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SPATIAL DECAY OF THE MULTI-SOLITONS OF THE GENERALIZED KORTEWEG-DE VRIES EQUATION

RAPHAËL CÔTE AND XAVIER FRIEDERICH

ABSTRACT. We study pointwise spatial decay of multi-solitons of the generalized Korteweg-de Vries equations. We obtain that, uniformly in time, these solutions and their derivatives decay exponentially in space on the left of and in the solitons region, and prove rapid decay on the right of the solitons.

1. INTRODUCTION

1.1. Multi-solitons and main result. We consider the generalized Korteweg-de Vries equations

$$(gKdV) \quad \partial_t u + \partial_x(\partial_x^2 u + u^p) = 0$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}$ and $p \geq 2$ is an integer.

Recall that (gKdV) admits a family of explicit traveling wave solutions R_{c_0, x_0} indexed by $(c_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R}$. Let Q be the unique (up to translation) positive solution in $H^1(\mathbb{R})$ (known also as *ground state*) to the following stationary elliptic problem associated with (gKdV)

$$Q'' + Q^p = Q,$$

given by the explicit formula

$$Q(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}}.$$

Then for all $c_0 > 0$ (velocity parameter) and $x_0 \in \mathbb{R}$ (translation parameter),

$$(1.1) \quad R_{c_0, x_0}(t, x) = Q_{c_0}(x - c_0 t - x_0)$$

is a global traveling wave solution of (gKdV) classically called the *soliton* solution, where $Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x)$.

We are interested here in qualitative properties of the *multi-solitons*, which are solutions to (gKdV) built upon solitons, and are defined as follows.

Definition 1.1. Let $N \geq 1$ and consider N solitons R_{c_i, x_i} as in (1.1) with speeds $0 < c_1 < \dots < c_N$. A *multi-soliton in $+\infty$* (resp. *in $-\infty$*) associated with the R_{c_i, x_i} is an H^1 -solution u of (gKdV) defined in a neighborhood of $+\infty$ (resp. $-\infty$) and such that

$$(1.2) \quad \left\| u(t) - \sum_{i=1}^N R_{c_i, x_i}(t) \right\|_{H^1} \rightarrow 0, \quad \text{as } t \rightarrow +\infty \text{ (resp. as } t \rightarrow -\infty).$$

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The study of multi-solitons is motivated by the soliton resolution conjecture, which asserts that generic solutions to nonlinear dispersive equations should behave as a sum of decoupled solitons for large times. Such a resolution was obtained for the original Korteweg-de Vries equation (KdV), corresponding to $p = 2$; and the modified Korteweg-de Vries equations (mKdV), corresponding to $p = 3$. We refer to [5, 24, 1] for instance.

In the context of (gKdV), multi-solitons were first constructed for (KdV) and (mKdV), via the inverse scattering transform. It provides explicit formulas: for (KdV), it writes

$$(1.3) \quad u = 6 \frac{\partial^2}{\partial x^2} \ln \det M,$$

where $M(t, x)$ is the $N \times N$ -matrix with generic entry

$$M_{(i,j)}(t, x) = \delta_{i,j} + 2 \frac{(c_i c_j)^{\frac{1}{4}}}{\sqrt{c_i} + \sqrt{c_j}} e^{\frac{1}{2}(\sqrt{c_i}(x-c_i t)+x_i+\sqrt{c_j}(x-c_j t)+x_j)}$$

and $\delta_{i,j}$ stands for the Kronecker symbol: see [8], [21, section 6], or [9]. We refer to Schuur [24, chapter 5, (5.5)] and to Lamb [13, chapter 5] for a formula for (mKdV). One can observe from these formulas that such a solution u in (1.3) is a multi-soliton both in $\pm\infty$ [21], with the same velocity parameters c_i in $\pm\infty$, but with distinct translation parameters whose shifts can be quantified in terms of the c_i .

The construction of multi-solitons was subsequently extended to many non integrable models, and (gKdV) is probably the equation for which their study has been most developed. One important result concerns the complete classification of the multi-solitons, depending on the value of p with respect to 5: recall that for $p < 5$, (gKdV) is L^2 -subcritical and solitons are stable, $p = 5$ is the L^2 critical equation, and for $p > 5$, (gKdV) is L^2 -supercritical and solitons are unstable, linearly and nonlinearly. Let us recall it below.

Theorem 1.2 (Martel [15]; Côte, Martel and Merle [4]; Combet [2]). *Let $p > 1$ be an integer and let $N \geq 1$, $0 < c_1 < \dots < c_N$, and $x_1, \dots, x_N \in \mathbb{R}$.*

If $p \leq 5$, there exists $T_0 \geq 0$ and a unique multi-soliton $u \in \mathcal{C}([T_0, +\infty), H^1(\mathbb{R}))$ associated with the R_{c_i, x_i} , $i \in \{1, \dots, N\}$.

If $p > 5$, there exists a one-to-one map Φ from \mathbb{R}^N to the set of all H^1 -solutions of (gKdV) defined in a neighborhood of $+\infty$ such that u is a multi-soliton in $+\infty$ associated with the R_{c_i, x_i} if and only if there exist $\lambda \in \mathbb{R}^N$ and $T_0 \geq 0$ such that $u|_{[T_0, +\infty)} = \Phi(\lambda)|_{[T_0, +\infty)}$.

Moreover, in each case, u belongs to $\mathcal{C}([T_0, +\infty), H^s(\mathbb{R}))$ for all $s \geq 0$, and there exist $\theta > 0$ (depending on the c_i but independent of s) and positive constants λ_s such that for all $s \geq 0$,

$$(1.4) \quad \forall t \geq T_0, \quad \left\| u(t) - \sum_{i=1}^N R_{c_i, x_i}(t) \right\|_{H^s} \leq \lambda_s e^{-\theta t}.$$

Remark 1.3. One can take

$$(1.5) \quad \theta := \frac{1}{32} \min \left\{ c_1, \min_{i=1, \dots, N-1} \{c_{i+1} - c_i\} \right\}^{\frac{3}{2}}$$

in the above theorem.

In this article, we go on studying qualitative properties of the multi-solitons and we are concerned with the behavior in space (at fixed time) of the multi-solitons of (gKdV). From formula (1.3), one can show that (KdV) multi-solitons are exponentially localized away from the centers of the involved solitons (as solitons are); this is also the case for (mKdV) multi-solitons. Our goal is to extend this property to multi-solitons of (gKdV), that is, non integrable equations.

Our main result is that there is indeed exponential decay in the solitons region and on the left of the train of solitons, and rapid decay on the right of it. Let us state this more precisely.

Theorem 1.4. *Fix the parameters $0 < c_1 < \dots < c_N$ and $x_1, \dots, x_N \in \mathbb{R}$. We consider $T \in \mathbb{R}$ and $u \in \mathcal{C}([T, +\infty), H^1(\mathbb{R}))$ a multi-soliton of (gKdV) associated with the solitons R_{c_i, x_i} , as given by Theorem 1.2 (u is unique if $p \leq 5$). Let $\beta > c_N$. Then there exist $T' > 0$ and $\kappa > 0$ such that for all $s \in \mathbb{N}$,*

- (1) (Exponential decay in the solitons region and to its left) *there exists $C_s > 0$ such that for all $t \geq T'$,*

$$(1.6) \quad \forall x \leq \beta t, \quad |\partial_x^s u(t, x)| \leq C_s \sum_{i=1}^N e^{-\kappa|x-c_i t|};$$

- (2) (Algebraic decay to the right of the last soliton) *for all $n \in \mathbb{N}$, there exists $C_{s,n} > 0$ such that for all $t \geq T'$,*

$$(1.7) \quad \forall x \geq \beta t, \quad |\partial_x^s u(t, x)| \leq \frac{C_{s,n}}{(x - \beta t)^n}.$$

Remark 1.5. Our bounds give $\kappa = O(1/\beta)$ as $\beta \rightarrow +\infty$, where the implicit constant depends on the c_i ; precise rates are stated in Proposition 2.1 and 2.5.

Theorem 1.4 shows in particular that for each fixed time $t \geq T'$, the multi-soliton $u(t)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. To our knowledge, this is the first result of quantitative spatial decay in a non integrable setting.

1.2. Comments and strategy of the proof. In [6] and [7, Section 3.1.2, (3.6) and (3.9)], the second author defines non dispersive solution of (gKdV) u at $+\infty$ by the property that for some $\rho > 0$,

$$\int_{x \leq \rho t} |u(t, x)|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

(such a notion was first developed by Martel and Merle [16, 17] in the vicinity of solitons). [6] showed that non dispersion is a dynamical characterization of multi-solitons: more precisely, a solution of (gKdV) which is non dispersive and remains close to a sum of N decoupled solitary waves for positive times is a multi-soliton in $+\infty$. For (KdV), the result is non perturbative: for any solutions (with sufficiently smooth initial data), non dispersion is equivalent to being a multi-soliton (for (mKdV), breathers may also occur). The convergence (1.4) shows on the other side that multi-solitons are non dispersive indeed. The decay obtained in Theorem 1.4 provides a quantitative version of this non dispersion.

To prove Theorem 1.4, we actually split space into three regions: the region to the left of the solitons, that is for $x \leq \alpha t$ for some $\alpha < c_1$; the solitons region $\alpha t \leq x \leq \beta t$; and the region to the right of the solitons $x \geq \beta t$.

Exponential decay of the multi-solitons of (gKdV) on the left of the solitons (for $x \leq \alpha t$) follows from revisiting a strong monotonicity argument set up in [6, section 2] (strengthened from Laurent and Martel [14]) and originally developed by Martel and Merle [16]. We take advantage here of the convergence (1.2) and the decay of the solitons, instead of a non dispersion assumption, as it is done in the mentioned references.

In the solitons region $\alpha t \leq x \leq \beta t$, estimate (1.6) is a direct consequence of the exponential convergence in (1.4).

