows to split a given family of "weakly" overlapping balls into a finite number of families of disjoint balls.

Lemma 3.2. *Let $0 < v < 1$ and $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a countable family of balls such that $\lim_{n \to +\infty} |B_n| = 0$, and for every $n \neq n' \in \mathbb{N}$, $vB_n \cap vB'_n = \emptyset$.*

There exists $\gamma_{d,v} + 1$ ($\gamma_{d,v}$ being the constant appearing in Lemma 3.3 below) sub-families of $\mathcal{B}$, $(\mathcal{F}_i)_{1 \leq i \leq \gamma_{d,v}+1}$, such that:

- *$\mathcal{B} = \bigcup_{1 \leq i \leq \gamma_{d,v}+1} \mathcal{F}_i$,*
- *$\forall\, 1 \leq i \leq \gamma_{d,v} + 1$, $\forall L \cap L' \in \mathcal{F}_i$, one has $L \cap L' = \emptyset$.*

Proof. The proof is based on the following lemma, whose proof can be found in [20], Lemma 2.7, pp.30 - there, the result is obtained for $v = 1/2$ but the proof remains valid for any $v < 1$.

Lemma 3.3. *For any $0 < v \leq 1$ there exists a constant $\gamma_{v,d} > 0$ depending only on v and the dimension d only, satisfying the following: if a family of balls $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ and a ball B are such that*

- *$\forall\ n \geq 1$, $|B_n| \geq \frac{1}{2}|B|$,*
- *$\forall\ n_1 \neq n_2 \geq 1$, $vB_{n_1} \cap vB_{n_2} = \emptyset$,*

then B intersects at most $\gamma_{v,d}$ balls of $\mathcal{B}$.

The families $\mathcal{F}_1, ..., \mathcal{F}_{\gamma_{d,v}+1}$ are built recursively.

For $k \in \mathbb{N}$, call $\mathcal{G}^{(k)} = \big\{L \in \mathcal{F} : 2^{-k-1} < |L| \leq 2^{-k}\big\}$. Notice that, because $\lim_{n \to +\infty} |B_n| = 0$, each $\mathcal{G}^{(k)}$ is empty or finite.

Observe first that for every $k \in \mathbb{N}$ and every ball $B \in \mathcal{G}^{(k)}$, and every pair of balls $B_1 \neq B_2 \in \bigcup_{k' \leq k} \mathcal{G}^{(k')} \setminus \{B\}$, one has $vB_1 \cap vB_2 = \emptyset$ and for $i = 1, 2$, $|B_i| \geq \frac{|B|}{2}$. By Lemma 3.3, this implies that B intersects at most $\gamma_{d,v}$ balls of $\bigcup_{k' \leq k} \mathcal{G}^{(k')} \setminus \{B\}$.

To get Lemma 3.2, we are going to sort the balls of $\bigcup_{k' \leq k} \mathcal{G}^{(k')}$ recursively on k into families $\mathcal{F}_1, ..., \mathcal{F}_{\gamma_{d,v}+1}$ of pairwise disjoint balls. At each step, a new ball B will be added to one of those families of balls $\mathcal{F}_i$ and the resulting family, $\mathcal{F}_i \bigcup \{B\}$ will be denoted again by $\mathcal{F}_i$.

Let k_0 be the smallest integer such that $\mathcal{G}^{(k_0)}$ is non-empty. Consider an arbitrary $L_0 \in \mathcal{G}^{(k_0)}$. By Lemma 3.3, L_0 intersects $n_0 \leq \gamma_{d,v}$ other balls of $\mathcal{G}^{(k_0)}$, that are denoted by $L_1, ..., L_{n_0}$. The sets $\mathcal{F}_i$ are then set as follows:

- *$\forall\, 1 \leq i \leq n_0$, $\mathcal{F}_i = \{L_i\}$,*
- *$\forall\ n_0 + 1 \leq i \leq \gamma_{d,v}$, $\mathcal{F}_i = \emptyset$,*
- *$\mathcal{F}_{\gamma_{d,v}+1} = \{L_0\}$.*

Further, consider $\widetilde{L} \notin \bigcup_{0\leq i\leq n_0}\{L_i\}$ (whenever such an $\widetilde{L}$ exists). The same argument (Lemma 3.3) ensures that $\widetilde{L}$ intersects at most $\gamma_{d,v}$ balls of $\mathcal{G}^{(k_0)}$.

In particular there must exists $1 \leq i \leq \gamma_{d,v}+1$ such that for every $L \in \mathcal{F}_i$, $\widetilde{L}\cap L = \emptyset$. Choosing arbitrarily one of those indices i, one adds $\widetilde{L}$ to $\mathcal{F}_i := \widetilde{L}\bigcup \mathcal{F}_i$ (we keep the same name for this new family).

The same argument remains valid for any other ball $L'' \notin \bigcup_{1\leq j\leq \gamma_{d,v}+1}\bigcup_{L\in\mathcal{F}_j}\{L\}$. Hence, proceeding recursively on all balls of $\mathcal{G}^{(k_0)}$ allows to sort the balls of $\mathcal{G}^{(k_0)}$ into $\gamma_{d,v}+1$ families $(\mathcal{F}_i)_{1\leq i\leq \gamma_{d,v}+1}$ of pairwise disjoint balls.

Next, let k_1 be the smallest integer such that $k_1 > k_0$ and $\mathcal{G}^{(k_1)}$ is non empty, take an arbitrary $L_0^{(1)} \in \mathcal{G}^{(k_1)}$. It is trivial to check that the family $\mathcal{G}^{(k_0)}\cup\mathcal{G}^{(k_1)}$ and the ball $L_0^{(1)}$ satisfy the conditions of Lemma 3.3. Subsequently, $L_0^{(1)}$ intersects at most $\gamma_{d,v}$ balls of $\mathcal{G}^{(k_0)}\bigcup\mathcal{G}^{(k_1)}$, and there must exist an integer $1 \leq i_0 \leq \gamma_{d,v}+1$ such that $L_0^{(1)}\cap\bigcup_{L\in\mathcal{F}_{i_0}} L = \emptyset$. As before, we add this ball $L_0^{(1)}$ to the family $\mathcal{F}_{i_0}$.

Consider $\widetilde{L} \in \mathcal{F}^{(k_1)}$ such that $\widetilde{L} \notin \bigcup_{1\leq i\leq\gamma_{d,v}+1}\mathcal{F}_i$ (whenever such a ball exists). The exact same argument shows the existence of an integer $1 \leq \widetilde{i} \leq \gamma_{d,v}+1$ such that $\widetilde{L}$ intersects at most $\gamma_{d,v}$ balls of $\mathcal{G}^{(k_0)}\bigcup\mathcal{G}^{(k_1)}$. One adds $\widetilde{L}$ to the family $\mathcal{F}_{\widetilde{i}}$, which remains composed only of pairwise disjoint balls.

One applies this argument to every ball of $\mathcal{F}^{(k_1)}$, hence finally sorting the balls of $\mathcal{F}^{(k_0)}\cup\mathcal{F}^{(k_1)}$ into $\gamma_{d,v}+1$ families of pairwise disjoint balls, as requested.

It is now easily seen that one can proceed recursively on $k \geq k_0$, ending up with the families $\mathcal{F}_1, ..., \mathcal{F}_{\gamma_{d,v}+1}$ fulfilling the desired properties. □

We are now ready to prove Proposition 3.1.

Proof. Fix $E \subset [0,1]^d$ and $\mathcal{F} = \big\{B(x, r_{(x)}) : x \in E, r_{(x)} > 0\big\}$.

One applies Besicovich's theorem (i.e. Proposition 3.1 with $v = 1$) to $\mathcal{F} = \big\{B(x, r_{(x)}) : x \in E\ r_{(x)} > 0\big\}$. This provides us with a finite set of families of balls $\mathcal{G}_1, ..., \mathcal{G}_{\gamma_{d,1}+1}$ composed of pairwise disjoint balls satisfying (15), i.e. $E \subset \bigcup_{1\leq i\leq Q_{\gamma_{d,1}+1}}\bigcup_{L\in\mathcal{G}_i} L$.

For every $1 \leq i \leq Q_{\gamma_{d,1}+1}$, one sets $\mathcal{G}_i^{(v)} = \big\{\frac{1}{v}L : L \in \mathcal{G}_i\big\}$, i.e. the sets of balls with same centers as $\mathcal{G}_i$ but with radii multiplied by $v^{-1} > 1$. Notice that by construction, $\forall\, 1 \leq i \leq Q_{\gamma_{d,1}+1}$, $\forall\, L \neq L' \in \mathcal{G}_i^{(v)}$, one has $vL \cap vL' = \emptyset$. Hence, Lemma 3.2 yields $\gamma_{d,v}+1$ sub-families $(\mathcal{G}_{i,j}^{(v)})_{1\leq j\leq\gamma_{d,v}+1}$ of $\mathcal{G}_i^{(v)}$ such that:

- $\forall\ 1 \leq j \leq \gamma_{d,v}+1$, $\forall\ L \neq L' \in \mathcal{G}_{i,j}^{(v)}$, one has $L \cap L' = \emptyset$,
- $\mathcal{G}_i^{(v)} = \bigcup_{1\leq j\leq\gamma_{d,v}+1}\mathcal{G}_{i,j}^{(v)}$.

Finally, we set for every $1 \leq i \leq Q_{d,1}$ and $1 \leq j \leq \gamma_{d,v}+1$

$$\mathcal{F}_{i,j} = \Big\{vL : L \in \mathcal{G}_{i,j}^{(v)}\Big\} \quad \text{and} \quad \mathcal{F}_i = \bigcup_{1\leq j\leq\gamma_{d,v}+1}\mathcal{F}_{i,j}.$$

These sets verify that:

- $\forall\ 1 \leq i \leq Q_{d,1}$, $\forall\ 1 \leq j \leq \gamma_{d,v}+1$, $\forall L \neq L' \in \mathcal{F}_{i,j}$, $\frac{1}{v}L \cap \frac{1}{v}L' = \emptyset$ (because the balls of $\mathcal{G}_{i,j}$ are pairwise disjoint),

- $E \subset \bigcup_{1\leq i\leq Q_{d,1}} \mathcal{G}_i = \bigcup_{1\leq i\leq Q_{d,1}} \bigcup_{1\leq j\leq \gamma_{d,v}+1} \mathcal{F}_{i,j}$.

