

# RELATIONS WITH A FIXED INTERVAL EXCHANGE TRANSFORMATION

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**ABSTRACT.** We study the group of all interval exchange transformations (IETs). We show that for every IET  $S$ , there exists a dense open set of admissible IETs that share a relation with  $S$ . This is an extension of a result published by Dahmani, Fujiwara and Guirardel in [DFG13]: the group generated by a generic pair of elements of  $\text{IET}([0;1[)$  is not free (assuming a suitable condition on the underlying permutation).

**Key words:** interval exchange transformations, free group of rank 2, interval exchange transformations with flips.

## 1. INTRODUCTION

We are interested in a question of Katok, namely whether there is a rank 2 free subgroup in the group of interval exchange transformations (IETs) or not. The group of IETs has been studied first by Keane and then more widely since 1980. While IETs with reducible underlying permutation are of course not minimal, Veech showed that there exist minimal IETs (hence with irreducible underlying permutation) which are not uniquely ergodic. This was the first examples of minimal and non uniquely ergodic maps.

IETs arise in many areas of mathematics such as dynamical systems, polygonal billiards, geometry and flows over flat surfaces. See [Via06] for a survey on the subject. They are well understood individually as maps, but the structure of their group still give rise to questions, such as Katok's one. For now, we do know some results about for exemple subgroup generated by torsion elements and rotations [Bos16], and about its commutator group. It is a proper normal subgroup (independently shown by Arnoux [Arn81b] and Sah [Sah81]). So the group of interval exchange transformations is not simple (contrary to the group of interval exchange transformations with flips, see part III.1 of [Arn81a]). Moreover, the commutator is exactly the subgroup generated by torsion elements [Vor17].

The main result of this paper give a partial answer to Katok's question:

**Theorem 1.1.** *Let  $S$  be any IET on  $[0;1[$ . There exists a dense open set  $\Omega_a(S) \subset \text{IET}_a([0;1[)$  such that for every  $T \in \Omega_a(S)$ ,  $\langle S, T \rangle$  is not free of rank 2.*

The set  $\text{IET}_a([0;1[)$  is the set (and not a subgroup) of admissible IETs (see Definition 3.2). It extends the following result, proved in [DFG13].

**Theorem 1.2** (Dahmani-Fujiwara-Guirardel). *There exists a dense open set  $\Omega \subset IET([0; 1[) \times IET_a([0; 1[)$  such that for every  $(S, T) \in \Omega$ ,  $\langle S, T \rangle$  is not free of rank 2.*

We present also some computations to know whether self-similar IET, such as Arnoux-Yoccoz example (see [AY81] and [Arn88]), can be in some  $\Omega_a(S)$ .

Note that Katok's question is easily answered in the group of affine interval exchange transformations (AIETs).

**Proposition 1.3.** *There exist  $f, g \in AIET([0; 1[)$  such that  $\langle f, g \rangle$  is a free group of rank 2.*

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## 2. OUTLINE OF THE PROOF

Before going to details we sum up the proof of Theorem 1.1:

- (1) (*Basic relations*). Pick  $T_0$  a  $q$ -rational IET (meaning that all its discontinuities are in rational points with denominator  $q$ ), and two obvious relations between  $S$  and  $T_0$ . Say  $r_1(S, T_0) = id = r_2(S, T_0)$ .
- (2) (*Small perturbations*). Pick  $T$  close to  $T_0$ . Then  $r_1(S, T)$  and  $r_2(S, T)$  induce translations with small translation length on every interval of continuity that are not too close to some finite set of points  $X_q(S)$  (see Definition 3.3).
- (3) (*IET with small support*). The commutator  $U = [r_1(S, T), r_2(S, T)]$  has a small support, located in a neighbourhood of  $X_q(S)$ .
- (4) (*Drifting the support*). With an additional condition on  $T_0$ , some power  $k$  of  $T$  is such that  $T^k(\text{supp}(U)) \cap \text{supp}(U) = \emptyset$ . We say that  $T$  *drifts* the support of  $U$ .
- (5) (*Relation*). Then  $U$  commutes with  $T^kUT^{-k}$ , which gives the non trivial relation  $[U, T^kUT^{-k}] = id$ .

Points (1) to (3) are done in section 4.1, point (4) in section 4.2 and point (5) in section 4.3. Before stating all lemmas that enable us to prove Theorem 1.1, we need to recall some classical notations and introduce some others.

## 3. BACKGROUND FOR THE PROOF

**3.1. Classical notations.** First of all we define a few linear maps, useful when speaking about IETs.

**Definition 3.1.** Let  $T$  be an IET on  $[0; 1[$ . We denote by  $\lambda(T) = (l_1, \dots, l_n)$  the set of lengths of its intervals of continuity, by  $\beta(T) = (b_1, \dots, b_{n-1})$  its points of discontinuity, by  $\pi(T)$  its underlying permutation and by  $\tau(T) = (t_1, \dots, t_n)$  its translation lengths.