The main novelty (and where most of our efforts are focused) concerns the region to the right of the solitons $x \geq \beta t$. The monotonicity argument, linked to the dynamic of the flow of (gKdV), does not apply anymore: indeed, it would require some knowledge (non dispersion) at

$t \rightarrow -\infty$ (or at least near the minimal existence time, as multi-solitons might blow up for the L^2 -supercritical (gKdV)). From this perspective, the point of Theorem 1.4 is actually to obtain some information of the behavior of multi-solitons for large decreasing times. Also notice that it would be sufficient to prove pointwise decay on the region $x \geq \beta t_0$ for one time t_0 , and then this information would easily be propagated for $t \geq t_0$. This is in line with general statements linked with persistence of regularity and decay of solutions to (gKdV), like Kato smoothing in [12] or Isaza, Linares and Ponce [10, 11]. Let us also mention [23]¹, where some polynomial decay was obtained (see Lemma 7.4).

Our strategy in the region $x \geq \beta t$ is as follows. We consider families of integrals of the form

$$I_{\varphi,s}(t) := \int_{x \geq \beta t} (\partial_x^s u)^2(t, x) \varphi(x - \beta t) dx$$

where φ is a suitable weight function. We show that variations of $I_{\varphi,s}$ are essentially controlled by the $I_{\varphi',s'}$ for $s' \in \{0, \dots, s+1\}$, under an exponential decay induction hypothesis. Then, by successive integrations, together with (1.4) (which provides the base case) and a triangular induction process, we show that we can bound $I_{\varphi,s}$ for $\varphi(y) = y^n$ for all $n \in \mathbb{N}$.

We of course expect that multi-solitons decay exponentially on the right as well, that is (1.6) holds without the restriction $x \leq \beta t$: this seems a natural conjecture as solitons are exponentially localized on both ends. Still, for the time being, estimate (1.7) is meaningful. As estimates of the form (1.4) are usual for multi-solitons, and as we use mildly the monotonicity property specific to (gKdV), we hope that our strategy may be extended to other models such as the nonlinear Schrödinger equations (see the main theorems in [3]) in order to prove algebraic decay (in the sense of (1.7)) for the corresponding multi-solitons.

In section 2, we consider the left region $x \leq \alpha t$ and the solitons region $\alpha t \leq x \leq \beta t$; and in section 3, we focus on the right region $x \geq \beta t$. In the appendix, we provide some bound on the H^s norm of solitons and multi-solitons, and in particular, track the constant λ_s in (1.4). Throughout the proofs, u is as in the statement of Theorem 1.4, and it will be convenient to denote

$$(1.8) \quad R_i := R_{c_i, x_i}, \quad R := \sum_{i=1}^N R_i, \quad \text{and} \quad z(t) := u(t) - R(t).$$

2. DECAY OF THE MULTI-SOLITONS ON THE LEFT

2.1. Decay of the multi-solitons on the left of the first soliton. The goal of this paragraph is to prove

Proposition 2.1 (Exponential decay in large time on the left of the first soliton). *Let $0 < \alpha < c_1$ and $\kappa_\alpha \in \left(0, \frac{\sqrt{\alpha}}{2}\right)$. There exists $T' \geq T$ such that for all $s \in \mathbb{N}$, there exists $C_s > 0$ such that for all $t \geq T'$,*

$$(2.1) \quad \forall x \leq \alpha t, \quad |\partial_x^s u(t, x)| \leq C_s e^{-\kappa_\alpha |x - \alpha t|}.$$

Remark 2.2. Using (1.4), one can easily see that the decay (2.1) implies that stated in (1.6) in the region $x \leq \alpha t$, with $\kappa = \kappa_\alpha$.

¹We thank Y. Martel for pointing to us this reference, upon completion of this work.

Proof. The proof follows the ideas of [6] and [14]. To reach the conclusion, we show the existence of $T' \in \mathbb{R}$ such that for each $s \in \mathbb{N}$, there exists $K_s > 0$ such that, with $\kappa := 2\kappa_\alpha$,

$$(2.2) \quad \forall t \geq T', \quad \int_{x \leq \alpha t} (\partial_x^s u(t, x))^2 e^{\kappa(\alpha t - x)} dx \leq K_s.$$

The first (and main) step is to obtain (2.2) for $s = 0$. For this, we claim a strong monotonicity property which is the purpose of Lemma 2.3 and Lemma 2.4 below.

Let us introduce, for some $\kappa > 0$ to be determined later, the function φ defined by

$$\varphi(x) = \frac{1}{2} - \frac{1}{\pi} \arctan(e^{\kappa x}).$$

It satisfies the following properties

$$(2.3) \quad \exists \lambda_0 > 0, \forall x \in \mathbb{R}, \quad \lambda_0 e^{-\kappa|x|} < -\varphi'(x) < \frac{1}{\lambda_0} e^{-\kappa|x|},$$

$$(2.4) \quad \forall x \in \mathbb{R}, \quad |\varphi^{(3)}(x)| \leq -\kappa^2 \varphi'(x).$$

$$(2.5) \quad \exists \lambda_1 > 0, \forall x \geq 0, \quad \lambda_1 e^{-\kappa x} \leq \varphi(x).$$

Moreover, let us observe that

$$(2.6) \quad \int_{x < \alpha t} u^2(t, x) e^{\kappa(\alpha t - x)} dx = \int_{x < 0} u^2(t, x + \alpha t) e^{-\kappa x} dx,$$

and that, for all $x_0 < 0$,

$$(2.7) \quad \begin{aligned} \int_{x_0 \leq x < 0} u^2(t, x + \alpha t) e^{-\kappa x} dx &\leq e^{-\kappa x_0} \int_{x \geq x_0} u^2(t, x + \alpha t) e^{-\kappa(x - x_0)} dx \\ &\leq \frac{1}{\lambda_1} e^{-\kappa x_0} \int_{\mathbb{R}} u^2(t, x + \alpha t) \varphi(x - x_0) dx. \end{aligned}$$

Since $\kappa^2 < \alpha$, one can choose $\delta \in (0, \alpha - \kappa^2)$. We consider $T' \in \mathbb{R}$ to be determined later. Then, for fixed $t_0 \geq T'$ and $x_0 \in \mathbb{R}$, we define

$$\begin{aligned} I_{(t_0, x_0)} : [T', +\infty) &\rightarrow \mathbb{R}^+ \\ t &\mapsto \int_{\mathbb{R}} u^2(t, x + \alpha t) \varphi(x - x_0 + \delta(t - t_0)) dx. \end{aligned}$$

We have

$$(2.8) \quad \forall t \geq T', \quad I_{(t_0, x_0)}(t) = \int_{\mathbb{R}} u^2(t, x) \varphi(x - x_0 + \delta(t - t_0) - \alpha t) dx,$$

so that by derivation with respect to t , we obtain

$$(2.9) \quad \begin{aligned} \frac{dI_{(t_0, x_0)}}{dt}(t) &= -3 \int_{\mathbb{R}} u_x^2(t, x) \varphi'(\tilde{x}) dx - (\alpha - \delta) \int_{\mathbb{R}} u^2(t, x) \varphi'(\tilde{x}) dx \\ &\quad + \int_{\mathbb{R}} u^2(t, x) \varphi^{(3)}(\tilde{x}) dx + \frac{2p}{p+1} \int_{\mathbb{R}} u^{p+1}(t, x) \varphi'(\tilde{x}) dx, \end{aligned}$$

where $\tilde{x} := x - x_0 + \delta(t - t_0) - \alpha t$. We then claim

Lemma 2.3. *There exists $C_0 > 0$ such that*

$$(2.10) \quad \forall x_0 \in \mathbb{R}, \forall t_0, t \geq T', \quad \frac{dI_{(t_0, x_0)}}{dt}(t) \geq -C_0 e^{-\kappa(-x_0 + \delta(t - t_0))}.$$

Proof. Due to property (2.4) of φ , we have

$$(2.11) \quad \left| \int_{\mathbb{R}} u^2(t, x) \varphi^{(3)}(\tilde{x}) dx \right| \leq -\kappa^2 \int_{\mathbb{R}} u^2(t, x) \varphi'(\tilde{x}) dx.$$

Furthermore we control the nonlinear part by considering

$$I_1(t) := \int_{|\tilde{x}| > -x_0 + \delta(t-t_0)} u^{p+1}(t, x) \varphi'(\tilde{x}) dx$$

and

$$I_2(t) := \int_{|\tilde{x}| \leq -x_0 + \delta(t-t_0)} u^{p+1}(t, x) \varphi'(\tilde{x}) dx.$$

On the one hand, we have due to (2.3)

$$(2.12) \quad |I_1(t)| \leq \frac{1}{\lambda_0} e^{-\kappa(-x_0 + \delta(t-t_0))} \int_{\mathbb{R}} |u|^{p+1}(t, x) dx \leq C e^{-\kappa(-x_0 + \delta(t-t_0))},$$

where we have used the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^{p+1}(\mathbb{R})$ and the fact that u belongs to $L^\infty([T', +\infty), H^1(\mathbb{R}))$. Note that $C > 0$ is independent of x_0 , t_0 , and t .

