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# Rapid adiabatic passage with two coupled pendula

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**Abstract** – We propose a classical analogy to illustrate the up  $\leftrightarrow$  down flip of a one-half spin submitted to a static and a rotating magnetic field, in the standard magnetic resonance configuration. Surprising though it may seem, it is possible to design a rather simple device – namely a set of two coupled pendula – to simulate and visualize the Larmor and Rabi precessions of the latter spin. In this analogy, the static field is associated with the angular frequency detuning between both eigenmodes of the two-degree-of-freedom harmonic oscillator constituted by the two pendula, whereas the rotating field is simulated by means of a parametric modulation of the length of one of the pendula. Under the circumstances, the up or down spin states are represented by either eigenmodes, and consequently the spin-flip is put in concrete form by the coupled pendula being driven from one eigenmode to the other. In the present paper, we experimentally demonstrate the possibility of triggering such a flip using a mechanical analogy of the so-called rapid adiabatic passage technique.

Let us consider a one-half spin, say a proton, with an angular momentum  $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ , a gyromagnetic factor  $\gamma$  and consequently a magnetic dipolar momentum  $\vec{M} = \gamma\vec{S}$ . The density operator  $\rho$  of this spin can be expanded on the  $\{\mathbb{1}, \sigma_X, \sigma_Y, \sigma_Z\}$  basis made up with the identity  $\mathbb{1}$  and the three Pauli operators  $\sigma_X$ ,  $\sigma_Y$  and  $\sigma_Z$ :

$$\rho = \frac{1}{2}(\mathbb{1} + m_X\sigma_X + m_Y\sigma_Y + m_Z\sigma_Z) = \frac{1}{2}(\mathbb{1} + \vec{m} \cdot \vec{\sigma}),$$

where  $\vec{m} = \text{Tr}\{\rho\vec{\sigma}\}$ . In presence of a magnetic field  $\vec{B}$ , the Hamiltonian of the spin is  $H = -\vec{M} \cdot \vec{B} = \frac{1}{2}\hbar\vec{\Omega} \cdot \vec{\sigma}$ , where  $\vec{\Omega} = -\gamma\vec{B}$  is an angular frequency. With these standard notations [1, 2], the Schrödinger equation  $i\hbar d\rho/dt = [H, \rho]$  simply reads, all simplifications carried out,

$$\frac{d\vec{m}}{dt} = \vec{\Omega} \wedge \vec{m}. \quad (1)$$

It is noteworthy that, despite its classical appearance, the above equation is in fact quantum. It indicates that the mean value  $\vec{m}$  of operator  $\vec{\sigma}$  moves in space with a constant modulus  $|\vec{m}|$  and an instantaneous rotation vector  $\vec{\Omega}$ . If the magnetic field is static, say  $\vec{B} = B_0\vec{e}_Z$ , this movement is the so-called Larmor precession [3], namely  $\vec{m}$  uniformly

rotates around  $\vec{B}_0$  with the constant angular velocity  $\vec{\Omega}_L = -\gamma\vec{B}_0$ .

In practice, due to various interactions with its surroundings, the spin undergoes an effective magnetic field  $\vec{B}(t) = \vec{B}_0 + \vec{b}_r(t)$ , where  $\vec{b}_r(t)$  is (at least partially) random. This random interaction results in a relaxation process, the Markovian treatment of which involves two relaxation times,  $T_1$  (longitudinal) and  $T_2$  (transverse). Within this framework, the motion equation (1) takes the form of the Bloch equation

$$\frac{d\vec{m}}{dt} = \vec{\Omega}_L \wedge \vec{m} - \frac{1}{T_1}(m_Z - m_Z^B)\vec{e}_Z - \frac{1}{T_2}\vec{m}_\perp,$$

where  $m_Z$  and  $\vec{m}_\perp$  respectively stand for the longitudinal (*i.e.*  $\parallel \vec{e}_Z$ ) and transverse (*i.e.*  $\perp \vec{e}_Z$ ) components of  $\vec{m}$ ,  $m_Z^B (= -\tanh(\hbar\Omega_L/2k_B T))$  at temperature  $T$ ) corresponding to the thermic Boltzmann equilibrium value of  $m_Z$ .

As well known from NMR [3, 5], it is possible to drive the protons off their thermal equilibrium state (characterized by  $\vec{m}^{\text{eq}} = m_Z^B\vec{e}_Z$ ) thanks to a wealth of radio-frequency short pulse sequences (by “short” we mean shorter than the relaxation times). One among them, widely used in the Magnetic Resonance Imaging (MRI) techniques [6, 7], is the so-called  $\pi$ -pulse. Let us recall

quickly what it is all about. In addition to the static field  $B_0 \vec{e}_Z$ , let us superimpose a rotating magnetic field  $\vec{b}_1(t) = b_1(\cos \omega t \vec{e}_X + \sin \omega t \vec{e}_Y)$ . Disregarding relaxation, the motion equation of the proton reads, allowing for (1),

$$\frac{d\vec{m}}{dt} = (\vec{\Omega}_L + \vec{\omega}_R(t)) \wedge \vec{m}, \quad (2)$$

