



**HAL**  
open science

## Mean-field limit of non-exchangeable systems

Pierre-Emmanuel Jabin, David Poyato, Juan Soler

► **To cite this version:**

Pierre-Emmanuel Jabin, David Poyato, Juan Soler. Mean-field limit of non-exchangeable systems. 2024. hal-03624464

**HAL Id: hal-03624464**

**<https://hal.science/hal-03624464v1>**

Preprint submitted on 3 Sep 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MEAN-FIELD LIMIT OF NON-EXCHANGEABLE SYSTEMS

PIERRE-EMMANUEL JABIN, DAVID POYATO, AND JUAN SOLER

ABSTRACT. This paper deals with the derivation of the mean-field limit for multi-agent systems on a large class of sparse graphs. More specifically, the case of non-exchangeable multi-agent systems consisting of non-identical agents is addressed. The analysis does not only involve PDEs and stochastic analysis but also graph theory through a new concept of limits of sparse graphs (extended graphons) that reflect the structure of the connectivities in the network and has critical effects on the collective dynamics. In this article some of the main restrictive hypothesis in the previous literature on the connectivities between the agents (dense graphs) and the cooperation between them (symmetric interactions) are removed.

## CONTENTS

1. Introduction. Multi-agent systems on graphs	1
1.1. Mean-field limits for identical weights	3
1.2. The use of graphons for multi-agent systems	4
1.3. Our new result	5
1.4. Example of limiting kernel and further comments	7
1.5. Sketch of the proof	9
2. A generalized McKean SDE and propagation of independence	11
3. A new hierarchy	14
3.1. Hierarchy of observables indexed by trees	14
3.2. A graphon-like representation of the system and its hierarchy	16
3.3. Stability on the hierarchy with artificial diffusion	21
3.4. Stability on the system without artificial diffusion	23
4. Extending graphons	26
4.1. Our goals	26
4.2. Defining $\tau(T, w, f)$ for unbounded $w$	27
4.3. The key compactness lemma	30
4.4. Proof of Theorem 4.1	36
5. Proof of the main result: Theorem 1.1	42
Appendix A. Models of biological neurons	43
A.1. Smooth neuron dynamics of the Fitzhugh-Nagumo or Hodgkin-Huxley type	43
A.2. Open problem: Integrate and fire models	45
A.3. Open problem: Learning mechanisms in neuron dynamics	45
Appendix B. Technical proofs from subsection 3.2	46
References	50

## 1. INTRODUCTION. MULTI-AGENT SYSTEMS ON GRAPHS

We derive in this article the mean-field limit for multi-agent systems posed on a large class of graphs. More precisely, we consider the dynamics of  $N$  agents, interacting pairwise through a complex map of connections between them. The information about this connectivity map is encompassed through

---

2010 *Mathematics Subject Classification.* 35Q49, 35Q83, 35R02, 35Q70, 05C90, 60G09, 35R06, 35Q89, 35Q92, 49N80, 92B20, 65N75.

*Key words and phrases.* Mean-field limits, Non-exchangeable multi-agent systems, Fully sparse graphs, Vlasov-type PDEs, Many particle complex systems, Synchronization, Emergent phenomena, Collective behaviour.

**Acknowledgment.** This work has been partially supported by the grants: DMS Grant 1908739, 2049020, 2205694 and 2219397 by the NSF (USA); RTI2018-098850-B-I00 from the MINECO-Feder (Spain), PY18-RT-2422, B-FQM-580-UGR20 & A-FQM-311-UGR18 from the Junta de Andalucia (Spain), and by the European Research Council (Europe) under the European Union's Horizon 2020 research and innovation program (grant agreement No 865711).

a weighted, a priori non-symmetric graph  $G_N$  where each vertex represents an agent. Denoting by  $X_i(t) \in \mathbb{R}^d$  the state of the  $i$ -th agent, and by  $K$  the interaction kernel, this leads to the system

$$\begin{cases} \frac{dX_i}{dt} &= \sum_{j=1}^N w_{ij} K(X_i - X_j), \\ X_i(0) &= X_i^0, \end{cases} \quad (1)$$

where the  $w_{ij} = w_{ij}^N \neq w_{ji}^N$  are the weight of the edge  $(i, j)$  of the graph.

While we consider for simplicity deterministic systems like (1), our analysis would extend in a straightforward manner to stochastic multi-agent systems with additive noise such as

$$dX_i = \sum_{j=1}^N w_{ij} K(X_i - X_j) dt + \sigma_N dW_i, \quad (2)$$

where the  $W_i^t$  are independent Wiener processes.

This paper focuses on the analysis of the behavior of the system as  $N \rightarrow \infty$ , for smooth Lipschitz kernels  $K$  but with minimal assumptions on the connectivities  $w_{ij}$ . We only impose scalings that naturally extend the classical notion of mean-field scaling: First, the total interaction felt by an agent should be of order 1, that is,

$$\max_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| = O(1), \quad \max_{1 \leq j \leq N} \sum_{i=1}^N |w_{ij}| = O(1), \quad \text{as } N \rightarrow \infty, \quad (3)$$

ensuring that complex dynamics emerge at time scales of order 1 as well. Secondly, no single agent is allowed to have a dominant role, that is,

$$\max_{1 \leq i, j \leq N} |w_{ij}| = o(1), \quad \text{as } N \rightarrow \infty, \quad (4)$$

which, together with (3), implies that a given agent is influenced by the average of a large number of other agents, so that we expect some concept of mean-field limit to persist.

System (1) is a canonical example of so-called non-exchangeable or non-interchangeable systems. This refers to the fact that (1) is not symmetric under permutation: If  $(X_i(t))_{1 \leq i \leq N}$  solves (1) and  $\sigma \in \mathcal{S}_N$  is any permutation, then  $(X_{\sigma(i)}(t))_{1 \leq i \leq N}$  is not in general a solution to (1). Instead the same permutation  $\sigma$  would need to be applied to the connectivities  $w_{ij}$ , thus modifying the system. Another way of formulating this is that agents are not identical because the effect of the  $j$ -th on the  $i$ -th agent may be very different from the effect on the  $k$ -th agent: One could have  $w_{ij} \gg 1$  but  $w_{ik} \approx 0$  for example.

Classically, studies of many-particle systems have often focused on identical agents and exchangeable systems which corresponds to choosing  $w_{ij} = 1/N$  in (1). The question of the mean-field limits for exchangeable systems where all agents are identical is now rather well understood for Lipschitz interaction kernels, with even significant progress on singular kernels; see for example the reviews [38, 44, 46]. The case of non-exchangeable systems and non-identical agents had remained comparatively less explored but recently started to receive a lot of attention because of their critical importance for many applications and their connections to diverse areas of Mathematics. The analysis of non-exchangeable multi-agent systems indeed not only involves PDEs and stochastic analysis but also graph theory. On the other hand, multi-agent systems are also found in a large number of fields from their classical applications in Physics to the Bio-sciences, Social Sciences or Economics. In many of those settings, agents naturally appear as non-identical: The structure of the connectivities in the network has proved to have critical effects on the dynamics, see among many [71, 72, 73, 83, 84], and the role of the network of interactions is often critical. We present one example of such application in Appendix A which is based on models for the dynamics of biological neurons for which we briefly refer for example to [23, 33, 40, 59, 68, 78, 85].

The main result of this paper is to incorporate the complex non-interchangeable graph structures defined by  $w_{ij}$  into a simple limiting Vlasov equation

$$\partial_t f(t, x, \xi) + \operatorname{div}_x \left( f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x - y) f(t, dy, \zeta) \right) = 0, \quad (5)$$

through the extended graphon  $w(\xi, d\zeta)$ .

In (5),  $f$  describes the probability of finding a single vertex/agent at position  $x \in \mathbb{R}^d$ , in a certain state of interaction with the network  $\xi \in [0, 1]$ , at time  $t > 0$ . This means that the role of an agent can be represented by adding an extra one-dimensional variable. In this way, we associate to the discrete connectivities  $w_{ij}$  a function  $w$  that accounts for the effect of the architecture of the complex original

system (1). This type of graphon-like representation had previously been explored in particular in [19, 20, 21, 49, 60, 61] or in [37, 48]. However those results required more stringent assumptions on the connectivities, so that for example [19, 20, 21, 49] only apply to dense graphs or symmetric interactions, whilst [37, 48] allow for sparse graphs but with a priori knowledge of some additional convergence of the coupling weights  $w_{ij}$ . The major contribution of the present article is to remove all such assumptions and only require the assumptions (3)-(4). This comes with considerable difficulties and impose a completely new strategy to obtain the limit equation.

**1.1. Mean-field limits for identical weights.** Before going further in the technical aspects of our work, we recall here the main notions for the idea of classical mean-field limit in exchangeable systems, corresponding to the simple case  $w_{ij} = \bar{w}/N$ ,

$$\begin{cases} \frac{dX_i}{dt} &= \frac{\bar{w}}{N} \sum_{j=1}^N K(X_i - X_j), \\ X_i(0) &= X_i^0. \end{cases} \quad (6)$$

A critical question for any multi-agent system is how to choose initial data for (6), since it is not feasible in almost any applications to individually identify or measure every single  $X_i^0$ . Classical assumptions rely on the notion of molecular chaos by imposing that the  $X_i^0$  be independent and identically distributed (*i.i.d.*) random variables (or almost *i.i.d.* in some appropriate sense) according to an initial probability distribution  $f^0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ .

This leads to the key notion of propagation of chaos, which consists in proving that the  $X_i(t)$  solving (6) still remain approximately *i.i.d.* (again in some appropriate measure). For exchangeable systems such as (6), propagation of chaos often then directly implies that the limit as  $N \rightarrow \infty$  of the 1-particle distribution of the system solves the mean-field equation

$$\partial_t f(t, x) + \operatorname{div}_x \left( \bar{w} f(t, x) \int_{\mathbb{R}^d} K(x - y) f(t, dy) \right) = 0. \quad (7)$$

We refer to the seminal results in [12, 31, 64] and later [77] for the derivation of (7) for  $K \in W^{1,\infty}(\mathbb{R}^d)$ . As mentioned earlier, the case of non Lipschitz kernels corresponds to many realistic applications, it is much more difficult and still poorly understood to some degree. An extensive discussion would however carry us too far from the main goals of this article and we only briefly refer again to [38, 44, 46].

A useful notion for system (6) is the so-called empirical measure which is given by,

$$\mu^N(t, x) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(x),$$

for every  $t \geq 0$ . In the deterministic case, the empirical measure is itself already a solution to the mean-field equation (7), which offers a straightforward approach to deriving the mean-field limit as  $N \rightarrow \infty$ , for Lipschitz kernels  $K$ , through a stability analysis of Eq. (7) in the space of measures. Specifically, the empirical measure offers a very simple way to formulate and understand the mean-field limit in terms of the weak convergence of the random measure  $\mu^N$  to the deterministic limit  $f$ . Classical results for Lipschitz kernels for example imply that

$$\|\mu_N - f\|_{W^{-1,1}} \leq \|\mu_N^0 - f\|_{W^{-1,1}} e^{t \bar{w} \|\nabla K\|_{L^\infty}}.$$

Some major differences are immediately apparent for non-exchangeable systems like (1). First of all, because agents are non-identical, it may not be realistic to assume that they are initially identically distributed. We will still assume that the  $X_i^0$  are independent though as this is essential for the reduction in complexity leading to (7). Instead of propagation of chaos, we consequently have to deal with the weaker notion of *propagation of independence*.

It is possible to extend some of the classical approaches to prove this propagation of independence (as we do in the proof later). Unfortunately propagation of independence no longer directly implies the limit (7) for general non-exchangeable systems, creating major difficulties. The one exception occurs in the special case where

$$\sum_{j=1}^N w_{ij} = \bar{w} \text{ is independent of } i.$$

In that setting, it is possible to still derive the classical mean-field limit (7); see [28] for *i.i.d* initial positions  $X_i^0$ , and to [24] for the stability for this initial distribution. For general connectivities  $w_{ij}$  however, new ideas and new methods seem to be needed.

**1.2. The use of graphons for multi-agent systems.** Graphons offer a non-parametric method of modeling and estimating large networks and are constructed as limits of sequences of dense graphs. There now exists a large literature dedicated to graphons for which we refer to the seminal [8, 9, 10, 11, 56].

To give a rough idea of how graphons play a critical role in non-exchangeable systems, consider a sequences of dense graphs  $G_N$  with nodes indexed from 1 to  $N$  and adjacency matrix  $w^N = (w_{ij}^N)_{1 \leq i, j \leq N}$  with the key assumption that

$$w_{ij}^N = \frac{\bar{w}_{ij}^N}{N} \quad \text{with} \quad \bar{w}_{ij}^N = O(1), \quad \text{as } N \rightarrow \infty. \quad (8)$$

For a fixed  $N$ , it is straightforward to define the graphon over  $[0, 1]^2$

$$\bar{w}_N(\xi, \zeta) = \sum_{i, j=1}^N \bar{w}_{ij} \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{I}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta). \quad (9)$$

One directly observes that  $\bar{w}_N$  is uniformly bounded in  $L^\infty([0, 1]^2)$  but a fundamental property behind graphons is that  $\bar{w}_N$  can be made compact in the appropriate distance after re-arrangements. More precisely, there exists a measure preserving map  $\phi_N$  on  $[0, 1]$  such that the rearranged object  $\bar{w}_N(\phi_N(\xi), \phi_N(\zeta))$  is compact in the norms of operators from  $L^\infty([0, 1])$  to  $L^1([0, 1])$ ; this norm is equivalent to the so-called *cut metric*, see [56]. The fact that such a limit exists was initially established in [57] using a weak version of the Szemerédi Regularity Lemma [79] and a martingale argument.

This turns out to be a key property for the purpose of deriving the mean-field limit. For example, it provides a natural way to obtain weak stability for Eq. (5). Namely, consider a sequence of solutions  $f_N$  to (5) with weights  $w_N$ , with  $w_N$  uniformly bounded in  $L^\infty([0, 1]^2)$  and  $f_N$  uniformly bounded in  $L^\infty([0, t_*] \times [0, 1], L^1 \cap W^{1, \infty}(\mathbb{R}^d))$ . Assume further that  $f$  is any other solution to (5) with weights  $w$  lying in the same regularity class as above. Then, we easily infer the stability estimate

$$\frac{d}{dt} \int_0^1 \int_{\mathbb{R}^d} |f_N - f| dx d\xi \leq C_1 \int_0^1 \int_{\mathbb{R}^d} |f_N - f| dx d\xi + C_2 \|w_N - w\|_{L^\infty \rightarrow L^1}, \quad (10)$$

where  $C_1, C_2$  are constants depending on the norms of  $f$ , the uniform bound of the norm of  $w_N$  and the  $L^1$  and Lipschitz norms of  $K$ . Consequently, using the above theory of dense graph limits, one can set the measure-preserving maps  $\phi_N$  on  $[0, 1]$  so that the re-arranged objects  $w_N(\phi_N(\xi), \phi_N(\zeta))$  converge (up to subsequence) in the operator norm, and also set  $w$  to be the obtained limit. By doing so, the second term in the right hand side of (10) converges to zero and, assuming well-prepared initial data, we can pass to the limit as  $N \rightarrow \infty$  in  $f_N(t, x, \phi_N(\xi))$  and obtain a solution  $f(t, x, \xi)$  to (5). As a consequence, we can determine the limit of the 1-particle distribution given by  $\int_0^1 f_N(t, x, \xi) d\xi$ :

$$\int_0^1 f_N(t, x, \xi) d\xi = \int_0^1 f_N(t, x, \phi_N(\xi)) d\xi \rightarrow \int_0^1 f(t, x, \xi) d\xi, \quad \text{as } N \rightarrow \infty.$$

We refer to [19, 20, 21, 49] in particular for a complete analysis that this brief sketch cannot do justice to. An alternative and more primitive approach to derive macroscopic limits for (1) was obtained in [60], again for graphons  $w \in L^\infty([0, 1]^2)$  and Lipschitz kernel  $K$ , in terms of the graph-limit equation

$$\partial_t X(t, \xi) = \int_0^1 w(\xi, \zeta) K(X(t, \xi) - X(t, \zeta)) d\zeta. \quad (11)$$

Here,  $X \in C^1(\mathbb{R}_+, L^\infty([0, 1]))$  can be interpreted as a time-evolving parametrization of the continuum of agents by a continuous index  $\xi \in [0, 1]$ . Whilst a large portion of the literature sticks to this simpler formulation, we remark that (11) corresponds to a special class of solutions to (5) having the form  $f(t, x, \xi) = \delta_{X(t, \xi)}(x)$ , but clearly not all solutions of (5) come from solutions to (11).