On the other, we observe that

$$(2.13) \quad \begin{aligned} |I_2(t)| &\leq \|u(t)\|_{L^\infty(x \leq \alpha t)}^{p-1} \int_{x \leq \alpha t} u^2(t, x) |\varphi'(\tilde{x})| dx \\ &\leq \sqrt{2}^{p-1} \|u(t)\|_{L^2(x \leq \alpha t)}^{\frac{p-1}{2}} \|u_x(t)\|_{L^2(x \leq \alpha t)}^{\frac{p-1}{2}} \int_{\mathbb{R}} u^2(t, x) |\varphi'(\tilde{x})| dx \\ &\leq \sqrt{2}^{p-1} \|u(t)\|_{L^2(x \leq \alpha t)}^{\frac{p-1}{2}} \sup_{t \geq T'} \|u(t)\|_{H^1}^{\frac{p-1}{2}} \int_{\mathbb{R}} u^2(t, x) |\varphi'(\tilde{x})| dx. \end{aligned}$$

Since u is a multi-soliton, we can choose $T' \geq 0$ such that for all $t \geq T'$,

$$(2.14) \quad \sqrt{2}^{p-1} \|u(t)\|_{L^2(x \leq \alpha t)}^{\frac{p-1}{2}} \sup_{t' \geq T'} \|u(t')\|_{H^1}^{\frac{p-1}{2}} \leq \frac{p+1}{2p} (\alpha - \delta - \kappa^2).$$

Let us justify it briefly (here lies the main change with respect to previous proofs based on non dispersion [6] or L^2 -compactness [14]): we have

$$\begin{aligned} \int_{x \leq \alpha t} u^2(t, x) dx &\leq 2 \int_{x \leq \alpha t} \left(u - \sum_{i=1}^N R_i \right)^2(t, x) dx + 2 \int_{x \leq \alpha t} \left(\sum_{i=1}^N R_i \right)^2(t, x) dx \\ &\leq 2C_0^2 e^{-2\theta t} + 2N \sum_{i=1}^N \int_{x \leq \alpha t} R_i^2(t, x) dx \end{aligned}$$

and for all $i = 1, \dots, N$, since $\alpha < c_i$, we have for $t \geq 0$:

$$\begin{aligned} \int_{x \leq \alpha t} R_i^2(t, x) dx &\leq C \int_{x \leq \alpha t} e^{-\sqrt{c_i}|x-c_i t-x_i|} e^{-\sqrt{c_i}|x-c_i t-x_i|} dx \\ &\leq C \int_{x \leq \alpha t} e^{-\sqrt{c_i}(c_i-\alpha)t} e^{-\sqrt{c_i}|x-c_i t-x_i|} dx \\ &\leq C e^{-\sqrt{c_i}(c_i-\alpha)t} \int_{\mathbb{R}} e^{-\sqrt{c_i}|x-c_i t-x_i|} dx \leq C e^{-\sqrt{c_i}(c_i-\alpha)t}. \end{aligned}$$

where C denotes a positive constant which can change from one line to the other and which only depends on c_i (see expression (1.1)).

Thus, we can pick up $C \geq 0$ such that for all $t \geq 0$,

$$\int_{x \leq \alpha t} u^2(t, x) dx \leq C \left(e^{-2\theta t} + \sum_{i=1}^N e^{-\sqrt{c_i}(c_i - \alpha)t} \right),$$

and then $T' \geq 0$ satisfying (2.14).

Taking into account (2.12), this eventually leads to the following estimate

$$(2.15) \quad \frac{2p}{p+1} \left| \int_{\mathbb{R}} u^{p+1}(t, x) \varphi'(\tilde{x}) dx \right| \leq -(\alpha - \delta - \kappa^2) \int_{\mathbb{R}} u^2(t, x) \varphi'(\tilde{x}) dx + C_0 e^{-\kappa(-x_0 + \delta(t-t_0))},$$

where $C_0 := \frac{2p}{p+1}C$ is independent of x_0 , t_0 , and t . Gathering (2.11) and (2.15) in (2.9), we finally deduce

$$\frac{dI_{(t_0, x_0)}(t)}{dt} \geq -3 \int_{\mathbb{R}} u_x^2(t, x) \varphi'(\tilde{x}) dx - C_0 e^{-\kappa(-x_0 + \delta(t-t_0))}.$$

This establishes Lemma 2.3. \square

As a consequence of the above lemma,

$$(2.16) \quad \exists C_1 > 0, \forall x_0 \in \mathbb{R}, \forall t \geq t_0, \quad I_{(t_0, x_0)}(t_0) \leq I_{(t_0, x_0)}(t) + C_1 e^{\kappa x_0},$$

with C_1 independent of the parameters x_0 and t_0 . Next, we claim the following:

Lemma 2.4. *For fixed $x_0 \in \mathbb{R}$ and $t_0 \geq T'$, $I_{(t_0, x_0)}(t) \rightarrow 0$ as $t \rightarrow +\infty$.*

Proof. This lemma is shown by adapting the proof in [14, paragraph 2.1, Step 2] and in [6]. Let ε be a positive real number. As in the previous proof, because u is a multi-soliton, we can find $T_1 \geq T'$ large such that for all $t \geq T_1$,

$$\int_{x < \alpha t} u^2(t, x) dx \leq \frac{\varepsilon}{2}.$$

Since $0 \leq \varphi \leq 1$, this enables us to see that

$$(2.17) \quad \int_{x < 0} u^2(t, x + \alpha t) \varphi(x - x_0 + \delta(t - t_0)) dx \leq \int_{x < \alpha t} u^2(t, x) dx \leq \frac{\varepsilon}{2}.$$

Now, recall that φ is decreasing so that

$$(2.18) \quad \int_{x \geq 0} u^2(t, x + \alpha t) \varphi(x - x_0 + \delta(t - t_0)) dx \leq \varphi(-x_0 + \delta(t - t_0)) \|u(t)\|_{L^2}^2 \leq \overline{C} \varphi(-x_0 + \delta(t - t_0)),$$

with $\overline{C} = \|u(t)\|_{L^2}^2$ for all $t \in J$. Moreover, since $\varphi(x) \rightarrow 0$ as $x \rightarrow +\infty$, there exists $T_2 \in \mathbb{R}$ such that for all $t \geq T_2$,

$$\overline{C} \varphi(-x_0 + \delta(t - t_0)) \leq \frac{\varepsilon}{2}.$$

Then, for all $t \geq \max\{T_1, T_2\}$,

$$I_{(t_0, x_0)}(t) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we have finished proving Lemma 2.4. \square

At this stage, we deduce from (2.16) and Lemma 2.4 that

$$(2.19) \quad \forall t_0 \geq T', \forall x_0 \in \mathbb{R}, \quad I_{(t_0, x_0)}(t_0) \leq C_1 e^{\kappa x_0}.$$

Thus, (2.7) leads to

$$\forall t \geq T', \quad \int_{x_0 \leq x < 0} u^2(t, x + \alpha t) e^{-\kappa x} dx \leq \frac{C_1}{\lambda_1}.$$

Letting $x_0 \rightarrow -\infty$, we infer that

$$\forall t \geq T', \quad \int_{x < 0} u^2(t, x + \alpha t) e^{-\kappa x} dx \leq \frac{C_1}{\lambda_1}$$

which proves (2.2) with $s = 0$.

Now, to conclude to (2.2) for all $s \in \mathbb{N}$, one actually proves by induction on $s \in \mathbb{N}$ the existence of $\tilde{K}_s \geq 0$ such that for all $t \geq T'$,

$$(2.20) \quad \int_{\mathbb{R}} (\partial_x^s u)^2(t, x + \alpha t) e^{-\kappa x} dx + \int_t^{t+1} \int_{\mathbb{R}} (\partial_x^s u)^2(\tau, x + \alpha \tau) e^{-\kappa x} dx d\tau \leq \tilde{K}_s.$$

For $s = 0$, this is in fact a consequence of (2.2) and of the following estimate: for all $t \geq t_0 \geq T'$,

$$I_{(t_0, x_0)}(t_0) - I_{(t_0, x_0)}(t) \leq \frac{C_1}{\lambda_1} e^{\kappa x_0} + 3 \int_{t_0}^t \int_{\mathbb{R}} u_x^2(\tau, x + \alpha \tau) \varphi'(x - x_0 + \delta(\tau - t_0)) dx d\tau$$

(which follows from the proof of Lemma 2.3). Indeed, we notice that by (2.3) and since φ is decreasing, for $\tau \in [t_0, t]$,

$$\lambda_0 e^{-\kappa|x-x_0|} < -\varphi'(x - x_0) \leq -\varphi'(x - x_0 + \delta(\tau - t_0))$$

so that for $t = t_0 + 1$ in particular, we have

$$\begin{aligned} \int_{t_0}^{t_0+1} \int_{x_0 < x} u_x^2(\tau, x + \alpha \tau) e^{-\kappa x} dx d\tau &\leq C e^{-\kappa x_0} (I_{(t_0, x_0)}(t) - I_{(t_0, x_0)}(t_0)) \\ &\leq C e^{-\kappa x_0} I_{(t_0, x_0)}(t_0 + 1) \leq C e^{-\kappa x_0} I_{(t_0+1, x_0)}(t_0 + 1) \leq C. \end{aligned}$$

where the last inequality results from (2.19). Taking the limit when $x_0 \rightarrow -\infty$, we obtain the desired inequality (2.20).

The rest of the induction argument closely follows [14, paragraph 2.3 and paragraph 2.2 Step 2]. Since it does not depend on the properties of the multi-soliton and for the sake of brevity, we will not detail the proof (2.20) for higher values of s . \square

2.2. Decay of the multi-solitons in the solitons region.

Proposition 2.5 (Exponential decay in the solitons region). *Let $0 < \alpha < c_1$ and $\beta > c_N$, and define*

$$(2.21) \quad \kappa_{\alpha, \beta} := \min \left\{ \sqrt{c_1}, \frac{\theta}{c_1 - \alpha}, \frac{\theta}{\beta - c_N}, \min_{i=1, \dots, N-1} \left\{ \frac{\theta}{c_{i+1} - c_i} \right\} \right\} > 0.$$

Then for all $s \in \mathbb{N}$, there exists $C_s > 0$ such that for all $t \geq T$,

$$(2.22) \quad \forall x \in [\alpha t, \beta t], \quad |\partial_x^s u(t, x)| \leq C_s \sum_{i=1}^N e^{-\kappa_{\alpha, \beta} |x - c_i t|}.$$

Proof. Recall the notation z given in (1.8). For all $s \in \mathbb{N}$, for all $t \geq T$, we have by (1.4) and the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$,

$$\|\partial_x^s z(t)\|_{L^\infty} \leq C \|\partial_x^s z(t)\|_{H^1} \leq C \|z(t)\|_{H^{s+1}} \leq \lambda_{s+1} e^{-\theta t}.$$

Fix $t \geq T$. For all $i = 1, \dots, N-1$ and $c_i t \leq x \leq c_{i+1} t$,

$$e^{-\theta t} \leq e^{-\kappa_{\alpha,\beta}(x-c_i t)} \quad \text{if and only if} \quad x \leq \left(c_i + \frac{\theta}{\kappa_{\alpha,\beta}}\right) t,$$

which is indeed satisfied since $c_{i+1} \leq c_i + \frac{\theta}{\kappa_{\alpha,\beta}}$ by the choice of $\kappa_{\alpha,\beta}$ (2.21).