If needed, we write  $b_0 = 0$  and  $b_n = 1$ . To sum up, here are the four maps we will use in this article:

$$\begin{aligned} \lambda : \text{IET}([0; 1[) &\rightarrow \left\{ (a_1, \dots, a_n) \mid \begin{array}{l} n \in \mathbb{N}, \quad \forall i \in \{1, \dots, n\}, a_i \in \mathbb{R}_+^*, \\ \sum_{i=1}^n a_i = 1 \end{array} \right\} \\ T &\mapsto \lambda(T) = (l_1, \dots, l_n) \\ \\ \beta : \text{IET}([0; 1[) &\rightarrow \{(a_1, \dots, a_{n-1}) \mid n \in \mathbb{N}, 0 < a_1 < \dots < a_{n-1} < 1\} \\ T &\mapsto \beta(T) = (b_1, \dots, b_{n-1}) \\ \\ \pi : \text{IET}([0; 1[) &\rightarrow \left\{ \sigma \in \mathfrak{S}_n \mid \begin{array}{l} n \in \mathbb{N}, \\ \forall i \in \{1, \dots, n-1\}, \sigma(i+1) \neq \sigma(i) + 1 \end{array} \right\} \\ T &\mapsto \pi(T) \\ \\ \tau : \text{IET}([0; 1[) &\rightarrow \{(a_1, \dots, a_n) \mid n \in \mathbb{N}, \forall i, a_i \in ]-1; 1[ \} \\ T &\mapsto \tau(T) = (t_1, \dots, t_n) \end{aligned}$$

We will denote by  $\Delta(T)$  the set of discontinuity points of an IET  $T$ . If  $X$  is a set, write  $\mathcal{N}_\epsilon(X) = \{y \mid d(y, X) < \epsilon\}$  the open  $\epsilon$ -neighbourhood of  $X$ .

We denote by  $\text{IET}_\sigma$  the set  $\pi^{-1}(\sigma)$  of IETs having  $\sigma$  as underlying permutation. Finally, we equip  $\text{IET}([0; 1[)$  with the following distance:

$$\begin{aligned} d : \text{IET}([0; 1[) \times \text{IET}([0; 1[) &\rightarrow \mathbb{R}_+ \cup \{\infty\} \\ (S, T) &\mapsto \begin{cases} \|\lambda(S) - \lambda(T)\|_1 & \text{if } \pi(S) = \pi(T) \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

**Definition 3.2.** A permutation  $\sigma \in \mathfrak{S}_n$  is said to be *non admissible* when it exists  $k \in \{1, \dots, n\}$ , such that  $\sigma(k) = k$  and  $\sigma(\{1, \dots, k\}) = \{1, \dots, k\}$ . An IET  $T$  is *admissible* when  $\sigma = \pi(T) \in \mathfrak{S}_n$  is an admissible permutation.

We denote by  $\text{IET}_a$  the set of admissible IETs on  $[0; 1[$ .

### 3.2. Specific notations.

**Definition 3.3.** Let  $S$  be any IET. Let  $q$  be a positive integer. Let us define

$$X_q(S) = \Delta(S^{-1}) \cup S \left( \frac{1}{q}\mathbb{N} \right) \cup \frac{1}{q}\mathbb{N}$$

and

$$Y_q(S) = \pi_q(X_q(S))$$

where  $\pi_q : [0; 1] \rightarrow \left[0; \frac{1}{q}\right[$  is the canonical projection modulo  $\frac{1}{q}$ .

Define also

$$Z_q(S) = Y_q(S) \sqcup \left( \frac{1}{q} + Y_q(S) \right) \sqcup \left( \frac{2}{q} + Y_q(S) \right) \sqcup \dots \sqcup \left( \frac{q-1}{q} + Y_q(S) \right) \sqcup \{1\}.$$

Denote by

$$\alpha_q(S) = \min_{C \text{ connected component of } [0; \frac{1}{q}[ \setminus Y_q(S)} \text{ diam}(C)$$

the length of the smallest interval of  $[0; \frac{1}{q}[ \setminus Y_q(S)$ .

And finally define

$$\mathcal{U}_\alpha^q = \{R \in IET([0; 1]) \mid \alpha_q(R) > \alpha\}.$$

Note that  $X_q(S) \subset Z_q(S)$  and that  $\alpha_q(S) > 0$  is well defined because the set  $X_q(S)$  is finite. See Figure 1.

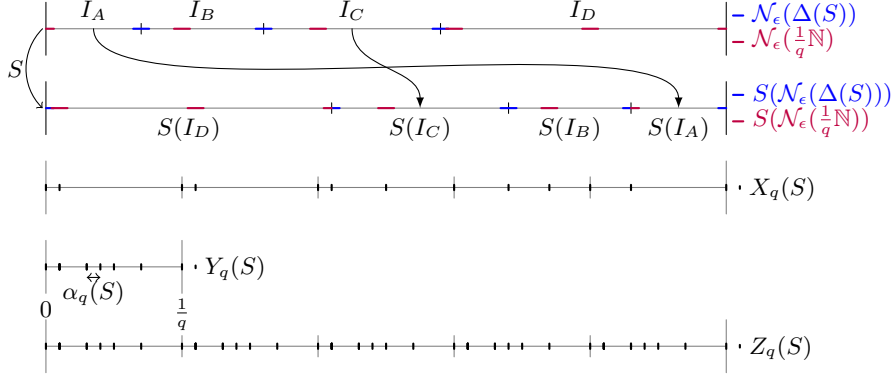


FIGURE 1. Illustration of the notations (here,  $q = 5$ )

*Remark 3.4.* Let  $S$  be an IET such that for every distinct  $x, y \in \Delta(S^{-1}) \cup \{0\}$ , one has  $x + \frac{1}{q}\mathbb{N} \neq y + \frac{1}{q}\mathbb{N}$ . If  $\alpha < \alpha_q(S)$ , then the set  $\mathcal{U}_\alpha^q$  is a neighbourhood of  $S$ .