The critical issue to implement a similar stability estimate like (10) in our case is that our assumptions (3)-(4) of course cannot ensure that the weights  $w_{ij}$  satisfy (8). As a matter of fact, we are specifically interested in sparse graphs where (8) does not hold.

Various extensions of graphons have been proposed to handle less stringent assumptions than (8). We mention in particular the extension to  $L^p$  graphons in [6, 7], the more general operator-based extension in [3] based on *graphops*, and the measure-theoretic extension in [50] based on *s-graphons*. Each of the three classes of continuum objects contain the family of finite graphs, and they can be endowed with a compact metric space structure under a certain topology, similarly to what it was done for graphons. We emphasize that the above assumption (3) only provides uniform bounds for  $w_N$  in the mixed spaces  $L_\xi^\infty \mathcal{M}_\zeta = L_\xi^\infty([0, 1], \mathcal{M}_\zeta([0, 1]))$  and  $L_\zeta^\infty \mathcal{M}_\xi = L_\zeta^\infty([0, 1], \mathcal{M}_\xi([0, 1]))$ . This scaling suggests defining a new class of continuum objects which we shall call *extended graphons*, see next section and Definition 3.6

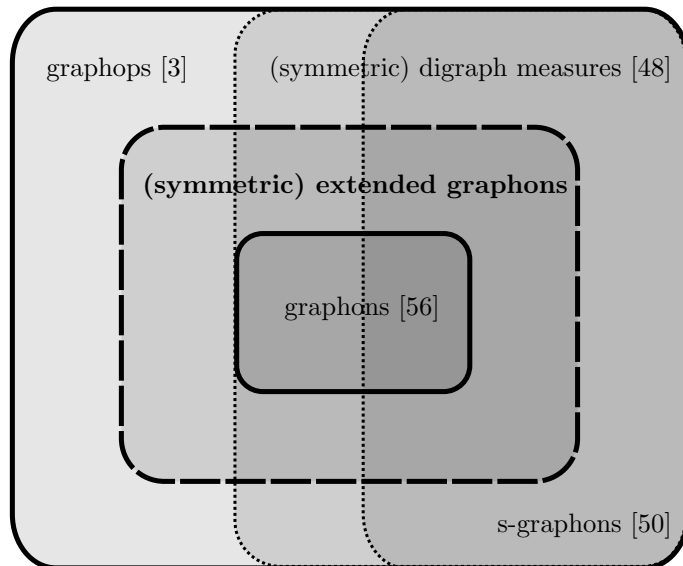


FIGURE 1. Comparison of the various graph limit theories.

below for further details. A similarly scaled extension was developed in [50], based on *digraph measures* or bounded everywhere defined families of measures, and is discussed more at length after Theorem 1.1. Closely related is also the scaling in [58], where the author considered limits of random graphs sampled from graphons not necessarily bounded but belonging to the mixed space  $L^\infty([0, 1], L^2([0, 1]))$ .

The goal of this paper is not to provide a complete classification of the existing theories of graph limits. However, the above classes of objects can be compared, and so has been done in more dedicated literature, *e.g.* [3, 50] and also [37, 48]. Without any intention of comparing their topologies, the previous classes of objects are related as sketched in Figure 1, at least when restricted to symmetric weights. We note that extended graphons account for an intermediate degree of sparsity as they occupy an intermediate position between the class of graphons and graphops. This fact will become clearer later in Section 3, and more particularly in the operator representation of extended graphons in Lemma 3.7. However, we do emphasize the topology that we use on those extended graphons is very distinct from other known objects.

**1.3. Our new result.** The main contribution of this paper is to provide the mean-field limit with only assumptions (3)-(4). This first requires a careful definition of the space for limiting kernel which we call *extended graphons* and we denote by  $L^\infty \mathcal{M}_\zeta \cap L^\infty \mathcal{M}_\xi$ ; namely  $w \in L^\infty \mathcal{M}_\zeta \cap L^\infty \mathcal{M}_\xi$  iff

$$w(\xi, d\zeta) \in L^\infty([0, 1], \mathcal{M}([0, 1])), \quad w(d\xi, \zeta) \in L^\infty([0, 1], \mathcal{M}([0, 1])), \quad (12)$$

where we define the space  $L^\infty([0, 1], \mathcal{M}([0, 1]))$  as the topological dual to the Banach space  $L^1([0, 1], C([0, 1]))$ , endowed with the corresponding weak-star topology.

We are now ready to state our main result,

**Theorem 1.1.** *Assume that the interaction kernel  $K$  belongs to  $W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)$  and consider a sequence  $(X_i(t))_{1 \leq i \leq N}$ , solving the multi-agent system (1) for connection weights satisfying only (3)-(4). Assume moreover that the initial data  $X_i^0$  are independent random variables, but not necessarily identically distributed, and denote their laws by  $f_i^0$ . Finally assume that*

$$\sup_{N \in \mathbb{N}} \sup_{1 \leq i \leq N} \mathbb{E}[|X_i^0|^2] < \infty, \quad \sup_{N \in \mathbb{N}} \sup_{1 \leq i \leq N} \|f_i^0\|_{W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)} < \infty.$$

*Then, there exist  $w \in L^\infty \mathcal{M}_\zeta \cap L^\infty \mathcal{M}_\xi$  and  $f \in L^\infty([0, t_*] \times [0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$ , for any  $t_* > 0$ , such that  $f$  solves (5) with  $w$  and, up to the extraction of a subsequence,*

$$\sup_{0 \leq t \leq T} \mathbb{E} W_1 \left( \int_0^1 f(t, \cdot, \xi) d\xi, \mu_N(t, \cdot) \right) \rightarrow 0,$$

*as  $N \rightarrow \infty$ , where  $W_1$  is the usual Wasserstein distance on  $\mathbb{R}^d$  and  $\mu^N(t, x) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(x)$  are the empirical measures associated to  $(X_i(t))_{1 \leq i \leq N}$ .*

Compared to the existing literature, Theorem 1.1 is the only result so far capable of handling the mean-field limit under the only assumptions (3)-(4). We do not obtain classical graphons since the limiting kernel  $w$  only belongs to  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , which is a natural space corresponding to the scaling in (3). As mentioned above, a space with a similar scaling is the space of digraph measures  $B([0, 1], \mathcal{M}([0, 1]))$ . This space was inspired in the s-graphons [50], specifically introduced to handle sparse graphs, and it was used in [48] for mean-field limits. There are a few key differences with the present analysis, as digraph measures  $B([0, 1], \mathcal{M}([0, 1]))$  consists of everywhere defined in  $\xi$  bounded measures in  $\zeta$  while our extended graphons in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  are only almost everywhere defined in  $\xi$  (or in  $\zeta$ ). The use of extended graphons offers several advantages when passing to the limit. In particular, in comparison [48] appears to require an improved continuous dependence in  $\xi$  of the digraph measures that further restricts the type of connectivities that can be used with respect to (3)-(4).

But our notion of extended graphons in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  does lead to some major issues, as it does not seem possible to pass to the limit in Eq. (5) directly. That imposes a new strategy briefly outlined in the sketch of proof in the next subsection. Some of that technical difficulty can be glimpsed from the fact that it is not yet clear in which sense Eq. (5) is posed with only  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ . The main issue is the non-linear term of course since  $f$  is only essentially bounded in  $\xi$  with no additional regularity, and only *a.e.* defined. More specifically, it is not immediately obvious how to make sense of terms like

$$\xi \in [0, 1] \mapsto \int_0^1 \phi(\zeta) w(\xi, d\zeta),$$

for  $\phi \in L^\infty([0, 1])$ . This is where the proper definition of  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  is needed and one reason why we require  $w$  to belong both to  $L_\xi^\infty \mathcal{M}_\zeta$  and to  $L_\zeta^\infty \mathcal{M}_\xi$  (which would be automatic if  $w$  was symmetric). A careful analysis then allows proving that the above function is correctly defined in  $L^\infty([0, 1])$  with

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L^\infty} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|\phi\|_{L^\infty},$$

as we prove later in Lemma 3.7. Note that the above also allow identifying any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  with a bounded linear operator  $L^\infty([0, 1]) \rightarrow L^\infty([0, 1])$ , and therefore this explains the above-mentioned relation in Figure 1 between extended graphons and graphops.

We also note that our proofs could imply an even more general formulation of Theorem 1.1. For example the assumptions  $f_i^0 \in L^1 \cap L^\infty(\mathbb{R}^d)$  would be enough, and it should even be possible to further relax some of the assumptions on the graphs with

$$\frac{1}{N} \sum_{i=1}^N \left( \sum_{j=1}^N |w_{ij}| \right)^p = O(1), \quad \frac{1}{N} \sum_{i=1}^N \sup_{1 \leq j \leq N} |w_{ij}|^{1/p} = o(1), \quad \text{as } N \rightarrow \infty,$$

for some  $p > 1$ , instead of (3)-(4).

As in classical mean-field limits, independence of initial data is the key to the reduction of complexity provided by Theorem 1.1. We assumed for simplicity that the  $X_i^0$  were fully independent but it would be straightforward to relax this condition by having them approximately independent.

Theorem 1.1 obtains the limit up to the extraction of a subsequence in  $N$ , since it makes no assumptions on the convergence of the initial data and it then amounts to a compactness result. Nevertheless, it is possible to derive the convergence of the whole sequence but the formulation is trickier than usual because of the role played by the connectivities  $w_{ij}$ . The first necessary assumption is naturally the convergence of the full sequence of empirical measures: There exists  $\bar{f}^0 \in W^{1,\infty}(\mathbb{R}^d)$  such that

$$\mathbb{E} W_1(\bar{f}^0, \mu_N^0) \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (13)$$

Alternatively, since  $X_i^0$  are independent, one may assume convergence of the 1-particle distribution

$$\mathbb{E} W_1 \left( \bar{f}^0, \frac{1}{N} \sum_{i=1}^N f_i^0 \right) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

However (13) is not enough and some additional convergence of the  $w_{ij}$  is also required.

Theorem 1.1 opens up new avenues to the analysis but still leaves some important questions unresolved. The main and obvious issue concerns the Lipschitz assumption on the kernel  $K$ . This is a critical issue for many applications which involve some form of singular interactions. Among several examples, we mention the integrate and fire system for biological neurons that we briefly describe in Appendix A. Integrate and fire models only very loosely fit within the multi-agent framework that we focus on in this

paper so that extending the present analysis to integrate and fire model would be a highly non trivial extension.

The Lipschitz regularity of  $K$  plays a critical role in the first step of our analysis when propagating independence. It does not appear as essential in the next, more complex steps of our proofs. Replacing the classical trajectory methods to obtain propagation of independence could allow obtaining an equivalent of Theorem 1.1 with less stringent assumptions on  $K$ .

**1.4. Example of limiting kernel and further comments.** There are compelling reasons why extended graphons  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  appear to be the correct scale for the type of complex structures of connectivities that we are interested in.

We first point out that our extended graphons occupy an intermediary place in the current hierarchy of graphon-like objects, see Figure 1. Obviously they are more general than classical graphons since they are not necessarily bounded, that is,  $w \notin L_{\xi, \zeta}^\infty$  in general. However as mentioned above, the analysis will show later that any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  can also be seen as the kernel of a bounded operator from  $L^\infty([0, 1]) \rightarrow L^\infty([0, 1])$ ; instead of an operator  $L^\infty([0, 1]) \rightarrow L^1([0, 1])$  for graphops. As such extended graphons appear as intermediary objects between classical graphons and other extensions such as graphops, s-graphons and digraph measures.

However the topology that we consider in our analysis for this space  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  is one of the major differences with respect to the existing literature. The proper convergence for our large-graph limit cannot hold in the classical cut distance, or any direct extensions such as the topology of bounded linear operators  $L^\infty([0, 1]) \rightarrow L^\infty([0, 1])$  to the best of our knowledge. This is clear from the stability estimate (10) for graphons, which inevitably breaks for unbounded graphons. Instead the key notion of convergence that we use is based on a new countable family of observables that mixes the connection kernel and the initial laws on each agent. Specifically, the observables can be defined as follows

$$\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})(t, x_1, \dots, x_{|T|}) = \frac{1}{N} \sum_{i_1, \dots, i_{|T|}=1}^N \prod_{(k, l) \in \mathcal{E}(T)} w_{i_k i_l} \prod_{m=1}^{|T|} f_{i_m}^0(x_m), \quad (14)$$

and they are indexed by finite trees  $T$ , where  $f_i^0$  denotes the initial law of the variable  $X_i^0$ . Those observables have a corresponding representation at the limit when the discrete weights  $(w_{i, j})_{1 \leq i, j \leq N}$  and laws  $f_i^0(x)$  are replaced by kernels  $w(\xi, \zeta)$  and initial data  $f^0(x, \xi)$  through the definition,

$$\tau(T, w, f)(t, x_1, \dots, x_{|T|}) = \int_{[0, 1]^{|T|}} \prod_{(k, l) \in \mathcal{E}(T)} w(\xi_k, \xi_l) \prod_{m=1}^{|T|} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}. \quad (15)$$

Those definitions allow defining precisely the convergence in Theorem 1.1: If we have the strong convergence in  $L^2$  of the observables  $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) \rightarrow \tau(T, w, f^0(\xi, x))$  as  $N \rightarrow \infty$ , for all trees  $T$ , then the dynamics of the  $X_i$  can effectively be represented at the limit by the solution  $f$  to (5),

$$\frac{1}{N} \sum_i \delta_{X_i(t)}(x) \rightarrow \int_0^1 f(t, x, d\xi).$$

As it can be seen in the sketch of proof below, the observables  $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})$  indeed play a critical role in our analysis as they entirely control the dynamics.

Note that since  $f_i^0$  are normalized probability measures, then by integrating the observables (14) on the spacial variables we are led to the well-known notion of moments on a graph, that is,

$$\int_{\mathbb{R}^{|T|}} \tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})(t, x_1, \dots, x_{|T|}) dx_1 \dots, dx_{|T|} = \tau(T, (w_{ij})_{1 \leq i, j \leq N}),$$

where  $\tau(T, (w_{ij})_{1 \leq i, j \leq N})$  is the *homomorphism density* of the finite tree  $T$  in the finite graph  $(w_{ij})_{1 \leq i, j \leq N}$ , which are also used in the classical theory of graphons. In a first major difference to graphons though, we can only define the  $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N})$  for trees  $T$  and not any arbitrary graph.

A second critical difference is that our notion of observables completely entangles the kernel with the initial conditions. In particular our main theorem does not provide independent convergence of the extended graphons and the initial data. In other words (see the example below as well), we could have that for some initial data  $f_i^0(x)$  and some connections  $w_{ij}$ , one has that  $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) \rightarrow \tau(T, w, f^0)$  for some limiting  $w(\xi, d\zeta)$  and  $f^0(x, \xi)$ . If one then considers different initial data  $\tilde{f}_i^0(x)$  but for the same connections  $w_{ij}$ , there is no particular reason why we would be able to use the same limiting kernel: We may need to derive a different limiting kernel  $\tilde{w}(\xi, d\zeta)$  together with a different limiting initial law  $\tilde{f}^0(x, \xi)$  to maintain the convergence  $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (\tilde{f}_i^0)_{1 \leq i \leq N}) \rightarrow \tau(T, \tilde{w}, \tilde{f}^0)$ .



While our main theorem ensures that there always exists a limiting representation, it is not necessarily unique. We can for instance find different kernels  $w(\xi, d\zeta) \neq \tilde{w}(\xi, d\zeta)$  and initial data  $f^0(x, \xi) \neq \tilde{f}^0(x, \xi)$  such that  $\tau(T, w, f^0) = \tau(T, \tilde{w}, \tilde{f}^0)$ ; see the example below again.