Similarly, for all $\alpha t \leq x \leq c_1 t$, we have $e^{-\theta t} \leq e^{-\kappa_{\alpha,\beta}(c_1 t - x)}$ because $c_1 \leq \alpha + \frac{\theta}{\kappa_{\alpha,\beta}}$. And for all $c_N t \leq x \leq \beta t$, we have $e^{-\theta t} \leq e^{-\kappa_{\alpha,\beta}(x - c_N t)}$ because $\beta \leq c_N + \frac{\theta}{\kappa_{\alpha,\beta}}$.

Thus, we obtain that for all $t \geq T$ and for all $\alpha t \leq x \leq \beta t$,

$$|\partial_x^s z(t, x)| \leq \|\partial_x^s z(t)\|_{L^\infty} \leq \lambda_{s+1} e^{-\theta t} \leq \lambda_{s+1} \sum_{i=1}^N e^{-\kappa_{\alpha,\beta}|x-c_i t|}.$$

Moreover, for all $i = 1, \dots, N$, $|\partial_x^s R_i(t, x)| \leq C_{i,s} e^{-\sqrt{c_i}|x-c_i t|}$ for some $C_{i,s} > 0$ depending on i and s . Hence, we conclude to (2.22) by the triangular inequality and by the fact that $\kappa_{\alpha,\beta} \leq \sqrt{c_1}$. \square

3. DECAY OF THE MULTI-SOLITONS ON THE RIGHT OF THE LAST SOLITON

In this subsection, we analyze the behavior of the multi-solitons on the right of the solitons region. We will focus on the proof of the following

Proposition 3.1 (Polynomial decay in large time on the right of the last soliton). *Let $\beta > c_N$. For all $s \in \mathbb{N}$ and for all $n \in \mathbb{N}$, there exists $C_{s,n} > 0$ such that for all $t \geq T$, for all $x > \beta t$,*

$$(3.1) \quad (\partial_x^s u(t, x))^2 \leq \frac{C_{s,n}}{(x - \beta t)^n}.$$

Notice that this statement will appear as a corollary of a more general result, which has its own interest (see Proposition 3.4 below) and which relies on a triangular induction process.

By Theorem 1.2, there exists $\theta > 0$ such that for all $s \in \mathbb{N}$, there exists $\lambda_s > 0$ such that for all $t \geq T$,

$$(3.2) \quad \|z(t)\|_{H^s} \leq \lambda_s e^{-\theta t}.$$

One can estimate the growth of λ_s with respect to s , and in fact, we will keep for the sequel

$$(3.3) \quad \lambda_s \leq C.2^{\mu_0^s},$$

with $\mu_0 > \max\{\sqrt{p}, \frac{p+1}{2}\}$ defined in Claim 3.2. The proof of estimate (3.3) is postponed to paragraph A.2 in the Appendix.

3.1. The key ingredient: stability by integration of a well-chosen set of weight functions. Fix

$$\eta \in (0, c_1)$$

The choice of η is made in order to obtain the interaction estimate of Claim 3.2 below, which roughly expresses that the growth of $x \mapsto e^{\sqrt{\eta}x}$ is weaker than the decay of the solitons.

Claim 3.2. For all $s \in \mathbb{N}$, there exists $C > 0$ and $\mu > \sqrt{p}$ such that for all $i = 1, \dots, N$ and for all $t \geq T$,

$$(3.4) \quad \int_{\mathbb{R}} |\partial_x^s R_i(t, x)| e^{\sqrt{\eta}(x-\beta t)} dx \leq C 2^{\mu^s}.$$

We will consider weight functions $\varphi \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ which satisfy the following assumptions:

$$\begin{aligned} (\mathbf{A}) \quad & (i) \quad \lim_{x \rightarrow -\infty} \varphi(x) = 0 \\ & (ii) \quad \exists \kappa_1 > 0, \forall x \in \mathbb{R}, \quad 0 \leq \varphi'(x) \leq \kappa_1 e^{\sqrt{\eta}x} \\ & (iii) \quad \exists \kappa_2 > 0, \forall x \in \mathbb{R}, \quad |\varphi^{(3)}(x)| \leq \kappa_2 \varphi'(x) \end{aligned}$$

and for given $\bar{s} \in \mathbb{N}$,

$$(\mathbf{B}(\bar{s})) \quad \exists C(\bar{s}, \varphi) > 0, \forall s \in \{0, \dots, \bar{s}\}, \forall t \geq T, \quad \int_{\mathbb{R}} (\partial_x^s z)^2(t, x) \varphi(x - \beta t) dx \leq C(\bar{s}, \varphi) e^{-\theta t}.$$

Let us define the following property which depends on \bar{s} and φ :

$$P(\bar{s}, \varphi) : \quad \varphi \text{ satisfies } (\mathbf{A}) \text{ and } (\mathbf{B}(\bar{s})).$$

Note that under assumptions (\mathbf{A}) (i) and (ii), φ is integrable in the neighborhood of $-\infty$ (see also (3.6) below); we can thus define on \mathbb{R}

$$\varphi_{[1]} : x \mapsto \int_{-\infty}^x \varphi(r) dr.$$

Next, we state the key ingredient, reflecting the triangular way of obtaining Proposition 3.1,

Proposition 3.3. For $\bar{s} \in \mathbb{N}$, $P(\bar{s}+1, \varphi) \Rightarrow P(\bar{s}, \varphi_{[1]})$ and moreover, for given $\mu_0 > \max\{\sqrt{p}, \frac{p+1}{2}\}$, there exists a constant $c(\eta, \kappa_1, \kappa_2) > 0$ (which depends only on η, κ_1 and κ_2) such that

$$(3.5) \quad C(\bar{s}, \varphi_{[1]}) \leq c(\eta, \kappa_1, \kappa_2) 2^{\mu_0 \bar{s}} C(\bar{s} + 1, \varphi).$$

As a corollary of the previous proposition, defining the set

$$\mathcal{E} := \{\varphi \in \mathcal{C}^3(\mathbb{R}, \mathbb{R}) \mid \varphi \text{ satisfies } (\mathbf{A}) \text{ and } (\mathbf{B}(s)) \text{ for all } s \in \mathbb{N}\},$$

we immediately obtain step by step, in a triangular way, the following

Proposition 3.4 (Stability by integration). *If $\varphi \in \mathcal{E}$, then $\varphi_{[1]} \in \mathcal{E}$.*

Remark 3.5. It results from the previous proposition that the multi-soliton and its derivatives have polynomial decay (see subsection 3.4). Note that if one could improve (3.3) and (3.5) by proving the existence of $C > 0$ and $c(\eta, \kappa_1, \kappa_2) > 0$ such that

$$\forall s \in \mathbb{N}, \quad \lambda_s \leq C^s$$

and

$$\forall s \in \mathbb{N}, \forall \varphi \in \mathcal{E}, \quad C(s, \varphi_{[1]}) \leq c(\eta, \kappa_1, \kappa_2) C(s + 1, \varphi),$$

we would deduce that the multi-soliton and all its derivatives decay exponentially on the domain $x > \beta t$ (see paragraph A.3 in the Appendix).

3.2. Proof of Claim 3.2.

Proof of Claim 3.2. On the one hand, we have

$$\int_{x \leq \beta t} e^{-\sqrt{c_i}|x-c_it|} e^{\sqrt{\eta}(x-\beta t)} dx \leq \int_{x \leq \beta t} e^{-\sqrt{c_i}|x-c_it|} dx \leq \int_{\mathbb{R}} e^{-\sqrt{c_i}|x|} dx \leq \frac{2}{\sqrt{c_i}}.$$

On the other hand, since $\beta > c_N$ and $\beta\sqrt{\eta} < c_i\sqrt{c_i}$,

$$\begin{aligned} \int_{x > \beta t} e^{-\sqrt{c_i}|x-c_it|} e^{\sqrt{\eta}(x-\beta t)} dx &\leq e^{(c_i\sqrt{c_i}-\beta\sqrt{\eta})t} \int_{x > \beta t} e^{(\sqrt{\eta}-\sqrt{c_i})x} dx \\ &\leq \int_{x > \beta t} e^{(\sqrt{\eta}-\sqrt{c_i})x} dx \leq \frac{e^{(\sqrt{\eta}-\sqrt{c_i})\beta t}}{\sqrt{c_i}-\sqrt{\eta}}. \end{aligned}$$

Hence, noticing that we have also $|\partial_x^k R_i(t, x)| \leq C_{i,k} e^{-\sqrt{c_i}|x-c_it|}$, where $C_{i,k}$ is a constant depending on the parameters of the soliton R_i and on k only, Claim 3.2 holds. \square

3.3. Proof of Proposition 3.3.

Proof. First of all, let us check that the properties gathered in **(A)** are stable by integration, that is, if we assume that φ satisfies **(A)**, then so does $\varphi_{[1]}$.

Assumption (ii) shows that φ' is integrable in the neighborhood of $-\infty$ and

$$\forall x \in \mathbb{R}, \quad 0 \leq \int_{-\infty}^x \varphi'(r) dr \leq \kappa_1 \int_{-\infty}^x e^{\sqrt{\eta}r} dr.$$

By (i), we thus obtain

$$(3.6) \quad \forall x \in \mathbb{R}, \quad 0 \leq \varphi(x) \leq \frac{\kappa_1}{\sqrt{\eta}} e^{\sqrt{\eta}x}.$$

Then (iii) implies that $\varphi^{(3)}$ is integrable in the neighborhood of $-\infty$ and so φ'' admits a limit in $-\infty$, which is necessarily 0 (since $\varphi'(x) \rightarrow 0$ as $x \rightarrow -\infty$). Finally, by integration, one obtains

$$\forall x \in \mathbb{R}, \quad |\varphi''(x)| \leq \kappa_2 \varphi(x).$$

Hence $\varphi_{[1]}$ indeed satisfies **(A)**.