This proves the statement and the fact that $Q_{d,v} = Q_{d,1}.(\gamma_{d,v}+1)$. □

3.2. **Consequences of the μ-asymptotic covering property.** One first shows that the constant C in Definition 2.4 can be replaced by 1 if infinite subsequences of balls are authorized. In fact Definition 2.4 ensures that any open set can be covered (with respect to the μ-measure) by disjoint balls B_n of arbitrary large indices.

Lemma 3.4. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ be a μ-a.c sequence of balls of $\mathbb{R}^d$ with $\lim_{n\to+\infty} r_n = 0$.*

Then for every open set Ω and every integer $g \in \mathbb{N}$, there exists a subsequence $(B^{(\Omega)}_{(n)}) \subset \{B_n\}_{n\geq g}$ such that:

(1) $\forall\, n \in \mathbb{N}$, $B^{(\Omega)}_{(n)} \subset \Omega$,

(2) $\forall\, 1 \leq n_1 \neq n_2$, $B^{(\Omega)}_{(n_1)} \cap B^{(\Omega)}_{(n_2)} = \emptyset$,

(3) $\mu\left(\bigcup_{n\geq 1} B^{(\Omega)}_{(n)}\right) = \mu(\Omega)$.

In addition, there exists an integer N_Ω such that for the balls $(B^{(\Omega)}_{(n)})_{n=1,...,N_\Omega}$, the conditions (1) and (2) are realized, and (3) is replaced by $\mu\left(\bigcup_{n=1}^{N_\Omega} B^{(\Omega)}_{(n)}\right) \geq \frac{3}{4}\mu(\Omega)$.

The last part of Lemma 3.4 simply follows from item (3) and the σ-additivity of μ.

Proof. The idea consists in covering Ω by pairwise disjoint balls amongst those balls of $\mathcal{B}$, such that their union has measure at least $C\mu(\Omega)$, then in covering the complementary of the union of those balls in Ω (that is still open) with at least a proportion C of its measure, and so on.

More precisely, this is achieved as follows:

- **Step 1:** By application of Definition 2.4 to $\Omega_0 := \Omega$ and $g \in \mathbb{N}$, there exists $C > 0$ and some integers $g \leq n_1 \leq ... \leq n_{N_0}$ so that the family of balls $\mathcal{F}_0 := \left\{B_{n_i} := B^{(0)}_i\right\}_{1\leq i\leq N_0}$ is pairwise disjoint and $\mu(\bigcup_{1\leq i\leq N_0} B_{n_i}) \geq C\mu(\Omega)$.

- **Step 2:** Setting $\Omega_1 = \Omega \setminus \bigcup_{L\in\mathcal{F}_0} L$, applying Definition 2.4 to Ω_1 with the integer g provides us with a family $\mathcal{F}_1$ of pairwise disjoint balls $B^{(1)}_1, ..., B^{(1)}_{N_1} \in \{B_n\}_{n\geq g}$ such that $\forall\ 1 \leq i \leq N_1$ $B^{(1)}_i \subset \Omega_1$ and

$$\mu(\bigcup_{1\leq i\leq N_2} B^{(1)}_i) \geq C\mu(\Omega_1).$$

One sets $\mathcal{F}_1 = \mathcal{F}_0 \bigcup \mathcal{F}_1$. One sees that

$$\begin{aligned}\mu\left(\bigcup_{L\in\mathcal{F}_1} L\right) &= \mu\left(\bigcup_{L\in\mathcal{F}_0} L\right) + \mu\left(\bigcup_{L\in\mathcal{G}_1} L\right)\\ &\geq \mu\left(\bigcup_{L\in\mathcal{F}_0} L\right) + C\left(\mu(\Omega) - \mu\left(\bigcup_{L\in\mathcal{F}_0} L\right)\right)\\ &\geq (1-C)\mu\left(\bigcup_{L\in\mathcal{F}_0} L\right) + C\mu\left(\Omega\right)\\ &\geq (C + C(1-C))\mu(\Omega).\end{aligned}$$

Observe that the balls of $\mathcal{F}_0$ and $\mathcal{F}_1$ are disjoint by construction.

• **Following steps :** Proceeding recursively, and applying the exact same argument as above, one constructs an increasing sequence of families $(\mathcal{F}_i)_{i\in\mathbb{N}}$ and a decreasing sequence of open sets Ω_i such that:

- $\forall\, i \in \mathbb{N}$, $L \in \{B_n\}_{n\geq g}$ and $\forall L \in \mathcal{F}_i$, $L \subset \Omega_i \subset \Omega$,
- $\forall\, i \in \mathbb{N}$, $\forall L \neq L' \in \mathcal{F}_i$, $L \cap L' = \emptyset$,
- $\forall\, i \neq j \in \mathbb{N}$, $\forall L \in \mathcal{F}_i$ and $\forall L \in \mathcal{F}_j$, $L \cap L' = \emptyset$,
- $\forall\, i \in \mathbb{N}$, $\mu\left(\bigcup_{L\in\mathcal{F}_i} L\right) \geq \mu(\Omega)\sum_{1\leq k\leq i} C(1-C)^{k-1}$.

Finally, setting $\mathcal{F} = \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$, one sees that $\mathcal{F}$ is constituted by pairwise disjoint balls chosen amongst $\{B_n\}_{n\geq g}$ satisfying

$$\mu(\Omega) \geq \mu\left(\bigcup_{L\in\mathcal{F}} L\right) \geq \mu(\Omega)\sum_{k\geq 1} C(1-C)^{k-1} = \mu(\Omega), \tag{16}$$

so that $\mathcal{F}$ fulfills the conditions of Lemma 3.4. □

An easy consequence is the following.

Corollary 3.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $(B_n)_{n\in\mathbb{N}}$ be a μ-a.c sequence of balls. Then for any Borel set E, for any $g \in \mathbb{N}$, there exists a sub-sequence of balls $(B^{(E)}_{(n)}) \subset \{B_n\}_{n\geq g}$ such that:*

(1) $\forall 1 \leq n_1 \neq n_2$, $B^{(E)}_{(n_1)} \cap B^{(E)}_{(n_2)} = \emptyset$,

(2) $\mu\Big(\bigcup_{n\in\mathbb{N}} B^{(E)}_{(n)} \cap E\Big) = \mu(E)$,

(3) $\mu\Big(\bigcup_{n\in\mathbb{N}} B^{(E)}_{(n)}\Big) \leq \mu(E) + \varepsilon$,

Proof. By outer regularity, there exists an open set Ω such that $E \subset \Omega$ and $m(\Omega) \leq \mu(E) + \varepsilon$. Applying Lemma 3.4 to Ω, the sequence $(B_n)_{n\in\mathbb{N}}$ fulfills the condition of Corollary 3.5. □

3.3. **Proof of Theorem 2.1. (1)** Assume first that $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ is μ-a.c, and let us prove that $\mu(\limsup_{n\to+\infty} B_n) = 1$.

For every $g \in \mathbb{N}$, applying Lemma 3.4, there exists a sub-family of balls, $\mathcal{F}_g \subset \{B_n\}_{n\geq g}$ such that $\mu(\bigcup_{L\in\mathcal{F}_g} L) = \mu(\mathbb{R}^d) = 1$. In particular, $\mu(\bigcup_{n\geq g} B_n) = 1$ for every $g \geq 1$, and $\mu(\limsup_{n\to+\infty} B_n) = \mu(\bigcap_{g\geq 1}\bigcup_{n\geq g} B_n) = 1$.

(2) Suppose next that there exists $v < 1$ such that $\mu(\limsup_{n\to+\infty} vB_n) = 1$, and let us show that $\mathcal{B}$ is μ-a.c.

Let Ω be an open set in $\mathbb{R}^d$. Our goal is to find a constant C such that the conditions of Definition 2.4 are realized.

Let $E = \Omega\cap\limsup_{n\to+\infty} vB_n$. For every $y \in E$, consider an integer $n_y \geq g$ large enough so that $y \in vB_{n_y}$ and $B(y, 2r_{n_y}) \subset \Omega$. This is possible since $\lim_{n\to+\infty} r_n = 0$.

Since $y \in vB_{n_y}$, one has

$$B(y,(1-v)r_{n_y}) \subset B_{n_y} \subset B(y,(1+v)r_{n_y}) \subset B(y,2r_{n_y}), \tag{17}$$

and the family $\mathcal{F} = \big\{B(y,(1-v)r_{n_y}) : y \in E\big\}$ covers E by balls centered on E.

Applying Proposition 3.1 with constant $v' = \frac{1-v}{2} < 1$ allows to extract from $\mathcal{F}$ finite or countable sub-families $\mathcal{F}_1, ..., \mathcal{F}_{Q_{d,v'}}$ such that:

- $\forall 1 \leq i \leq Q_{d,v'}$, $L \neq L' \in \mathcal{F}_i$, one has $\frac{1}{v'}L \cap \frac{1}{v'}L' = \emptyset$.
- E is covered by the families $\mathcal{F}_i$, i.e. (15) holds true.

Now, $\mu(\Omega) = \mu(E) \leq \mu\left(\bigcup_{i=1}^{Q_{d,v'}}\bigcup_{L\in\mathcal{F}_i} L\right)$. There must exist $1 \leq i_0 \leq Q_{d,v'}$ such that

$$\mu\left(\bigcup_{L\in\mathcal{F}_{i_0}} L\right) \geq \frac{1}{Q_{d,v'}}\mu(E) = \frac{1}{Q_{d,v'}}\mu(\Omega).$$

There exist L_1, L_2, ... L_N balls of $\mathcal{F}_{i_0}$ such that

$$\mu\left(\bigcup_{1\leq k\leq N} L_k\right) \geq \frac{1}{2Q_{d,v'}}\mu(\Omega),$$

Notice the following facts:

- $\forall 1 \leq i \leq Q_{d,v'}$, every $L \in \mathcal{F}_i$ is naturally associated with some $y \in E$ and some ball B_{n_y}, with $L \subset B_{n_y} \subset \Omega$,
- $\forall 1 \leq i \leq Q_{d,v'}$, if $L \in \mathcal{F}_i$ is associated with $y \in E$ and B_{n_y} and $L' \in \mathcal{F}_i$ is associated with $y' \in E$ and $B_{n_{y'}}$, then $\frac{1}{v'}L \cap \frac{1}{v'}L' = \emptyset$ implies by (17) that $B_{n_y} \cap B_{n_{y'}} = \emptyset$.