where  $\vec{\omega}_R(t) = -\gamma \vec{b}_1(t)$ . The above equation can be solved for  $\vec{m}$  using a well known trick. Let us indeed consider a frame rotating around  $\vec{e}_Z$  with the angular velocity  $\vec{\omega} = \omega \vec{e}_Z$ . In this frame, let us introduce the basis  $\{\vec{e}_{X'} = \cos \omega t \vec{e}_X + \sin \omega t \vec{e}_Y, \vec{e}_{Y'} = -\sin \omega t \vec{e}_X + \cos \omega t \vec{e}_Y, \vec{e}_{Z'} = \vec{e}_Z\}$ . Field  $\vec{b}_1$  is static (as well as  $\vec{B}_0$ ) in the rotating frame:  $\vec{b}_1 = b_1 \vec{e}_{X'}$ . Denoting  $(d\vec{m}/dt)'$  the time-derivative of vector  $\vec{m}$  in the latter frame, and observing that  $(d\vec{m}/dt)' = (d\vec{m}/dt) - \vec{\omega} \wedge \vec{m}$ , we are left with

$$\left(\frac{d\vec{m}}{dt}\right)' = [(\Omega_L - \omega)\vec{e}_{Z'} + \omega_R \vec{e}_{X'}] \wedge \vec{m} = \vec{\Omega}_{\text{eff}} \wedge \vec{m}. \quad (3)$$

Consequently, in the rotating frame, the motion of vector  $\vec{m}$  is a Larmor precession with an effective angular velocity  $\vec{\Omega}_{\text{eff}} = (\Omega_L - \omega)\vec{e}_{Z'} + \omega_R \vec{e}_{X'}$ . This situation is illustrated in figure 1. Interestingly, if the angular frequency  $\omega$  of the rotating field is chosen equal to  $\Omega_L$ , then the spin will precess around  $\vec{e}_{X'}$  at the so-called Rabi [8] angular frequency  $\omega_R$ . As a consequence, a resonant (*i.e.*  $\omega = \Omega_L$ ) rotating field pulse with duration  $\Delta t = \pi/\omega_R$  will flip an initially thermalized proton magnetic moment from parallel to  $\vec{B}_0$  to antiparallel. Observe that, since  $\omega_R$  is proportional to the amplitude  $b_1$  of the radiofrequency pulse, it is in principle always possible to manage to have  $\Delta t \ll T_1, T_2$ , so that neglecting relaxation during the  $\pi$ -pulse is relevant.

The  $\pi$ -pulse is selective in the sense that, among a bunch of protons, only the resonant ones (those with  $\Omega_L = \omega$ ) will be flipped by the pulse. Should (for any reason) the static field  $\vec{B}_0$  be inhomogeneous, then only a fraction of the spins will respond. This feature is turned to account in medical imaging, for instance to “slice” the tissue to be imaged and thus reconstruct a 3D representation. Nevertheless, it can also be a drawback in some other situation, and it could be interesting to obtain an overall nonselective spin flip. It is possible, by means of a hardly more complicated pulse sequence, as explained hereafter.

To begin with, let us come back to equation (1) and consider the case of a slowly varying  $\vec{\Omega}(t)$ . Let us set  $\vec{\Omega}(t) = \Omega(t)\vec{e}(t)$ , with vector  $\vec{e}(t)$  unitary, and  $\vec{\omega}(t) = \vec{e} \wedge \dot{\vec{e}}$ ; therefore  $\dot{\vec{e}} = \vec{\omega} \wedge \vec{e}$ . Let us now introduce the moving frame whose instantaneous rotation velocity is  $\vec{\omega}(t)$  and denote (as above) the vector time-derivative in this frame with a prime ( $'$ ). Splitting  $\vec{m}$  in its longitudinal  $\vec{m}_{\parallel}$  (parallel to  $\vec{e}$ ) and transverse  $\vec{m}_{\perp}$  (orthogonal to  $\vec{e}$ ) parts, we have  $\vec{m} = \vec{m}_{\parallel} + \vec{m}_{\perp}$  with

$$m_{\parallel} = \vec{m} \cdot \vec{e} \quad \text{and} \quad \vec{m}_{\perp} = \vec{e} \wedge (\vec{m} \wedge \vec{e}). \quad (4)$$

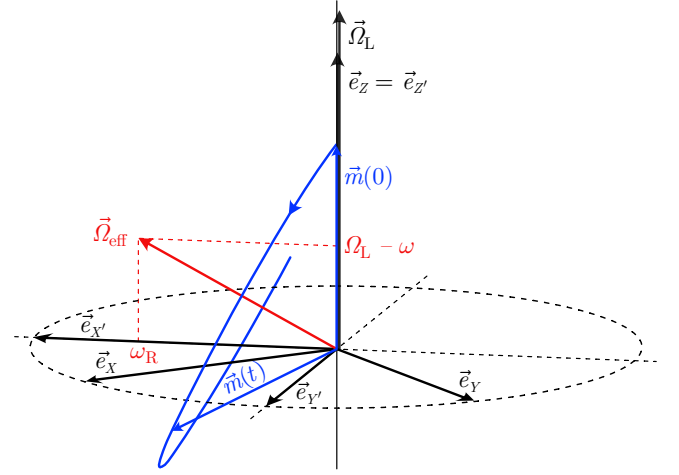


Fig. 1: The motion of vector  $\vec{m}$  in the rotating frame  $\{\vec{e}_{X'}, \vec{e}_{Y'}, \vec{e}_{Z'}\}$  is a Larmor precession with the angular velocity  $\vec{\Omega}_{\text{eff}} = (\Omega_L - \omega)\vec{e}_{Z'} + \omega_R \vec{e}_{X'}$ . If  $\omega$  is tuned equal to  $\Omega_L$ , then this precession occurs around the horizontal axis  $\vec{e}_{X'}$  at the Rabi angular frequency  $\omega_R$ : after a rotating field pulse with a duration  $\Delta t = \pi/\omega_R$ , a magnetic momentum  $\vec{m}_0$  initially aligned with the static field  $\vec{B}_0$  is turned into its opposite  $-\vec{m}_0$ , *i.e.* antiparallel to  $\vec{B}_0$ . This spin flip is a resonant effect: if  $|\Omega_L - \omega| \gg \omega_R$ ,  $\vec{\Omega}_{\text{eff}}$  is roughly parallel to axis  $\vec{e}_{Z'}$ , and the pulse does not affect the spin, the state of which remains consequently unchanged.