Now take $\bar{s} \in \mathbb{N}$ and let us show that $\varphi_{[1]}$ verifies **B**(\bar{s}) if one assumes that φ satisfies **B**($\bar{s}+1$).

We define for all $s \in \mathbb{N}$, for all $x_0 \geq 0$ and for all $t \geq T$:

$$J_{s,x_0}(t) := \int_{\mathbb{R}} (\partial_x^s z)^2(t, x) \varphi(x - x_0 - \beta t) dx.$$

For ease of reading, we will denote $\tilde{x} = \tilde{x}(t) := x - x_0 - \beta t$ (for $x_0 > 0$ and $t \geq T$).

We first show the following recurrence formula which makes the link between the functions J_{s,x_0} , $s \in \mathbb{N}$.

Lemma 3.6. *For all $s \in \mathbb{N}$, there exists $C_s \geq 0$ (independent of x_0) such that*

$$(3.7) \quad \left| \frac{d}{dt} J_{s,x_0}(t) \right| \leq C_s \int_{\mathbb{R}} \sum_{k=1}^{s+1} (\partial_x^k z)^2(t, x) \varphi'(\tilde{x}) dx + C_s e^{-\theta t} \sum_{k=0}^{s-1} J_{k,x_0}(t) + C_s e^{-\sqrt{\eta}x_0} e^{-\theta t}.$$

In addition, for all $\mu_1 > \max\{\sqrt{p}, \frac{p+1}{2}\}$, there exists $\gamma_1 > 0$ independent of s such that for all s ,

$$(3.8) \quad C_s \leq \gamma_1 2^{\mu_1 s}.$$

Proof. Let us compute

$$\begin{aligned} \frac{d}{dt} J_{s,x_0}(t) &= -3 \int_{\mathbb{R}} (\partial_x^{s+1} z)^2 \varphi'(\tilde{x}) dx + \int_{\mathbb{R}} (\partial_x^s z)^2 \varphi^{(3)}(\tilde{x}) dx \\ &\quad - \beta \int_{\mathbb{R}} (\partial_x^s z)^2 \varphi'(\tilde{x}) dx + 2 \int_{\mathbb{R}} \partial_x^s \left((z+R)^p - \sum_{i=1}^N R_i^p \right) (\partial_x^s z \varphi)_x dx. \end{aligned}$$

By **(A)** (iii), we have

$$(3.9) \quad \left| \int_{\mathbb{R}} (\partial_x^s z)^2 \varphi^{(3)}(\tilde{x}) dx - \beta \int_{\mathbb{R}} (\partial_x^s z)^2 \varphi'(\tilde{x}) dx \right| \leq (\kappa_2 + \beta) \int_{\mathbb{R}} (\partial_x^s z)^2 \varphi'(\tilde{x}) dx.$$

We now control the nonlinear term $\int_{\mathbb{R}} \partial_x^s (z^p) (\partial_x^s z \varphi)_x dx$ which does not contain any soliton. If $s = 0$, we observe that

$$\int_{\mathbb{R}} z^p (z\varphi)' dx = \frac{p}{p+1} \int_{\mathbb{R}} z^{p+1} \phi'(\tilde{x}) dx;$$

thus

$$(3.10) \quad \left| \int_{\mathbb{R}} z^p (z\varphi)' dx \right| \leq \frac{p}{p+1} \|z(t)\|_{L^\infty}^{p-1} \int_{\mathbb{R}} z^2 \phi'(\tilde{x}) dx \leq C \int_{\mathbb{R}} z^2 \phi'(\tilde{x}) dx.$$

If $s \geq 1$, we can write

$$\int_{\mathbb{R}} \partial_x^s (z^p) (\partial_x^s z \varphi)_x dx = \int_{\mathbb{R}} \partial_x^s (z^p) \partial_x^s z \phi'(\tilde{x}) dx - \int_{\mathbb{R}} \partial_x^{s-1} (z^p) (\partial_x^{s+2} z \phi + \partial_x^{s+1} z \varphi') dx.$$

We have

$$\partial_x^k (z^p) = \sum_{i_1 + \dots + i_p = k} \binom{k}{i_1, \dots, i_p} \partial_x^{(i_1)} z \dots \partial_x^{(i_p)} z$$

so that

$$\begin{aligned} \left| \int_{\mathbb{R}} \partial_x^{s-1} (z^p) \partial_x^{s+2} z \varphi dx \right| &\leq \|\partial_x^{s+2} z(t)\|_{L^\infty} \sum_{i_1 + \dots + i_p = s-1} \int_{\mathbb{R}} \left| \partial_x^{(i_1)} z \right| \dots \left| \partial_x^{(i_p)} z \right| \varphi(\tilde{x}) dx \\ &\leq C \|z(t)\|_{H^{s+3}} \frac{1}{p} \sum_{i_1 + \dots + i_p = s-1} \sum_{k=1}^p \int_{\mathbb{R}} \left| \partial_x^{(i_k)} z \right|^p \varphi(\tilde{x}) dx \\ &\leq C \|z(t)\|_{H^{s+3}} \|z(t)\|_{H^s}^{p-2} \sum_{i_1 + \dots + i_p = s-1} \sum_{k=1}^p \int_{\mathbb{R}} \left(\partial_x^{(i_k)} z \right)^2 \varphi(\tilde{x}) dx \\ (3.11) \quad &\leq Cp^s \lambda_{s+3} \lambda_s^{p-2} e^{-(p-1)\theta t} \sum_{k=0}^{s-1} \int_{\mathbb{R}} (\partial_x^k z)^2 \varphi(\tilde{x}) dx, \end{aligned}$$

where we used (3.2). Similarly we obtain

$$(3.12) \quad \left| \int_{\mathbb{R}} \partial_x^{s-1} (z^p) \partial_x^{s+1} z \varphi' dx \right| \leq Cp^s \lambda_{s+2} \lambda_s^{p-2} e^{-(p-1)\theta t} \int_{\mathbb{R}} \sum_{k=0}^{s-1} (\partial_x^k z)^2 \varphi'(\tilde{x}) dx$$

and

$$(3.13) \quad \left| \int_{\mathbb{R}} \partial_x^s (z^p) \partial_x^s z \varphi' dx \right| \leq Cp^s \lambda_{s+2} \lambda_{s+1}^{p-2} e^{-(p-1)\theta t} \int_{\mathbb{R}} \sum_{k=0}^s (\partial_x^k z)^2 \varphi'(\tilde{x}) dx.$$

Hence we can take

$$C_s \leq Cp^s \lambda_{s+3}^{p-1} \leq 2^{\mu_1^s}$$

for all $\mu_1 > \mu_0 > \max \left\{ \sqrt{p}, \frac{p+1}{2} \right\}$ and s large enough.

Moreover,

$$\int_{\mathbb{R}} \partial_x^s \left((z + R)^p - \sum_{i=1}^N R_i^p - z^p \right) (\partial_x^s z \varphi)_x dx = I_1 + I_2,$$

with

$$I_1 = \int_{\mathbb{R}} \partial_x^s \left(R^p - \sum_{i=1}^N R_i^p \right) (\partial_x^{s+1} z \varphi + \partial_x^s z \varphi') dx$$

and

$$I_2 = \sum_{k=1}^{p-1} \binom{p}{k} \sum_{i_1 + \dots + i_p = s} \binom{s}{i_1, \dots, i_p} \int_{\mathbb{R}} \partial_x^{i_1} z \dots \partial_x^{i_k} z \partial_x^{i_{k+1}} R \dots \partial_x^{i_p} R (\partial_x^{s+1} z \varphi + \partial_x^s z \varphi') dx.$$

We have

$$\begin{aligned} |I_1| &\leq \left(\int_{\mathbb{R}} (\partial_x^{s+1} z)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\partial_x^s \left(R^p - \sum_{i=1}^N R_i^p \right) \varphi \right)^2 dx \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{\mathbb{R}} (\partial_x^s z)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(\partial_x^s \left(R^p - \sum_{i=1}^N R_i^p \right) \varphi' \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq C \|z\|_{H^{s+1}} p^{s+1} 2^{\mu^s} e^{-2\theta t} e^{-\sqrt{\eta} x_0}. \end{aligned}$$

Moreover

(3.14)

$$\begin{aligned} |I_2| &\leq \sum_{k=1}^{p-1} \binom{p}{k} \sum_{i_1 + \dots + i_p = s} \binom{s}{i_1, \dots, i_p} \|z\|_{H^{s+2}} \|z\|_{H^{s+1}}^k \|R\|_{H^{s+1}}^{p-k-1} \int_{\mathbb{R}} (|\partial_x^{i_p} R \varphi| + |\partial_x^{i_p} R \varphi'|) dx \\ &\leq C 2^p p^s \|z\|_{H^{s+2}}^{k+1} \|R\|_{H^{s+1}}^{p-k-1} p^{s+1} 2^{\mu^s} e^{-2\theta t} e^{-\sqrt{\eta} x_0}, \end{aligned}$$

where the second inequality is a consequence of Claim 3.2. Indeed, Claim 3.2 rewrites as follows: for all $x_0 \geq 0$,

$$\int_{\mathbb{R}} |\partial_x^k R_i(t, x)| e^{\sqrt{\eta}(x-x_0-\beta t)} dx \leq C e^{-\sqrt{\eta} x_0}.$$

Thus, by property **(A)** (ii) satisfied by φ , we infer

$$\int_{\mathbb{R}} |\partial_x^k R_i(t, x)| (\varphi(\tilde{x}) + \varphi'(\tilde{x})) dx \leq C e^{-\sqrt{\eta} x_0}.$$

Now, we obtain Lemma 3.6 by gathering the above estimates. We can find a constant γ_1 independent of s and depending only on η , κ_1 and κ_2 such that for s sufficiently large,

$$C_s \leq \gamma_1 2^{\mu^s}.$$

Even if it means taking γ_1 greater, we can assume that the above estimate holds for all s . \square

Remark 3.7. Let us observe that we could obtain sharper estimates than (3.11), (3.12), (3.13), and (3.14) due to integrations by parts. But this would have only little impact on the growth rate in s at this stage, and in fact it would not affect (3.5) by the same argument as that given at the end of the previous proof.