The first fact implies that there exist N integers n_1, ..., n_N such that $B_{n_k} \subset \Omega$ and

$$\mu\left(\bigcup_{1\leq k\leq N} B_{n_k}\right) \geq \frac{1}{2Q_{d,v'}}\mu(\Omega),$$

The second fact implies that these balls B_{n_k}, $k = 1, ..., N$ are pairwise disjoint.

This exactly proves that $\mathcal{B}$ is μ-a.c.

3.4. **A version of Borel-Cantelli Lemma.** In this manuscript, one mainly focuses on establishing Hausdorff dimension of limsup sets knowing that a certain limsup set of balls has full measure. In many situation, proving that those limsup sets have full measure is straightforward. When it is not, it is convenient to have a tool at our disposal to be able to determine whether or not it is the case. In the case where the measure involved is doubling is treated by Beresnevich-Velani.

Theorem 3.6 ([5])**.** *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a doubling measure and $(B_n)_{n\in\mathbb{N}}$ a sequence of balls centered in* $\mathrm{supp}(\mu)$ *such that $|B_n| \to 0$. Then $\mu(\limsup_{n\in\mathbb{N}} B_n) = 1 \Leftrightarrow$ $\exists C > 1$ such that for any open ball B centered on* $\mathrm{supp}(\mu)$*, there exists a sub-sequence $(L_{B,n})_{n\in\mathbb{N}}$ of $(B_n)_{n\in\mathbb{N}}$ satisfying:*

- $L_{B,n} \subset B$,
- $\sum_{n=0}^{+\infty} \mu(L_{B,n}) = +\infty$,
- *for infinitely many $Q \in \mathbb{N}$,*

$$\sum_{s,t=1}^{Q} \mu(L_{B,s} \cap L_{B,t}) \leq \frac{C}{\mu(B)} \left(\sum_{n=1}^{Q} \mu(L_{B,n}) \right)^2 . \tag{18}$$

Thanks to Theorem 2.1, one can complete this Theorem and remove the doubling assumption.

Proposition 3.7. *Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of closed balls satisfying $|B_n| \to 0$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure.*

(A): *Assume that $(B_n)_{n\in\mathbb{N}}$ is μ-a.c, then, there exists $C > 1$ such that for any open ball B, there exists a sub-sequence of $(B_n)_{n\in\mathbb{N}}$, $(L_{n,B})_{n\in\mathbb{N}}$ satisfying, for any $n \in \mathbb{N}$, $L_{B,n} \subset B$ and*

$$\sum_{n\geq 0} \mu(L_{B,n}) = +\infty \tag{19}$$

and for infinitely many Q,

$$\sum_{s,t=1}^{Q} \mu(L_{B,s} \cap L_{B,t}) \leq \frac{C}{\mu(B)} \left(\sum_{n=1}^{Q} \mu(L_{B,n}) \right)^2 . \tag{20}$$

(B): *Assume that there exists $C > 1$ such that for any open ball B, there exists a sub-sequence of $(B_n)_{n\in\mathbb{N}}$ $(L_{n,B})_{n\in\mathbb{N}}$ with, for any $n \in \mathbb{N}$, $L_{n,B} \subset B$, satisfying* (19) *and* (20)*, then $\mu(\limsup_{n\to+\infty} B_n) = 1$, so that, for any $\kappa > 1$, $(\kappa B_n)_{n\in\mathbb{N}}$ is μ-a.c.*

Proof. Item A is proved in [5] (this part of the proof does not use the doubling property of the measure in [5]). Moreover, it is also proved in [5] that, if there exists $C > 0$ such that for any open ball B, there exists a sub-sequence of $(B_n)_{n\in\mathbb{N}}$ $(L_{n,B})_{n\in\mathbb{N}}$ with, for any $n \in \mathbb{N}$, $L_{n,B} \subset B$, satisfying (19) and (20), then

$$\mu(\limsup_{n\to+\infty} B_n \cap B) \geq \frac{1}{C}\mu(B).$$

The following lemma combined with Theorem 2.1 finishes the proof of Proposition 3.7.

Lemma 3.8. *Let $E \subset \mathbb{R}^d$. Assume that there exists $0 < c < 1$ such that for any open ball B, $\mu(E \cap B) \geq c\mu(B)$, then $\mu(E) = 1$.*

Proof. Assume that $\mu(E) < 1$ and set $A = \mathbb{R}^d \setminus E$. By hypothesis, $\mu(A) > 0$.

Let us recall the following density-lemma (which holds in metric sapces in which Besicovitch's theorem holds).

Lemma 3.9. [7] *Let $m \in \mathcal{M}(\mathbb{R}^d)$, $0 < c < 1$ and A be a Borel set with $m(A) > 0$. For every $r > 0$, set*

$$A(r) = \{x \in A \ : \ \forall \tilde{r} \leq r, \ m(B(x, \tilde{r}) \cap A) \geq c.m(B(x, \tilde{r}))\} \tag{21}$$

Then

$$m\left(\bigcup_{r>0} A(r)\right) = m(A). \tag{22}$$

By Lemma 3.9, there exists an open ball B such that $\mu(B) > 0$ and

$$\mu(B \cap A) \geq (1 - \frac{c}{2})\mu(B).$$

This yields

$$\begin{aligned}\mu(E \cap B \cap A) &= \mu(E \cap B) + \mu(A \cap B) - \mu\left((E \cap B) \cup (A \cap B)\right) \\ &\geq (c + 1 - \frac{c}{2} - 1)\mu(B) = \frac{c}{2}\mu(B) > 0,\end{aligned}$$

which implies $\mu(E \cap A) > 0$, which is a contradiction. □

Taking $c = \frac{1}{C}$ and applying Lemma 3.8 finishes the proof of Proposition 3.7. □

Remark 3.10. *A version of Proposition 3.7 might also be useful in more general metric spaces. The only geometric property we used to prove Proposition 3.7 is actually Proposition 3.1 (which also implies Lemma 3.9), so that Proposition 3.7 actually holds in any direction-limited spaces as defined in* [11].

4. Proof of Theorem 2.3

The following section is dedicated to the study of the properties one can ask an μ-a.c sequence $(B_n)_{n\in\mathbb{N}}$ to verify, up to an μ-a.c extraction.

The concept of conditioned ubiquity was introduced by Barral and Seuret in [1]. It consists in asking the balls of the sequence (B_n) to verify some specific properties with respect to the measure μ. When investigating the Hausdorff dimension of some sets $(U_n)_{n\in\mathbb{N}}$, where $U_n \subset B_n$, in practical cases (when the measure carries some self-similarity), it turns out that when a lower-bound is found for $(U_n)_{n\in\mathbb{N}}$ using the fact that the sequence $(B_n)_{n\in\mathbb{N}}$ is of limsup of full μ-measure, it is often quite easy to prove that $\limsup_{n\to+\infty} U_n$ has precisely the expected measure provided that the sequence (B_n) verifies some specific properties with respect to μ.

Note that in full generality, understanding the optimality of a bound as mentioned above, means understanding very finely the behavior of the measure μ on the sets U_n (the sequence (B_n) being μ-a.c). It will be proved in this article that, under mild conditions on the sequence $(B_n)_{n\in\mathbb{N}}$, it is always possible to give a natural upper-bound for $\dim_H(\limsup_{n\to+\infty} U_n)$. This upper-bound turns out to be optimal when the measure carries enough self-similarity (in particular it works for quasi-Bernoulli measures or fully supported self-similar measures).

In this section, the balls $(B_n)_{n\in\mathbb{N}}$ are supposed to be pairwise distinct and such that $|B_n| \underset{n\to+\infty}{\to} 0$.

4.1. **Extraction of weakly redundant μ-a.c subsequences.** The main result of this section is stated here.

Proposition 4.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $(B_n)_{n\in\mathbb{N}}$ be a μ-a.c sequence of balls $\lim_{n\to+\infty} |B_n| = 0$. There exists a subsequence $(B_{\psi(n)})_{n\in\mathbb{N}}$ of $(B_n)_{n\in\mathbb{N}}$ which is weakly redundant and μ-a.c.*

Proof. Let $g_k \in \mathbb{N}$ be large enough so that $\forall n \geq g_k$, $|B_n| \leq 2^{-k}$. By Proposition 3.4, applied with the sequence $(B_n)_{n\in\mathbb{N}}$, $\Omega = \mathbb{R}^d$ for any $k \in \mathbb{N}$, there exists a sub-sequence $(B_{(n,k)})$ of $\{B_n\}_{n\geq g_k}^{\mathbb{N}}$ satisfying

(1) $\forall 1 \leq n_1 \neq n_2$, $B_{(n_1,k)} \cap B_{(n_2,k)} = \emptyset$,

(2) $\mu\Big(\bigcup_{n\in\mathbb{N}} B_{(n,k)}\Big) = 1$.

Define $\mathcal{B}_\psi = (B_{\psi(n)})_{n\in\mathbb{N}}$ as the sub-sequence of balls corresponding to $\bigcup_{k\in\mathbb{N}} \left\{B_{(n,k)}\right\}_{n\in\mathbb{N}}$.