All of this makes the notion of convergence that is developed in the present paper very different from most of mean-field limit results based on graphons or graphon-like objects. Because our result applies to any connections with only the bounds (3)-(4) without any additional convergence assumptions, it is unclear whether it would even be possible to identify a specific topology on the connections  $w_{ij}$  or kernels  $w$  without mixing them with the initial data.

Several other open questions concern the new system of observables that we introduce in (15). Those are conjectured to naturally extend the notion of marginals, and hierarchy of marginals to non-exchangeable systems. The structure of the hierarchy of observables (15) remains rather poorly understood however. A first example is the range of possible functions that one can reach through (15): What would be the conditions on a sequence  $\alpha_T(x_1, \dots, x_{|T|})$  indexed by trees  $T$  so that we can find  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $f \in L^\infty(\mathbb{R}^d \times [0, 1])$ , such that

$$\tau(T, w, f) = \alpha(T), \quad \forall T?$$

To illustrate those remarks, we discuss in some depth a natural example of connection weights, based on a sparse graph. Choose  $M \ll 1$ , and separate  $\{1, \dots, N\}$  into  $N/M$  subsets  $E_1, \dots, E_{N/M}$ , with  $|E_k| = M$  for each  $k \leq N/M$ . Introduce further a permutation  $\phi \in \mathcal{S}_{N/M}$  which lets us define

$$w_{ij} = \begin{cases} \frac{1}{M} & \text{if } i \in E_k \text{ and } j \in E_{\phi(k)} \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

This choice of connection has a very simple interpretation. Agents are divided in  $N/M$  distinct classes  $E_k$ ,  $k = 1 \dots N/M$ . Each agent  $i$  in a class  $E_k$  is connected with exactly  $M$  other agents that all belong to the class  $E_{\phi(k)}$ . By a combinatorial argument note that the density of those graphs, that is, the proportion of existing edges over the maximum number of possible edges that they could, behaves as  $M/N$ . Because  $M$  is much smaller than  $N$  this does lead to a sparse graph of connections.

Note that those connections satisfy (3)-(4) as long as  $M \gg 1$ . In particular if  $i \in E_k$

$$\sum_{j=1}^N |w_{ij}| = \sum_{j \in E_{\phi(k)}} \frac{1}{M} = 1,$$

with a similar bound on  $\sum_{i=1}^N |w_{ij}|$  since  $\phi$  is a permutation.

It is possible to try to connect more directly the connections to a tentative limiting kernel through

$$w_N(\xi, \zeta) = \sum_{i,j=1}^N N w_{ij} \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{I}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta),$$

which is a formula that we are making use of in the proof.

However this may not have a simple expression in general, even within the limited framework of this example. Instead to derive a more explicit formula for  $w_N$ , it is necessary to re-arrange the indices  $i$  in a proper order. Here, it would be useful to list all indices in a given class  $E_k$  consecutively with for instance

$$E_k = \{(k-1)M + 1, \dots, kM\}, \quad k = 1 \dots N/M.$$

If one performs a permutation to have such a formula, this leads to

$$w_N(\xi, \zeta) = \frac{N}{M} \mathbb{I}_{\zeta \in [\phi_N(\xi), \phi_N(\xi) + M/N)},$$

provided that we define  $\phi_N$  appropriately as per

$$\phi_N(\xi) = (\phi(k) - 1) \frac{M}{N}, \quad \text{if } \xi \in [(k-1)M/N, kM/N).$$

This expression of  $w_N$  of course amounts to a discretization of

$$\bar{w}_N(\xi, \zeta) = \delta_{\phi_N(\xi)}(\zeta),$$

where we observe that  $\phi_N$  is a *a.e.* defined measure-preserving map on  $[0, 1]$ . This kernel  $\bar{w}_N$  is hence an example of an extended graphon in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , that cannot be represented by classical graphons, any  $L^p$  extensions, or through objects in the space  $B([0, 1], \mathcal{M}([0, 1]))$ . However, this does not directly imply that the limiting dynamics must follow some kernel

$$\bar{w}(\xi, \zeta) = \delta_{\bar{\phi}(\xi)}(\zeta). \tag{16}$$

First of all, note that even the convergence  $\bar{w}_N \rightarrow \bar{w}$  in the weak-\* topology generally would require that  $\phi_N$  converges to  $\bar{\phi}$  strongly in  $L^1$ . But, as we point out above, what is actually needed is the strong convergence of the  $\tau(T, (w_{ij})_{1 \leq i, j \leq M}, (f_i)_{1 \leq i \leq N})$  for all finite trees  $T$ .

Then, it is more convenient to define

$$f_N(t, x, \xi) = \sum_{i=1}^N \bar{f}_i(t, x) \mathbb{I}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}(\xi),$$

such that we have the identity  $\tau(T, (w_{ij})_{1 \leq i, j \leq N}, (f_i^0)_{1 \leq i \leq N}) = \tau(T, w_N, f_N^0)$ . For example if  $T = T_2$  the only tree with two vertices then we simply have

$$\tau(T_2, w_N, f_N) = \int_{[0, 1]^2} w_N(\xi_1, \xi_2) f_N^0(x_1, \xi_1) f_N^0(x_2, \xi_2) d\xi_1 d\xi_2.$$

If we approximate  $w_N$  by  $\bar{w}_N$ , which requires some regularity in  $\xi$  of  $f_N^0$ , we have the simple

$$\tau(T_2, w_N, f_N) = \int_0^1 f_N^0(x_1, \xi) f_N^0(x_2, \phi_N(\xi)) d\xi + o(1) \quad \text{as } N \rightarrow \infty.$$

This correctly suggests that to easily identify the limiting kernel  $\bar{w}$  in that setting as (16), there would need to exist a permutation of the indices (or equivalent a measure-preserving transform in  $\xi$ ) such that both  $\phi_N$  and  $f_N^0$  are strongly compact in  $\xi$  after the permutation.

This is obviously a very restrictive set of assumptions. The fact that we do not impose anything like this in our theorem is a good indication of the difficulty of the analysis to prove that there nevertheless always exists some limiting kernel among extended graphons.

We can also observe, even in this simple example, that the limiting kernel is not necessarily unique. For instance if we take all  $X_i^0$  *i.i.d.*, this implies that all  $f_i^0$  are identical. We could then simply take  $\bar{w} = 1$  as we fall in the setting mentioned earlier in the introduction.

**1.5. Sketch of the proof.** We briefly explain here some of the main ideas and steps that are used in the proof of the main Theorem 1.1.

◇ *Step 1: Propagation of independence.*

This is done in Section 2 by introducing the independent system of coupled PDEs (19), restated here

$$\begin{cases} \partial_t \bar{f}_i + \operatorname{div}_x \left( \bar{f}_i(t, x) \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(x-y) \bar{f}_j(t, dy) \right) = 0, \\ \bar{f}_i(t=0) = f_i^0, \end{cases} \quad (17)$$

and it proves (*cf.* Proposition 2.2) that  $\bar{f}_i(t, x)$  are correct approximations for the law of  $X_i(t)$  under the assumption (4). As mentioned earlier, this relies on a straightforward extension to the classical trajectorial methods for mean-field limits, see [80].

For general connectivities  $w_{ij}$ , this is only the very beginning as it does not allow directly obtaining the limit of the corresponding 1-particle distribution given by

$$\frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x).$$

The one exception concerns the special case where  $\sum_{j=1}^N w_{ij} = \bar{w}$  is independent of  $i$ . If moreover all  $f_i^0$  are equal to  $\bar{f}^0$ , then it can be seen that  $\bar{f}_i = \bar{f}$  solves (17) provided that  $\bar{f}$  is a solution to the classical mean-field limit.

◇ *Step 2: Introducing extended graphons.*

The next step in our proof is to define our extended empirical graphons for a fixed  $N$ , which is still done through the equivalent of formula (9), namely

$$\begin{aligned} w_N(\xi, \zeta) &= \sum_{i, j=1}^N N w_{ij} \mathbb{I}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}(\xi) \mathbb{I}_{\left[\frac{j-1}{N}, \frac{j}{N}\right)}(\zeta), \\ f_N(t, x, \xi) &= \sum_{i=1}^N \bar{f}_i(t, x) \mathbb{I}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}(\xi). \end{aligned}$$

It is straightforward to check that, since the  $\bar{f}_i$  solve (17), then  $w_N$  and  $f_N$  solve the limiting equation (5), namely we have

$$\partial_t f_N(t, x, \xi) + \operatorname{div}_x \left( f_N(t, x, \xi) \int_0^1 w_N(\xi, \zeta) \int_{\mathbb{R}^d} K(x-y) f_N(t, y, \zeta) dy d\zeta \right) = 0.$$

This is performed at the beginning of Section 3 together with the study of some basic properties of our extended graphons under the only scaling  $w_N \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , which include the aforementioned

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_\xi^\infty} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|\phi\|_{L^\infty}.$$

For a fixed  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , this allows for example obtaining the existence and uniqueness of a solution  $f \in L^\infty([0, T] \times [0, 1], W^{1,\infty}(\mathbb{R}^d))$  to (5), see Proposition 3.11. However, we are unfortunately incapable of passing to the limit directly in the above equation for  $f_N$  to derive (5). The main obstruction is the lack of compactness in  $\xi$  on  $f_N$  combined with the very weak topology in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ .

◇ *Step 3: Introducing and studying the new observables.*

The latter difficulty is what leads to the introduction of the observables, that can be defined at any time  $t$  through the formula (15) that we repeat here,

$$\tau(T, w_N, f_N)(t, x_1, \dots, x_{|T|}) = \int_{[0, 1]^{|T|}} \prod_{(k,l) \in \mathcal{E}(T)} w_N(\xi_k, \xi_l) \prod_{m=1}^{|T|} f_N(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}.$$

Those observables include the 1-particle distribution which simply corresponds to choosing the tree  $T = T_1$  with only one vertex:

$$\tau(T_1, w_N, f_N)(t, x) = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x).$$

The fact that the  $\tau(T, w_N, f_N)$  are well defined follows from the basic analysis of our extended graphons in the previous point.

A critical point is that those observables solve an independent hierarchy of equations, which reads

$$\partial_t \tau(T, w_N, f_N) + \sum_{i=1}^{|T|} \operatorname{div}_{x_i} \left( \int_{\mathbb{R}^d} K(x_i - z) \tau(T+i, w_N, f_N)(t, x_1, \dots, x_{|T|}, z) dz \right) = 0,$$

where  $T+i$  denotes the tree obtained from  $T$  by adding a leaf on the  $i$ -th vertex.

It turns out that we can pass to the limit in the above hierarchy and even obtain the uniqueness of solutions upon appropriate assumptions on the  $\tau(T, w_N, f_N)$  as per Theorem 3.23. The analysis is performed in the second part of Section 3 and is one of the major contribution of the paper.

Independently to the proof in this paper, we mention the very recent [53] that obtains uniqueness on the classical exchangeable BBGKY or Vlasov hierarchies. The basic idea in that proof is reminiscent of the strategy that we develop here for Theorem 3.23, even if [53] uses relative entropies while we base our estimates on  $L^2$  bounds.

◇ *Step 4: Identifying the limit.*

Theorem 3.23 allows passing to the limit in all  $\tau(T, w_N, f_N)$ , after extracting a sub-sequence, to some hierarchy of  $h_T$ . It remains to identify the  $h_T$  by finding  $w$  and  $f^0$  such that if  $f$  solves the limiting equation (5) with  $w$  and  $f^0$  as initial data then we have the representation

$$\tau(T, w, f) = h_T.$$

This is done in Section 4 in Theorem 4.1 and relies on a modified regularity lemma for graphons stated in Lemma 4.7. This also requires the definition of the  $\tau(T, w, f)$  in general with only  $f \in L^\infty([0, T] \times [0, 1], W^{1,\infty}(\mathbb{R}^d))$  and  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , which is based on an abstract algebra that encompasses the sequence of operations on the tree that are performed to produce  $\tau(T, w, f)$ . Putting all estimates together finally allows to conclude.

As one can see, this does not prove the convergence of  $f_N$  to  $f$  in any direct sense. Instead it only shows the convergence of all  $\tau(T, w_N, f_N)$  to the  $\tau(T, w, f)$ . It implies the claimed result by taking  $T = T_1$  the trivial tree with only one node and no edge, which in particular yields

$$\int_0^1 f_N(t, x, \xi) d\xi = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x) \rightarrow \int_0^1 f(t, x, \xi) d\xi.$$

## 2. A GENERALIZED MCKEAN SDE AND PROPAGATION OF INDEPENDENCE

The aim of this section is to show that independence of events is propagated in our system (1) in an appropriate sense. It will be the equivalent of the aforementioned classical concept of propagation of chaos (for uniform weights  $w_{ij} = \frac{1}{N}$ ), but it is not and it requires a careful extension of the usual arguments. To that end, we shall propose an extension of the classical [80] McKean SDE associated with the particles system to our current case with non-uniform weights.

**Lemma 2.1** (McKean SDE). *Consider the nonlinear system of SDE for  $(\bar{X}_1, \dots, \bar{X}_N)$  given by*

$$\begin{cases} \frac{d\bar{X}_i}{dt} = \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(\bar{X}_i - y) \bar{f}_j(t, dy), \\ \bar{X}_i(t=0) = \bar{X}_i^0, \end{cases} \quad (18)$$

where  $K \in W^{1,\infty}$  and we denote  $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$ . Then, for any random initial data  $(\bar{X}_1^0, \dots, \bar{X}_N^0)$  such that  $\mathbb{E}[|\bar{X}_i^0|] < \infty$  there is existence and uniqueness, trajectorial and in law of solutions of (18). In addition, if  $\bar{X}_i^0$  are independent then  $\bar{X}_i(t)$  are also independent for each  $t \in \mathbb{R}_+$ .

The proof is an elementary extension of the classical case with uniform weights, see [80, Theorem 1.1]. We notice that the independence of initial data is crucial in the previous result. However,  $\bar{X}_i^0$  are not necessarily identically distributed, although we could have also assumed so. Unfortunately, notice that the presence of non-uniform weights makes particles non-exchangeable and there is a rupture of symmetry. Thus, the necessity for the use of the different laws  $\bar{f}_j$  in the SDE is justified. This was not the case for uniform weights, where symmetry is propagated and for each  $\bar{X}_i$  the McKean SDE can be closed in terms of its law  $\bar{f}_i$  itself.

**Proposition 2.2** (Propagation of independence). *Let  $(X_1, \dots, X_N)$  be solution to (1) with  $K \in W^{1,\infty}$  and consider the associated laws  $f_i(t, \cdot) = \text{Law}(X_i(t))$ . Assume that  $X_i^0$  are independent random variables such that  $\mathbb{E}[|X_i^0|^2] < \infty$ , and the following uniform estimates hold*

$$\sup_{1 \leq i \leq N} \sqrt{\mathbb{E}[X_i^0]^2} \leq M, \quad \sup_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| \leq C,$$

for every  $N \in \mathbb{N}$  and appropriate  $M, C \in \mathbb{R}_+$ . Consider the solution  $\bar{f}_i$  to the coupled PDE system

$$\begin{cases} \partial_t \bar{f}_i + \text{div}_x \left( \bar{f}_i(t, x) \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(x - y) \bar{f}_j(t, dy) \right) = 0, \\ \bar{f}_i(0, x) = f_i(0, x). \end{cases} \quad (19)$$

Then, the following estimate holds

$$\sup_{1 \leq i \leq N} W_1(f_i(t, \cdot), \bar{f}_i(t, \cdot)) \leq C_1(t) \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}, \quad (20)$$

for every  $t \in \mathbb{R}_+$ , where  $W_1$  is the 1-Wasserstein distance. We also have a direct control on the empirical measure of the system, namely

$$\mathbb{E} W_1 \left( \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}, \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, \cdot) \right) \leq \frac{\tilde{C} C_2(t)}{N^\theta} + C_1(t) \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}, \quad (21)$$

for every  $t \in \mathbb{R}_+$  and appropriate constants  $\tilde{C}, \theta > 0$  depending only on the dimension  $d$ . In addition,  $C_1(t)$  and  $C_2(t)$  depend on  $M, C, \|K\|_{W^{1,\infty}}$  and  $t$ , and they can be made explicit by

$$\begin{aligned} C_1(t) &:= \sqrt{\frac{2}{C}} (e^{2Ct \|K\|_{W^{1,\infty}}} - 1), \\ C_2(t) &:= (2M^2 + 2C^2 \|K\|_{L^\infty} t^2)^{1/2}. \end{aligned}$$

We notice that if weights relax uniformly to zero, *i.e.*,  $\lim_{N \rightarrow \infty} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2} = 0$ , then the above estimate (20) guarantees that the creation of correlation between particles of system (1) is weak and relax to zero as  $N \rightarrow \infty$ . Indeed, the relaxation rate extends the usual  $N^{-1/2}$  of the classical propagation of chaos for uniform weights  $w_{ij} = \frac{1}{N}$ , and the laws  $f_i$  of each particle is approximated by the independent laws  $\bar{f}_i$  in the McKean SDE (18) in the mean field limit  $N \rightarrow \infty$ , see [80, Theorem 1.4].