Let us give an example of an IET  $S$  such that  $\mathcal{U}_\alpha(S)$  is not a neighbourhood of  $S$  to show the importance of the hypothesis on  $S$ .

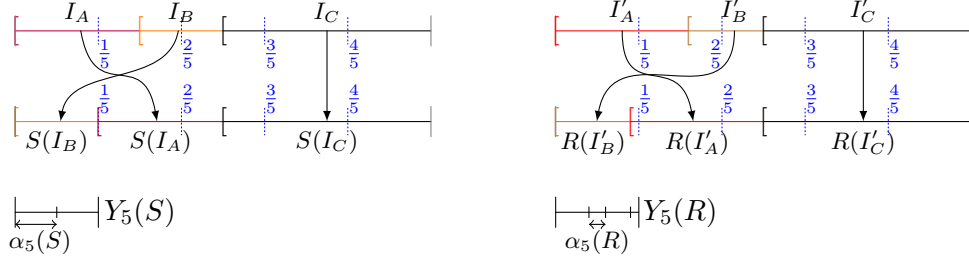
**Example 3.5.** Let  $q \geq 3$  and consider  $S$  defined by  $\pi(S) = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$

and  $\lambda(S) = \left(\lambda_A = \frac{3}{2q}, \lambda_B = \frac{1}{q}, \lambda_C = \frac{2q-5}{2q}\right)$ .

Let  $\alpha \in ]0; \frac{1}{2q}[$ ,  $\epsilon \in ]0; \alpha[$  and define  $R$  by  $\lambda(R) = \lambda(S) + (\epsilon, -\epsilon, 0)$  and  $\pi(R) = \pi(S)$ . See Figure 2 (where  $q = 5$ ).

Then  $\alpha_q(R) < \epsilon < \alpha < \frac{1}{2q} = \alpha_q(S)$ , so  $R \notin \mathcal{U}_\alpha$  though  $R$  is at distance  $2\epsilon$  from  $S$ .

*Remark 3.6.* The condition (written in Remark 3.4) on  $S$  corresponds to being outside a finite union of hyperplans. This means that for every  $S$  in a full-measure set  $\mathcal{A}$ , for every  $\alpha < \alpha_q(S)$ ,  $\mathcal{U}_\alpha(S)$  is a neighbourhood of  $S$ .


 FIGURE 2. Example where  $\mathcal{U}_\alpha$  is not a neighbourhood of  $S$ 

**Lemma 3.7.** *Let  $q \in \mathbb{N}$ ,  $\epsilon > 0$  and  $S$  be an IET in  $\mathcal{U}_\epsilon^q$ . Then:*

$$S \left( \mathcal{N}_\epsilon \left( \Delta(S) \cup \frac{1}{q}\mathbb{N} \right) \right) \subset \mathcal{N}_\epsilon(X_q(S)).$$

*Proof.* Note that  $S(\mathcal{N}_\epsilon(\Delta(S))) = \mathcal{N}_\epsilon(S(\Delta(S)) \cup \{1\})$  and that  $S(\mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})) \subset \mathcal{N}_\epsilon(S(\frac{1}{q}\mathbb{N}))$ . Hence:

$$\begin{aligned} S \left( \mathcal{N}_\epsilon \left( \Delta(S) \cup \frac{1}{q}\mathbb{N} \right) \right) &\subset \mathcal{N}_\epsilon(S(\Delta(S))) \cup \mathcal{N}_\epsilon(\{1\}) \cup \mathcal{N}_\epsilon \left( S \left( \frac{1}{q}\mathbb{N} \right) \right) \\ &\subset \mathcal{N}_\epsilon \underbrace{(S(\Delta(S)) \cup \{0, 1\})}_{=\Delta(S^{-1}) \cup \{0, 1\}} \cup \mathcal{N}_\epsilon \left( S \left( \frac{1}{q}\mathbb{N} \right) \right) \\ &\subset \mathcal{N}_\epsilon(X_q(S)). \end{aligned}$$

□

#### 4. PROOF OF THE MAIN RESULT

We are now ready to prove Theorem 1.1. Let us start to build an IET with small support.

**4.1. Building an IET with small support.** The following lemmas are quite the same as the ones of [DFG13], with little changes to overcome the non  $q$ -rationality of  $S$ .

**Lemma 4.1.** *For all  $\epsilon > 0$ ,  $q \in \mathbb{N}$ , there exists  $\eta > 0$  such that if  $T_0$  is  $q$ -rational and if  $T$  is  $\eta$ -close to  $T_0$ , then for all IET  $S$ , the IET  $ST^{q!}S^{-1}$  induces translations on  $[0; 1[ \setminus S(\mathcal{N}_\epsilon(\Delta(S) \cup \frac{1}{q}\mathbb{N}))$  with translation lengths strictly less than  $\epsilon$ .*

*Proof.* Let  $\eta < \frac{\epsilon}{2q!}$  and  $x \notin S(\mathcal{N}_\epsilon(\Delta(S) \cup \frac{1}{q}\mathbb{N}))$ .