Since the following inclusion holds

$$\bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} B_{(n,k)} \subset \limsup_{n\to+\infty} B_{\psi(n)}, \tag{23}$$

by item (2) one has

$$\mu(\limsup_{n\to+\infty} B_{\psi(n)}) = 1.$$

Note that, for all $k \in \mathbb{N}$, all $B \in \left\{B_{(n,k)}\right\}_{n\in\mathbb{N}}$, $|B| \leq 2^{-k}$. Following the notation of Definition 2.3, for any $k \in \mathbb{N}$, $\mathcal{T}_k(\mathcal{B}_\psi)$ can contain only balls of the sequence of the k first families $\left\{B_{(n,k)}\right\}_{n\in\mathbb{N}}$, which are composed of pairwise disjoint balls. This proves that $\mathcal{T}_k(\mathcal{B}_\psi)$ can be sorted in at most $k+1$ families of pairwise disjoint balls. In particular, $\mathcal{B}_\psi$ is weakly redundant.

It remains to show that $(B_{\psi(n)})_{n\in\mathbb{N}}$ is μ-a.c.

Let Ω be an open set and $g \in \mathbb{N}$. One will extract from $\mathcal{B}_\psi$ a finite number of balls satisfying the condition of Definition 2.4.

There exists k_0 so large that, $\mu\big(\left\{x \in \Omega(x, 2^{-k_0+1})\right\}\big) \geq \frac{3\mu(\Omega)}{4}$ for any $k \geq k_0$, $\left\{B_{(n,k)}\right\}_{n\in\mathbb{N}} \subset \left\{B_{\psi(n)}\right\}_{n\geq g}$ and $\mu(\limsup_{n\to+\infty} B_{\psi(n)} \cap \Omega) \geq \frac{3\mu(\Omega)}{4}$. Setting

$$\widehat{E} = \left\{x \in \limsup_{n\to+\infty} B_{\psi(n)} \cap \Omega : B(x, 2^{-k_0+1}) \subset \Omega\right\},$$

it holds that

$$\mu(\widehat{E}) \geq \frac{1}{2}\mu(\Omega).$$

Recalling (23), for every $x \in \widehat{E}$, consider B_x, the ball of $\left\{B_{(n,k_0)}\right\}_{n\in\mathbb{N}}$ containing x. Note that, because for $B \in \left\{B_{(n,k_0)}\right\}_{n\in\mathbb{N}}$, $|B| \leq 2^{-k_0}$, one has $B_x \subset B(x, 2^{-k_0+1}) \subset \Omega$. Set $\mathcal{F}_1 = \left\{B_x : x \in \widehat{E}\right\}$. The set $\mathcal{F}_1$ is composed of pairwise disjoint balls (by item (1) above) of $\left\{B_{\psi(n)}\right\}_{n\geq g}$ included in Ω and such that

$$\mu\Big(\bigcup_{L\in\mathcal{F}_1} L\Big) \geq \mu(\widehat{E}) \geq \frac{1}{2}\mu(\Omega). \tag{24}$$

Using the σ-additivity of μ concludes the proof. □

4.2. **Extraction of sub-sequences of balls with conditioned measure.** Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $(B_n)_{n\in\mathbb{N}}$ be an μ-a.c sequence of balls.

This part aims to understand what condition can be assumed about the measure of the ball of the sequence $(B_n)_{n\in\mathbb{N}}$ in general under the μ-a.c condition.

More precisely, item (2) of Theorem 2.3 is proved.

Proposition 4.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. For any sequence of balls $(B_n)_{n\in\mathbb{N}}$ satisfying $\mu(\limsup_{n\to+\infty} vB_n) = 1$ for some $0 < v < 1$, there exists an μ-a.c sub-sequence $(B_{\phi(n)})_{n\in\mathbb{N}}$ verifying*

$$\underline{\dim}_H(\mu) \leq \liminf_{n\to+\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \limsup_{n\to+\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \overline{\dim}_P(\mu).$$

Remark 4.3. *For the left part of (4.2), the proof actually only uses the fact that $\mu(\limsup_{n\to+\infty} B_n) = 1$.*

Let us introduce some useful sets to prove Lemma 4.5 and Lemma 4.6, which are key in order to prove (4.2).

Definition 4.1. *Let $0 \leq \alpha \leq \gamma$ be real numbers, $\mu \in \mathcal{M}(\mathbb{R}^d)$, and $\varepsilon, \rho > 0$ two positive real numbers. Then define*

(25)
$$E_\mu^{[\alpha,\gamma],\rho,\varepsilon} = \left\{x \in \mathbb{R}^d : \ \underline{\dim}(\mu, x) \in [\alpha, \gamma] \text{ and } \forall r \leq \rho, \ \mu(B(x, r)) \leq r^{\underline{\dim}(\mu,x)-\varepsilon}\right\},$$

(26)
$$F_\mu^{[\alpha,\beta],\rho,\varepsilon} = \left\{x \in \mathbb{R}^d : \overline{\dim}(\mu, x) \in [\alpha, \beta] \text{ and } \forall r < \rho, \ \mu(B(x, r)) \geq r^{\overline{\dim}(\mu,x)+\varepsilon}\right\}.$$

and

$$E_\mu^{[\alpha,\gamma],\varepsilon} = \bigcup_{n\geq 1} E_\mu^{[\alpha,\gamma],\frac{1}{n},\varepsilon} \quad \text{and} \quad F_\mu^{[\alpha,\gamma],\varepsilon} = \bigcup_{n\geq 1} F_\mu^{[\alpha,\gamma],\frac{1}{n},\varepsilon}. \tag{27}$$

The following statements are easily deduced from Definition 2.2.

Proposition 4.4. *For every $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\rho > 0$, every $0 \leq \alpha \leq \gamma$ and $\varepsilon > 0$,*

$$\mu(E_\mu^{[\alpha,\gamma],\varepsilon}) = \mu(\{x : \underline{\dim}(\mu, x) \in [\alpha, \gamma]\})$$
$$\mu(F_\mu^{[\alpha,\gamma],\varepsilon}) = \mu(\left\{x : \overline{\dim}(\mu, x) \in [\alpha, \gamma]\right\})$$

and

$$E_\mu^{[\alpha,\gamma],\rho,\varepsilon} \subset \left\{x \in \mathbb{R}^d : \ \forall r \leq \rho, \ \mu(B(x, r)) \leq r^{\alpha-\varepsilon}\right\} \tag{28}$$
$$F_\mu^{[\alpha,\gamma],\rho,\varepsilon} \subset \left\{x \in \mathbb{R}^d : \ \forall r \leq \rho, \ \mu(B(x, r)) \geq r^{\gamma+\varepsilon}\right\}.$$

Furthermore, for $\alpha_1 = \underline{\dim}_H(\mu)$ and $\gamma_1 = supess_\mu(\underline{\dim}(\mu, x))$, one has

$$\mu(E_\mu^{[\alpha_1,\gamma_1],\varepsilon}) = 1. \tag{29}$$

Similarly, for $\alpha_2 = infess_\mu(\overline{\dim}(\mu, x))$ and $\gamma_2 = \overline{\dim}_P(\mu)$, one has

$$\mu\left(F_\mu^{[\alpha_2,\gamma_2],\varepsilon}\right) = 1. \tag{30}$$

Proof. For any $x \in \mathbb{R}^d$, for any $\varepsilon > 0$, there exists $r_x > 0$ such that, $\forall r \leq r_x$, $r^{\overline{\dim}(\mu,x)+\varepsilon} \leq \mu(B(x, r)) \leq r^{\underline{\dim}(\mu,x)-\varepsilon}$. This implies

$$F_\mu^{[\alpha,\gamma],\rho,\varepsilon} \subset \left\{x \in \mathbb{R}^d : \ \forall r \leq \rho, \ \mu(B(x, r)) \geq r^{\gamma+\varepsilon}\right\},$$

$$E_\mu^{[\alpha,\gamma],\rho,\varepsilon} \subset \left\{x \in \mathbb{R}^d : \ \forall r \leq \rho, \ \mu(B(x,r)) \leq r^{\alpha-\varepsilon}\right\},$$

and

$$E_\mu^{[\alpha,\gamma],\varepsilon} = \{x : \underline{\dim}(\mu,x) \in [\alpha,\gamma]\} \text{ and } F_\mu^{[\alpha,\gamma],\varepsilon} = \left\{x : \overline{\dim}(\mu,x) \in [\alpha,\gamma]\right\}.$$

Since

$$\mu([\text{infess}_\mu(\underline{\dim}(\mu,x)), \text{supess}_\mu(\underline{\dim}(\mu,x))]) = 1$$

and

$$\mu([\text{infess}_\mu(\overline{\dim}(\mu,x)), \text{supess}_\mu(\overline{\dim}(\mu,x))]) = 1,$$

recalling Definition 2.2, it holds that, following the notation of Proposition 4.4,

$$\mu\left(E_\mu^{[\alpha_1,\gamma_1],\varepsilon}\right) = \mu\left(F_\mu^{[\alpha_2,\gamma_2],\varepsilon}\right) = 1.$$

□

Before showing Proposition 4.2, let us start by the two following Lemmas 4.5 and 4.6. The first one will be used to prove the left part of the inequality (9) while the second one will be useful to prove the right part.

Lemma 4.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ be a μ-a.c sequence of balls of $\mathbb{R}^d$ with $\lim_{n\to+\infty} r_n = 0$.*

For any $\varepsilon > 0$, there exists a μ-a.c subsequence $(B_{\phi(n)})_{n\in\mathbb{N}}$ of $\mathcal{B}$ such that for every $n \in \mathbb{N}$, $\mu(B_{\phi(n)}) \leq (r_{\phi(n)})^{\underline{\dim}_H(\mu)-\varepsilon}$.

Proof. Set $\alpha = \underline{\dim}_H(\mu)$ and $\gamma = \textit{suppess}_\mu(\underline{\dim}(\mu,x))$.

Let Ω be an open set and $\varepsilon > 0$. By (29), $\mu(E_\mu^{[\alpha,\gamma],\frac{\varepsilon}{2}}) = 1$ and $\mu(\Omega \cap E_\mu^{[\alpha,\gamma],\frac{\varepsilon}{2}}) = \mu(\Omega)$.