*Proof.* Let  $(\bar{X}_1, \dots, \bar{X}_N)$  be the unique solution to (18) with initial data  $\bar{X}_i^0 = X_i^0$ , according to Lemma 2.1. We note first that we can simply propagate the second moments,

$$\mathbb{E}|X_i|^2 \leq \mathbb{E}(|X_i^0| + C \|K\|_{L^\infty} t)^2 \leq C_2(t)^2.$$

In addition, let us denote the laws  $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$  and consider the sub- $\sigma$ -algebras generated by  $\bar{X}_i$ , i.e.,  $\mathcal{F}_i(t) = \sigma(\bar{X}_i(t))$ , for each  $t \in \mathbb{R}_+$  and  $i = 1, \dots, N$ . Since the variables  $\bar{X}_i$  and  $\bar{X}_j$  are independent by Lemma 2.1, for any  $j = 1, \dots, N$  with  $j \neq i$ , then we have

$$\int_{\mathbb{R}^d} K(\bar{X}_i - y) \bar{f}_j(t, dy) = \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)],$$

where  $\mathbb{E}_i = \mathbb{E}_i[\cdot | \mathcal{F}_i(t)]$  denotes the conditional expectation given  $\mathcal{F}_i(t)$ . Therefore, taking the difference in equations (1) and (18) we obtain the SDE

$$\begin{aligned} \frac{d}{dt}(X_i - \bar{X}_i) &= \sum_{j=1}^N w_{ij} (K(X_i - X_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \\ &= \sum_{j=1}^N w_{ij} (K(X_i - X_j) - K(\bar{X}_i - \bar{X}_j)) + \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]), \end{aligned}$$

for every  $t \in \mathbb{R}_+$ . Hence, taking total expectation and using the Lipschitz continuity of  $K$  we have that

$$\frac{d}{dt} \mathbb{E}|X_i - \bar{X}_i| \leq 2C[K]_{\text{Lip}} \sup_{1 \leq j \leq N} \mathbb{E}|X_j - \bar{X}_j| + \mathbb{E} \left[ \left| \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right| \right]. \quad (22)$$

We remark that all the expectations are finite by the assumption on  $X_i^0$ . Our goal now is to derive a bound from (22). Notice that the key point is to control the relaxation of the second term of the right hand side as  $N \rightarrow \infty$  in terms of weights  $w_{ij}$ . Such a step is reminiscent of the classical propagation of chaos with uniform weights  $w_{ij} = \frac{1}{N}$ , see [80, Theorem 1.4]. However, the presence of non-uniform weights makes the argument more subtle. By Jensen's inequality we shall control the second order moment instead. Indeed, expanding the square we obtain that

$$\begin{aligned} &\mathbb{E} \left[ \left| \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right|^2 \right] \\ &= \sum_{j,k=1}^N w_{ij} w_{ik} \mathbb{E} \left[ (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \cdot (K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]) \right]. \end{aligned}$$

Since  $K(0) = 0$ , then only the terms with  $j \neq i$  and  $k \neq i$  persist in the above sum. We shall prove that all the terms with  $j \neq k$  also disappear. This will be a consequence of the statistical independence of  $\bar{X}_i$ . Indeed, take  $j \neq k$  in the sum and notice that  $(\bar{X}_i, \bar{X}_j, \bar{X}_k)$  are independent. Therefore,

$$\begin{aligned} &\mathbb{E} \left[ (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \cdot (K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]) \right] \\ &= \mathbb{E} \left\{ \mathbb{E}_i \left[ (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right] \mathbb{E}_i \left[ (K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]) \right] \right\} = 0. \end{aligned}$$

Here, we have used again the law of total expectation  $\mathbb{E} = \mathbb{E} \mathbb{E}_i$  and the fact that the random variables  $K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]$  and  $K(\bar{X}_i - \bar{X}_k) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_k)]$  are conditionally independent given  $\mathcal{F}_i$ . This means that only the terms  $j = k$  persist in the sum and

$$\begin{aligned} &\mathbb{E} \left[ \left| \sum_{j=1}^N w_{ij} (K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]) \right|^2 \right] = 2 \sum_{j=1}^N w_{ij}^2 |K(\bar{X}_i - \bar{X}_j) - \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]|^2 \\ &\leq 8C \|K\|_{L^\infty}^2 \sup_{1 \leq j \leq N} |w_{ij}|, \end{aligned}$$

for each  $i = 1, \dots, N$  and  $t \in \mathbb{R}_+$ . Finally, putting everything together into (22), using Grönwall's inequality and noting that  $\mathbb{E}[|X_i(0) - \bar{X}_i(0)|] = 0$  (by definition), we obtain the following estimate

$$\sup_{1 \leq i \leq N} \mathbb{E}|X_i - \bar{X}_i| \leq C_1(t) \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2},$$

for every  $t \in \mathbb{R}_+$ .

To conclude the proof, it only remains to check that  $\bar{f}_i = \text{Law}(\bar{X}_i)$  are precisely the solution to the PDE system (19), and that the above error estimate for the processes implies the corresponding  $W_1$  error bounds (20) and (21) for the laws. Let us assume the former and let us prove the later first. Since  $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$ , this directly implies (20), since

$$\begin{aligned} W_1(f_i(t, \cdot), \bar{f}_i(t, \cdot)) &= \sup_{\|\nabla\phi\|_{L^\infty} \leq 1} \int_{\mathbb{R}^d} \phi(x) (f_i - \bar{f}_i)(t, dx) \\ &= \sup_{\|\nabla\phi\|_{L^\infty} \leq 1} \mathbb{E}[\phi(X_i(t)) - \phi(\bar{X}_i(t))] \leq \mathbb{E}|X_i(t) - \bar{X}_i(t)|. \end{aligned}$$

Similarly, denoting the two empirical measures

$$\mu_N(t, dx) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(dx), \quad \bar{\mu}_N(t, dx) := \frac{1}{N} \sum_{i=1}^N \delta_{\bar{X}_i(t)}(dx),$$

we have that

$$\begin{aligned} \mathbb{E}W_1(\mu_N(t, \cdot), \bar{\mu}_N(t, \cdot)) &= \mathbb{E} \left[ \sup_{\|\nabla\phi\|_{L^\infty} \leq 1} \frac{1}{N} \sum_{i=1}^N (\phi(X_i(t)) - \phi(\bar{X}_i(t))) \right] \\ &\leq \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N |X_i(t) - \bar{X}_i(t)| \right] \leq \sup_{1 \leq i \leq N} \mathbb{E}|X_i(t) - \bar{X}_i(t)|. \end{aligned}$$

Furthermore, by the independence of the  $\bar{X}_i$ , we have, through a straightforward extension of the proofs in [30] or [5] (see also Theorem 1 in [34]) which consider the exchangeable case that for some  $\tilde{C}, \theta > 0$ ,

$$\mathbb{E}W_1 \left( \bar{\mu}_N(t, \cdot), \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, \cdot) \right) \leq \frac{\tilde{C}}{N^\theta} \sup_{1 \leq i \leq N} \mathbb{E}[|X_i(t)|^2]^{1/2} \leq \frac{\tilde{C}C_2(t)}{N^\theta},$$

provided for example that the second moments  $\mathbb{E}|\bar{X}_i|^2$  are bounded uniformly in  $N$ , which we proved earlier. Combining the above estimates and using the triangle inequality allows deducing (21).

Finally, for completeness we show that the laws  $\bar{f}_i(t, \cdot) = \text{Law}(\bar{X}_i(t))$  verify (19). Set any  $\varphi \in C_c^1(\mathbb{R}^{2d})$ . Then, we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x_i) \bar{f}_i(t, x) dx = \frac{d}{dt} \mathbb{E}[\varphi(\bar{X}_i)] = \mathbb{E} \left[ \nabla\varphi(\bar{X}_i) \cdot \frac{d\bar{X}_i}{dt} \right] = \sum_{j=1}^N w_{ij} \mathbb{E} \left[ \nabla\varphi(\bar{X}_i) \cdot \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)] \right].$$

We notice that  $\mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)] = \mathbb{E}[K(\bar{X}_i - \bar{X}_j) | \bar{X}_i] \circ \bar{X}_i$ , where  $\mathbb{E}[\cdot | \bar{X}_i]$  denotes the conditional expectation with respect to the random variable  $\bar{X}_i : v \in \Upsilon \rightarrow \bar{X}_i(v) \in \mathbb{R}^d$ . Then, we can write

$$\begin{aligned} \mathbb{E}[\nabla\varphi(\bar{X}_i) \cdot \mathbb{E}_i[K(\bar{X}_i - \bar{X}_j)]] &= \int_{\Upsilon} \nabla\varphi(\bar{X}_i(v)) \cdot \mathbb{E}[K(\bar{X}_i(v) - \bar{X}_j) | \bar{X}_i = \bar{X}_i(v)] \mathbb{P}(dv) \\ &= \int_{\Upsilon} \nabla\varphi(\bar{X}_i(v)) \cdot \mathbb{E}[K(\bar{X}_i(v) - X_j)] \mathbb{P}(dv) \\ &= \int_{\Upsilon} \nabla\varphi(\bar{X}_i(v)) \cdot \left( \int_{\mathbb{R}^d} K(\bar{X}_i(v) - y) \bar{f}_j(t, dy) \right) \mathbb{P}(dv) \\ &= \int_{\mathbb{R}^{2d}} \nabla\varphi(x) \cdot K(x - y) \bar{f}_i(t, dx) \bar{f}_j(t, dy). \end{aligned}$$

Putting everything together and noticing that  $\varphi$  is arbitrary, we deduce that  $\bar{f}_i$  verifies (19) in weak form, for each  $i = 1, \dots, N$ .  $\square$

The above result is only the beginning to prove our main result because deriving the system (19) is not enough to conclude. Indeed, so far we have only replaced a coupled system of ODEs by a couple system of PDEs. We do not even have any compactness with respect to  $N$ . There is one big exception:

**Remark 2.3** (Exchangeable case). *Assume that  $\sum_{j=1}^N w_{ij} = \bar{w}$  and  $\bar{f}_i^0 = \bar{f}^0$  for every  $i = 1, \dots, N$  and some  $\bar{w} \in \mathbb{R}$  and  $\bar{f}^0 \in \mathcal{P}(\mathbb{R}^d)$ . Then, there is usual propagation of chaos. Specifically, we have that  $\bar{f}_i(t, x) = \bar{f}(t, x)$  for every  $i = 1, \dots, N$ , where  $\bar{f}$  solves the usual Vlasov equation:*

$$\partial_t \bar{f} + \bar{w} \operatorname{div}_x \left( \bar{f}(t, x) \int_{\mathbb{R}^d} K(x - y) \bar{f}(t, dy) \right) = 0.$$

### 3. A NEW HIERARCHY

To make some of the calculations in this section rigorous, let us introduce a variant of the original system (19) for the  $\bar{f}_i$  which includes an artificial diffusion term:

$$\partial_t \bar{f}_i + \operatorname{div}_x \left( \bar{f}_i(t, x) \sum_{j=1}^N w_{ij} \int_{\mathbb{R}^d} K(x-y) \bar{f}_j(t, dy) \right) = \nu \Delta_x \bar{f}_i. \quad (23)$$

With identical arguments as in Proposition 2.2 introducing a Wiener process in (18) with variance  $2\nu t$  we can deduce (23). This can be done starting from system (1) with a noise  $dW_i$ , where the  $W_i$  are  $N$  independent Wiener processes

$$dX_i = \sum_{j=1}^N w_{ij} K(X_i - X_j) dt + \sqrt{2\nu} dW_i.$$

Of course (19) is just a special case of (23) with  $\nu = 0$ . As in the case without diffusion, in (23) there is an abrupt rupture of symmetry caused by the presence of potentially heterogeneous weights  $w_{ij}$ . Indeed, even for identical initial data  $\bar{f}_i^0$ , the intrinsic dynamics instantaneously yields non-exchangeable distributions  $\bar{f}_i(t)$  for any  $t > 0$ . This prevents us from reducing the system (23) of  $N$  coupled PDEs to a single Vlasov-McKean type PDEs in the usual way as in the classical setting with uniform weights. In addition, there is no realistic hope to identify the limit of each individual  $\bar{f}_i$  as  $N \rightarrow \infty$ . However, we may still be able to study the limit of appropriate averaged statistics. In this section, we shall introduce a natural family of observables that will allow determining a closed averaged description of (23) in terms of an infinite hierarchy of PDEs. For such a novel hierarchy we study uniqueness and stability under the presence of artificial diffusion ( $\nu > 0$ ). Although an analogous result is still open for the hierarchy in the absence of artificial diffusion ( $\nu = 0$ ), we shall prove that the stability can still be recovered for the full initial system (19) under appropriate regularity of the initial data.

**3.1. Hierarchy of observables indexed by trees.** For simplicity of the presentation, in this part we shall restrict to the starting case (19) without artificial diffusion ( $\nu = 0$ ). Associated with any solution of (19) we shall define an infinite family of observables indexed by the set of trees containing an averaged information of  $\bar{f}_i$  and  $w_{ij}$ . The role of trees is suggested by the construction and will become apparent in a moment. For clarity, we recall some necessary notation of graph theory that will be used throughout the paper.

**Definition 3.1** (Graphs and trees).

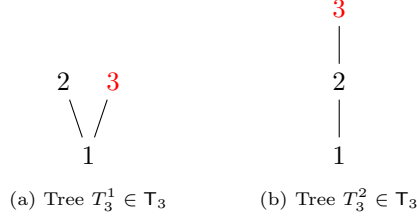
- (i) A **(simple) graph** is a pair  $G = (V, E)$  where  $V$  is a finite set (vertices) and  $E \subseteq \{(i, j) \in V \times V : i \neq j\}$  is a symmetric subset (edges). A graph  $G = (V, E)$  is called a **weighted graph** if it is endowed with a (weight) function  $W : V \times V \rightarrow \mathbb{R}$  such that  $E = \{(i, j) \in V \times V : W(i, j) \neq 0\}$ . The amount of vertices (or order) will be denoted by  $|G| := \#V$ .
- (ii) A **tree** is a graph  $T = (V, E)$  so that any couple of different vertices  $i, j \in V$  are connected by exactly one path (concatenation of edges) and each vertex  $i \in V$  cannot be connected with itself.
- (iii) We define the family  $\mathbb{T}_n$  of **labeled trees** of order  $n$  by the following recursive formula

$$\begin{aligned} \mathbb{T}_1 &:= \{T_1\}, \\ \mathbb{T}_{n+1} &:= \{T + i : T \in \mathbb{T}_n, i \in \{1, \dots, n\}\}, \quad n \in \mathbb{Z}_0^+, \end{aligned}$$

where  $T_1$  is the only tree with one vertex  $\{1\}$ . Here, for any tree  $T$  with vertices  $\{1, \dots, n\}$  and any such vertex  $i$ , we will denote by  $T + i$  to the new tree with vertices  $\{1, \dots, n+1\}$  after adding the leaf  $n+1$  at node  $i$ . The family of all labeled trees of arbitrary order is then defined by  $\mathbb{T} := \cup_{n=1}^{\infty} \mathbb{T}_n$ . Note that, in particular,  $\mathbb{T}$  contains all possible trees modulo graph isomorphism.