Then  $S^{-1}(x) \notin \mathcal{N}_\epsilon(\Delta(S) \cup \frac{1}{q}\mathbb{N})$ . Since  $T^{q!}$  is a translation on  $[0; 1[ \setminus \mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$  with translation length strictly less than  $\epsilon$  (Lemma 5.5 of [DFG13]), one gets that  $|T^{q!}S^{-1}(x) - S^{-1}(x)| < \epsilon$ . Since  $d(S^{-1}(x), \Delta(S)) \geq \epsilon$ , this gives that  $T^{q!}S^{-1}(x)$  is in the same interval of continuity of  $S$  than  $S^{-1}(x)$ . Applying

the local isometry  $S$  to the inequality  $|T^{q!}S^{-1}(x) - S^{-1}(x)| < \epsilon$  one gets  $|ST^{q!}S^{-1}(x) - x| < \epsilon$ .  $\square$

**Lemma 4.2.** *For all  $\epsilon > 0$ ,  $q \in \mathbb{N}$ , there exists  $\eta > 0$  such that if  $T_0$  is  $q$ -rational and if  $T$  is  $\eta$ -close to  $T_0$ , then for all IET  $S$ , the IET  $[T^{q!}, ST^{q!}S^{-1}]$  induces the identity on each interval of  $[0; 1[\setminus \mathcal{N}_\epsilon(X_q(S))]$ .*

*Proof.* Let us write  $A_\epsilon = [0; 1[\setminus \mathcal{N}_\epsilon(X_q(S))]$  and  $A_{\frac{\epsilon}{2}} = [0; 1[\setminus \mathcal{N}_{\frac{\epsilon}{2}}(X_q(S))]$ .

Note that  $\mathcal{N}_{\frac{\epsilon}{2}}(\frac{1}{q}\mathbb{N}) \subset \mathcal{N}_\epsilon(X_q(S))$  and  $S(\mathcal{N}_{\frac{\epsilon}{2}}(\Delta(S) \cup \frac{1}{q}\mathbb{N})) \subset \mathcal{N}_\epsilon(X_q(S))$ .

Apply Lemma 4.1 with  $\frac{\epsilon}{2}$ : whenever  $T$  is closer than  $\eta = \frac{\epsilon}{4q!}$  to  $T_0$ , the IET  $ST^{q!}S^{-1}$  induces translations of translation lengths strictly smaller than  $\frac{\epsilon}{2}$  on each connected component of  $[0; 1[\setminus S(\mathcal{N}_{\frac{\epsilon}{2}}(\Delta(S) \cup \frac{1}{q}\mathbb{N}))] \supset A_{\frac{\epsilon}{2}}$ . Moreover, the IET  $T^{q!}$  induces translations of translation lengths strictly smaller than  $\frac{\epsilon}{2}$  on each connected component of  $[0; 1[\setminus \mathcal{N}_{\frac{\epsilon}{2}}(\frac{1}{q}\mathbb{N})] \supset A_{\frac{\epsilon}{2}}$  (Lemma 5.5 of [DFG13]).

Let  $x \in A_\epsilon$ . Denote by  $I = [a_1; a_2]$  the connected component of  $x$  in  $A_{\frac{\epsilon}{2}}$  and by  $t, t' \in [-\frac{\epsilon}{2}; \frac{\epsilon}{2}]$  the translation lengths on  $I$  of  $T^{q!}$  and  $ST^{q!}S^{-1}$  respectively. Both  $x+t$  and  $x+t'$  are in  $I$  so  $T^{q!} \circ ST^{q!}S^{-1}(x) = T^{q!}(x+t') = x+t'+t$  and  $ST^{q!}S^{-1} \circ T^{q!}(x) = ST^{q!}S^{-1}(x+t) = x+t+t'$ .

Hence  $T^{q!}$  and  $ST^{q!}S^{-1}$  commute on  $A_\epsilon = [0; 1[\setminus \mathcal{N}_\epsilon(X_q(S))]$ . We conclude that  $[T^{q!}, ST^{q!}S^{-1}]$  induces the identity on each interval of  $[0; 1[\setminus \mathcal{N}_\epsilon(X_q(S))]$ .  $\square$

In other words, the IET  $U = [T^{q!}, ST^{q!}S^{-1}]$  has its support included in  $\mathcal{N}_\epsilon(X_q(S))$  and so in  $\mathcal{N}_\epsilon(Z_q(S))$ .

This means that the support of  $U$  is both small and included in the "periodic" set  $\mathcal{N}_\epsilon(Z_q(S)) = \mathcal{N}_\epsilon(Y_q(S)) \sqcup \left(\mathcal{N}_\epsilon(Y_q(S)) + \frac{1}{q}\right) \sqcup \dots \sqcup \left(\mathcal{N}_\epsilon(Y_q(S)) + \frac{q-1}{q}\right)$ . This periodicity is important since the only control we have on the drift (as we will see in section 4.2) is modulo  $\frac{1}{q}$ .

## 4.2. Drifting the support.

**Definition 4.3** (drifting vector and drifting direction). Let  $\sigma \in \mathfrak{S}_n$ , and  $T \in IET_\sigma$ . Write  $(l_1, \dots, l_n) = \lambda(T)$  and  $(t_1, \dots, t_n) = \tau(T)$ .

We say that  $\sigma$  is *driftable* if there exist  $(dl_1, \dots, dl_n)$ , where  $\sum dl_i = 0$ , such that the IET  $T'$  defined by  $\lambda(T') = (l_1 + dl_1, \dots, l_n + dl_n)$  and  $\pi(T') = \pi(T)$  has its translation lengths  $\tau(T') = (t_1 + dt_1, \dots, t_n + dt_n)$  with all  $dt_i$  positive.

In this case,  $(dl_1, \dots, dl_n)$  is called a *drifting direction* and  $(dt_1, \dots, dt_n)$  a *drifting vector*.

*Remark 4.4.* A permutation is driftable if and only if it is admissible. See Proposition 5.12 of [DFG13] for a proof.