For every $x \in \Omega \cap E_\mu^{[\alpha,\gamma],\frac{\varepsilon}{2}}$, there exists $r_x > 0$ such that $B(x,r_x) \subset \Omega$ and $x \in E_\mu^{[\alpha,\gamma],r_x,\frac{\varepsilon}{2}}$.

Recall (27) and that the sets $E_\mu^{[\alpha,\gamma],\rho,\varepsilon}$ are non-increasing in ρ. In particular there exists $\rho_\Omega > 0$ such that the set $E_\Omega := \left\{x \in \Omega \cap E_\mu^{[\alpha,\gamma],\rho_\Omega,\frac{\varepsilon}{2}} : \ r_x \leq \rho_\Omega\right\}$ verifies

$$\mu(E_\Omega) \geq \frac{3\mu(\Omega)}{4}. \tag{31}$$

Let $g \in \mathbb{N}$. Applying Lemma 3.4 to Ω, the sequence (B_n) and the measure m, there exists N_Ω as well as $g \leq n_1 \leq ... \leq n_{N_\Omega}$ verifying:

(1) for every $1 \leq i \neq j \leq N_\Omega$, $B_{n_i} \cap B_{n_j} = \emptyset$,
(2) for every $1 \leq i \leq N_\Omega$, $2r_{n_i} \leq \rho_\Omega$ and $2^{\alpha-\frac{\varepsilon}{2}} \leq r_{n_i}^{-\frac{\varepsilon}{2}}$,
(3) $\mu(\bigcup_{1\leq i\leq N_\Omega} B_{n_i}) \geq \frac{\mu(\Omega)}{2}$.

We may assume that $\mu(B_{n_i}) > 0$ for every i, otherwise B_{n_i} does not play any role.

Item (3) together with (31) implies that

$$\mu\left(\bigcup_{1\leq i\leq N_\Omega} B_{n_i} \cap E_\Omega\right) \geq \frac{\mu(\Omega)}{4}.$$

Furthermore, for every $1 \leq i \leq N_\Omega$ verifying $B_{n_i} \cap E_\Omega \neq \emptyset$, it holds that $0 < \mu(B_{n_i}) \leq (r_{n_i})^{\alpha-\varepsilon}$. Indeed, let $x \in B_{n_i} \cap E_\Omega$. By item (2), $B_{n_i} \subset B(x, 2r_{n_i})$, and by (25) , item (2), and (28), it holds that

$$\mu(B_{n_i}) \leq \mu(B(x,2r_{n_i})) \leq (2r_{n_i})^{\alpha-\frac{\varepsilon}{2}} \leq (r_{n_i})^{\alpha-\varepsilon}.$$

Writing $\mathcal{B}' = \{B_n : \mu(B_n) \leq r_n^{\alpha-\varepsilon}\}$, the argument above shows that only balls of $\mathcal{B}'$ have been used to cover Ω . This is satisfied for every open set Ω, so that $\mathcal{B}'$ is a sub-sequence of $\mathcal{B}$ satisfying the condition of Definition 2.4, which concludes the proof of Lemma 4.5. □

Lemma 4.6. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $v < 1$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ a sequence of balls of $\mathbb{R}^d$ verifying $\mu(\limsup_{n\to+\infty} vB_n) = 1$.*

For all $\varepsilon > 0$, there exists a sub-sequence $(B_{\phi(n)})_{n\in\mathbb{N}}$ of $\mathcal{B}$ as well as $0 < v' < 1$ such that $\mu(\limsup_{n\to+\infty} v'B_{\phi(n)}) = 1$ and for all $n \in \mathbb{N}$, one has $\mu(B_{\phi(n)}) \geq (r_{\phi(n)})^{\overline{\dim}_H(\mu)+\varepsilon}$.

Remark 4.7. *The sequence $(B_{\phi(n)})_{n\in\mathbb{N}}$ found in Lemma 4.6 is in particular μ-a.c by Theorem 2.1.*

Proof. Let $\alpha = \text{infess}_\mu(\overline{\dim}(\mu, x))$ and $\gamma = \overline{\dim}_P(\mu)$. Let $\varepsilon > 0$ and $v < v' < 1$.

By (30) and Theorem 2.1, $\mu(\limsup_{n\to+\infty} vB_n \cap F_\mu^{[\alpha,\gamma],\frac{3\varepsilon}{2}}) = 1$. For all $x \in \limsup_{n\to+\infty} vB_n \cap F_\mu^{[\alpha,\gamma],\frac{3\varepsilon}{2}}$, there exists $r_x > 0$ small enough so that

$$r_x^{\frac{\varepsilon}{2}} \leq (v'-v)^{\gamma+\frac{3\varepsilon}{2}} \text{ and } \forall 0 < r \leq r_x,\ \mu(B(x,r)) \geq r^{\gamma+\frac{3\varepsilon}{2}}. \tag{32}$$

Since $x \in \limsup_{n\to+\infty} vB_n$, for all $n \in \mathbb{N}$, there exists $n_x \geq n$ such that $x \in vB_{n_x}$ and $(v'-v)r_{n_x} \leq r_x$. Note that $B(x,(v'-v)r_{n_x}) \subset v'B_{n_x}$. This implies the following inequalities:

$$\mu(B_{n_x}) \geq \mu(v'B_{n_x}) \geq \mu(B(x,(v'-v)r_{n_x}) \geq ((v'-v)r_{n_x})^{\gamma+\frac{3\varepsilon}{2}} \geq r_{n_x}^{\gamma+2\varepsilon}.$$

Set $\mathcal{B}_{\gamma,2\varepsilon} = \{B_n : \mu(B_n) \geq r_n^{\gamma+2\varepsilon}\}$. One just showed that

$$\limsup_{n\to+\infty} vB_n \cap F^{[\alpha,\gamma],\frac{\varepsilon}{2}} \subset \limsup_{B\in\mathcal{B}_{\gamma,2\varepsilon}} v'B.$$

This proves that $\mu(\limsup_{B\in\mathcal{B}_{\gamma,2\varepsilon}} v'B) = 1$.

Since $\varepsilon > 0$ was arbitrary, the results also holds with $\frac{\varepsilon}{2}$, which proves Lemma 4.6. □

We are now ready to prove Proposition 4.2.

Proof. Set $\alpha = \underline{\dim}_H(\mu)$ and $\beta = \overline{\dim}_H(\mu)$.

Let us fix $(\varepsilon_n)_{n\in\mathbb{N}} \in (\mathbb{R}^{*+})^{\mathbb{N}}$ verifying $\lim_{n\to+\infty} \varepsilon_n = 0$.

The strategy of the proof consists in constructing recursively coverings of the cube $\mathbb{R}^d$ by using Lemma 4.5 and Lemma 4.6 and a diagonal argument (on the choice of ε) at each step.

More precisely, at step 1, one will build a family of balls $(\mathcal{F}_{1,i})_{i\in\mathbb{N}}$ verifying:

- for all $i,j \geq 1$, $\forall L \in \mathcal{F}_{1,i}$, $\forall L' \in \mathcal{F}_{1,j}$ such that $L \neq L'$, one has $L \cap L' = \emptyset$,
- for all $i \geq 1$, $\mathcal{F}_{1,i}$ is a finite sub-family of $\{B_n\}_{n\geq 1}$,
- for all $i \geq 1$, for all $L \in \mathcal{F}_{1,i}$, $|L|^{\beta+\varepsilon_i} \leq \mu(L) \leq |L|^{\alpha-\varepsilon_i}$,
- $\mu\left(\bigcup_{i\in\mathbb{N}} \bigcup_{L\in\mathcal{F}_{1,i}} L\right) = 1$.

Note that for each $i \in \mathbb{N}$, only a finite number of balls $L \in \mathcal{F}_1 := \bigcup_{j\in\mathbb{N}} \mathcal{F}_{1,j}$ verifies (for that ε_i naturally associated with those balls) $|L|^{\beta+\varepsilon_i} \leq \mu(L) \leq |L|^{\alpha-\varepsilon_i}$.

At step 2, a family of balls $(\mathcal{F}_{2,i})_{i\in\mathbb{N}}$ will be constructed such that:

- for all $i,j \geq 1$, $L \in \mathcal{F}_{2,i}$, $L' \in \mathcal{F}_{2,j}$, $L \neq L' = \emptyset$,

- for all $i \geq 1$, $\mathcal{F}_{2,i}$ is a finite sub-family of $\{B_n\}_{n\geq 2}$,
- for all $i \geq 1$, for all $L \in \mathcal{F}_{2,i}$, $|L|^{\beta+\varepsilon_{i+1}} \leq \mu(L) \leq |L|^{\alpha-\varepsilon_{i+1}}$,
- one has

$$\mu\left(\bigcup_{i\in\mathbb{N}}\bigcup_{L\in\mathcal{F}_{2,i}} L\right) = 1. \tag{33}$$

Write $\mathcal{F}_2 = \bigcup_{i\geq 1}\mathcal{F}_{2,i}$. Note that the family of balls $\mathcal{F}_2$ verifies, by construction, that any $L \in \mathcal{F}_2$ the natural ε_i associated with L is never equal to ε_1, so that only some balls constructed in step 1 are associated with ε_1 .

The other steps are achieved following the same scheme.

The construction is detailed below:

Step 1:

Let $\Omega_{1,1} = \mathbb{R}^d$.

Sub-step 1.1:

By Lemma 4.5 and Lemma 4.6 applied to $\varepsilon = \varepsilon_1$, there exists a μ-a.c sub-sequence $(B_{\psi_{1,1}(n)})_{n\in\mathbb{N}}$, satisfying, for every $n \in \mathbb{N}$,

$$|B_{\psi_{1,1}(n)}|^{\beta+\varepsilon_1} \leq \mu(B_{\psi_{1,1}(n)}) \leq |B_{\psi_{1,1}(n)}|^{\alpha-\varepsilon_1}.$$

By Lemma 3.4 applied to $\Omega_{1,1}$, the sequence $(B_{\psi_{1,1}(n)})_{n\in\mathbb{N}}$ and $g = 1$, there exists an integer $N_{1,1}$ as well as some balls $L_{1,1,1}, ..., L_{1,1,N_{1,1}} \in \{B_n\}_{n\geq 1}$ verifying:

- for all $1 \leq i < j \leq N_{1,1}$, $L_{1,1,i} \cap L_{1,1,j} = \emptyset$,
- for all $1 \leq i \leq N_{1,1}$, $|L_{1,1,i}|^{\beta+\varepsilon_1} \leq \mu(L_{1,1,i}) \leq |L_{1,1,i}|^{\alpha-\varepsilon_1}$,
- $\mu(\bigcup_{1\leq i\leq N_{1,1}} L_{1,1,i}) \geq \frac{1}{2}$.