**Remark 3.2** (Directed graphs and directed trees.). Our graphs  $G = (V, E)$  are a priori assumed directed so that  $(i, j) \in E$  and  $(j, i) \in E$  do not in general represent the same edge. Since we work with labeled trees, those are imbued with a natural orientation starting from the root. When  $T$  is a labeled tree, we shall for this reason write  $(i, j) \in T$  to indicate such an oriented edge:  $(i, j) \in T$  if there exists an edge between vertices  $i$  and  $j$  and if  $i$  is closer to the root than  $j$ .

With this notation, we introduce our family of observables  $\{\mathcal{O}_N^T : T \in \mathbb{T}\}$  as follows.


 FIGURE 2. Trees in  $\mathbb{T}_3$ 

**Definition 3.3** (Observables). *Consider any solution  $(\bar{f}_i)_{i=1,\dots,N}$  of (19) with weights  $(w_{ij})_{i,j=1,\dots,N}$ . Then, we define the family of time-dependent Radon measures:*

$$\mathcal{O}_N^T(t, x_1, \dots, x_{|T|}) := \frac{1}{N} \sum_{i_1, \dots, i_{|T|=1}}^N \prod_{(k,l) \in T} w_{i_k i_l} \prod_{m=1}^{|T|} \bar{f}_{i_m}(t, x_m), \quad (24)$$

for any  $t \geq 0$ , every  $(x_1, \dots, x_{|T|}) \in \mathbb{R}^{d|T|}$  and each tree  $T \in \mathbb{T}$ .

Let us note that  $\mathcal{O}_N^T(t, \cdot) \in \mathcal{M}(\mathbb{R}^{d|T|})$  are finite Radon measures for each  $t \geq 0$ . Unfortunately, the heterogeneity of weights still implies that  $\mathcal{O}_N^T$  are not symmetric. In the next section we shall show that they indeed satisfy an appropriate hierarchy of PDEs containing an averaged information of (19). Just to illustrate the underlying idea, let us explore some examples. First, consider the only existing tree  $T_1 \in \mathbb{T}_1$  of order 1 and note that (24) simply reduces to  $\mathcal{O}_N^{T_1}(t, x) = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x)$ . Summing in (19) over  $i = 1, \dots, N$  yields

$$\partial_t \mathcal{O}_N^{T_1}(t, x) + \operatorname{div}_x \left( \int_{\mathbb{R}^d} K(x-y) \frac{1}{N} \sum_{i,j=1}^N w_{ij} \bar{f}_i(t, x) \bar{f}_j(t, y) dy \right) = 0, \quad (25)$$

which cannot be closed in terms of  $\mathcal{O}_N^{T_1}$  only. In fact, setting the only existing tree  $T_2 \in \mathbb{T}_2$  of order 2 allows identifying the new observable  $\mathcal{O}_N^{T_2}(t, x, y) = \frac{1}{N} \sum_{i,j=1}^N w_{ij} \bar{f}_i(t, x) \bar{f}_j(t, y)$  inside the divergence of (25). In other words, (25) can be readily written as

$$\partial_t \mathcal{O}_N^{T_1}(t, x) + \operatorname{div}_x \left( \int_{\mathbb{R}^d} K(x-y) \mathcal{O}_N^{T_2}(t, x, y) dy \right) = 0. \quad (26)$$

Since (26) requires a priori knowledge of  $\mathcal{O}_N^{T_2}$  we may seek for an equation of such a new observable. Indeed, from (19) we obtain again that

$$\begin{aligned} \partial_t \mathcal{O}_N^{T_2}(t, x, y) + \operatorname{div}_x \left( \int_{\mathbb{R}^d} K(x-z) \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{ik} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z) dz \right) \\ + \operatorname{div}_y \left( \int_{\mathbb{R}^d} K(y-z) \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{jk} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z) dz \right) = 0. \end{aligned} \quad (27)$$

Note that in this case two different divergence terms appear. Accordingly, note that  $\mathbb{T}_3$  also contains two possible trees  $T_3^1 = T_2 + 1$  and  $T_3^2 = T_2 + 2$  which respectively arise by adding the new leaf 3 at either of the vertices 1 and 2 of  $T_2 \in \mathbb{T}_2$  (see Figure 2) By construction

$$\begin{aligned} \mathcal{O}_N^{T_3^1}(t, x, y, z) &= \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{ik} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z), \\ \mathcal{O}_N^{T_3^2}(t, x, y, z) &= \frac{1}{N} \sum_{i,j,k=1}^N w_{ij} w_{jk} \bar{f}_i(t, x) \bar{f}_j(t, y) \bar{f}_k(t, z). \end{aligned}$$

Again, we identify the above observables in the divergences of (27), so that the latter can be restated as

$$\partial_t \mathcal{O}_N^{T_2}(t, x, y) + \operatorname{div}_x \left( \int_{\mathbb{R}^d} K(x-z) \mathcal{O}_N^{T_3^1}(t, x, y, z) dz \right) + \operatorname{div}_y \left( \int_{\mathbb{R}^d} K(y-z) \mathcal{O}_N^{T_3^2}(t, x, y, z) dz \right) = 0. \quad (28)$$

It is now apparent that an analogous recursive argument would lead to similar equations like (26) and (28) involving observables indexed by all other higher order trees. In the following section we will make



this argument rigorous and will recover the full hierarchy of equations for the observables  $\mathcal{O}_N^T$  with  $T \in \mathbb{T}$ . To do so, we shall rely on a more general formulation of the system (19) and the observables  $\mathcal{O}_N^T$  in (24) inspired in the treatment of graphons which is formally independent of the decomposition into agents.

**3.2. A graphon-like representation of the system and its hierarchy .** Let us remark that in the limit  $N \rightarrow \infty$  we must control two different processes simultaneously: a many-particles limit and a large-graph limit. Unfortunately, the explicit dependence on the discrete labels  $i = 1, \dots, N$  of agents still persists both in the expression of the observables in Definition 3.3 and in the systems (19) and (23) with and without artificial diffusion. This is reminiscent of the theory of graphons (*cf.* [56, 57]). Specifically, graphons have proved a suitable tool to identify the limit of dense graph limits by replacing discrete indices by continuous ones. Accordingly, we expect to recover integrals as  $N \rightarrow \infty$  that replace the various finite sums over particles' labels. However, for our graphs, we need to develop an appropriate extension of the theory of graphons that allows for the joint treatment of the large-graph limit for non necessarily dense graphs and the many-particles limit.

**Definition 3.4** (Empirical graphons). *Consider weights  $(w_{ij})_{i,j=1,\dots,N} \subseteq \mathbb{R}$  and probability distributions  $(\bar{f}_i)_{i=1,\dots,N} \subseteq \mathcal{P}(\mathbb{R}^d)$ . Then, we define  $w_N$  and  $f_N$  as follows*

$$w_N(\xi, \zeta) := \sum_{i,j=1}^N N w_{ij} \mathbb{I}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}(\xi) \mathbb{I}_{\left[\frac{j-1}{N}, \frac{j}{N}\right)}(\zeta), \quad \xi, \zeta \in [0, 1],$$

$$f_N(x, \xi) := \sum_{i=1}^N \bar{f}_i(x) \mathbb{I}_{\left[\frac{i-1}{N}, \frac{i}{N}\right)}(\xi), \quad x \in \mathbb{R}^d, \xi \in [0, 1].$$
(29)

Note that Definition 3.4 allows representing discrete weights  $(w_{ij})_{i,j=1,\dots,N}$  as scalar functions of two continuous variables  $\xi, \eta \in [0, 1]$ . Similarly, distributions  $(\bar{f}_i)_{i=1,\dots,N}$  can be represented as a family of probability measures in  $\mathcal{P}(\mathbb{R}^d)$  parametrized by the continuous variable  $\xi \in [0, 1]$ . Using such a notation, we can rewrite the coupled system of PDEs (19) and (23) with and without artificial diffusion in a more compact form that forgets labels of the specific individuals. Specifically, we equivalently obtain

$$\partial_t f_N(t, x, \xi) + \operatorname{div}_x \left( f_N(t, x, \xi) \int_0^1 \int_{\mathbb{R}^d} w_N(\xi, \zeta) K(x-y) f_N(t, y, \zeta) dy d\zeta \right) = 0, \quad (30)$$

$$\partial_t f_N(t, x, \xi) + \operatorname{div}_x \left( f_N(t, x, \xi) \int_0^1 \int_{\mathbb{R}^d} w_N(\xi, \zeta) K(x-y) f_N(t, y, \zeta) dy d\zeta \right) = \nu \Delta_x f_N(t, x, \xi), \quad (31)$$

in the sense of distributions. Our ultimate goal will be to identify the limit of the above systems as  $N \rightarrow \infty$ . For later use, we specify minimal assumptions on the space which will contain the limiting structures.

**Definition 3.5.** *We define the weak-star Bochner space  $L_\xi^\infty \mathcal{M}_\zeta$  as the topological dual of  $L_\xi^1 C_\zeta$ ; namely those elements consist of the weak-star measurable essentially bounded maps  $\xi \in [0, 1] \mapsto w(\xi, d\zeta) \in \mathcal{M}([0, 1])$  which make the following seminorm finite*

$$\|w\|_{L_\xi^\infty \mathcal{M}_\zeta} := \sup_{\|\phi\|_{C([0, 1])} \leq 1} \operatorname{ess\,sup}_{\xi \in [0, 1]} \left| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right|.$$

Similarly, we define  $L_\zeta^\infty \mathcal{M}_\xi$ , or more generally  $L_\xi^\infty B_\zeta^*$  for any dual Banach space  $B^*$ .

As usual, we will identify  $L_\xi^\infty \mathcal{M}_\zeta$  with its Kolmogorov quotient under the above semi-norm  $\|\cdot\|_{L_\xi^\infty \mathcal{M}_\zeta}$ , which is a Banach space. Specifically, we shall identify elements  $w_1 \approx w_2$  when for any  $\phi \in C([0, 1])$  the scalar functions  $\xi \in [0, 1] \mapsto \int_{\mathbb{R}^d} \phi(\zeta) w_i(\xi, d\zeta)$  are equal almost everywhere for  $i = 1, 2$ . As mentioned above, the main feature of such a Banach space is its duality representation  $L_\xi^\infty \mathcal{M}_\zeta \equiv (L_\xi^1 C_\zeta)^*$ , where  $L_\xi^1 C_\zeta$  stands for the (strong) Bochner space, see [43]. Although this property is desirable, the weak-star Bochner spaces raise several analytical difficulties. For instance, strong-measurability of the representatives is not granted because  $\mathcal{M}([0, 1])$  fails the Radon-Nikodym property. Fortunately, in our special case  $C([0, 1])$  is separable so that, at least,  $\xi \in [0, 1] \mapsto \|w(\xi, \cdot)\|_{\mathcal{M}([0, 1])}$  is measurable and essentially bounded. For the same reason, the above equivalence relation  $\approx$  reduces to the usual *a.e.* identity of functions. Therefore, a simpler expression of the norm in the separable case is available:

$$\|w\|_{L_\xi^\infty \mathcal{M}_\zeta} = \operatorname{ess\,sup}_{\xi \in [0, 1]} \|w(\xi, \cdot)\|_{\mathcal{M}([0, 1])}.$$

**Definition 3.6.** *We shall say that  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  is an (extended) graphon.*

For our purpose we will need to integrate extended graphons against functions  $\phi$  that are only in  $L^\infty$ . This leads to the straightforward issue as to whether  $\xi \in [0, 1] \mapsto \int_0^1 \phi(\zeta) w(\xi, d\zeta)$  can be well defined if  $\phi \in L^\infty([0, 1])$ . In fact if we only had  $w \in L_\xi^\infty \mathcal{M}_\zeta$ , then the above integral would be highly sensitive to modifying  $\phi(\zeta)$  over the atoms of  $w(\xi, d\zeta)$ , preventing any straightforward definition.

The key point for our investigations is that our extended graphons belongs to both  $L_\xi^\infty \mathcal{M}_\zeta$  and  $L_\zeta^\infty \mathcal{M}_\xi$ . This yields two weakly-star measurable and bounded families of measures  $\xi \in [0, 1] \mapsto w(\xi, d\zeta) \in \mathcal{M}([0, 1])$  and  $\zeta \in [0, 1] \mapsto w(d\xi, \zeta) \in \mathcal{M}([0, 1])$  that induce the same bi-variate measure  $w(\xi, d\zeta) d\xi = w(d\xi, \zeta) d\zeta \in \mathcal{M}([0, 1]^2)$ . This implies that  $w$  is a transition kernel in the full sense and the following continuity of the associated linear operator on  $L^\infty$  holds true.

**Lemma 3.7** (Operator representation of kernels). *Consider the following bounded bilinear operator*

$$\begin{aligned} (L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times C_\zeta &\longrightarrow L_\xi^\infty, \\ (w, \phi) &\longmapsto \int_0^1 \phi(\zeta) w(\cdot, d\zeta). \end{aligned}$$

*This operator extends into a bounded operator from  $(L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times L_\zeta^\infty$  to  $L_\xi^\infty$ . Specifically, we have the following estimates*

$$\begin{aligned} \left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_\xi^1} &\leq \|w\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi\|_{L^1}, \\ \left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_\xi^\infty} &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\phi\|_{L^\infty}, \end{aligned} \quad (32)$$

for any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $\phi \in L^\infty$ . Moreover, the following continuity property holds: if  $w_n \xrightarrow{*} w$  weakly-star in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ ,  $\phi_n \rightarrow \phi$  strongly in  $L^1$  and the  $\phi_n$  are uniformly bounded in  $L^\infty$ , then

$$\int_0^1 \phi_n(\zeta) w_n(\cdot, d\zeta) \rightarrow \int_0^1 \phi(\zeta) w(\cdot, d\zeta), \quad (33)$$

weakly-star in  $L^\infty$ .

The proof of Lemma 3.7 relies on a careful but straightforward density argument, which we provide in Appendix B. Lemma 3.7 immediately implies the following version with test functions  $\phi$  depending on  $x$ .

**Lemma 3.8.** *Consider the following bounded bilinear operator*

$$\begin{aligned} (L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times C_\zeta B_x^* &\longrightarrow L_\xi^\infty B_x^*, \\ (w, \phi) &\longmapsto \int_0^1 \phi(\cdot, \zeta) w(\cdot, d\zeta), \end{aligned}$$

where  $B_x^*$  stands for any Banach space. Then it is also bounded from  $(L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times L_\zeta^\infty B_x^*$  to  $L_\xi^\infty B_x^*$ , and from  $(L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi) \times L_\zeta^\infty B_x^*$  (with  $L_\zeta^\infty B_x^*$  endowed with the  $L_\zeta^1 B_x^*$  norm) to  $L_\xi^1 B_x^*$ . Specifically, we have the following estimates

$$\begin{aligned} \left\| \int_0^1 \phi(x, \zeta) w(\xi, d\zeta) \right\|_{L_\xi^1 B_x^*} &\leq \|w\|_{L_\zeta^\infty \mathcal{M}_\xi} \|\phi\|_{L_\zeta^1 B_x^*}, \\ \left\| \int_0^1 \phi(x, \zeta) w(\xi, d\zeta) \right\|_{L_\xi^\infty B_x^*} &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\phi\|_{L_\zeta^\infty B_x^*}, \end{aligned} \quad (34)$$

for any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $\phi \in L_\zeta^\infty B_x^*$ ,

Note that  $w_N$  and  $f_N$  in Definition 3.4 must belong to the previous spaces. Then, the study of the limit of those structures as  $N \rightarrow \infty$  and the identification of the limiting equations of (30) and (31) is in order. Namely, we expect to obtain  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $f \in L^\infty([0, t_*], L_\xi^1 L_x^1)$  with  $t_* > 0$  verifying either of the following equations (with or without artificial diffusion) in a weak sense:

$$\partial_t f(t, x, \xi) + \operatorname{div}_x \left( f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) = 0, \quad (35)$$

$$\partial_t f(t, x, \xi) + \operatorname{div}_x \left( f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) = \nu \Delta_x f(t, x, \xi). \quad (36)$$

In (35) and (36) we recognize a nonlinear drift, which is fibered as there is no transport on the variable  $\xi$ . For later use, we define formally the associated velocity field.