**Definition 4.5.** Let  $T_0$  be an IET and  $\sigma = \pi(T_0) \in \mathfrak{S}_n$  be its underlying permutation. Let  $dl \in \mathbb{R}^n$ . We define  $T_\theta \in IET_\sigma$  by  $\lambda(T_\theta) = \lambda(T_0) + \theta dl$ ,

where  $\theta$  is small enough to ensure that all the lengths  $l_i(T_\theta)$  of the intervals of continuity of  $T_\theta$  are positive.

**Lemma 4.6.** *Let  $\sigma \in \mathfrak{S}_n$  be a driftable permutation, and  $dl$  and  $dr$  be the associated drifting direction and drifting vector. Let  $q \in \mathbb{N}$  and  $T_0 \in IET_\sigma([0; 1])$  be  $q$ -rational. Let  $\theta$  be such that  $0 < \theta dr_{\max} < \frac{1}{q}$  and small enough so that  $T_\theta$  is well defined. Let  $\rho = \frac{dt_{\max}}{dt_{\min}}$ ,  $\alpha < \frac{1}{2q}$ ,  $\epsilon < \frac{\alpha}{11\rho}$ ,  $\theta < \frac{\epsilon}{dt_{\min}}$  and  $\mu < \frac{\theta dt_{\min}}{4}$ . Let  $T$  be  $\mu$ -close to  $T_\theta$ .*

*Then there exists  $k \in \mathbb{N}$  such that all the translation lengths of  $T^k$  are in  $[2\epsilon; \alpha - 2\epsilon] \bmod \frac{1}{q}$ .*

*Proof.* First, one has

$$\theta dt_{\min} < \epsilon < \frac{\alpha}{11\rho} \leq \frac{1}{22q\rho} < \frac{1}{q}$$

and

$$2\mu < \frac{1}{2}\theta dt_{\min} < \theta dt_{\min}$$

and

$$\theta dt_{\max} + 2\mu < \theta dt_{\max} + \theta dt_{\min} \leq 2\theta dt_{\max} = 2\theta dt_{\min}\rho < 2\epsilon\rho < \frac{1}{11q} < \frac{1}{q}.$$

We deduce (Lemma 5.9 of [DFG13]) that all the translation lengths of  $T$  are in the interval  $[\theta dt_{\min} - 2\mu; \theta dt_{\max} + 2\mu] \bmod \frac{1}{q}$ , which is included in  $[\frac{1}{2}\theta dt_{\min}; \frac{3}{2}\theta dt_{\max}]$ .

For every  $k \in \mathbb{N}$  such that  $\frac{3k}{2}\theta dt_{\max} < \frac{1}{q}$ , the IET  $T^k$  has all its translation lengths in  $[\frac{k}{2}\theta dt_{\min}; \frac{3k}{2}\theta dt_{\max}] \bmod \frac{1}{q}$ . We want an integer  $k$  such that both  $2\epsilon < \frac{k}{2}\theta dt_{\min}$  and  $\frac{3k}{2}\theta dt_{\max} < \alpha - 2\epsilon$ . Take  $k$  such that  $4\epsilon < k\theta dt_{\min} \leq 5\epsilon$  (this is possible because  $\theta dt_{\min} < \epsilon$ ). Then one has:

$$\frac{3k}{2}\theta dt_{\max} = \frac{3k}{2}\theta dt_{\min}\rho \leq \frac{3}{2} \times 5\epsilon\rho < \frac{15}{2} \frac{\alpha}{11} = \alpha - \frac{7\alpha}{22} < \alpha - \frac{2\alpha}{11} < \alpha - 2\epsilon.$$

So  $T^k$  has all its translation lengths in  $[2\epsilon; \alpha - 2\epsilon] \bmod \frac{1}{q}$ .  $\square$

**4.3. Conclusion.** We are in the last step: ready to build a non trivial relation.

**Proposition 4.7.** *Assume  $T_0$  to be  $q$ -rational and its underlying permutation  $\sigma \in \mathfrak{S}_n$  to be driftable. Let  $S$  be any IET.*

*Then there exist a set  $\mathcal{U} \subset IET([0; 1])$  that contains  $S$  and an open set  $\mathcal{V} \subset IET_\sigma([0; 1])$  which accumulates on  $T_0$  such that  $\langle S, T \rangle$  is not free of rank 2 whenever  $T \in \mathcal{V}$ .*

*Moreover if  $S$  is such that for every distinct  $x, y \in \Delta(S)$ , one has  $x + \frac{1}{q}\mathbb{N} \neq y + \frac{1}{q}\mathbb{N}$ , then the set  $\mathcal{U}$  is a neighbourhood of  $S$ .*

*Proof.* Let  $\alpha < \alpha_q(S)$ . Note that  $\alpha_q(S) \leq \frac{1}{2q}$  because  $S$  is not  $q$ -rational.

Let  $dl$  and  $dt$  be the drifting direction and vector for  $\sigma$ . Let  $\rho = \frac{dt_{\max}}{dt_{\min}}$  and take

$$\epsilon < \frac{\alpha}{11\rho}; \quad \eta < \frac{\epsilon}{4(q!+2)}; \quad \theta < \min\left(\frac{\epsilon}{dt_{\min}}, \frac{\eta}{2\|dl\|_1}\right); \quad \mu < \min\left(\frac{\theta dt_{\min}}{4}, \frac{\eta}{2}\right).$$

Take  $\mathcal{U} = \mathcal{U}_\alpha^q$ . If  $S$  is such that for every distinct  $x, y \in \Delta(S)$ , one has  $x + \frac{1}{q}\mathbb{N} \neq y + \frac{1}{q}\mathbb{N}$ , then the set  $\mathcal{U}$  is a neighbourhood of  $S$  (Remark 3.4).