Set $\mathcal{F}_{1,1} = \{L_{1,1,i}\}_{1\leq i\leq N_{1,1}}$.

Sub-step 1.2:

Let $\Omega_{1,2} = \Omega_{1,1} \setminus \bigcup_{L\in\mathcal{F}_{1,1}} L$.

By Lemma 4.5 and Lemma 4.6 with $\varepsilon = \varepsilon_2$, there exists a μ-a.c sub-sequence $(B_{\psi_{1,2}(n)})_{n\in\mathbb{N}}$ satisfying

$$|B_{\psi_{1,2}(n)}|^{\beta+\varepsilon_2} \leq \mu(B_{\psi_{1,2}(n)}) \leq |B_{\psi_{1,2}(n)}|^{\alpha-\varepsilon_2}.$$

One applies Lemma 3.4 to the open set $\Omega_{1,2}$, the sub-sequence of balls $(B_{\psi_{1,2}(n)})_{n\in\mathbb{N}}$ and $g = 1$. There exists $N_{1,2} \in \mathbb{N}$ such that $L_{1,2,1}, ..., L_{1,2,N_{1,2}}$ verifies:

- for all $1 \leq i < j \leq N_{1,2}$, $L_{1,2,i} \cap L_{1,2,j} = \emptyset$,
- for all $1 \leq i \leq N_{1,2}$, $|L_{1,2,i}|^{\beta+\varepsilon_2} \leq \mu(L_{1,2,i}) \leq |L_{1,2,i}|^{\alpha-\varepsilon_2}$,
- $\mu(\bigcup_{1\leq i\leq N_{1,2}} L_{1,2,i}) \geq \frac{1}{2}\mu(\Omega_{1,2})$.

The family $\mathcal{F}_{1,2}$ is defined as $\mathcal{F}_{1,2} = \{L_{1,2,i}\}_{1\leq i\leq N_{1,2}}$.

Proceeding iteratively as Sub-step 1.1 and Sub-step 1.2, for any $i \in \mathbb{N}$, at Sub-step 1.i a family of balls $(\mathcal{F}_{1,i})_{i\in\mathbb{N}}$ is constructed so that it verifies:

- for all $i, j \geq 1$, $L \in \mathcal{F}_{1,i}$, $L' \in \mathcal{F}_{1,j}$, if $L \neq L'$, then $L \cap L' = \emptyset$,
- for all $i \geq 1$, $\mathcal{F}_{1,i}$ is a finite subset of $\{B_n\}_{n\geq 1}$,
- for all $i \geq 1$, for all $L \in \mathcal{F}_{1,i}$, $|L|^{\beta+\varepsilon_i} \leq \mu(L) \leq |L|^{\alpha-\varepsilon_i}$,

- $\mu\left(\bigcup_{i\in\mathbb{N}}\bigcup_{L\in\mathcal{F}_{1,i}} L\right)=1$.

Recall that, to justify the last item, this recursive scheme allows to cover $\mathbb{R}^d$, up to a set of μ-measure 0 (the argument is similar to the one developed at the end of the proof of Lemma 3.4 to obtain (16)).

Set $\mathcal{F}_1=\bigcup_{i\geq 1}\mathcal{F}_{1,i}$. With each ball $L\in\mathcal{F}$ is naturally associated a positive real number $\varepsilon(L)$, such that $\varepsilon(L)=\varepsilon_i$ if $L\in\mathcal{F}_{1,i}$.

Let us notice that the construction of the family $\mathcal{F}_2$ does not rely on the existence of the family $\mathcal{F}_1$, so that the families $\mathcal{F}_k$ can actually be built independently, following the same scheme, as described below.

Step k:

As in step 1, one constructs a family of balls $(\mathcal{F}_{k,i})_{i\geq 1}$ verifying:

- for all $i,j\geq 1$, $L\in\mathcal{F}_{k,i}$, $L'\in\mathcal{F}_{k,j}$, $L\neq L'$, $L\cap L'=\emptyset$,
- for all $i\geq 1$, $\mathcal{F}_{k,i}$ is a finite subset of $\{B_n\}_{n\geq k}$,
- for all $i\geq 1$, for all $L\in\mathcal{F}_{k,i}$, $|L|^{\beta+\varepsilon_{i+k}}\leq\mu(L)\leq|L|^{\alpha-\varepsilon_{i+k}}$,
- one has

$$\mu\left(\bigcup_{i\in\mathbb{N}}\bigcup_{L\in\mathcal{F}_{k,i}} L\right)=1. \tag{34}$$

Set $\mathcal{F}_k=\bigcup_{i\geq 1}\mathcal{F}_{k,i}$ and $\mathcal{F}=\bigcup_{k\geq 1}\mathcal{F}_k$.

Denote by $(B_{\phi(n)})_{n\in\mathbb{N}}$ the sub-sequence of balls that constitutes the family $\mathcal{F}$.

By construction, for all $i\in\mathbb{N}$, only a finite number of balls $L\in\mathcal{F}$ verifies $\varepsilon(L)=\varepsilon_i$ (and $|L|^{\beta+\varepsilon_i}\leq\mu(L)\leq|L|^{\alpha-\varepsilon_i}$). In particular, for all $\varepsilon>0$, there exists N large enough so that, for every $n\geq N$, $\varepsilon_n\leq\varepsilon$. Similarly, there exists $N'\in\mathbb{N}$ so large that for every $n'\geq N'$,

$$|B_{\phi(n')}|^{\beta+\varepsilon}\leq\mu(B_{\phi(n')})\leq|B_{\phi(n')}|^{\alpha-\varepsilon}.$$

It follows that

$$\alpha-\varepsilon\leq\liminf_{n\to+\infty}\frac{\log\mu(B_{\phi(n)})}{\log|B_{\phi(n)}|}\leq\limsup_{n\to+\infty}\frac{\log\mu(B_{\phi(n)})}{\log|B_{\phi(n)}|}\leq\beta+\varepsilon.$$

Letting $\varepsilon\to 0$ shows that

$$\underline{\dim}_H(\mu)\leq\liminf_{n\to+\infty}\frac{\log\mu(B_{\phi(n)})}{\log|B_{\phi(n)}|}\leq\limsup_{n\to+\infty}\frac{\log\mu(B_{\phi(n)})}{\log|B_{\phi(n)}|}\leq\overline{\dim}_H(\mu).$$

It only remains to prove that $(B_{\phi(n)})_{n\in\mathbb{N}}$ is μ-a.c.

Let Ω be an open set and $g\in\mathbb{N}$. We find a finite family of balls $\{L\}_{i\in\mathcal{I}}\subset\left\{B_{\phi(n)}\right\}_{n\geq g}$ satisfying the conditions of Definition 2.4.

Note that, by (33),

$$\text{setting } E=\bigcap_{k\geq 1}\bigcup_{L\in\mathcal{F}_k} L,\quad\text{then }\ \mu(E)=1.$$

Let $x\in\Omega\cap E$ and $r_x>0$ small enough so that $B(x,r_x)\subset\Omega$. Consider $k_x\geq\phi(g)\geq g$ large enough so that, for all $n\geq k_x$, $|B_n|\leq 2r_x$. Recall that $\mathcal{F}_{k_x}\subset\{B_n\}_{n\geq k_x}$. Finally, let us fix k large enough so that $\mu(\widehat{E})\geq\frac{\mu(\Omega)}{2}$, where $\widehat{E}=\{x\in E\ :\ k_x\leq k\}$. For $x\in\widehat{E}$, let $L_x\in\mathcal{F}_k$ be the ball that contains x (the balls

of $\mathcal{F}_k$ being pairwise disjoint, L_x is well defined) and $\{L_i\}_{i\geq 1} = \left\{L_x : x \in \widehat{E}\right\}$. One has

- for all $1 \leq i < j$, $L_i \cap L_j = \emptyset$,
- for all $i \in \mathbb{N}$, $L_i \in \left\{B_{\phi(n)}\right\}_{n\geq g}$ and $L_i \subset \Omega$,
- $\mu(\bigcup_{i\geq 1} L_i) \geq \mu(\widehat{E}) \geq \frac{\mu(\Omega)}{2}$.

By σ-additivity, there exists $N \in \mathbb{N}$ such that $\mu(\bigcup_{1\leq i\leq N} L_i) \geq \frac{\mu(\Omega)}{4}$, which proves that $(B_{\phi(n)})_{n\in\mathbb{N}}$ satisfies Definition 2.4 with $C = \frac{1}{4}$ and is indeed μ-a.c. □

One finishes this section with the following proposition, which supports the idea that, roughly speaking, for an α exact-dimensional measure μ and a μ-a.c sequence of balls (B_n), considering balls $(B_n)_{n\in\mathbb{N}}$ which does not verify $\mu(B_n) \approx |B_n|^\alpha$ is not relevant from the μ-standpoint.

Proposition 4.8. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be an α exact-dimensional measure and $(B_n)_{n\in\mathbb{N}}$ a sequence of balls satisfying $|B_n| \to 0$. Let $\varepsilon > 0$. Let is also define $\mathcal{B}^\varepsilon_> = \{B_n : \mu(B_n) \leq |B_n|^{\alpha+\varepsilon}\}$ and $\mathcal{B}^\varepsilon_< = \{B_n : \mu(B_n) \geq |B_n|^{\alpha-\varepsilon}\}$. Then*

(1) for any $v < 1$, $\mu(\limsup_{B\in\mathcal{B}^\varepsilon_>} vB) = 0$,

(2) $\mu(\limsup_{B\in\mathcal{B}^\varepsilon_<} B) = 0$.