**Definition 3.9** (Velocity field). *Consider any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ ,  $f \in L_{t,\xi}^\infty L_x^1$  and  $K \in L^\infty$ . Let us set  $\phi_f(t, x, \zeta) := \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy$  which is well defined in  $L_{\xi,t,x}^\infty$  by the usual convolution estimates. We define the associated velocity field  $\mathcal{V}[f]$  as follows,*

$$\mathcal{V}_f(x, \xi) := \int_0^1 w(\xi, d\zeta) \phi_f(t, x, \zeta) = \int_{\mathbb{R}^d} K(x-y) f(\zeta, y) dy, \quad (x, \xi) \in \mathbb{R}^d \times [0, 1],$$

which is well defined in  $L_{\xi,t,x}^\infty$  by Lemma 3.8.

We may then define the weak solutions to (35)-(36) in the usual way.

**Definition 3.10** (Weak solutions). *Consider any pair consisting of a  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and any  $f \in L^\infty([0, t_*], L_\xi^\infty L_x^1)$ . We say that  $(w, f)$  is a weak solution of (36) for some  $K \in L^\infty$  and for any  $\nu \geq 0$  if the following weak formulation is verified*

$$\begin{aligned} & \int_0^{t_*} \int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(t, x, \xi) f(t, x, \xi) dx d\xi dt \\ & + \int_0^{t_*} \int_0^1 \int_{\mathbb{R}^d} \nabla_x \varphi(t, x, \xi) \cdot \mathcal{V}_f(t, x, \xi) f(t, x, \xi) dx d\xi dt \\ & + \nu \int_0^{t_*} \int_0^1 \int_{\mathbb{R}^d} \Delta_x \varphi(t, x, \xi) f(t, x, \xi) dx d\xi dt = 0, \end{aligned} \quad (37)$$

for any  $\varphi \in C_c^2((0, t_*) \times [0, 1] \times \mathbb{R}^d)$ .

From those estimates, one can obtain the existence of weak solutions.

**Proposition 3.11.** *Consider any  $K \in L^1 \cap L^\infty$  with  $\operatorname{div} K \in L^1$ , any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and any  $f^0 \in L^\infty([0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$ . Then for any  $\nu \geq 0$ , there exists  $f \in L^\infty([0, t_*] \times [0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d))$  such that  $(w, f)$  is a weak solution to (36) (or (35) if  $\nu = 0$ ) for every  $t_* > 0$  as per Definition 3.10.*

The proof is given in Appendix B and follows a straightforward fixed point argument from classical a priori estimates for advection-diffusion equations.

We finish this short analysis of the space  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  by noting the following density result

**Lemma 3.12.** *For any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , there exists  $w_n \in L_{\xi,\zeta}^\infty$ , uniformly bounded in  $L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ , converging to  $w$  in  $L_\xi^1 H_\zeta^{-1} \cap L_\zeta^1 H_\xi^{-1}$ .*

We also provide the proof of Lemma 3.12 in Appendix B.

Unfortunately we do not know how to pass to the limit directly in Eqs. (30) or (31), as we lack some compactness in the  $\xi$  variable either for  $w_N$  or for  $f_N$ . Indeed, from our uniform bounds, we could extract converging subsequences  $w_N \overset{*}{\rightharpoonup} w$  and  $f_N \overset{*}{\rightharpoonup} f$  respectively in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $L_{t,\xi}^\infty(L_x^1 \cap W_x^{1,\infty})$ . But this gives only weak convergence on  $w_N$  and weak convergence in  $\xi$  on  $f_N$  whereas Lemma 3.7 for example requires some strong convergence on  $f_N$ . To further emphasize the point, the existence result that we just stated does rely on strong convergence of  $f$  through the contraction argument.

To bypass this issue, we will look at an analogous family of observables providing an averaged information of the system, since it is much easier to obtain compactness on those observables as we will see in the next section. Inspired in Definition 3.3 we set the following operator acting on abstract elements  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $f \in L_\xi^\infty L_x^1$ .

**Definition 3.13** ( $\tau$  operator). *Consider any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and any  $f \in L_\xi^\infty L_x^1$ . Then, we (formally) define the operator*

$$\tau(T, w, f)(x_1, \dots, x_{|T|}) := \int_{[0, 1]^{|T|}} \prod_{(k,l) \in T} w(\xi_k, \xi_l) \prod_{m=1}^{|T|} f(x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}, \quad (38)$$

for every tree  $T \in \mathbb{T}$ .

This notation allows removing again the explicit discrete indices for the agents' labels in Definition 3.3 of the observables  $\mathcal{O}_N^T$ . Specifically, note that the latter enjoy the following more universal representation

$$\mathcal{O}_N^T = \tau(T, w_N, f_N), \quad T \in \mathbb{T}.$$

We also note that, at the present, it is not clear why  $\tau(T, w, f)$  given by (38) is well defined with only  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $f \in L_\xi^\infty L_x^1$ . This requires more work and will be performed in the next section (cf. Definition 4.5). Instead, for the moment it is enough to observe that  $\tau(T, w_N, f_N)$  is

indeed well defined, and more generally,  $\tau(T, w, f)$  is also well defined as soon as  $w$  has a density, *i.e.*  $w \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ . Moreover, we can then obtain uniform bounds in terms of only the  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  norms as follows.

**Lemma 3.14** (Homomorphism densities). *Consider any  $w \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$  and define*

$$\tau(T, w) := \int_{[0, 1]^{|T|}} \prod_{(k, l) \in T} w(\xi_k, \xi_l) d\xi_1 \dots d\xi_{|T|}, \quad T \in \mathbb{T}. \quad (39)$$

Then,  $\tau(T, w)$  is finite and the following relation holds true

$$|\tau(T, w)| \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1}. \quad (40)$$

*Proof.* We argue by induction. First, note the only tree  $T_1 \in \mathbb{T}_1$  actually consists of only one vertex and no edges. Then, we trivially obtain that  $\prod_{(k, l) \in T_1} |w(\xi_k, \xi_l)| = 1$  so that  $\tau(T_1, |w|) = 1$ . By the induction hypothesis, let us assume now that we have

$$\tau(|T_n|, w) \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T_n|-1},$$

for any  $T_n \in \mathbb{T}_n$ . Consider any  $T_{n+1} \in \mathbb{T}_{n+1}$  and find a tree  $T_n \in \mathbb{T}_n$  and a vertex  $i \in \{1, \dots, n\}$  so that we can write  $T_{n+1} = T_n + i$ . Hence, we obtain

$$\begin{aligned} \tau(T_{n+1}, |w|) &= \int_{[0, 1]^{n+1}} \prod_{(k, l) \in T_{n+1}} |w(\xi_k, \xi_l)| d\xi_1 \dots d\xi_{n+1} \\ &= \int_{[0, 1]^n} \left( \int_0^1 |w(\xi_i, \xi_{n+1})| d\xi_{n+1} \right) \prod_{(k, l) \in T_n} |w(\xi_k, \xi_l)| d\xi_1 \dots d\xi_n, \end{aligned}$$

where we have used that  $\xi_{n+1}$  does not appear in the above product over edges of  $T_n$ . Hence, by our hypothesis on  $w$  and by definition (39) of  $\tau(T_n, |w|)$  we obtain

$$\tau(T_{n+1}, |w|) \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \tau(T_n, |w|).$$

We then conclude by applying the induction hypothesis to  $\tau(T_n, |w|)$ .  $\square$

We can obtain as a straightforward consequence of Lemma 3.14,

**Lemma 3.15.** *Consider any  $w \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ , any  $f \in L_\xi^\infty L_x^1$  and  $T \in \mathbb{T}$ . Then  $\tau(T, w, f) \in L^1(\mathbb{R}^{d|T|})$  and the following estimate is fulfilled*

$$\|\tau(T, w, f)\|_{L^1(\mathbb{R}^{d|T|})} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi}^{|T|-1} \|f\|_{L_\xi^\infty L_x^1}^{|T|}.$$

A similar result holds with the space  $L^1$  replaced by any other function Banach space which is stable under tensorization and admits a Minkowski integral inequality. In particular, if in addition  $f \in L_\xi^\infty W_x^{k, p}$  for some  $k \geq 0$  and  $p \in [1, \infty]$ , then  $\tau(T, w, f) \in W^{k, p}(\mathbb{R}^{d|T|})$  and

$$\|\tau(T, w, f)\|_{W^{k, p}(\mathbb{R}^{d|T|})} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi}^{|T|-1} \|f\|_{L_\xi^\infty W_x^{k, p}}^{|T|}.$$

Formula (39) for  $\tau(T, w)$  in Lemma 3.15 is reminiscent of the usual extension of the homomorphism density for bounded graphons  $w \in L^\infty([0, 1]^2)$  [56, 57]. However, our theory must account for eventually unbounded graphons so that a more delicate treatment is required. Namely, note that in (39) we have restricted to trees  $T \in \mathbb{T}$  and graphons  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ . Indeed, in such a class of graphons, if  $T$  is replaced by a generic graph  $G$ , then the above proof breaks down due to the eventual presence of cycles in the graph. This leads to eventual infinite values of  $\tau(G, w)$ .

We note that we prove in section 4 that the operator  $\tau(T, w, f)$  can be rigorously defined for any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ . We refer more precisely to Definition 4.5 and in the present context to Lemma 4.3 which implies that if  $w_n \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$  converges to  $w$  in  $L_\xi^1 H_\zeta^{-1} \cap L_\zeta^1 H_\xi^{-1}$  then  $\tau(T, w, f)$  can be obtained as

$$\tau(T, w, f) = \lim_{n \rightarrow \infty} \tau(T, w_n, f).$$

Of course Lemma 3.12 exactly allows constructing an appropriate sequence of  $w_n$ . But deriving the limit of  $\tau(T, w_n, f)$  as above requires the precise construction of an algebra describing the sequence of operations involved in obtaining  $\tau(T, w, f)$ . This algebra plays a key role in Section 4 and, for this reason, the corresponding results are stated there.

Assuming for the time being that type of property holds, it implies an immediate extension of Lemma 3.15 to more generic kernels.

**Corollary 3.16.** *Consider any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , any  $f \in L_\xi^\infty W_x^{k,p}$  for any  $k \geq 0$ ,  $p \in [1, \infty]$ , and  $T \in \mathbb{T}$ . Then  $\tau(T, w, f) \in W^{k,p}(\mathbb{R}^{d|T|})$*

$$\|\tau(T, w, f)\|_{W^{k,p}(\mathbb{R}^{d|T|})} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi}^{|T|-1} \|f\|_{L_\xi^\infty W_x^{k,p}}^{|T|}.$$

In the following result, we derive the full hierarchy of observables presented in the previous section under our graphon representation. Specifically, we now provide the system solved by  $\tau(T, w, f)$  whenever  $(w, f)$  is a weak solution to either (35) or (36) in the sense of Definition 3.10.

**Proposition 3.17.** *Consider any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $f \in L^\infty([0, t_*], L_\xi^\infty L_x^1)$  so that  $(w, f)$  is a weak solution to (36) in the sense of Definition 3.10 for some  $K \in L^\infty$  and for any  $\nu \geq 0$ . Then,  $\tau(T, w, f)$  solves the generalized, linear, non-exchangeable Vlasov hierarchy*

$$\partial_t \tau(T, w, f) + \sum_{i=1}^{|T|} \operatorname{div}_{x_i} \left( \int_{\mathbb{R}^d} K(x_i - z) \tau(T + i, w, f)(t, x_1, \dots, x_{|T|}, z) dz \right) = \nu \sum_{i=1}^{|T|} \Delta_{x_i} \tau(T, w, f), \quad (41)$$

for any tree  $T \in \mathbb{T}$  in the sense of distributions.

*Proof.* First of all, we notice that from the properties of  $w$  and  $f$  we can guarantee that  $\tau(T, w, f) \in L^\infty([0, t_*], L^1(\mathbb{R}^{d|T|}))$  thanks to Corollary 3.16. Since  $K \in L^\infty$ , then there is no issue to define the various terms in the equation and justify our various calculations in the distributional sense. For simplicity of the presentation, we avoid using weak formulations here but a straightforward adaptation yields the rigorous argument. Also, we shall shorten notation by denoting  $\tau(T) \equiv \tau(T, w, f)$ , for any  $T \in \mathbb{T}$ , when there is no confusion. First, we differentiate  $\tau(T, w, f)$  in time

$$\partial_t \tau(T) = \sum_{n=1}^{|T|} \int_{[0, 1]^{|T|}} \prod_{(k,l) \in T} w(\xi_k, \xi_l) \partial_t f(t, x_n, \xi_n) \prod_{m \neq n} f(x_m, \xi_m) d\xi_1 \dots d\xi_{|T|}.$$

Using (36) on  $\bar{f}(t, x_n, \xi_n)$  yields

$$\begin{aligned} \partial_t \tau(T) &= - \sum_{n=1}^{|T|} \int_{[0, 1]^{|T|}} \prod_{(k,l) \in T} w(\xi_k, \xi_l) \operatorname{div}_{x_n} \left( \int_0^1 \int_{\mathbb{R}^d} K(x_n - y) w(\xi_n, \zeta) f(t, x_n, \xi_n) f(t, y, \zeta) dy d\zeta \right) \\ &\quad \times \prod_{m \neq n} f(x_m, \xi_m) d\xi_1 \dots d\xi_{|T|} \\ &\quad + \nu \sum_{n=1}^{|T|} \int_{[0, 1]^{|T|}} \prod_{(k,l) \in T} w(\xi_k, \xi_l) \Delta_{x_n} f(t, x_n, \xi_n) \prod_{m \neq n} f(x_m, \xi_m) d\xi_1 \dots d\xi_{|T|} \\ &= - \sum_{n=1}^{|T|} \operatorname{div}_{x_n} \left( \int_{[0, 1]^{|T|}} \prod_{(k,l) \in T} w(\xi_k, \xi_l) \int_0^1 \int_{\mathbb{R}^d} K(x_n - y) w(\xi_n, \zeta) f(t, x_n, \xi_n) f(t, y, \zeta) dy d\zeta \right. \\ &\quad \left. \times \prod_{m \neq n} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|} \right) + \nu \sum_{n=1}^{|T|} \Delta_{x_n} \tau(T), \end{aligned}$$

where in the last line we have used that no other  $f(t, x_m, \xi_m)$  depends on  $x_n$  as we exclude  $m = n$  from the product. Hence, we find

$$\begin{aligned} \partial_t \tau(T) &= - \sum_{n=1}^{|T|} \operatorname{div}_{x_n} \int_{\mathbb{R}^d} K(x_n - z) \int_{[0, 1]^{|T|}} \int_0^1 \left( w(\xi_n, \zeta) \prod_{(k,l) \in T} w(\xi_k, \xi_l) \right. \\ &\quad \left. \times f(t, z, \zeta) \prod_{m \neq n} f(t, x_m, \xi_m) \right) d\zeta d\xi_1 \dots d\xi_{|T|} dz + \nu \sum_{n=1}^{|T|} \Delta_{x_n} \tau(T). \end{aligned}$$

We recall that for any  $n = 1, \dots, |T|$  the tree  $T + n$  contains exactly the same edges as  $T$  plus a new edge from the vertex  $n$  to the new vertex  $|T| + 1$ . Thus, recalling that we only consider edges  $(k, l) \in T + n$  that run from the root to the leaves, we have that

$$\tau(T + n) = \int_{[0, 1]^{|T|+1}} w(\xi_n, \xi_{|T|+1}) \prod_{(k,l) \in T} w(\xi_k, \xi_l) f(t, x_{|T|+1}, \xi_{|T|+1}) \prod_{m=1}^{|T|} f(t, x_m, \xi_m) d\xi_1 \dots d\xi_{|T|+1}.$$

Changing variables  $\xi_{|T|+1}$  with  $\zeta$  and  $x_{|T|+1}$  with  $z$ , we arrive at

$$\partial_t \tau(T) = - \sum_{n=1}^{|T|} \operatorname{div}_{x_n} \int_{\mathbb{R}^d} K(x_n - z) \tau(T+n)(x_1, \dots, x_{|T|}, z) dz + \nu \sum_{n=1}^{|T|} \Delta_{x_n} \tau(T),$$

thus, concluding the proof.  $\square$

Proposition 3.17 shows that we can study directly the propagation of the observables  $\tau(T, w, f)$ . Our next step is naturally to analyze the stability on the hierarchy with diffusion (41).