Take  $T_\theta \in IET_\sigma$  such that  $\lambda(T_\theta) = \lambda(T_0) + \theta dl$ . Define  $\mathcal{V}_{dl, \theta}$  the set of IETs that are at distance strictly less than  $\mu$  from  $T_\theta$ . Then every  $T \in \mathcal{V}_{dl, \theta}$  verifies:

$$d(T, T_0) \leq d(T, T_\theta) + d(T_\theta, T_0) < \mu + \theta \|dl\|_1 < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

From Lemma 4.2 one knows that for every  $(R, T) \in \mathcal{U} \times \mathcal{V}_{dl, \theta}$  the IET  $U = [T^{q!}, RT^{q!}R^{-1}]$  has its support included in  $\mathcal{N}_\epsilon(Z_q(R))$ . By definition of  $\alpha_q(R)$ , the smallest connected component of  $[0; 1[ \setminus \mathcal{N}_\epsilon(Z_q(R))$  has length at least  $\alpha_q(R) - 2\epsilon$ , hence at least  $\alpha - 2\epsilon$  because  $R \in \mathcal{U} = \mathcal{U}_\alpha^q$ .

So it is enough to drift the support of  $U$  with a drift in  $[2\epsilon, \alpha - 2\epsilon]$  modulo  $\frac{1}{q}\mathbb{Z}$ . And from Lemma 4.6, there exists  $k \in \mathbb{N}$  such that  $[U, T^k U T^{-k}] = id$ .

It remains to check that this relation is not trivial. Denote by  $u = t^{q!} r t^{q!} r^{-1} t^{-q!} r t^{-q!} r^{-1}$  the (non-trivial) word over the letters  $r^{\pm 1}$  and  $t^{\pm 1}$  such that  $u(S, T) = U$ . The word  $w = u t^k u t^{-k} u^{-1} t^k u^{-1} t^{-k}$  is equal to:

$$\begin{aligned} & (t^{q!} r t^{q!} r^{-1} t^{-q!} r t^{-q!} r^{-1}) \cdot t^k \cdot (t^{q!} r t^{q!} r^{-1} t^{-q!} r t^{-q!} r^{-1}) \cdot t^{-k} \\ & (r t^{q!} r^{-1} t^{q!} r t^{-q!} r^{-1} t^{-q!}) \cdot t^k \cdot (r t^{q!} r^{-1} t^{q!} r t^{-q!} r^{-1} t^{-q!}) \cdot t^{-k} \end{aligned}$$

which is reduced, except if  $k = q!$  in which case it reduces to the reduced word:

$$t^{q!} r t^{q!} r^{-1} t^{-q!} r t^{-q!} r^{-1} t^k t^{q!} r t^{q!} r^{-1} t^{-q!} r t^{-q!} r^{-1} t^{-k} r t^{q!} r^{-1} t^{2q!} r t^{-q!} r^{-1} t^{-q!} t^{-k}.$$

So the relation is not trivial. Hence for every  $(R, T) \in \mathcal{U} \times \mathcal{V}_{dl, \theta}$  the subgroup  $\langle R, T \rangle$  is not free of rank 2.

In order to have an open set  $\mathcal{V}$  which accumulates on  $T_0$  we take the union of all the convenient  $\mathcal{V}_{dl, \theta}$ . More precisely, we define:

$$\mathcal{V}_{dl} = \bigcup_{\epsilon < \frac{\alpha}{11\rho}} \bigcup_{\eta < \frac{\epsilon}{4(q!+2)}} \bigcup_{\theta < \min\left(\frac{\epsilon}{dt_{\min}}, \frac{\eta}{2\|dl\|_1}\right)} \mathcal{V}_{dl, \theta} \quad \text{and} \quad \mathcal{V} = \bigcup_{dl \text{ drifting direction}} \mathcal{V}_{dl}.$$

□

*Proof of Theorem 1.1.* Let  $S$  be an IET on  $[0; 1[$ . For every  $q$ -rational driftable IET  $T_0$ , fix an  $\alpha < \alpha_q(S)$  and denote by  $\mathcal{V}(T_0)$  the open sets given by the previous proposition and define:

$$\Omega_{irred}(S) = \bigcup_{q \geq 2} \bigcup_{2 \leq n \leq q} \bigcup_{\sigma \in \mathfrak{S}_n \text{ driftable}} \bigcup_{T_0 \in IET_\sigma \text{ } q\text{-rational}} \mathcal{V}(T_0).$$

The set  $\Omega_{irred}(S)$  is a union of open sets hence is open, it is dense in  $IET_{irred}([0; 1[)$  because of the density of  $\mathbb{Q}$  in  $\mathbb{R}$ . And the proof of the previous proposition shows that for every  $T \in \Omega_{irred}(S)$ ,  $\langle S, T \rangle$  is not free of rank 2.

□



**4.4. Looking for smaller relations.** We have shown that given any IET  $S$ , there is a dense subset of admissible IETs sharing a relation with  $S$ . The next natural question is the measure of this set (in each connected components of  $IET([0; 1])$ ). It is non-zero but seems very small. Indeed the open set  $\mathcal{V}(T_0)$  that we built around a  $q$ -rational IET  $T_0$  has a diameter which is bounded by  $\frac{1}{q^3}$  times a constant (depending only on the underlying permutation  $\sigma$  of  $T_0$ ). For instance, there does not exist a  $T_0$  such that the example of Arnoux-Yoccoz is in  $\mathcal{V}(T_0)$ . This example is interesting because it is self-similar (applying a certain number of times Rauzy induction gives the same IET).