Observe that, since  $(d\vec{e}/dt)' = 0$  by definition, the same splitting is available too for the time-derivative of  $\vec{m}$ , with  $(d\vec{m}/dt)'_{\parallel} = (dm_{\parallel}/dt)'$  and  $(d\vec{m}/dt)'_{\perp} = (d\vec{m}_{\perp}/dt)'$ . Since  $(d\vec{m}/dt)' = (\vec{\Omega} - \vec{\omega}) \wedge \vec{m}$ , we easily derive

$$\frac{dm_{\parallel}}{dt} = -\vec{e} \cdot (\vec{\omega} \wedge \vec{m}_{\perp}), \quad \left(\frac{d\vec{m}_{\perp}}{dt}\right)' = \vec{\Omega} \wedge \vec{m}_{\perp} - (\vec{\omega} \wedge \vec{m})_{\perp}. \quad (5)$$

The above equation set is exact; it can nevertheless be simplified if  $|\vec{\omega}| \ll |\vec{\Omega}|$ , *i.e.* if the direction  $\vec{e}$  of vector  $\vec{\Omega}$  varies *adiabatically* (in the Ehrenfest sense). In the latter case, we can neglect the  $(\vec{\omega} \wedge \vec{m})_{\perp}$  term in the right-hand side of the second equation (5). The movement of  $\vec{m}_{\perp}$  can then be approximated by a rotation of angle  $\int^t \Omega(t') dt'$  around axis  $\vec{e}$ . Since this rotation is fast compared to that of axis  $\vec{e}$  itself, the  $(\vec{e}, \vec{\omega}, \vec{m}_{\perp})$  term brings no secular contribution to the time-evolution of  $m_{\parallel}$  (see the first equation (5)):  $m_{\parallel}$  is an adiabatic invariant of the motion of  $\vec{m}$ . In other words,  $\vec{m}$  makes a constant angle  $\psi$  with  $\vec{e}$ . *Inter alia*, if  $\vec{m}$  is initially parallel (or antiparallel) to  $\vec{e}$ , it remains so in the further course of its movement. This result is a particular case of the so-called “adiabatic theorem”. It can be turned into account to implement a nonresonant inversion of the spin population, as explained below.

Let us come back to the Rabi configuration of the  $\pi$ -pulse sequence, namely a static field  $\vec{B}_0$  plus a rotating field  $\vec{b}_1(t)$ . But now, the angular frequency  $\omega$  of the latter is time-dependent: it is deliberately chosen smaller

than the Larmor angular frequency  $\Omega_L$  at the beginning of the sequence, then progressively increased so as to end up at a value larger than  $\Omega_L$ . Notwithstanding this peculiarity, equation (2) is still valid and can be solved for  $\vec{m}$  using the rotating frame trick adapted as follows. Setting  $\varphi(t) = \int_0^t \omega(t') dt'$  and introducing the basis  $\{\vec{e}_{X'} = \cos \varphi \vec{e}_X + \sin \varphi \vec{e}_Y, \vec{e}_{Y'} = -\sin \varphi \vec{e}_X + \cos \varphi \vec{e}_Y, \vec{e}_{Z'} = \vec{e}_Z\}$ , we recover equation (3) with  $\vec{\Omega}_{\text{eff}}$  now time-dependent:  $\vec{\Omega}_{\text{eff}} = (\Omega_L - \omega(t))\vec{e}_{Z'} + \omega_R(t)\vec{e}_{X'}$ . A stylized sequence is displayed in figure 2. Untill  $t = t_A$  (point A) the rotating field's amplitude  $b_1$  is off:  $\omega_{RA} = 0$  and  $\vec{\Omega}_{\text{eff}}(t_A) = \vec{\Omega}_{\text{effA}} = (\Omega_L - \omega_A)\vec{e}_{Z'}$ . Then, from  $t = t_A$  on,  $b_1$  is progressively increased until  $t = t_B$  (point B), the angular frequency  $\omega$  being fixed at a constant value  $\omega_A = \omega_B < \Omega_L$ :  $\vec{\Omega}_{\text{eff}}(t_B) = \vec{\Omega}_{\text{effB}} = (\Omega_L - \omega_B)\vec{e}_{Z'} + \omega_{RB}\vec{e}_{X'}$ . Next, between  $t = t_B$  and  $t = t_C$  (point C), the angular frequency  $\omega$  is swept from  $\omega_B (= \omega_A)$  to  $\omega_C > \Omega_L$ , while the Rabi angular frequency  $\omega_R$  is kept constant:  $\omega_{RB} = \omega_{RC}$  and  $\vec{\Omega}_{\text{eff}}(t_C) = \vec{\Omega}_{\text{effC}} = (\Omega_L - \omega_C)\vec{e}_{Z'} + \omega_{RC}\vec{e}_{X'}$ . At last, between  $t = t_C$  and  $t = t_D$  (point D), the amplitude of the rotating field is decreased to zero at the constant angular frequency  $\omega_C (= \omega_D)$ ; thus,  $\omega_{RD} = 0$  and  $\vec{\Omega}_{\text{eff}}(t_D) = \vec{\Omega}_{\text{effD}} = (\Omega_L - \omega_D)\vec{e}_{Z'}$ . As can be seen in the figure, the tip of vector  $\vec{\Omega}_{\text{eff}}$  moves on rectangle ABCD. If each of the three legs AB, BC and CD is travelled through slowly enough, then the adiabatic theorem is applicable in the frame (referred to as the rotating frame above) associated with basis  $\{\vec{e}_{X'}, \vec{e}_{Y'}, \vec{e}_{Z'}\}$ . As a consequence, if  $\vec{m}(t_A) = \vec{m}_A$  is parallel to  $\vec{e}_Z$ , then  $\vec{m}(t_D) = \vec{m}_D = -\vec{m}_A$ . Of course the overall duration  $t_D - t_A$  should be shorter than the relaxation times  $T_1$  and  $T_2$ . Hence the somehow oxymoric ‘‘rapid adiabatic passage’’ [9, 10] coined to qualify this sequence.