**3.3. Stability on the hierarchy with artificial diffusion.** Along this section, we will study a stability property of generic solutions to the previous hierarchy of non-exchangeable Vlasov-type equations (41) which are not necessarily parametrized as  $\tau(T, w, f)$  for a weak solution  $(w, f)$  of (36). Specifically, we shall consider any sequence  $h = (h_T)_{T \in \mathbb{T}}$  with  $h_T \in L^\infty([0, t_*], L^1(\mathbb{R}^{d|T|}))$  for any  $T \in \mathbb{T}$  that solves the analogous hierarchy of non-exchangeable Vlasov equations

$$\partial_t h_T + \sum_{i=1}^{|T|} \operatorname{div}_{x_i} \left( \int_{\mathbb{R}^d} K(x_i - z) h_{T+i}(t, x_1, \dots, x_{|T|}, z) dz \right) = \nu \sum_{i=1}^{|T|} \Delta_{x_i} h_T, \quad (42)$$

in the sense of distributions, for some  $K \in L^\infty$  and any  $\nu > 0$ .

We do need to emphasize that this section does not provide a well-posedness theory for (42). While we do obtain uniqueness on solutions with enough a priori estimates, this does not directly yield the existence of any solutions in that class. This does not impact the present paper as we only the uniqueness result for the special type of solutions parametrized as  $h_T = \tau(T, w, f)$  with  $(w, f)$  a weak solution of (36) (cf. Proposition 3.17).

We also emphasize that the presence of artificial diffusion is needed in the method of proof in this subsection and a similar result with  $\nu = 0$  is still unknown. To analyze stability we shall define the following norms.

**Definition 3.18** (Norm of the hierarchy). *Consider any family  $h = (h_T)_{T \in \mathbb{T}}$  such that  $h_T \in L^2(\mathbb{R}^{d|T|})$  for every  $T \in \mathbb{T}$ . Then, we define the following norms:*

$$\|h\|_\lambda = \sup_{T \in \mathbb{T}} \lambda^{|T|/2} \|h_T\|_{L^2(\mathbb{R}^{d|T|})}, \quad (43)$$

for any  $\lambda > 0$ .

The norm (43) is natural since, for any  $f \in L^2(\mathbb{R}^d)$  and the special sequence  $h = (h_T)_{T \in \mathbb{T}}$  given by  $h_T = \tau(T, w, f)$  for any  $T \in \mathbb{T}$ , we obtain the following bound

$$\|h\|_\lambda \leq \sup_{T \in \mathbb{T}} \lambda^{|T|/2} \|w\|_{L^\infty_{\xi} \mathcal{M}_\zeta}^{|T|-1} \|f\|_{L^2}^{|T|},$$

for any  $\lambda > 0$  thanks to Corollary 3.16. Hence, setting  $\lambda > 0$  sufficiently small we obtain a finite quantity. Indeed, it provides the following stability result for solutions of (42).

**Theorem 3.19.** *Consider any solution  $h = (h_T)_{T \in \mathbb{T}}$  in the sense of distribution to (42) for some  $\nu > 0$  and  $K \in L^2$  such that  $h_T \in L^\infty([0, t_*], (L^1 \cap L^2)(\mathbb{R}^{d|T|}))$  for any  $T \in \mathbb{T}$ . Assume that there exists some  $\lambda > 0$  such that  $C_\lambda := \sup_{t \in [0, t_*]} \|h(t, \cdot)\|_\lambda < \infty$ . Then, for any  $p > 1$  and any  $\theta \in (0, 2^{-p'})$  there exists a constant  $C_{p, \theta} \in \mathbb{R}_+$  such that*

$$\|h(t, \cdot)\|_{\theta\lambda} \leq C_\lambda C_{p, \theta} \exp \left( p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu}} t \log \frac{\|h^0\|_{\theta\lambda}}{C_\lambda} \right), \quad (44)$$

for any  $t \in [0, t_*]$ .

**Remark 3.20.** *Taking  $p > 1$  sufficiently large, note that any  $\theta \in (0, \frac{1}{2})$  provides an estimate (44). However, such a constraint on the size of  $\theta$  is clearly an artifact of the method of proof since a similar estimate can be obtained for any  $\theta \in (0, 1)$  by interpolation. Namely, set any  $p_0 > 1$  and any  $\theta_0 \in (0, 2^{-p_0})$  so that (44) is fulfilled. Consider any arbitrary  $\theta \in [\frac{1}{2}, 1)$ . Then, note that  $\theta_0\lambda < \theta\lambda < \lambda$  and*

$$\|h(t, \cdot)\|_{\theta\lambda} \leq \|h(t, \cdot)\|_{\theta_0\lambda}^\alpha \|h(t, \cdot)\|_\lambda^{1-\alpha},$$

for any  $t \in [0, t_*]$ , where we have set  $\alpha := \log \theta / \log \theta_0$ . Then, we obtain

$$\|h(t, \cdot)\|_{\theta\lambda} \leq C_\lambda C_{p_0, \theta_0}^\alpha \exp \left( \alpha p_0^{-\frac{\|K\|_{L^2}^2}{2\theta_0\lambda\nu}} t \log \frac{\|h^0\|_{\theta\lambda}}{C_\lambda} \right), \quad (45)$$

for any  $t \in [0, t_*]$ .

**Remark 3.21.** *Thanks to the linearity of the hierarchy, Theorem 3.19 provides uniqueness among weak solutions  $h = (h_T)_{T \in \mathbb{T}}$  to (42) for some  $\nu > 0$ , such that  $h_T \in L^\infty([0, t_*], (L^1 \cap L^2)(\mathbb{R}^{d|T|}))$  for any  $T \in \mathbb{T}$  and  $\sup_{t \in [0, t_*]} \|h(t, \cdot)\|_\lambda < \infty$  for some  $\lambda > 0$ . Indeed, assume that  $h^0 = 0$ . Then,  $\|h^0\|_{\theta\lambda} = 0$  for any  $\theta \in (0, 1)$  so that (44) trivially implies that  $\|h(t, \cdot)\|_{\theta\lambda} = 0$  (i.e.,  $h(t, \cdot) = 0$ ) at any later time  $t \in (0, t_*]$ . Unfortunately, it does not however provide a bounded or Markov semi-group for those norms and in fact semi-group like estimates blow-up in finite time as it will be apparent in the proof.*

*Proof of Theorem 3.19.* We restrict to the special case  $C_\lambda = 1$ , all the other cases following by linearity of the hierarchy (42) and homogeneity of the norms (43). First, for a given  $T \in \mathbb{T}$ , notice that (42) implies

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{d|T|}} |h_T|^2 dx_1 \dots dx_{|T|} &= 2 \sum_{i=1}^{|T|} \int_{\mathbb{R}^{d(|T|+1)}} \nabla_{x_i} h_T K(x_i - z) h_{T+i} dx_1 \dots x_{|T|} dz \\ &\quad - 2\nu \sum_{i=1}^{|T|} \int_{\mathbb{R}^{d|T|}} |\nabla_{x_i} h_T|^2 dx_1 \dots dx_{|T|}. \end{aligned}$$

To justify the above formal calculation, we note that

$$\partial_t h_T = \nu \Delta h_T - \operatorname{div} j_T,$$

where  $j_T = \int_{\mathbb{R}^d} K(x_i - z) h_{T+i} dz$ . From our assumption on  $K$  and  $h_{T+i}$ , we immediately have that  $j_T \in L^\infty([0, t_*], L^2)$ . Namely, we obtain

$$\|j_T(t, \cdot)\|_{L^2(\mathbb{R}^{d|T|})} \leq \|K\|_{L^2} \|h_{T+i}(t, \cdot)\|_{L^2(\mathbb{R}^{d(|T|+1)})}.$$

Standard properties of the heat kernel then prove that  $h_T \in L^2([0, t_*], H^1(\mathbb{R}^{d|T|}))$ .

By the Cauchy–Schwartz inequality and Young’s inequality we further obtain

$$\frac{d}{dt} \|h_T(t, \cdot)\|_{L^2(\mathbb{R}^{d|T|})}^2 \leq \frac{\|K\|_{L^2}^2}{2\nu} \sum_{i=1}^{|T|} \|h_{T+i}(t, \cdot)\|_{L^2(\mathbb{R}^{d(|T|+1)})}^2.$$

For this reason, we introduce the family of intermediary norms

$$\|h(t, \cdot)\|_n := \sup_{|T|=n} \|h_T(t, \cdot)\|_{L^2(\mathbb{R}^{d|T|})},$$

for any tree order  $n \in \mathbb{N}$ , which readily satisfies

$$\frac{d}{dt} \|h(t, \cdot)\|_n^2 \leq n \frac{\|K\|_{L^2}^2}{2\nu} \|h(t, \cdot)\|_{n+1}^2.$$

By induction this estimate yields

$$\begin{aligned} \|h(t, \cdot)\|_n^2 &\leq \left( \frac{\|K\|_{L^2}^2}{2\nu} \right)^{m-n} \int_s^t \frac{(m-1)! (t-r)^{m-n-1}}{(n-1)! (m-n-1)!} \|h(r, \cdot)\|_m^2 dr \\ &\quad + \sum_{k=n}^{m-1} \left( \frac{\|K\|_{L^2}^2}{2\nu} \right)^{k-n} \frac{(k-1)! (t-s)^{k-n}}{(n-1)! (k-n)!} \|h(s, \cdot)\|_k^2. \end{aligned}$$

for any  $m > n$ . Using the assumption on  $h$  one has

$$\|h(r, \cdot)\|_m \leq \lambda^{-m},$$

for any  $r \in [s, t]$ . We then infer

$$\begin{aligned} \|h(t, \cdot)\|_n^2 &\leq \left( \frac{\|K\|_{L^2}^2}{2\nu} \right)^{m-n} \binom{m-1}{n-1} (t-s)^{m-n} \lambda^{-m} \\ &\quad + \sum_{k=n}^{m-1} \left( \frac{\|K\|_{L^2}^2}{2\nu} \right)^{k-n} \binom{k-1}{n-1} (t-s)^{k-n} \|h(s, \cdot)\|_k^2. \end{aligned} \tag{46}$$

Obviously, (46) can only provide smallness for short time intervals  $(t-s) \sim \nu$ , if used directly. As we advanced in Remark 3.21, we cannot bound a semi-group corresponding to the hierarchy for the norm  $\|h(t, \cdot)\|_\lambda$  on all times. Instead we fix now the time step

$$\delta := \frac{2\theta\lambda\nu}{\|K\|_{L^2}^2},$$

and consider the associated sequence of discrete times  $t_i = \delta i$  with  $i = 1, \dots, N$ . Applying (46) to  $t = t_{i+1}$  and  $s = t_i$ , one obtains

$$\sup_{t \in [t_i, t_{i+1}]} (\theta\lambda)^n \|h(t, \cdot)\|_n^2 \leq \theta^m \binom{m-1}{n-1} + \sum_{k=n}^{m-1} (\theta\lambda)^k \binom{k-1}{n-1} \|h(t_i, \cdot)\|_k^2.$$

We now interpolate the last factors

$$(\theta^{1/p}\lambda)^k \|h(t_i, \cdot)\|_k^2 = ((\theta\lambda)^k \|h(t_i, \cdot)\|_k^2)^{1/p} (\lambda^k \|h(t_i, \cdot)\|_k^2)^{1/p'} \leq \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p} \|h(t_i, \cdot)\|_{\lambda}^{2/p'} \leq \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p},$$

where  $1/p + 1/p' = 1$  and we have used the assumption  $\|h(t_i, \cdot)\|_{\lambda} \leq 1$ . This yields

$$\sup_{t \in [t_i, t_{i+1}]} (\theta\lambda)^n \|h(t, \cdot)\|_n^2 \leq \theta^m \binom{m-1}{n-1} + \sum_{k=n}^{m-1} \theta^{k/p'} \binom{k-1}{n-1} \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p}.$$

Since  $\sum_{n=1}^m \binom{m-1}{n-1} = 2^{m-1}$ , we have that  $\binom{m-1}{n-1} \leq 2^{m-1}$ . Similarly, summing over  $k$ , we obtain

$$\sup_{t \in [t_i, t_{i+1}]} (\theta\lambda)^n \|h(t, \cdot)\|_n^2 \leq \frac{1}{2} (2\theta)^m + \frac{1}{2} \sum_{k=n}^{m-1} (2\theta^{1/p'})^k \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p} \leq \frac{1}{2} (2\theta)^m + \frac{1}{2(1-2\theta^{1/p'})} \|h(t_i, \cdot)\|_{\theta\lambda}^{2/p},$$

where we have used that  $2\theta^{1/p'} < 1$  by hypothesis to guarantee the summability of the last factor. Indeed, since  $2\theta < 1$  then taking limits  $m \rightarrow \infty$  shows that

$$\sup_{t \in [t_i, t_{i+1}]} \|h(t, \cdot)\|_{\theta\lambda} \leq \left( \frac{1}{2(1-2\theta^{1/p'})} \right)^{1/2} \|h(t_i, \cdot)\|_{\theta\lambda}^{1/p},$$

and therefore

$$\sup_{t \in [t_i, t_{i+1}]} \|h(t, \cdot)\|_{\theta\lambda} \leq \left( \frac{1}{2(1-2\theta^{1/p'})} \right)^{p'/2} \|h^0\|_{\theta\lambda}^{p^{-i}}.$$

For any  $t \in [0, t_*)$  set  $i$  with  $t \in [t_i, t_{i+1})$ . Since  $\|h^0\|_{\theta\lambda} \leq \|h^0\|_{\lambda} \leq 1$ , the relation

$$i \leq \frac{t}{\delta} = \frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t$$

allows concluding the proof.  $\square$

We emphasize that Theorem 3.19 only requires  $K \in L^2$  instead of  $K \in W^{1,\infty}$ , but it only provides stability on the *viscous* hierarchy (42). Indeed, we do not know if any similar result could hold on the non-viscous hierarchy (*i.e.*  $\nu = 0$ ). However Theorem 3.19 does imply a comparable stability result on the solutions  $f$  of the starting graphon equation (35) without diffusion, provided that some added regularity is available on the  $f$  or on  $K$ . The strategy is straightforward and consists in adding some artificial viscosity, which is performed in the next subsection.

**3.4. Stability on the system without artificial diffusion.** Our stability result relies on stronger regularity for weak solutions  $(w, f)$  to (35). Before stating it, it is useful to observe that smoothness indeed propagates in time by a similar argument as in Proposition 3.11.

**Lemma 3.22.** *Consider any  $K \in W^{1,1}$  with  $\operatorname{div} K \in L^\infty$ , any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and any that  $f^0 \in L_\xi^\infty (L_x^1 \cap L_x^\infty \cap H_x^1)$ . Assume that  $(w, f)$  is a weak solution to (35). Then,  $f \in L^\infty([0, t_*], L_\xi^\infty (L_x^1 \cap L_x^\infty \cap H_x^1))$ , for any  $t_* < \infty$ , and it satisfies*

$$\|f(t, \cdot, \cdot)\|_{L_\xi^\infty (L_x^1 \cap L_x^\infty \cap H_x^1)} \leq C e^{e^C t - 1},$$

for some  $C \in \mathbb{R}_+$  depending only on  $\|w\|_{L_\xi^\infty \mathcal{M}_\zeta}$ ,  $\|K\|_{W^{1,1}}$ ,  $\|\operatorname{div} K\|_{L^\infty}$  and  $\|f^0\|_{L_\xi^\infty (L_x^1 \cap L_x^\infty \cap H_x^1)}$ .