But maybe we could have built a bigger set  $\Omega(S)$ . We discuss quickly this issue in the following subsection.

**4.4.1. Arnoux-Yoccoz's example.** Let us present an example from Arnoux and Yoccoz (see [AY81] and [Arn88]). Let  $a$  be the only real number such that  $a^3 + a^2 + a = 1$ . Define the IET  $g$  by

$$\lambda(g) = \left( \lambda_A = \frac{a}{2}, \lambda_{A'} = \frac{a}{2}, \lambda_B = \frac{a^2}{2}, \lambda_{B'} = \frac{a^2}{2}, \lambda_C = \frac{a^3}{2}, \lambda_{C'} = \frac{a^3}{2} \right)$$

and

$$\pi(g) = \begin{pmatrix} A & A' & B & B' & C & C' \\ A' & A & B' & B & C' & C \end{pmatrix}.$$

Let  $h$  be the rotation of angle  $\frac{1}{2}$ , namely  $\lambda(h) = (\lambda_A = \frac{1}{2}, \lambda_B = \frac{1}{2})$  and  $\pi(h) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ . Define finally  $f = h \circ g$ . One has:

$$\lambda(f) = \left( \lambda_A = \frac{1-a}{2}, \lambda_B = a - \frac{1}{2}, \lambda_C = \frac{a}{2}, \lambda_D = \frac{a^2}{2}, \lambda_E = \frac{a^2}{2}, \lambda_F = \frac{a^3}{2}, \lambda_G = \frac{a^3}{2} \right)$$

and

$$\pi(f) = \begin{pmatrix} A & B & C & D & E & F & G \\ B & E & D & G & F & C & A \end{pmatrix}.$$

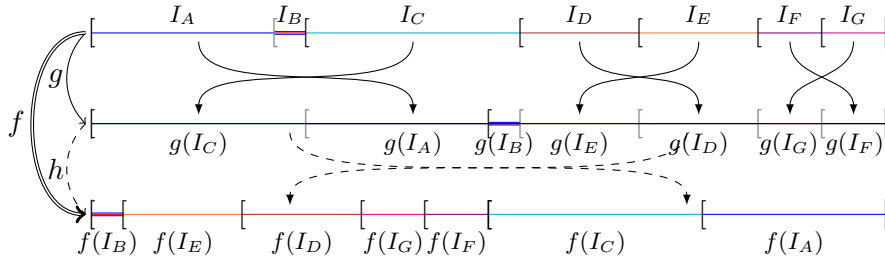


FIGURE 3. The example of the IET  $f$  from Arnoux and Yoccoz

For every  $X \in \{A, B, \dots, G\}$ ,  $\lambda_X$  is an algebraic number of degree 3, so there exist  $A(\lambda_X)$  such that for every rational number  $\frac{p}{q}$ ,  $\left| \lambda_X - \frac{p}{q} \right| \geq \frac{A(\lambda_X)}{q^3}$

(by Liouville's Theorem). Moreover we easily compute such  $A(\lambda_X)$ . We conclude that a  $q$ -rational IET  $T_0$  is far from  $f$  by at least:

$$d(f, T_0) \geq \sum_{X \in \{A, B, \dots, G\}} \frac{A(\lambda_X)}{q^3} \geq \frac{282}{q^2} > \frac{1}{q!}.$$

On the other hand  $f$  should be  $\eta$ -close to  $T_0$  to be in  $\mathcal{V}(T_0)$ , where  $\eta$  is less than  $\frac{1}{q!}$ . This proves that  $f$  is in none of the  $\mathcal{V}(T_0)$ .

The factor  $\frac{1}{q!}$  comes from the length of the basic relation we used ( $T_0^{q!} = id$ ). To increase the size of  $\mathcal{V}(T_0)$ , we can take a smaller basic relation like  $ST_0^k S^{-1} = id$  where  $k$  is the order of  $T_0$ . This gives bigger open sets  $\mathcal{V}(T_0)$  around  $q$ -rational IETs  $T_0$  that have "small" order.

We want to know if it is enough. For every  $q$  between 20 and 20000, we have computed the closest  $q$ -rational IET to  $f$ , its distance  $\delta$  to  $f$ , its order  $\mathfrak{o}$  and the bound  $\mathfrak{b} = 40q(\mathfrak{o}+2)\delta$ . The condition  $\mathfrak{b} < 1$  is necessary so that  $f$  is in an enlarged  $\mathcal{V}(T_0)$ .

The distance  $\delta(q)$  between  $f$  and the closest  $q$ -rational IET seems to decrease approximately like  $\frac{1}{q}$  (and not faster). See Figure 4.

*Remark 4.8.* The distance between  $f$  and the closest  $q$ -rational IET to  $f$  is at most equal to  $\frac{5}{q}$ . Computations suggest that this distance do not decrease faster.