Now that the stage is set in the NMR framework, let us come to the very point of the present letter: what does this quantum behaviour have to do with a classical motion in general and how the rapid adiabatic passage of a one-half spin can be simulated by a two-pendulum device in particular?

To answer this *prima facie* puzzling question, let us consider a two-degree-of-freedom harmonic oscillator (HO2) made of two simple pendula (in the small oscillation limit) coupled by means of a torsion spring. Our experimental apparatus was chosen by reason of its simplicity and visibility. It is displayed in figure 3. Both pendula can rotate freely around axis (A). They are made of a mass  $M$  guided by a couple of practically massless rods. The length  $l_2$  of pendulum 2 is fixed, say  $l_2 = l_0$ , whereas the length  $l_1(t)$  of pendulum 1 can be modulated around its mean value  $l_0$ , say  $l_1(t) = l_0(1 + \varepsilon(t) \cos \varphi(t))$  with  $|\varepsilon| \ll 1$ : mass  $M$  is held – and can slid along its rods – thanks to a cable through a grooved pulley, in such a way that no torque at all is exerted upon pendulum 1 with respect to axis (A) when the cable is dragged back and forth by an electric engine. In this sense, modulating length  $l_1$  effects a *parametric excitation* of the whole HO2.

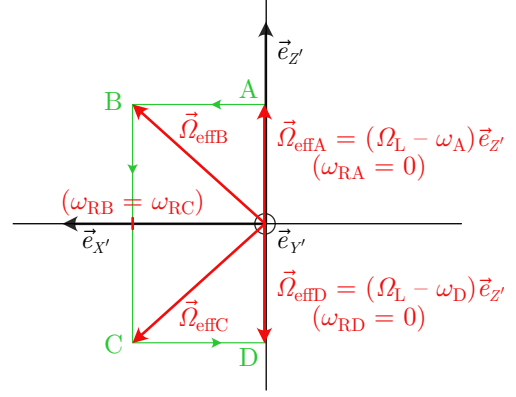


Fig. 2: Stylized sequence of the effective field  $\vec{\Omega}_{\text{eff}} = (\Omega_L - \omega(t))\vec{e}_{Z'} + \omega_R(t)\vec{e}_{X'}$  in the rotating frame associated with basis  $\{\vec{e}_{X'}, \vec{e}_{Y'}, \vec{e}_{Z'}\}$ . Leg AB (resp. CD): adiabatic switching on (resp. off) of the Rabi amplitude  $\omega_R$  at constant angular frequency  $\omega_A = \omega_B$  (resp.  $\omega_D = \omega_C$ ). Leg BC: adiabatic sweeping of the rotating field's angular frequency  $\omega$  from  $\omega_B < \Omega_L$  to  $\omega_C > \Omega_L$  at constant Rabi amplitude  $\omega_{RB} = \omega_{RC}$ . The overall sequence ABCD should be shorter than the spin's relaxation times.

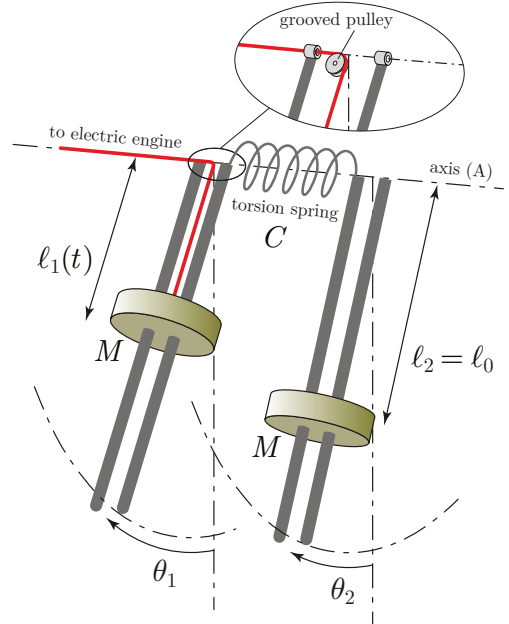


Fig. 3: Experimental device. Both pendula can rotate freely around axis (A) and are coupled by means of a torsion spring with angular stiffness  $C$ . Thanks to a grooved pulley and a traction wire, the left mass  $M$  can be slid along a couple of rods in such a way that no torque at all is exerted upon (A): sliding this mass thus implements a pure parametric excitation of the two-pendula system. Pendulum 1's length modulation is  $l_1(t) = l_0(1 + \varepsilon(t) \cos \varphi(t))$  with  $|\varepsilon| \ll 1$ .