Proof. (1) Suppose that there exists $0 < v < 1$ such that $\mu(\limsup_{B\in\mathcal{B}^\varepsilon_>} vB) > 0$. Then, since μ is assumed to be exact-dimensional, there exists $x \in \limsup_{B\in\mathcal{B}^\varepsilon_>} vB$ such that $\lim_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} = \alpha$.

Consider $r_x > 0$ small enough so that, for any $0 < r \leq r_x$, $\mu(B(x,r)) \geq r^{\dim(\mu)+\frac{\varepsilon}{2}}$ and $(\frac{1-v}{2})^{\dim(\mu)+\frac{\varepsilon}{2}} \geq r_x^{\frac{\varepsilon}{4}}$. Let also n be large enough so that $x \in B_n$ and $|B_n| \leq r_x$. Then $B(x, \frac{(1-v)}{2}|B_n|) \subset B_n$, so that

$$\mu(B_n) \geq \mu(B(x, \frac{1-v}{2}|B_n|)) \geq |B_n|^{\alpha+\frac{\varepsilon}{2}}(\frac{1-v}{2})^{\alpha+\frac{\varepsilon}{2}} \geq |B_n|^{\alpha+\frac{3\varepsilon}{4}}. \tag{35}$$

This contradicts the definition of $\mathcal{B}^\varepsilon_>$.

(2) Assume that $\mu(\limsup_{B\in\mathcal{B}^\varepsilon_<} B) > 0$. Then, again, there exists $x \in \limsup_{B\in\mathcal{B}^\varepsilon_<} B$ so that $\lim_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} = \alpha$. Consider $r_x > 0$ small enough so such that, for any $0 < r \leq r_x$, $\mu(B(x,r)) \leq r^{\alpha-\frac{\varepsilon}{2}}$. Consider $n \in \mathbb{N}$ large enough so that $x \in B_n$ and $|B_n| \leq r_x$. One has $B_n \subset B(x, |B_n|)$, hence

$$\mu(B_n) \leq \mu(B(x, |B_n|)) \leq |B_n|^{\alpha-\frac{\varepsilon}{2}}.$$

This contradicts the definition of $\mathcal{B}^\varepsilon_<$. □

Remark 4.9. *For doubling measures, it is straightforward that item* (1) *can be replaced by simply $\mu(\limsup_{B\in\mathcal{B}^\varepsilon_>} B) = 0$. It can be proved that this is also the case for 1-average d-1 unrectifiable measures (as a consequence of* [18, Theorem 2.11]*). Some self-similar measures with open set condition satisfies this property (see* [18] *again for more details).*

5. Some explicit examples

In this section, applications of Theorem 2.3 are given.

5.1. **Rational approximation.** Let us recall the following result from Hurwitz see [14], p 219, for more details.

Theorem 5.1. *Let $x \in [0,1] \setminus \mathbb{Q}$. There exists an infinite number of pairs $(p,q) \in \mathbb{N} \times \mathbb{N}^*$ with $p \wedge q = 1$ and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5} q^2}. \tag{36}$$

An immediate corollary of Theorem 2.1, Theorem 2.3 and Theorem 5.1 is the following:

Corollary 5.2. *Let $\mu \in \mathcal{M}([0,1])$ be any diffuse measure. Then the sequence of balls $\left(B(\frac{p}{q}, \frac{1}{q^2}) \right)_{0 \leq p \leq q, q \in \mathbb{N}^*, p \wedge q = 1}$ is weakly redundant (see [4]) and μ-a.c. In particular, if μ is α-exact-dimensional, for $0 \leq \alpha \leq d$, then there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ with $\lim_{n \to +\infty} \varepsilon_n = 0$ and an infinite number of pairs $(p_n, q_n) \in \Big(\mathbb{N} \times \{0, ..., q_n\} \Big)^{\mathbb{N}}$ such that $p_n \wedge q_n = 1$ and*

$$\Big(\frac{1}{q_n^2} \Big)^{\alpha + \varepsilon_n} \leq \mu\Big(B\Big(\frac{p_n}{q_n}, \frac{1}{q_n^2} \Big) \Big) \leq \Big(\frac{1}{q_n^2} \Big)^{\alpha - \varepsilon_n}.$$

Moreover, writing

$$\mathcal{B}_\mu = \left\{ B(\frac{p_n}{q_n}, \frac{1}{q_n^2}) : \Big(\frac{1}{q_n^2} \Big)^{\alpha + \varepsilon_n} \leq \mu\Big(B\Big(\frac{p_n}{q_n}, \frac{1}{q_n^2} \Big) \Big) \leq \Big(\frac{1}{q_n^2} \Big)^{\alpha - \varepsilon_n} \right\}_{n \in \mathbb{N}},$$

one has

$$\dim_H (\limsup_{B \in \mathcal{B}_\mu} B) = \alpha.$$

5.2. **Application to Random balls.** Let us recall Shepp's Theorem of Shepp [22].

Theorem 5.3. *Let $(l_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ and $(X_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d uniformly distributed random variables on $[0,1]$. Then*

$$\limsup_{n \to +\infty} B(X_n, l_n) = [0,1] \Leftrightarrow \sum_{n \geq 0} \frac{1}{n^2} \exp(l_1 + ... + l_n) = +\infty.$$

From Theorem 2.1, Theorem 2.3 and Theorem 5.3, the following corollary is deduced.

Corollary 5.4. *For any α exact-dimensional measure $\mu \in \mathcal{M}([0,1])$, $0 \leq \alpha \leq d$, for almost any i.i.d sequence of random variables uniformly distributed on $[0,1]$, $(X_n)_{n \in \mathbb{N}}$, there exists a sequence of positive real numbers $(\varepsilon_k)_{k \in \mathbb{N}}$ with $\varepsilon_k \to 0$ and a subsequence $(n_k)_{k \in \mathbb{N}} \to +\infty$ satisfying*

$$\Big(\frac{2}{n_k} \Big)^{\alpha + \varepsilon_k} \leq \mu\Big(B\Big(X_{n_k}, \frac{2}{n_k} \Big) \Big) \leq \Big(\frac{2}{n_k} \Big)^{\alpha - \varepsilon_k}.$$

Writing again

$$\mathcal{B}_\mu = \left\{ B\Big(X_{n_k}, \frac{2}{n_k} \Big) : \Big(\frac{2}{n_k} \Big)^{\alpha + \varepsilon_k} \leq \mu\Big(B\Big(X_{n_k}, \frac{2}{n_k} \Big) \Big) \leq \Big(\frac{2}{n_k} \Big)^{\alpha - \varepsilon_k} \right\}_{k \in \mathbb{N}},$$

one has

$$\dim_H (\limsup_{B \in \mathcal{B}_\mu} B) = \alpha.$$

5.3. **Examples in dynamical systems.** Let us introduce some notation.

Let $m \geq 2$ and $S = \{f_1, ..., f_m\}$ be a system of m similarities of $\mathbb{R}^d \to \mathbb{R}^d$ of ratio of contraction $0 < c_1 < 1, ..., 0 < c_m < 1$.

Let us also write $\Lambda = \{1, ..., n\}$, $\Lambda^* = \bigcup_{k\geq 0} \Lambda^k$ and for $\underline{i} = (i_1, ..., i_k) \in \Lambda^k$,

- $f_{\underline{i}} = f_{i_1} \circ ... \circ f_{i_k}$,
- $X_{\underline{i}} = f_{\underline{i}}([0,1]^d)$,
- $c_{\underline{i}} = c_{i_1} \times ... \times c_{i_k}$.

Let us fix also $(p_1, ..., p_m) \in \mathbb{R}_+^{\mathbb{N}}$ a probability vector, i.e a vector verifying $\sum_{1\leq i\leq m} p_i = 1$.

Remark 5.5. *Let μ defined by (11) and $x \in K_S$. Then, for any $k \in \mathbb{N}$, the balls $\{B(f_{\underline{i}}(x), 2|K_S|c_{\underline{i}})\}_{\underline{i}\in\Lambda^k}$ covers K_S. In particular, by Theorem 2.1 $\Big(B(f_{\underline{i}}(x), 3|K_S|c_{\underline{i}})\Big)_{\underline{i}\in\Lambda^*}$ is μ-a.c.*

As a consequence of Theorem 2.3 and Lemma 5.5, one gets:

Corollary 5.6. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a measure defined by (11) and $x \in K_S$. There exists a μ-a.c weakly redundant sub-sequence of balls $(B_n)_{n\in\mathbb{N}}$ extracted from $\Big(B(f_{\underline{i}}(x), 3c_{\underline{i}})\Big)_{\underline{i}\in\Lambda^*}$ such that, for all $n \in \mathbb{N}$ and for some sequence $(\varepsilon_k)_{k\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ verifying $\varepsilon_k \to 0$,*

$$|B_n|^{\dim(\mu)+\varepsilon_n} \leq \mu(B_n) \leq |B_n|^{\dim(\mu)-\varepsilon_n},$$

and $\mathcal{B}_\mu = \{B_n\}_{n\in\mathbb{N}}$ satisfies

$$\dim_H(\limsup_{n\to+\infty} B_n) = \dim(\mu).$$

6. Proofs of Theorem 2.6

The proof strongly relies on the following result proved in [9].

Theorem 6.1. *Let S be a self-similar IFS of $\mathbb{R}^d$. Let K be the attractor of S. Let μ be a self-similar measure associated with S. For any $0 \leq s < \dim(\mu)$, there exists a constant $c = c(d, \mu, s) > 0$ depending on the dimension d, μ and s only, such that for any ball $B = B(x, r)$ centered on K and $r \leq 1$, any open set Ω, one has*

$$c(d,\mu,s)|B|^s \leq \mathcal{H}^{\mu,s}_\infty(\mathring{B}) \leq \mathcal{H}^{\mu,s}_\infty(B) \leq |B|^s \text{ and}$$
$$c(d,\mu,s)\mathcal{H}^s_\infty(\Omega \cap K) \leq \mathcal{H}^{\mu,s}_\infty(\Omega) \leq \mathcal{H}^s_\infty(\Omega \cap K). \tag{37}$$

For any $s > \dim(\mu)$, $\mathcal{H}^{\mu,s}_\infty(\Omega) = 0$.