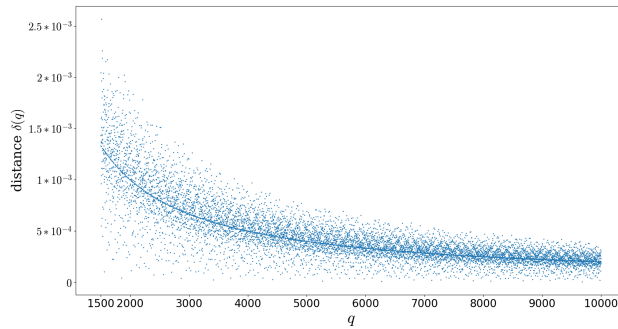


FIGURE 4. Distance  $\delta(q)$  between  $f$  and the closest  $q$ -rational IET to  $f$

For some values of  $q$ , the order  $\mathfrak{o}(q)$  of this  $q$ -rational IET (the closest to  $f$ ) increases linearly. See Figure 5.

As a result, the bound  $\mathfrak{b}(q)$  increases at least linearly and it seems that there is no rational IET  $T_0$  close enough to  $f$  so that  $f \in \mathcal{V}(T_0)$ . See Figure 6.

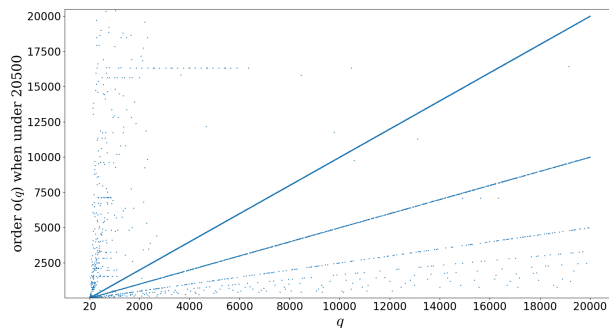


FIGURE 5. Order  $\mathfrak{o}(q)$  of the closest  $q$ -rational IET to  $f$

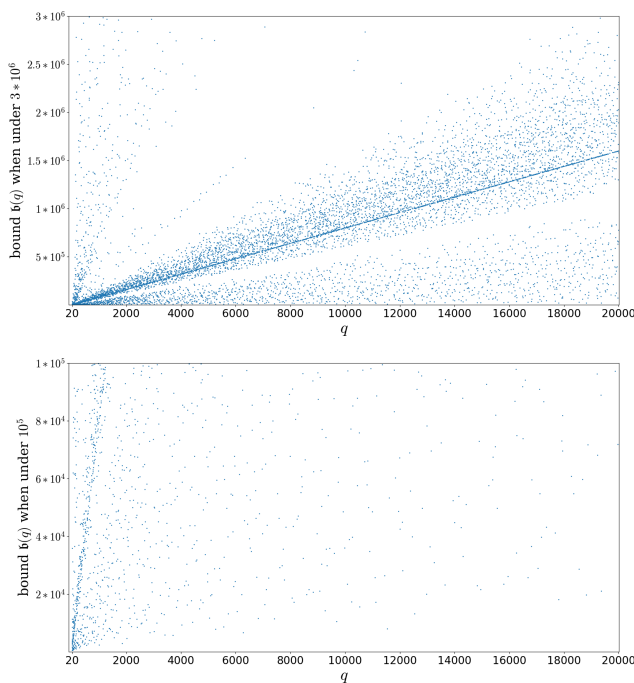


FIGURE 6. Bound  $\mathfrak{b}(q)$

5. FREE GROUP IN THE GROUP OF AFFINE INTERVAL EXCHANGE TRANSFORMATIONS

**Definition 5.1** (AIET). An *affine interval exchange transformation* (AIET) on  $[0; 1[$  is a bijection from  $[0; 1[$  onto itself which is everywhere continuous on the right, continuous except on a finite number of points and differentiable but on a finite set and with constant differential on every interval where it is defined.

One denotes by  $\Delta(T)$  the union of discontinuities of  $T$  and of the finite set where its differential  $T'$  is not defined.

On the following figures, we represent in deep (resp. light) colors the intervals that are expanded (resp. contracted) by the AIET and their images in light (resp. deep) colors.

Contrary to what happens in the group of IETs, one can easily find a free subgroup in the group of AIETs because its elements can contract and expand some parts of the domaine of definition.

*Proof of Proposition 1.3.* Let  $V, W, X, Y$  be four disjoint subintervals of  $[0; 1[$  that do not cover  $[0; 1[$ . Let  $f$  be an AIET that sends  $V$  onto  $[0; 1[ \setminus W$  (and  $[0; 1[ \setminus V$  onto  $W$ ). This implies that  $f(X)$  and  $f(Y)$  are included in  $W$  and  $f^{-1}(X)$  and  $f^{-1}(Y)$  are included in  $V$ . Let  $g$  be an AIET that sends  $X$  onto  $[0; 1[ \setminus Y$  (and  $[0; 1[ \setminus X$  onto  $Y$ ). This implies that  $g(V)$  and  $g(W)$  are included in  $Y$  and  $g^{-1}(V)$  and  $g^{-1}(W)$  are included in  $X$ . See Figure 7. Applying ping-pong lemma, we see that  $\langle f, g \rangle$  is a free group of rank 2.

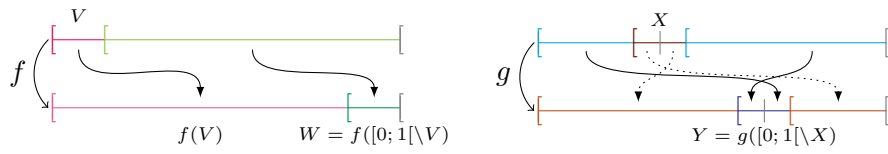


FIGURE 7. Example of AIETs  $f$  and  $g$  spanning a free group of rank 2

□

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