Let us begin with considering the *free* HO2 (no modulation implemented). Denoting by  $(\theta_1, \theta_2)$  the angles the pendula rods make with their (vertical) rest position, and introducing the column matrix  $\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ , the Lagrangian of the system reads, superscript  $\dagger$  indicating matrixial transposition,

$$L_0(\Theta, \dot{\Theta}) = \frac{1}{2}(\dagger\dot{\Theta}J_0\dot{\Theta} - \dagger\Theta K_0\Theta), \quad (6)$$

where  $J_0 = M\ell_0^2$  and  $K_0 = \begin{pmatrix} Mg\ell_0 + C & -C \\ -C & Mg\ell_0 + C \end{pmatrix}$  respectively stand for the pendula inertia momentum and the stiffness matrix,  $C$  being the angular stiffness of the coupling spring. From (6), the Lagrange equations yield the differential system  $J_0\ddot{\Theta} + K_0\Theta = 0$ . Although this system could be handled as it is, it is more convenient to introduce a few simplifying notations. Let us set  $\omega_0^2 = \frac{g}{\ell_0} + \frac{C}{J_0}$  and  $\kappa = C/\omega_0^2 J_0$ . The matrix  $\Omega_0^2 = K_0/J_0$  then reads  $\Omega_0^2 = \omega_0^2 \begin{pmatrix} 1 & -\kappa \\ -\kappa & 1 \end{pmatrix}$  and can be diagonalized by means of the orthogonal basis change matrix  $P_e = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ . We consequently have  $P_e^{-1}\Omega_0 P_e = \Omega_e = \begin{pmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{pmatrix}$ , with the angular eigenfrequencies  $\omega_+$  and  $\omega_-$  given by  $\omega_{\pm}^2 = (1 \pm \kappa)\omega_0^2$ . Next, let us introduce the so-called standard dynamic variables  $Q = \Omega_0^{1/2} J_0^{1/2} \Theta$ . With  $P = \partial L_0 / \partial \dagger\dot{\Theta}$  standing for the conjugate momenta of variables  $Q$ , the usual Legendre transformation of  $L_0$  ends up with the free Hamiltonian

$$H_0(Q, P) = -L_0 + \dagger P \dot{Q} = \frac{1}{2}(\dagger P \Omega_0 P + \dagger Q \Omega_0 Q). \quad (7)$$

Now, falling into step with R. J. Glauber's definition in his quantization of the electromagnetic field [11, 12], let us gather the two sets of real canonic variables  $(Q, P)$  in a unique set of complex variables  $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\sqrt{2\hbar}}(Q + iP)$  (the quantum constant  $\hbar$  is kept here for analogy purposes; its numerical value will of course play no role in our purely classical problem). The motion equations derived from Hamiltonian (7), namely  $\{\dot{Q} = \Omega_0 P, \dot{P} = -\Omega_0 Q\}$  simply read  $\dot{A} = -i\Omega_0 A$ . Setting  $A_e = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = P_e^{-1} A$ , the latter equation is diagonalized in  $\dot{A}_e = -i\Omega_e A_e$ . The eigenmodes Glauber variables  $\alpha_+$  and  $\alpha_-$  consequently evolve independently from each other:  $\alpha_{\pm} = \alpha_{\pm}(0) e^{-i\omega_{\pm} t}$ . Observe by the way that the free Hamiltonian (7) equally reads, superscript  $\dagger$  indicating transconjugation,  $H_0 = A^\dagger \hbar \Omega_0 A = A_e^\dagger \hbar \Omega_e A_e = \hbar\omega_+ |\alpha_+|^2 + \hbar\omega_- |\alpha_-|^2$ , and that the Glauber variables commutation relations are given by the Poisson brackets  $\{\alpha_+, \alpha_+^*\} = \{\alpha_-, \alpha_-^*\} = 1/i\hbar$ , all other brackets being zero. In this respect, it is then tempting to draw one's inspiration from the Schwinger representation of  $SU(2)$  [13, 14]. Expanding the pseudo density

matrix  $D = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} (\alpha_+^* \ \alpha_-^*)$  on the  $\{\mathbb{1}, \sigma_X, \sigma_Y, \sigma_Z\}$  basis, we have  $D = \frac{1}{2}(N\mathbb{1} + \vec{m} \cdot \vec{\sigma})$  with

$$\begin{aligned} N &= \text{Tr}(D) = |\alpha_+|^2 + |\alpha_-|^2, \quad \vec{m} = \text{Tr}(D\vec{\sigma}) \\ &\rightsquigarrow m_X = 2\text{Re}(\alpha_+^* \alpha_-), \\ m_Y &= 2\text{Im}(\alpha_+^* \alpha_-), \\ m_Z &= |\alpha_+|^2 - |\alpha_-|^2, \end{aligned} \quad (8)$$

where  $N$ , which can be regarded as the semiclassical quanta number, is the norm of vector  $\vec{m}$  (it is easy to check that  $N^2 = m_X^2 + m_Y^2 + m_Z^2$ ). With these notations,  $H_0 = \frac{1}{2}N\hbar(\omega_+ + \omega_-) + \frac{1}{2}\hbar\Omega_L m_Z$ , where  $\Omega_L = \omega_+ - \omega_-$ . Observing at last that the dynamic variables  $(N, \vec{m})$  obey the commutation relations

$$\{N, \vec{m}\} = 0, \quad \{m_X, m_Y\} = \frac{2}{\hbar}m_Z \text{ and cyclic shift}, \quad (9)$$

the Hamilton equations of motion read

$$\frac{dN}{dt} = \{N, H_0\} = 0, \quad \frac{d\vec{m}}{dt} = \Omega_L \vec{e}_Z \wedge \vec{m}. \quad (10)$$

In the above equations, we have considered the variables  $m_X, m_Y$  and  $m_Z$  as the three components of some vector  $\vec{m} = m_X \vec{e}_X + m_Y \vec{e}_Y + m_Z \vec{e}_Z$ , where  $\{\vec{e}_X, \vec{e}_Y, \vec{e}_Z\}$  should be regarded as a (direct) orthonormal basis in a 3-dimension  $\mathbb{R}^3$ -isomorphous Euclidean space that we shall henceforth refer to as the ‘‘Larmor space’’. It is noteworthy that, contrary to our everyday-life  $\mathbb{R}^3$  space in which the *true* NMR processes recalled in the beginning of this letter take place, the Larmor space is fictitious. Notwithstanding the latter remark, equation (10) shows that the *free* motion of our HO2 can be represented as a Larmor precession of vector  $\vec{m}$  with angular velocity  $\Omega_L \vec{e}_Z$ .