Then, we obtain the following control of $J_{s, x_0}(t)$:

Lemma 3.8. *For all $s \in \mathbb{N}$, there exists a constant $K_s \geq 1$ such that for all $t \geq T$:*

$$(3.15) \quad J_{s,x_0}(t) \leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \left(\sum_{k=1}^{s+1} (\partial_x^k z)^2(t', x) \varphi'(\tilde{x}(t')) \right) dx dt' + K_s e^{-\sqrt{\eta}x_0} e^{-\theta t}.$$

In addition, given $\mu_2 > \max\{\sqrt{p}, \frac{p+1}{2}\}$, there exists $\gamma_2 > 0$ such that for all s ,

$$K_s \leq \gamma_2 2^{\mu_2^s}.$$

Proof. It follows from (3.7) and an induction argument. Notice that for all $s \in \mathbb{N}$, $J_{s,x_0}(t) \rightarrow 0$ as $t \rightarrow +\infty$. Thus, for $s = 0$, (3.15) follows by integration of (3.7) between t and $+\infty$. Now assume that (3.15) is proved for $0, \dots, s-1$ for some particular $s \geq 1$. Then, by integration of (3.7) between t and $+\infty$ (for $t \geq T$), it results:

$$\begin{aligned} J_{s,x_0}(t) &\leq C_s \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=1}^{s+1} (\partial_x^k z)^2(t', x) \varphi'(\tilde{x}) dx dt' + C_s e^{-\sqrt{\eta}x_0} \int_t^{+\infty} e^{-\theta t'} dt' \\ &\quad + C_s \sum_{s'=0}^{s-1} K_{s'} \int_t^{+\infty} e^{-\theta t'} \int_{t'}^{+\infty} \int_{\mathbb{R}} \sum_{k=1}^{s'+1} (\partial_x^k z)^2(t'', x) \varphi'(\tilde{x}) dx dt'' dt' \\ &\quad + C_s \sum_{s'=0}^{s-1} K_{s'} e^{-\sqrt{\eta}x_0} \int_t^{+\infty} e^{-\theta t'} dt' \\ &\leq C_s \int_t^{+\infty} \int_{\mathbb{R}} \left(\sum_{k=1}^{s+1} (\partial_x^k z)^2(t', x) \varphi'(\tilde{x}) \right) dx dt' \\ &\quad + \sum_{s'=0}^{s-1} C_s K_{s'} \left(\int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{s'+1} (\partial_x^k z)^2(t'', x) \varphi'(\tilde{x}) dx dt'' \right) \int_t^{+\infty} e^{-\theta t'} dt' \\ &\quad + \frac{C_s}{\theta} \max\{K_{s'}, s' = 0, \dots, s-1\} e^{-\sqrt{\eta}x_0} e^{-\theta t}. \end{aligned}$$

Hence there exists $K_s \geq 1$ such that

$$(3.16) \quad K_s \leq C_s + \sum_{s'=0}^{s-1} C_s K_{s'} \leq 2C_s \sum_{s'=0}^{s-1} K_{s'}$$

and for which

$$J_{s,x_0}(t) \leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \left(\sum_{k=1}^{s+1} (\partial_x^k z)^2(t', x) \varphi'(\tilde{x}) \right) dx dt' + K_s e^{-\sqrt{\eta}x_0} e^{-\theta t}.$$

From the inequality (3.16) and an induction argument, we can bound

$$\begin{aligned} K_s &\leq 2C_s \left(\sum_{s'=0}^{s-2} K_{s'} + K_{s-1} \right) \leq 2C_s (1 + 2C_{s-1}) \sum_{s'=0}^{s-2} K_{s'} \\ &\leq 2C_s (1 + 2C_{s-1}) \dots (1 + 2C_1) K_0 \\ &\leq 2C_s \times 4C_{s-1} \times \dots \times 4C_1 C_0 \leq 2^{2s-1} \prod_{i=0}^s C_i \leq 2^{\mu_2^s}, \end{aligned}$$

for all $\mu_2 > \mu_1$ and s sufficiently large (see (3.8) in Lemma 3.6). \square

Let us now conclude the proof of $P(\bar{s}, \varphi_{[1]})$. We integrate estimate (3.15) provided by Lemma 3.8 on $[0, +\infty)$ with respect to x_0 . We obtain by Fubini theorem: for $t \geq T$,

$$\begin{aligned} & \int_{\mathbb{R}} (\partial_x^{\bar{s}} z)^2(t, x) \int_0^{+\infty} \varphi(x - x_0 - \beta t) dx_0 dx \\ & \leq K_{\bar{s}} \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{\bar{s}+1} (\partial_x^k z)^2(t', x) \int_0^{+\infty} \varphi'(x - x_0 - \beta t') dx_0 dx dt' + \frac{K_{\bar{s}}}{\sqrt{\eta}} e^{-\theta t} \end{aligned}$$

and then by an affine change of variable

$$\int_{\mathbb{R}} (\partial_x^{\bar{s}} z)^2(t, x) \varphi_{[1]}(x - \beta t) dx \leq K_{\bar{s}} \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{\bar{s}+1} (\partial_x^k z)^2(t', x) \varphi(x - \beta t') dx dt' + \frac{K_{\bar{s}}}{\sqrt{\eta}} e^{-\theta t}.$$

Considering that φ satisfies $(\mathbf{B}(\bar{s} + 1))$, this finally shows that

$$\int_{\mathbb{R}} (\partial_x^{\bar{s}} z)^2(t, x) \varphi_{[1]}(x - \beta t) dx \leq \frac{K_{\bar{s}}}{\theta} \sum_{k=0}^{\bar{s}+1} C(k, \varphi) e^{-\theta t} + \frac{K_{\bar{s}}}{\sqrt{\eta}} e^{-\theta t}$$

Hence, $\varphi_{[1]}$ satisfies $(\mathbf{B}(\bar{s}))$ and one can take

$$C(\bar{s}, \varphi_{[1]}) \leq C(\bar{s} + 2) K_{\bar{s}} C(\bar{s} + 1, \varphi).$$

Thus we obtain (3.5), which finishes proving Proposition 3.3. \square

3.4. Rapid decrease on the right: proof of Proposition 3.1.

Proof. Now, we show polynomial decay of z and its derivatives. This consists in an application of Proposition 3.4 and is the object of Claim 3.9 and Claim 3.10 below.

Set $\eta \in (0, c_1)$ and introduce the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi(x) := \frac{2}{\pi} \arctan(e^{\sqrt{\eta}x}).$$

Observe that $\varphi \in \mathcal{E}$, in view of (3.2) and due to φ being bounded. We define a sequence $(\varphi_{[n]})_{n \in \mathbb{N}}$ of functions $\mathbb{R} \rightarrow \mathbb{R}$ as follows: $\varphi_{[0]} := \varphi$ and for all $n \in \mathbb{N}^*$, for all $x \in \mathbb{R}$,

$$\varphi_{[n]}(x) := \int_{-\infty}^x \varphi_{[n-1]}(y) dy.$$

By Proposition 3.4, we have

$$\forall n \in \mathbb{N}, \quad \varphi_{[n]} \in \mathcal{E}.$$

The following claim motivates the introduction of this sequence $(\varphi_{[n]})_n$.

Claim 3.9 (Polynomial growth of $\varphi_{[n]}$). We have for all $n \in \mathbb{N}$

$$(3.17) \quad \forall x \leq 0, \quad 0 \leq \varphi_{[n]}(x) \leq \frac{1}{\sqrt{\eta}^n} e^{\sqrt{\eta}x}$$

and

$$(3.18) \quad \forall x \geq 0, \quad \frac{1}{2} \frac{x^n}{n!} \leq \varphi_{[n]}(x) \leq \sum_{k=0}^n \frac{1}{\sqrt{\eta}^{n-k}} \frac{x^k}{k!}.$$

Proof. We argue by induction on n .

Note that $\varphi_{[0]} = \varphi$ is an increasing function and that

$$\forall t \geq 0, \quad \arctan t \leq t.$$

Thus

$$\begin{aligned} \forall x \leq 0, \quad 0 \leq \varphi_{[0]}(x) &\leq \frac{2}{\pi} e^{\sqrt{\eta}x} \leq e^{\sqrt{\eta}x} \\ \forall x \geq 0, \quad \frac{1}{2} = \varphi(0) &\leq \varphi_{[0]}(x) \leq 1. \end{aligned}$$

Now assume that (3.17) and (3.18) hold for some $n \in \mathbb{N}$ being fixed.

By definition of $\varphi_{[n+1]}$ and by the induction assumption, we have for all $x \leq 0$

$$0 \leq \varphi_{[n+1]}(x) \leq \int_{-\infty}^x \frac{1}{\sqrt{\eta}^n} e^{\sqrt{\eta}t} dt \leq \frac{1}{\sqrt{\eta}^{n+1}} e^{\sqrt{\eta}x}.$$

In particular,

$$0 \leq \varphi_{[n+1]}(0) \leq \frac{1}{\sqrt{\eta}^{n+1}}.$$

By the induction assumption, we then infer that for all $x \geq 0$,

$$\varphi_{[n+1]}(x) = \varphi_{[n+1]}(0) + \int_0^x \varphi_{[n]}(t) dt$$

satisfies

$$0 + \int_0^x \frac{1}{2} \frac{t^n}{n!} dt \leq \varphi_{[n+1]}(x) \leq \frac{1}{\sqrt{\eta}^{n+1}} + \int_0^x \sum_{k=0}^n \frac{1}{\sqrt{\eta}^{n-k}} \frac{t^k}{k!} dt.$$

Thus for all $x \geq 0$,

$$\frac{1}{2} \frac{x^{n+1}}{(n+1)!} \leq \varphi_{[n+1]}(x) \leq \frac{1}{\sqrt{\eta}^{n+1}} + \sum_{k=0}^n \frac{1}{\sqrt{\eta}^{n-k}} \frac{x^{k+1}}{(k+1)!} = \sum_{k=0}^{n+1} \frac{1}{\sqrt{\eta}^{n+1-k}} \frac{x^k}{k!}.$$