Recall that the sequence $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ is assumed to be weakly redundant. In such a case, for any $\varepsilon > 0$, following the notation involved in Definition 2.3, it holds that

$$\sum_{n\geq 0} |B_n|^\varepsilon \mu(B_n) = \sum_{k\geq 0} \sum_{B\in\mathcal{T}_k(\mathcal{B})} |B|^\varepsilon \mu(B) \leq \sum_{k\geq 0} \sum_{1\leq j\leq J_k} 2^{-k\varepsilon} \sum_{B\in T_{k,j}(\mathcal{B})} \mu(B).$$

Since for every (k,j) the family $T_{k,j}$ is composed of pairwise disjoint balls, this yields

$$\sum_{n\geq 0}|B_n|^{\varepsilon}\mu(B_n)\leq \sum_{k\geq 0}\sum_{1\leq j\leq J_k}2^{-k\varepsilon}=\sum_{k\geq 0}J_k2^{-k\varepsilon}<+\infty. \tag{38}$$

Now, for $n\in\mathbb{N}$, consider a sequence of balls $(A_k^n)_{k\in\mathbb{N}}$, with $|A_k^n|\leq |B_n|$ and such that $U_n\cap K\subset\bigcup_{k\geq 0}A_k^n$. Recall Theorem 6.1 and its notations. One has.

$$\mathcal{H}^s_\infty(U_n)\leq\sum_{k\geq 0}|A_k^n|^s\leq 2\mathcal{H}^s_\infty(U_n)\leq\frac{2}{c(d,\mu,s)}\mathcal{H}^{\mu,s}_\infty(U_n)\leq\frac{2}{c(d,\mu,s)}\mu(B_n). \tag{39}$$

Since for each $n\in\mathbb{N}$, $U_n\cap K\subset\bigcup_{k\geq 0}A_k^n$, it holds that $\limsup_{n\to+\infty}U_n\cap K\subset\limsup_{k,n\to+\infty}A_k^n$. For any $\varepsilon>0$, one gets

$$\sum_{n\geq 0}\sum_{k\geq 0}|A_k^n|^{s+\varepsilon}\leq\sum_{n\geq 0}|B_n|^{\varepsilon}\frac{2}{c(d,\mu,s)}\mu(B_n).$$

In particular, by (38),

$$\sum_{n\geq 0}\sum_{k\geq 0}|A_k^n|^{s+\varepsilon}<+\infty. \tag{40}$$

One concludes that

$$\mathcal{H}^{s+\varepsilon}(\limsup_{n\to+\infty}U_n\cap K)\leq\mathcal{H}^{s+\varepsilon}(\limsup_{k,n\to+\infty}A_k^n)<+\infty.$$

This implies that $\dim_H(\limsup_{n\to+\infty}U_n\cap K)\leq s+\varepsilon$ and ε being arbitrary,

$$\dim_H(\limsup_{n\to+\infty}U_n\cap K)\leq s.$$

Remark 6.2.

- *An important fact to underline here is that the convergences established in* (38) *and* (39) *do not rely on the fact that the measure is self-similar, but hold for any measure μ. One could state a comparable upper-bound Theorem for any measure μ by replacing K by a G_δ set of full measure in* (12).

  These computations also have the following straightforward consequence for a measure $\mu\in\mathcal{M}(\mathbb{R}^d)$ without the self-similarity assumption: Assume that, for n large enough, $\mathcal{H}^{\mu,s}_\infty(U_n)\leq\mu(B_n)$. If the sequence $(U_n)_{n\in\mathbb{N}}$ verifies that for any ball $B_i\subset U_n$ one also has $B_i\subset\bigcup_{k\geq n}A_k^n$ (where the balls $(A_{k,n})_{k\in\mathbb{N}}$ are chosen as in the proof of Theorem 2.6), then $\dim_H(\limsup_{n\to+\infty}U_n)\leq s$. In particular if this holds for any $s>s(\mu,\mathcal{B},\mathcal{U})$, then $\dim_H(\limsup_{n\to+\infty}U_n)=s(\mu,\mathcal{B},\mathcal{U})$.

- *When the self-similar measure verifies $\operatorname{supp}(\mu)=[0,1]^d$, the existence of s_0 as in Corollary 2.7 is ensured as soon as the shapes of the sets U_n are "uniform" in n. For instance, consider the case where $\mu=\mathcal{L}^d$ and $(U_n=R_n)_{n\in\mathbb{N}}$, where R_n is an open rectangle associated with some vector $\boldsymbol{\tau}=(\tau_1,...,\tau_d)$ defined as in Theorem 2.10. Recall that by Theorem 6.1, the Lebesgue essential Hausdorff content and the classical Hausdorff content are equivalent. It is easily verified that, for any $n\in\mathbb{N}$, $\mathcal{H}^s_\infty(R_n)=|B_n|^{g_{\boldsymbol{\tau}}(s)}$, for the mapping $g_{\boldsymbol{\tau}}:\mathbb{R}^+\to\mathbb{R}^+$, defined as (see* [17]*)*

$$g_{\boldsymbol{\tau}}(s)=\max_{1\leq k\leq d}\left\{s\tau_k-\sum_{1\leq i\leq k}\tau_k-\tau_i\right\}.$$

Note that $g_{\boldsymbol{\tau}}(s)$ does not depend on n. Corollary can therefore be applied with $s_0 = \min_{s:g_{\boldsymbol{\tau}}(s)\geq d}\{s\}$.

- *Unfortunately, when such an s_0 does not exist, the Hausdorff dimension of $\limsup_{n\to+\infty} U_n$ has to depend on the structure of the sequence (U_n) itself.*

Consider $0 < s_1 < s_2 \leq d$ and two vectors $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ such that $s_1 = \min_{s:g_{\boldsymbol{\tau_1}}(s)\geq d}\{s\}$ and $s_2 = \min_{s:g_{\boldsymbol{\tau_2}}(s)\geq d}\{s\}$. Consider a weakly redundant sequence of balls $(B_n)_{n\in\mathbb{N}}$ of $[0,1]^d$ and a sequence of open sets $(U_n)_{n\in\mathbb{N}}$, $U_n \subset B_n$ satisfying:

- $|B_n| \to 0$,
- $\mathcal{L}^d(\limsup_{n\to+\infty} B_n) = 1$,
- *for any $n \in \mathbb{N}$, $B_n \subset [0,\frac{1}{2}) \times \prod_{i=2}^d [0,1]$ or $B_n \subset (\frac{1}{2},1] \times \prod_{i=2}^d [0,1]$,*
- *for any $n \in \mathbb{N}$ such that $B_n \subset [0,\frac{1}{2}) \times \prod_{i=2}^d [0,1]$, $U_n = (R_n)$ with R_n an open rectangle associated with $\boldsymbol{\tau_1}$ as in Theorem 2.10,*
- *for any $n \in \mathbb{N}$ such that $B_n \subset (\frac{1}{2},1] \times \prod_{i=2}^d [0,1]$, $U_n = R_n$ with R_n an open rectangle associated with $\boldsymbol{\tau_2}$.*

Then smallest real number such that the condition of Theorem 2.6 holds is s_2, the largest real number such that the condition of Theorem 2.5 holds is s_1 and $\dim_H(\limsup_{n\to+\infty} U_n) = s_2$.

On the other hand, following the scheme of example 3.5 in [17], it is also possible to construct two weakly redundant sequences of balls $(B_{n,1})_{n\in\mathbb{N}}$ and $(B_{n,2})_{n\in\mathbb{N}}$ such that:

- $|B_{n,1}| \to 0$ *and* $|B_{n,2}| \to 0$,
- $0 < \mathcal{L}^d(\limsup_{n\to+\infty} B_{n,2}) < 1$,
- $\limsup_{n\to+\infty} R_{n,2} = \emptyset$, *where $R_{n,2} \subset B_{n,2}$ is an open rectangle associated with $\boldsymbol{\tau_2}$,*
- $\limsup_{n\to+\infty} B_{n,1} \subset [0,1]^d \setminus \limsup_{n\to+\infty} B_{n,2}$ *and*
$$\mathcal{L}^d(\limsup_{n\to+\infty} B_{n,1}) = 1 - \mathcal{L}^d(\limsup_{n\to+\infty} B_{n,2}).$$
For any $n \in \mathbb{N}$, denote by $R_{n,1} \subset B_{n,1}$ an open rectangle associated with $\boldsymbol{\tau_1}$.

Those properties implies that that any sequence $(B_n)_{n\to+\infty}$ corresponding to the family $\{B_{n,i}\}_{n\in\mathbb{N},i\in\{1,2\}}$ is weakly redundant and satisfies $\mathcal{L}^d(\limsup_{n\to+\infty} B_n) = 1$.

Again, the smallest real number such that the condition of Theorem 2.10 holds is s_2, the largest real number such that the condition of Theorem 2.5 holds is s_1 but this time, $\dim_H(\limsup_{n\to+\infty} U_n) = s_1$.

7. Conclusion and perspectives

The properties stated in Theorem 2.3 (the prescribed measure and the weak redundancy) are of course non exhaustive and maybe more can be imposed to well chosen subsequences of μ-.a.c sequences of balls. It turns out that in the quasi-Bernoulli case it is enough to get condition under which the lower-bound found in [8] is also an upper-bound , but it is likely that in some other cases, one needs to ask the sequence to verify more properties to ensure the sharpness of a certain lower-bound. In particular, it can be proved that, under very weak hypothesis on

a μ-a.c sequence $(B_n)_{n\in\mathbb{N}}$, given a G_δ set of full measure G, it is possible to assume (up to an extraction) that the set $G' = \limsup_{n\to+\infty} B_n$ is a G_δ set of full measure with $G' \subset G$ (so that one can assume that $\limsup_{n\to+\infty} B_n$ is always included in any G_δ of full measure if needed).

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