So far, we have been able to simulate the Larmor precession of a one-half spin in a static magnetic field  $\vec{B}_0 = B_0 \vec{e}_Z$ . How could we simulate a time-dependent field  $\vec{b}_1(t)$  rotating around  $\vec{e}_Z$ ? To take up this challenge, let us slightly modulate pendulum 1's length as indicated above in the description of figure 3: its inertia momentum with respect to axis (A) now reads  $J_1(t) = M\ell_1^2(t) \simeq J_0(1 + 2\varepsilon(t) \cos \varphi(t))$  so that, in Lagrangian (6), the scalar  $J_0$  and the stiffness matrix  $K_0$  should be substituted by the time-dependent inertia and stiffness matrices  $J(t) = J_0\mathbb{1} + \delta J(t)$  and  $K(t) = K_0 + \delta K(t)$  with, at the first order in  $\varepsilon$  (assumed  $|\varepsilon| \ll 1$ ),

$$\begin{aligned} \delta J(t) &= 2\varepsilon(t) \cos \varphi(t) J_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \delta K(t) &= 2\varepsilon(t) \cos \varphi(t) Mg\ell_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (11)$$

Next, substituting variables  $Q = \Omega_0^{1/2} J_0^{1/2} \Theta$  for  $\Theta$  (as in the free case) in the completed Lagrangian and performing the usual Legendre transformation, we obtain  $H(t) = H_0 + \delta H(t)$ . In the free eigenmodes  $(+, -)$  representation,  $H_0$  reads  $A_e^\dagger \hbar \Omega_0 A_e$  (as already mentioned)

whereas perturbation  $\delta H(t)$  is quadratic in Glauber variables ( $A_e, A_e^*$ ). From now on, let us focus on the quasi-degeneracy limit, assuming that our pendula are weakly coupled:  $\kappa \ll 1 \rightsquigarrow \omega_{\pm} \simeq (1 \pm \frac{\kappa}{2})\omega_0$  (whence  $\Omega_L \simeq \kappa\omega_0 \ll \omega_0$ ) with  $\omega_0^2 \simeq g/\ell_0$ .

In our experimental implementation, the measured angular frequencies were:  $\omega_0 = 0.623$  rad/s,  $\omega_+ = 0.637$  rad/s and  $\omega_- = 0.608$  rad/s, hence  $\Omega_L = 0.029$  rad/s and  $\kappa = 0.047 \ll 1$ . We are clearly in the quasidegeneracy framework. Within this framework, the Hamilton equations of motion simplify in

$$\dot{A}_e = -i\Omega_e A_e + \frac{i\omega_0}{4}\varepsilon(t)\cos\varphi(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}(A_e - 3A_e^*). \quad (12)$$

Assuming in addition that angle  $\varphi(t)$  varies slowly compared to  $\omega_0 t$  ( $\dot{\varphi} = \omega(t)$  being of order  $\Omega_L \ll \omega_0$ ), we can tackle the above equations (12) in a perturbative way. At order zero,  $A_e(t) = e^{-i\Omega_e t} A_e(0)$  oscillates *roughly* as  $e^{-i\omega_0 t}$ . As a consequence, the  $A_e^*$  term in the right-hand side varies as  $e^{+i\omega_0 t}$  and brings no secular contribution to the time-evolution of  $A_e$ , so it can be neglected (Secular Approximation). It is then noteworthy that the remaining equation can be derived from the ‘‘secularized’’ Hamiltonian  $H^{\text{sec}}(t) = A_e^\dagger \hbar \left[ \Omega_e - \frac{1}{4}\varepsilon(t)\cos\varphi(t)\omega_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] A_e$ , which can in turn be expressed in terms of variables  $(N, \vec{m})$ , resulting in the motion equations in the Larmor space:

$$\begin{aligned} \frac{dN}{dt} &= 0, & \frac{d\vec{m}}{dt} &= \{\vec{m}, H^{\text{sec}}(t)\} = \vec{\Omega}(t) \wedge \vec{m}, \\ & & \text{with } \vec{\Omega}(t) &= \Omega_L \vec{e}_Z - \frac{1}{2}\varepsilon(t)\cos\varphi(t)\omega_0 \vec{e}_X. \end{aligned} \quad (13)$$

Observe that the time-dependent part of  $\vec{\Omega}(t)$  is polarized along axis  $\vec{e}_X$ , which *prima facie* does not match the expected rotating field  $\vec{\omega}_R(t)$  in equation (2). In fact, it nearly does, provided that the modulation depth  $\varepsilon$  should be chosen smaller than the dimensionless coupling constant  $\kappa$ , as explained below. To begin with, let us write

$$\cos\varphi\vec{e}_X = \frac{1}{2}(\cos\varphi\vec{e}_X + \sin\varphi\vec{e}_Y) + \frac{1}{2}(\cos\varphi\vec{e}_X - \sin\varphi\vec{e}_Y), \quad (14)$$