This finishes the induction argument, hence the proof of Claim 3.9. \square

Claim 3.10. For all $n \in \mathbb{N}$, for all $s \in \mathbb{N}$, there exists $K_{s,n} \geq 0$ such that for all $t \geq T$,

$$\int_{x \geq \beta t} (\partial_x^s z)^2 (x - \beta t)^n dx \leq K_{s,n} e^{-\theta t}.$$

Proof. We have already observed that $\varphi_{[n]}$ belongs to \mathcal{E} . Using Claim 3.9, we deduce that for all $s \in \mathbb{N}$,

$$\int_{x \geq \beta t} (\partial_x^s z)^2 (x - \beta t)^n dx \leq 2n! \int_{\mathbb{R}} (\partial_x^s z)^2 \varphi_{[n]}(x - \beta t) dx \leq K_{s,n} e^{-\theta t},$$

where $K_{s,n} := 2n!C(s, \varphi_{[n]})$. \square

Proposition 3.1 follows now from Claim 3.10, from the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and the decay of all derivatives of the R_i . \square

Remark 3.11. For all $s \in \mathbb{N}$, there exists φ satisfying **(A)** and **(B(s))** and which grows faster than each polynomial function as $x \rightarrow +\infty$. This follows from Proposition 3.3: given $\tilde{\mu}_0 > \max\{\sqrt{p}, \frac{p+1}{2}\}$, the sum of the series of functions

$$\sum_{n \geq 0} \frac{x^n}{2\tilde{\mu}_0^{s+n}} \in \mathcal{E}.$$

See paragraph A.3 in the Appendix for details.

A. APPENDIX

A.1. Growth of the H^s norms of the solitons.

Proposition A.1. *For all $\mu > \sqrt{p}$, there exists s_0 such that for all $s \geq s_0$,*

$$(A.1) \quad \|Q\|_{H^s} \leq 2^{\mu^s}.$$

Proof. Differentiating the fundamental equation satisfied by Q , that is $Q'' + Q^p = Q$, we obtain the following recurrence formula:

$$\forall s \in \mathbb{N}, \quad Q^{(s+2)} = Q^{(s)} - \sum_{i_1 + \dots + i_p = s} \binom{s}{i_1, \dots, i_p} Q^{(i_1)} \dots Q^{(i_p)}.$$

Let us observe that for $i_1, \dots, i_p \in \mathbb{N}$ such that $i_1 + \dots + i_p = s$,

- if there exists $j \in \{1, \dots, p\}$ such that $i_j = s$, then

$$\begin{aligned} \int_{\mathbb{R}} \left(Q^{(i_1)} \dots Q^{(i_p)} \right)^2 dx &= \int_{\mathbb{R}} \left(Q^{(s)} \right)^2 Q^{2(p-1)} dx \\ &\leq \|Q\|_{L^\infty}^{2(p-1)} \int_{\mathbb{R}} \left(Q^{(s)} \right)^2 dx \leq C \|Q\|_{H^1}^{2(p-1)} \|Q\|_{H^s}^2 \end{aligned}$$

(C being a constant depending only on p);

- if for all $j \in \{1, \dots, p\}$ such that $i_j \leq s-1$, then

$$\begin{aligned} \int_{\mathbb{R}} \left(Q^{(i_1)} \dots Q^{(i_p)} \right)^2 dx &\leq \prod_{k=2}^p \|Q^{(i_k)}\|_{L^\infty}^2 \int_{\mathbb{R}} \left(Q^{(i_1)} \right)^2 dx \\ &\leq C \prod_{k=2}^p \|Q\|_{H^{i_k+1}}^2 \|Q\|_{H^{i_1}}^2 \leq C \|Q\|_{H^s}^{2(p-1)} \|Q\|_{H^{s-1}}^2, \end{aligned}$$

(C being again a constant depending only on p , which can change from one line to the other).

Thus for $s \in \mathbb{N}^*$,

$$\|Q^{(s+2)}\|_{L^2} \leq C \left(\|Q^{(s)}\|_{L^2} + p^s \|Q\|_{H^s}^p \right),$$

which implies

$$\|Q\|_{H^{s+2}} \leq Cp^s \|Q\|_{H^s}^p,$$

with a constant C depending only on p .

We finally obtain

$$\|Q\|_{H^s} \leq C^{\frac{p^{\lfloor \frac{s}{2} \rfloor} - 1}{p-1}} p^{p^{\lfloor \frac{s}{2} \rfloor} s} \|Q\|_{H^1} \leq 2^{\mu^s},$$

with $\mu > \sqrt{p}$ and s sufficiently large. □

Remark A.2. As a corollary of Proposition A.1, we also obtain the existence of a constant C depending on p and the soliton parameters such that for all $s \in \mathbb{N}$,

$$(A.2) \quad \left\| \left(\sum_{i=1}^N R_i(t) \right)^p - \sum_{i=1}^N R_i(t)^p \right\|_{H^s} \leq Cp^s \max_{i=1, \dots, N} \|R_i\|_{H^s}^p e^{-2\theta t}.$$

A.2. Proof of estimate (3.3). The goal is here to make explicit the constants appearing in the computations done by Martel [15, Section 3.4] in the proof of smoothness of the multi-solitons.

We thus repeat the arguments developed by Martel (presented slightly differently), keeping track of the growth of the constant λ_s (with respect to s) such that

$$\forall t \geq T, \quad \|z(t)\|_{H^s} \leq \lambda_s e^{-\theta t}.$$

Proof. We consider regularity indices $s \geq 5$, as that case makes the argument easier: the point being that the exponential decay rate θ does not change when we go from the estimation of $\|z\|_{H^{s-1}}$ to that of $\|z\|_{H^s}$; there is a loss for $s = 2$ (treated in detail in [15]), which can be avoided for $s = 3, 4$ using an extra argument, see the footnote below in the proof.

The starting point is to study the variations of

$$\frac{d}{dt} \int_{\mathbb{R}} (\partial_x^s z)^2 dx = 2 \int_{\mathbb{R}} \partial_x^s \left((z + R)^p - \sum_{i=1}^N R_i^p \right) \partial_x^{s+1} z dx.$$

Thus, the terms which have to be controlled are the source term (involving z only linearly)

$$2 \int_{\mathbb{R}} \partial_x^{s+1} \left(R^p - \sum_{i=1}^N R_i^p \right) \partial_x^s z dx.$$

and

$$-2 \int_{\mathbb{R}} \partial_x^{s+1} \left(\sum_{k=1}^p \binom{p}{k} z^k R^{p-k} \right) \partial_x^s z dx = -2 \sum_{k=1}^p \binom{p}{k} \sum_{i_1 + \dots + i_p = s+1} \binom{s+1}{i_1, \dots, i_p} I_{k, i_1, \dots, i_p},$$

$$\text{where } I_{k, i_1, \dots, i_p} = \int_{\mathbb{R}} \partial_x^{i_1} z \dots \partial_x^{i_k} z \partial_x^{i_{k+1}} R \dots \partial_x^{i_p} R \partial_x^s z dx.$$

(i) For $k = 1$, integrating by parts, we have

$$\begin{aligned} & \sum_{i_1 + \dots + i_p = s+1} \binom{s+1}{i_1, \dots, i_p} \int_{\mathbb{R}} \partial_x^{i_1} z \partial_x^{i_2} R \dots \partial_x^{i_p} R \partial_x^s z dx \\ &= \int_{\mathbb{R}} \partial_x^{s+1} z R^{p-1} \partial_x^s z dx + (s+1)(p-1) \int_{\mathbb{R}} \partial_x^s z \partial_x R R^{p-2} \partial_x^s z dx \\ & \quad + \sum_{\substack{i_1 + \dots + i_p = s+1 \\ \forall j, i_j \leq s-1}} \int_{\mathbb{R}} \partial_x^{i_1} z \partial_x^{i_2} R \dots \partial_x^{i_p} R \partial_x^s z dx \\ \text{(A.3)} \quad &= \frac{2s+1}{2} \int_{\mathbb{R}} (\partial_x^s z)^2 \partial_x (R^{p-1}) dx + \sum_{\substack{i_1 + \dots + i_p = s+1 \\ \forall j, i_j \leq s-1}} \int_{\mathbb{R}} \partial_x^{i_1} z \partial_x^{i_2} R \dots \partial_x^{i_p} R \partial_x^s z dx. \end{aligned}$$

The first integral

$$\text{(A.4)} \quad \frac{2s+1}{2} \int_{\mathbb{R}} (\partial_x^s z)^2 \partial_x (R^{p-1}) dx$$

can not be bounded directly in a suitable way. The key idea in [15] is to add a lower order term

$$\frac{2s+1}{3} p \int_{\mathbb{R}} (\partial_x^{s-1} z)^2 R^{p-1} dx$$

whose variation at leading order will precisely cancel (A.4). Thus we are lead to consider

$$F_s(t) := \int_{\mathbb{R}} (\partial_x^s z)^2 dx - \frac{2s+1}{3} p \int_{\mathbb{R}} (\partial_x^{s-1} z)^2 R^{p-1} dx.$$

Going back to (A.3), we bound

$$(A.5) \quad \left| \sum_{\substack{i_1+\dots+i_p=s+1 \\ \forall j, i_j \leq s-1}} \int_{\mathbb{R}} \partial_x^{i_1} z \partial_x^{i_2} R \dots \partial_x^{i_p} R \partial_x^s z \, dx \right| \leq Cp^{s+1} \|z\|_{H^{s-1}}^2 \|R\|_{H^s}^{p-1}.$$

(ii) Let us now consider the case where $2 \leq k \leq p$. If there exists $j \in \{1, \dots, k\}$ such that $i_j = s+1$, then integrating by parts,

$$(A.6) \quad \begin{aligned} |I_{k, i_1, \dots, i_p}| &= \frac{1}{2} \left| \int_{\mathbb{R}} (\partial_x^s z)^2 \partial_x (z^{k-1} R^{p-k}) \, dx \right| \\ &\leq \frac{1}{2} \|\partial_x (R^{p-k} z^{k-1})\|_{L^\infty} \|z\|_{H^s}^2 \leq \frac{1}{2} \|R\|_{H^2}^{p-k} \|z\|_{H^2}^{k-1} \|z\|_{H^s}^2. \end{aligned}$$