*i.e.* regard the linearly polarized term  $\cos\varphi\vec{e}_X$  as the sum of two counter-rotating terms. Setting  $\vec{e}_{X'} = \cos\varphi\vec{e}_X + \sin\varphi\vec{e}_Y$  and  $\omega_R(t) = \varepsilon(t)\omega_0/4$ , we can write the precession equation (13) in the frame with basis  $\{\vec{e}_{X'}, \vec{e}_{Y'}, \vec{e}_{Z'} = \vec{e}_Z\}$  rotating around  $\vec{e}_Z$  at the angular frequency  $\omega(t) = \dot{\varphi}$  and get

$$\left(\frac{d\vec{m}}{dt}\right)' = [(\vec{\Omega}_L - \vec{\omega}(t))\vec{e}_{Z'} + \omega_R(t)\vec{e}_{X'} + \delta\vec{\omega}_R(t)] \wedge \vec{m}, \quad (15)$$

where  $\delta\vec{\omega}_R(t) = \omega_R(\cos 2\varphi\vec{e}_{X'} - \sin 2\varphi\vec{e}_{Y'})$  rotates around  $\vec{e}_{Z'}$  at the angular velocity  $-2\omega(t)$ . Now, in the rapid adiabatic passage sequence,  $|\omega(t)|$  is close to  $\Omega_L = \kappa\omega_0$  and  $|\vec{\Omega}_{\text{eff}}| = \sqrt{(\Omega_L - \omega)^2 + \omega_R^2}$  is consequently of the order of  $\omega_R$ , as well as  $|\delta\vec{\omega}_R|$ . If  $\omega_R \ll \omega \simeq \Omega_L$  (*i.e.*

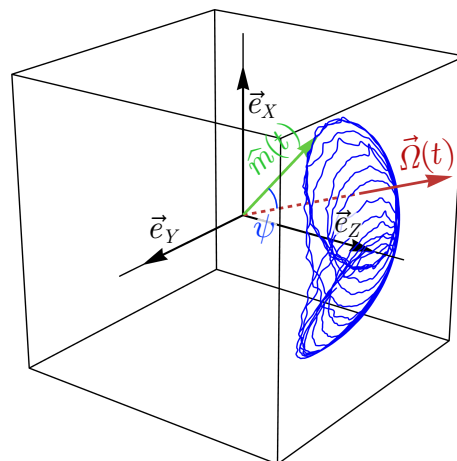


Fig. 4: Illustration of the adiabatic theorem. Vector  $\hat{m}$  precesses with the angular velocity  $\vec{\Omega}(t) = \Omega(t)\vec{e}(t) = \omega_0(\kappa\vec{e}_Z - \frac{1}{2}\varepsilon(t)\vec{e}_X)$ , as explained in the text. If direction  $\vec{e}(t)$  varies slowly enough, the angle  $\psi$  it makes with  $\hat{m}$  keeps a constant value over time (adiabatic invariant).

$\varepsilon \ll \kappa$ ), then it can be shown that equation (3) is still available with the effective field  $\vec{\Omega}_{\text{eff}}$  simply changed in  $\vec{\Omega}'_{\text{eff}} = (\Omega_L + \frac{\omega_R^2}{4\omega} - \omega)\vec{e}_{Z'} + \omega_R\vec{e}_{X'}$ , where  $\frac{\omega_R^2}{4\omega}$  is the so-called Bloch-Siegert shift [15] of the resonance. Neglecting this shift is known as the Rotating Wave Approximation (RWA). It is equivalent to dropping the second (counter-rotating) term in the right-hand side of equation (14). We shall henceforth make the RWA: save for the latter approximation, our experimental mechanical simulation fulfils the validity conditions of equation (3).

In a couple of foregoing papers, the free Larmor [16] and the Rabi [17] precessions mechanical simulations with two coupled pendula have been described and discussed at some length. In the present letter, we will not resume those discussions but rather focus on the new experimental results obtained concerning the adiabatic theorem and the rapid adiabatic passage. The angles  $\theta_1(t)$  and  $\theta_2(t)$  being acquired thanks to a (frictionless) optical detection device, the Glauber variables  $(\alpha_1, \alpha_2)$  are determined (see (7) and below). Then we derive the eigenmodes Glauber variables  $(\alpha_+, \alpha_-)$  using the change of basis matrix  $P_e$  and we calculate the dynamic variables  $(N, \vec{m})$  as displayed in (8) ( $\hbar$  taken equal to unity in these calculations). Despite the high ( $> 1000$ ) quality factor of our HO2 which commonly allows 20 minutes long free oscillations,  $|\vec{m}| (= N)$  decreases in the course of time during our experimentation. To (artificially) get rid of this decay, we normalize  $\vec{m}$  by  $N$ , so that the tip of the vector  $\hat{m} = \vec{m}/N$  always moves on a sphere with radius unity, the exact analog of the so-called Bloch sphere. Figures 4 and 5 display (in blue) the trajectory of the latter tip on the Bloch sphere in the Larmor space.

Figure 4 illustrates the free Larmor precession of  $\hat{m}$  around a slowly varying field:  $\vec{\Omega}(t) = \omega_0(\kappa\vec{e}_Z - \frac{1}{2}\varepsilon(t)\vec{e}_X)$ ,



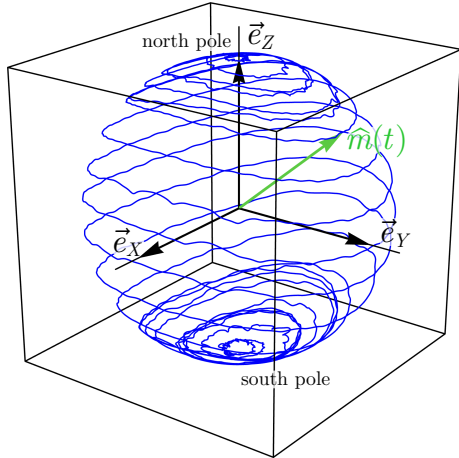


Fig. 5: Illustration of the rapid adiabatic passage. The two-pendulum system is initially prepared in its symmetric eigenmode ( $-$ ): vector  $\hat{m}$  then points towards the south pole of the Bloch sphere. Due to the slow switchings (on and off) of the modulation  $\varepsilon(t)$  and to the slow sweeping of the instantaneous angular velocity  $\omega(t) = \dot{\varphi}(t)$ , it spirals on the Bloch sphere so as to end up at its north pole, which corresponds to the antisymmetric eigenmode ( $+$ ) of the two-pendulum system.

where the  $\kappa\vec{e}_Z$  and the  $-\frac{1}{2}\varepsilon(t)\vec{e}_X$  terms respectively correspond to the coupling between the two pendula and to the detuning of their proper angular frequencies when pendulum 1's length is modulated ( $\ell_1 = (1 + \varepsilon)\ell_0$ ,  $\ell_2 = \ell_0$ ). In this experiment, the modulation depth  $\varepsilon$  was varied from  $-0.051$  to  $+0.051$ . As recalled above,  $\hat{m}$  moves on a cone, hence the circular trajectory of its tip. When  $\ell_1$  is modulated, the axis of the cone moves in the  $(\vec{e}_X, \vec{e}_Z)$  plane and the latter circle "slips" on the Bloch sphere. If the modulation is slow enough to fulfil the adiabatic criterion, the angle  $\psi$  between  $\hat{m}$  and  $\vec{\Omega}$  is constant, and so is the radius of the circle, as observed in figure 4.

Figure 5 displays a full spinflip process. Vector  $\hat{m}$  is initially pointing towards the south pole of the Bloch sphere. Then a modulation sequence as described in figure 2 is applied and  $\hat{m}$ 's tip progressively winds round the sphere (hence its spiral trajectory) so as to end up at the north pole. The experiment is particularly visual: the initial south pole corresponds to the lower angular frequency ( $\omega_-$ ) mode for which both pendula oscillate in phase; the final north pole corresponds to the higher angular frequency ( $\omega_+$ ) mode for which both pendula are  $180^\circ$  out of phase. The spinflip is thus illustrated by a continuous transition of the two-pendulum system from its symmetric eigenmode ( $-$ ) to its antisymmetric eigenmode ( $+$ ).

In our experiment, the full duration of the spinflip sequence was 800 s and the maximum value of the depth modulation  $\varepsilon$  (corresponding to leg BC of rectangle ABCD in figure 2) was compatible with our using the RWA.

In conclusion, we have presented a mechanical analogy of a long known technique used in the NMR domain to

perform an overall population inversion (spin flip) on a bunch of one-half spins immersed in a nonhomogeneous magnetic field, in conditions where the standard Rabi  $\pi$ -pulse sequence is not available. By the way, we have illustrated the so-called adiabatic theorem. It is noteworthy that, although generally regarded as a quantum object, our simulated one-half spin is utterly classical and that, contrary to the heavy true NMR experimental device, our illustration setup is remarkably simple.

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#### REFERENCES

- [1] Cohen-Tannoudji C., Diu B. and Laloë F., *Quantum Mechanics*, New York, John Wiley, 1977
- [2] Merzbacher E., *Quantum Mechanics*, New York, J. Wiley & Sons, 1998
- [3] Abragam A., *The principles of nuclear magnetism*, Clarendon Press, 1961
- [4] Bloch F., *Phys. Rev* **70**, 4604, 1946
- [5] Norberg R. E., *Am. J. Phys.* **33**, 71 (1965)
- [6] Brown R. W., Norman Cheng Y.-C., Mark Haacke E., Thompson M. P., Venkatesan R., *Magnetic Resonance Imaging: physical principles and sequence design*, New York, J. Wiley & Sons, 1999
- [7] Fukushima E., *NMR in biomedicine: the physical basis*, Springer Science & Business Media, 1989
- [8] Rabi I. I., Ramsey N. F. and Schwinger J., *Rev. Mod. Phys.* **26**, 167 (1954)
- [9] Levitt M. H., Freeman R., *Journal of Magnetic Resonance* **33**, 473 (1979)
- [10] Hardy C. J., Edelstein W. A. and Vatis D., *Journal of Magnetic Resonance* **66**, 470 (1986)
- [11] Glauber R. J., 1963, *Phys. Rev.* **131**, 2766-2788
- [12] Glauber R. J., 1965, *Quantum optics and electronics* (New-York: Gorda and Breach)
- [13] Schwinger J., *On angular momentum*, unpublished report number NYO-3071 (1952)
- [14] Chaturvedi S., Marmo G., Mukunda N., Simon R. and Zampini A., *The Schwinger representation of a group: concept and applications* arXiv : quant-ph/0505012v1 (2005)
- [15] Bloch F. and Siegert A., *Phys. Rev.* **57**, 522 (1940)
- [16] Leroy V., Bacri J.-C., Hocquet T. and Devaud M., *Eur. J. Phys* **27**, 1363 (2006)
- [17] Leroy V., Bacri J.-C., Hocquet T. and Devaud M., *Eur. J. Phys* **31**, 157 (2010)