If there exists $j \in \{1, \dots, k\}$ such that $i_j = s$, then

$$(A.7) \quad |I_{k, i_1, \dots, i_p}| \leq C \|z\|_{H^2}^{k-1} \|R\|_{H^2}^{p-k} \int_{\mathbb{R}} (\partial_x^s z)^2 \, dx.$$

If there exists $j \in \{1, \dots, k\}$ such that $i_j = s-1$, then for all $j' \in \{1, \dots, k\}$ such that $j' \neq j$, $i_{j'} \leq \sum_{l=1}^p i_l - s + 1 \leq 2 \leq s-3$ (by the choice of $s \geq 5$)². Hence, integrating by parts,

$$(A.8) \quad \begin{aligned} |I_{k, i_1, \dots, i_p}| &= \left| -\frac{1}{2} \int_{\mathbb{R}} (\partial_x^{s-1} z)^2 \partial_x (\partial_x^{i'_1} z \dots \partial_x^{i'_{k-1}} z \partial_x^{i_{k+1}} R \dots \partial_x^{i_p} R) \, dx \right| \\ &\leq \frac{1}{2} \|z\|_{H^{s-1}}^2 \left\| \partial_x (\partial_x^{i'_1} z \dots \partial_x^{i'_{k-1}} z \partial_x^{i_{k+1}} R \dots \partial_x^{i_p} R) \right\|_{L^\infty} \\ &\leq C \|z\|_{H^{s-1}}^{k+1} \|R\|_{H^{s+1}}^{p-k}, \end{aligned}$$

where C is a universal constant depending only on p .

In the other cases, $i_1, \dots, i_k \leq s-2$ and so

$$(A.9) \quad |I_{k, i_1, \dots, i_p}| \leq \left| -\int_{\mathbb{R}} \partial_x^{s-1} z \partial_x (\partial_x^{i_1} z \dots \partial_x^{i_k} z \partial_x^{i_{k+1}} R \dots \partial_x^{i_p} R) \, dx \right| \leq \|z\|_{H^{s-1}}^{k+1} \|R\|_{H^{s-1}}^{p-k}.$$

(iii) For the source term, by integration by parts,

$$\begin{aligned} \left| 2 \int_{\mathbb{R}} \partial_x^{s+1} \left(R^p - \sum_{i=1}^N R_i^p \right) \partial_x^s z \, dx \right| &\leq C \|z\|_{H^{s-1}} \left\| R^p - \sum_{i=1}^N R_i^p \right\|_{H^{s+2}} \\ &\leq C \lambda_{s-1} e^{-\theta t} \left\| R^p - \sum_{i=1}^N R_i^p \right\|_{H^{s+2}}. \end{aligned}$$

(iv) When computing the time differential of $\int_{\mathbb{R}} (\partial_x^{s-1} z)^2 R^{p-1} \, dx$, as mentioned (and by construction), one term cancels (A.4) and the others are bounded as in (i) and (ii).

²If $s = 3$ or 4 , a term which is cubic in $\partial_x^{s-1} z$, of the type

$$\int_{\mathbb{R}} (\partial_x^{s-1} z)^3 P(R_i, z, \partial_x z) \, dx$$

can occur, where P is some function (but there are no terms with higher power of $\partial_x^{s-1} z$). Via the Gagliardo-Nirenberg inequality, it can be bounded by

$$\left(\|z\|_{H^s}^{1/6} \|z\|_{H^{s-1}}^{5/6} \right)^3 \|P(R_i, z, \partial_x z)\|_{L^\infty} \lesssim C(1 + \|z\|_{H^2})^{p-2} e^{-5\theta/2t} \|z\|_{H^s}^{1/2},$$

the point being that the decay rate in θ is greater than 2: one can then complete the estimates as written here for $s \geq 5$.

Gathering the bounds (A.5), (A.6), (A.7), (A.8), (A.9), it results that

$$\begin{aligned}
\left| \frac{d}{dt} F_s(t) \right| &\leq C \lambda_{s-1} e^{-\theta t} \left\| R^p - \sum_{i=1}^N R_i^p \right\|_{H^{s+1}} + C \sum_{k=2}^p \binom{p}{k} \|z\|_{H^s}^2 p^{s+1} \|z\|_{H^2}^{k-1} \|R\|_{H^s}^{p-k} \\
&\quad + C \sum_{k=2}^p \binom{p}{k} p^{s+1} \|z\|_{H^{s-1}}^{k+1} \|R\|_{H^s}^{p-k} \\
&\leq C \lambda_{s-1} e^{-\theta t} p^{s+2} 2^{\mu^{s+2}} p e^{-2\theta t} + C \|z\|_{H^s}^2 e^{-2\theta t} p^{s+1} 2^p 2^{\mu^s (p-2)} \\
&\quad + C e^{-3\theta t} p^{s+1} 2^p 2^{\mu^s (p-2)} \lambda_{s-1}^{p+1} \\
(A.10) \quad &\leq C \left(p^s 2^{\mu^s p} e^{-2\theta t} \|z\|_{H^s}^2 + p^s 2^{\mu^s p} \lambda_{s-1}^{p+1} e^{-3\theta t} \right).
\end{aligned}$$

Moreover, we have by definition of F_s ,

$$\int_{\mathbb{R}} (\partial_x^s z)^2 dx \leq |F_s| + C s \|R\|_{H^2}^{p-1} \|\partial_x^{s-1} z\|_{L^2}^2 \leq |F_s| + C s \lambda_{s-1}^2 e^{-2\theta t}.$$

Hence,

$$\left| \frac{d}{dt} F_s(t) \right| \leq C p^s 2^{\mu^s p} e^{-2\theta t} (|F_s(t)| + s \lambda_{s-1}^2 e^{-2\theta t}) + C p^s 2^{\mu^s p} \lambda_{s-1}^{p+1} e^{-3\theta t}.$$

By integration we finally obtain

$$|F_s(t)| \leq C p^s 2^{\mu^s p} \lambda_{s-1}^{p+1} e^{-2\theta t}$$

and thus

$$\|z\|_{H^s}^2 \leq C p^s 2^{\mu^s p} \lambda_{s-1}^{p+1} e^{-2\theta t}.$$

We can therefore take

$$(A.11) \quad \lambda_s \leq C p^{\frac{s}{2}} 2^{\frac{p}{2} \mu^s} \lambda_{s-1}^{\frac{p+1}{2}},$$

which yields via an induction

$$\lambda_s \leq C \sum_{k=0}^{s-1} \left(\frac{p+1}{2}\right)^k p^{\sum_{k=0}^{s-1} \frac{s-k}{2}} \left(\frac{p+1}{2}\right)^k \lambda_0^{\left(\frac{p+1}{2}\right)^s} 2^{\frac{p}{2} \sum_{k=0}^{s-1} \mu^{s-k} \left(\frac{p+1}{2}\right)^k},$$

and so,

$$(A.12) \quad \lambda_s \leq C 2^{\left(\frac{p+1}{2}\right)^s} p^{\frac{s}{2} \left(\frac{p+1}{2}\right)^{s-1}} 2^{(\max\{\mu, \frac{p+1}{2}\})^s} \lambda_0^{\left(\frac{p+1}{2}\right)^s} \leq 2^{\mu_0^s},$$

with $\mu_0 > \max\{\mu, \frac{p+1}{2}\} > \max\{\sqrt{p}, \frac{p+1}{2}\}$ and s sufficiently large (depending on μ_0). \square

Remark A.3. Note that we can refine a bit (A.8) but the final estimate concerning λ_s would not be better than (A.12), considering that μ_0 is to be chosen strictly greater than $\max\{\sqrt{p}, \frac{p+1}{2}\}$.

A.3. Details concerning Remarks 3.5 and 3.11. Let $s \in \mathbb{N}$. In order to ensure that $\int_{x>\beta t} (\partial_x^s z)^2 e^{\sigma(x-\beta t)} dx$ is finite for some $\sigma > 0$, it suffices by (3.17) that

$$\sum_{n=0}^{+\infty} \int_{\mathbb{R}} (\partial_x^s z)^2 \sigma^n \varphi_{[n]} dx$$

is finite. Thus it suffices that the series $\sum_{n \geq 0} C(s, \varphi_{[n]}) \sigma^n$ converges.

This condition is satisfied under the following assumptions:

$$(A.13) \quad C(s, \varphi_{[1]}) \leq c_0 C(s+1, \varphi) \quad \text{and} \quad \lambda_s \leq \tilde{c}_0^s.$$

Indeed, if we assume (A.13), we obtain

$$C(s, \varphi_{[n]}) \leq \tilde{c}_0^n \tilde{c}_0^{2(s+n)} \leq \tilde{c}_0^{2s} (c_0 \tilde{c}_0^2)^n,$$

which guarantees the existence of $\sigma > 0$ such that the series $\sum_{n \geq 0} C(s, \varphi_{[n]}) \sigma^n$ converges.

From Proposition 3.3, we also deduce, proceeding step by step, that:

$$\begin{aligned} C(s, \varphi_{[n]}) &\leq c(\eta, \kappa_1, \kappa_2) 2^{\mu_0^s} C(s+1, \varphi) \\ &\leq c(\eta, \kappa_1, \kappa_2)^n 2^{\mu_0^s + \dots + \mu_0^{s+n-1}} C(s+n, \varphi) \\ &\leq c(\eta, \kappa_1, \kappa_2)^n 2^{n\mu_0^{s+n-1}} \lambda_{s+n}^2. \end{aligned}$$

Taking $\tilde{\mu}_0 > \mu_0$, this shows that the integral

$$\int_{x \geq \beta t} (\partial_x^s z)^2(t, x) \left(\sum_{n=0}^{+\infty} \frac{(x - \beta t)^n}{2^{\tilde{\mu}_0^{s+n}}} \right) dx$$

is finite.

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