*Proof.* The following estimates can be made rigorous through a regularization process, for example adding viscosity as in the equation (36), together with a classical iterative delay in time leading to a linear system. We first observe that by integration by parts we obtain the *a priori* bounds

$$\begin{aligned} \frac{d}{dt} \|f(t, \cdot, \cdot)\|_{L_x^p}^p &= -(p-1) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x-y) f(t, x, \xi)^p f(t, y, \zeta) dx dy \\ &\leq (p-1) \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|f(t, \cdot, \cdot)\|_{L_x^p}^p, \end{aligned}$$



for  $t \in [0, t_*]$  and  $\xi \in [0, 1]$ , where above we have used again Lemma 3.8. Since  $f(t, \cdot, \xi)$  preserves mass for each  $\xi \in [0, 1]$ , one has that  $\|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} = \|f^0\|_{L_\xi^\infty L_x^1}$  for each  $t \in [0, t_*]$  and this allows easily propagating any  $L_\xi^\infty L_x^p$  bound of  $f$  by Gronwall's inequality, leading to the following estimate

$$\|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^p} \leq C_p e^{C_p t}.$$

Similarly, differentiating (19) with respect to  $x$  we now get

$$\begin{aligned} \partial_t \nabla_x f(t, x, \xi) + \operatorname{div}_x \left( \nabla_x f(t, x, \xi) \otimes \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) \\ + \operatorname{div}_x \left( f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} \nabla K(x-y)^\top f(t, y, \zeta) dy \right) = 0. \end{aligned}$$

Again, by integrating by parts we easily get the following decomposition

$$\frac{d}{dt} \frac{1}{2} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2 = I_1 + I_2 + I_3,$$

where each term takes the form

$$\begin{aligned} I_1 &:= \frac{1}{2} \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x-y) |\nabla_x f(t, x, \xi)|^2 f(t, y, \zeta) dx dy, \\ I_2 &:= - \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x-y) \nabla_x f(t, x, \xi) \cdot \nabla_y f(t, y, \zeta) f(t, x, \xi) dx dy, \\ I_3 &:= - \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \nabla_x f(t, x, \xi)^\top \cdot \nabla K(x-y) \cdot \nabla_x f(t, x, \xi) f(t, y, \zeta) dx dy. \end{aligned}$$

By the hypothesis on  $K$ , we have

$$\begin{aligned} |I_1| &\leq \frac{1}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2, \\ |I_2| &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^1} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^\infty} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2, \\ |I_3| &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\nabla K\|_{L^1} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^\infty} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2, \end{aligned}$$

where we have used again Lemma 3.8. Putting everything together, using the previous  $L^p$  estimates and the fact that  $\|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} = \|f^0\|_{L_\xi^\infty L_x^1}$  for all  $t \in [0, t_*]$ , one obtains

$$\frac{d}{dt} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2 \leq C e^{Ct} \|\nabla_x f(t, \cdot, \xi)\|_{L_x^2}^2.$$

Therefore, we conclude by means of Gronwall's lemma.  $\square$

With this additional regularity, in the context of the preceding subsections, and from Theorem 3.19 and Proposition 3.17, we may derive the following uniqueness result.

We do note that in the following theorem, we use the  $\tau(T, w, f)$  for generic  $w, \tilde{w} \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ . While the operator  $\tau$  is trivially well defined if  $w, \tilde{w} \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ , we rigorously justify the definition of  $\tau(T, w, f)$  for kernels in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  in the next Section 4 around Definition 4.5. We refer to the more extensive discussion just after Lemma 3.15.

**Theorem 3.23.** *Consider any couple of weak solution to (35)  $f, \tilde{f} \in L^\infty([0, t_*], L_\xi^\infty(L_x^1 \cap L_x^\infty \cap H_x^1))$  with associated  $w, \tilde{w} \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , for some  $K \in L^\infty \cap W^{1,1}$  with  $\operatorname{div} K \in L^\infty$ . Then, we have*

$$\left\| \int_0^1 (f - \tilde{f})(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \frac{C}{(\log |\log \|\tau(\cdot, w, f^0) - \tau(\cdot, \tilde{w}, \tilde{f}^0)\|_\lambda|)^{1/2}},$$

for any  $t \in [0, t_*]$ , and some constants  $C, \lambda > 0$ , which only depend on  $t_*$ , the various norms of  $K$ , the norm of  $w, \tilde{w}$  in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and the norm of the initial data  $f^0, \tilde{f}^0$  in  $L_\xi^\infty(L_x^1 \cap L_x^\infty \cap H_x^1)$ .

**Remark 3.24.** *A direct  $L^p$  estimate would only provide the following trivial estimate*

$$\|f - \tilde{f}\|_{L_\xi^\infty L_\zeta^\infty L_x^2} \leq e^{Ct} \left( \|f^0 - \tilde{f}^0\|_{L_\xi^\infty L_x^2} + C \|w - \tilde{w}\|_{L_\xi^\infty \mathcal{M}_\zeta} \right).$$

*Note that such estimate would require a very precise strong control on the difference  $w - \tilde{w}$ , which is not available to us. However, as we will see in the next section, we can obtain good estimates on the weaker objects  $\tau(T, w, f^0) - \tau(T, \tilde{w}, \tilde{f}^0)$ .*

**Remark 3.25.** *If we assume more regularity on  $K$ , for example  $K \in W^{1,\infty}$ , then it is possible to obtain stability with instead less regularity on  $f$  and  $\tilde{f}$  (typically  $f^0 \in L_\xi^\infty L_x^\infty$ , for example).*

Before turning to the proof of Theorem 3.23, let us emphasize some of its main consequences. The key point is that it offers a direct way to obtain some form of compactness on our solutions. More specifically, we can extract strong convergence in  $L^2$  on each of the  $\tau(T, w_N, f_N^0)$  through a classical diagonal extraction process since each of them is bounded in  $L^1 \cap W^{1,\infty}$  by Corollary 3.16. From Theorem 3.23, this implies that  $\int_0^1 f_N(t, x, \xi) d\xi$  is strongly compact in  $L_x^2$ . However, we are still not able to identify the corresponding limit in terms of a solution to (35). This will require an appropriate extension of graphons that is fully developed in the next section.

*Proof of Theorem 3.23.* For some  $\nu > 0$ , let us consider the solution  $f^\nu$  to the system (36) with artificial diffusion with the same initial data  $f^0$  and same weights  $w$  as  $f$ . We define similarly  $\tilde{f}^\nu$  solution to (36) with the same initial data  $\tilde{f}^0$  and weights  $\tilde{w}$  as  $\tilde{f}$ . The existence of both  $f^\nu$  and  $\tilde{f}^\nu$  is provided by Proposition 3.11.

The strategy of the proof is classical: First use the regularity of  $f$  (resp.  $\tilde{f}$ ) to compare them with  $f^\nu$  (resp.  $\tilde{f}^\nu$ ). Then use Theorem 3.19 to compare the observables from  $f^\nu$  and  $\tilde{f}^\nu$ . To compare  $f$  with  $f^\nu$ , we first derive  $H^1$  estimates for  $f^\nu$  in the usual way

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|f^\nu(t, \cdot, \xi)\|_{L_x^2}^2 &= -\nu \int_{\mathbb{R}^d} |\nabla_x f^\nu(t, x, \xi)|^2 dx \\ &\quad + \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} f^\nu(t, x, \xi) \nabla f^\nu(t, x, \xi) \cdot K(x-y) f^\nu(t, y, \zeta) dy dx. \end{aligned}$$

Integrating by parts in the second term yields

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|f^\nu(t, \cdot, \xi)\|_{L_x^2}^2 + \nu \int_{\mathbb{R}^d} |\nabla_x f^\nu(t, x, \xi)|^2 dx \\ = -\frac{1}{2} \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^{2d}} \operatorname{div} K(x-y) |f^\nu(t, x, \xi)|^2 f^\nu(t, y, \zeta) dy dx \\ \leq \frac{1}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|f^\nu(t, \cdot, \xi)\|_{L_x^2}^2, \end{aligned}$$

where we have used Lemma 3.8 in the last step. Again, notice that  $\|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} = \|f^0\|_{L_\xi^\infty L_x^1}$  for all  $t \in [0, t_*]$ . Using Gronwall's lemma we obtain

$$\|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2}^2 + \nu \int_0^t \int_{\mathbb{R}^d} |\nabla_x f^\nu(s, x, \xi)|^2 dx ds \leq \|f^0\|_{L_\xi^\infty L_x^2}^2 e^{\frac{t}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f^0\|_{L_\xi^\infty L_x^1}}. \quad (47)$$

Now, observe that the difference  $f^\nu - f$  satisfies the following equation

$$\begin{aligned} \partial_t (f^\nu - f)(t, x, \xi) + \operatorname{div}_x \left( (f^\nu - f)(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f^\nu(t, y, \zeta) dy \right) \\ + \operatorname{div}_x \left( f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) (f^\nu - f)(t, y, \zeta) dy \right) \\ - \nu \Delta_x (f^\nu - f)(t, x, \xi) = \nu \Delta_x f(t, x, \xi). \end{aligned}$$

Therefore, similar arguments as above lead to analogous  $H^1$  estimates

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2}^2 + \nu \int_{\mathbb{R}^d} |\nabla_x (f^\nu - f)(t, x, \xi)|^2 dx \\ \leq \frac{1}{2} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^\infty} \|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^1} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \\ + \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|K\|_{L^2} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty H_x^1} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \|(f^\nu - f)(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2} \\ + \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\operatorname{div} K\|_{L^1} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^\infty} \|(f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \|(f^\nu - f)(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2} \\ + \nu \|\nabla_x (f^\nu - f)(t, \cdot, \xi)\|_{L_x^2} \|f(t, \cdot, \cdot)\|_{L_\xi^\infty H_x^1}. \end{aligned}$$

Using the additional regularity  $f, \tilde{f} \in L^\infty([0, t_*], L_\xi^\infty(L_x^1 \cap L_x^\infty \cap H_x^1))$  from Lemma 3.22, applying Young's inequality in the last term, and recalling that  $f(0, \cdot, \cdot) = f^\nu(0, \cdot, \cdot)$  we obtain by Gronwall's lemma that

$$\|(f^\nu - f)(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2} \leq C(t) \sqrt{\nu}, \quad (48)$$

for some continuous and non-decreasing function  $C = C(t) \in \mathbb{R}_+$  that only depends on the various norms of  $K$ , the norm of  $w$  in  $L_\xi^\infty \mathcal{M}_\zeta$  and the norm of the initial datum  $f^0$  in  $L_\xi^\infty(L_x^1 \cap L_x^\infty \cap H_x^1)$ . Note that a similar estimate can be obtained for  $\tilde{f}^\nu - \tilde{f}$ . The function  $C$  is actually a triple exponential.

By Proposition 3.17, the observables  $\tau(T, w, f^\nu)$  and  $\tau(T, \tilde{w}, \tilde{f}^\nu)$  with  $T \in \mathbb{T}$  both solve the same hierarchy (42) (since  $w, \tilde{w}$  do not appear explicitly in that formulation). Therefore,  $h = (h_T)_{T \in \mathbb{T}}$ , with  $h_T = \tau(T, w, f^\nu) - \tau(T, \tilde{w}, \tilde{f}^\nu)$ , again solves (42) by linearity. In addition, Corollary 3.16 implies that

$$\begin{aligned} \|h_T(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{d|T|})} &\leq \|\tau(T, w, f^\nu(t, \cdot, \cdot))\|_{L^2(\mathbb{R}^{d|T|})} + \|\tau(T, \tilde{w}, \tilde{f}^\nu(t, \cdot, \cdot))\|_{L^2(\mathbb{R}^{d|T|})} \\ &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|f^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2}^{|T|} + \|\tilde{w}\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|\tilde{f}^\nu(t, \cdot, \cdot)\|_{L_\xi^\infty L_x^2}^{|T|} \\ &\leq \frac{2}{w_{\min}} \left( w_{\max} \frac{\|f^0\|_{L_\xi^\infty L_x^2} + \|\tilde{f}^0\|_{L_\xi^\infty L_x^2}}{2} e^{\frac{t_*}{4} w_{\max} \|\operatorname{div} K\|_{L^\infty} \|f^0\|_{L_\xi^\infty L_x^2}} \right)^{|T|}, \end{aligned}$$

where  $w_{\min} = \min\{\|w\|_{L_\xi^\infty \mathcal{M}_\zeta}, \|\tilde{w}\|_{L_\xi^\infty \mathcal{M}_\zeta}\}$  and  $w_{\max} = \max\{\|w\|_{L_\xi^\infty \mathcal{M}_\zeta}, \|\tilde{w}\|_{L_\xi^\infty \mathcal{M}_\zeta}\}$ . Consequently, we obtain that  $\sup_{t \in [0, t_*]} \|h(t, \cdot, \cdot)\|_\lambda < 1$ , for any  $\lambda > 0$  such that

$$\sqrt{\lambda} < \min \left\{ \frac{2}{w_{\max}}, \frac{w_{\min}}{w_{\max}} \right\} \frac{e^{-\frac{t_*}{4} w_{\max} \|\operatorname{div} K\|_{L^\infty} \|f^0\|_{L_\xi^\infty L_x^2}}}{(\|f^0\|_{L_\xi^\infty L_x^2} + \|\tilde{f}^0\|_{L_\xi^\infty L_x^2})}.$$

We can now apply Theorem 3.19 for this  $\lambda$ , any  $p > 1$  and  $\theta \in (0, 2^{-p'})$ , and obtain

$$\sup_{t \in [0, t_*]} \|h(t, \cdot, \cdot)\|_{\theta\lambda} \leq C_{p,\theta} \exp \left( p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t} \log \|h^0\|_{\theta\lambda} \right),$$

for any  $T \in \mathbb{T}$  and some constant  $C_{p,\theta} > 0$  depending only on  $p$  and  $\theta$ . In the special case of the trivial tree  $T_1 \in \mathbb{T}_1$  with only one vertex, we obtain

$$\|h\|_{\theta\lambda} \geq \lambda \|h_{T_1}\|_{L^2} = \sqrt{\lambda} \left\| \tau(T_1, w, f^\nu) - \tau(T_1, \tilde{w}, \tilde{f}^\nu) \right\|_{L_x^2} = \sqrt{\lambda} \left\| \int_0^1 (f^\nu - \tilde{f}^\nu)(t, \cdot, \xi) d\xi \right\|_{L_x^2}.$$

Therefore, we deduce that

$$\left\| \int_0^1 (f^\nu - \tilde{f}^\nu)(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \frac{C_{p,\theta}}{\sqrt{\lambda}} \exp \left( p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t} \log \|h^0\|_{\theta\lambda} \right), \quad (49)$$

for any  $t \in [0, t_*]$ . By (48) and Minkowski's integral inequality, we also have that

$$\left\| \int_0^1 (f^\nu - f)(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \|f^\nu - f\|_{L_\xi^\infty L_x^2} \leq C(t) \sqrt{\nu},$$

and similarly for  $\tilde{f}^\nu - \tilde{f}$ , where the function  $C = C(t)$  is given above. This combined with (49) immediately yields that

$$\left\| \int_0^1 (f - \tilde{f})(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \frac{C_{p,\theta}}{\sqrt{\lambda}} \exp \left( p^{-\frac{\|K\|_{L^2}^2}{2\theta\lambda\nu} t} \log \|h^0\|_{\theta\lambda} \right) + C(t) \sqrt{\nu}.$$

Since the above is valid for any  $\nu > 0$ , it only remains to optimize in  $\nu$  appropriately. For example, considering  $p = e$  for simplicity and setting

$$\nu = \frac{\|K\|_{L^2}^2}{\theta\lambda} t (\log |\log \|h^0\|_{\theta\lambda}|)^{-1},$$

ends the proof.  $\square$

#### 4. EXTENDING GRAPHONS

**4.1. Our goals.** This section is centered on the process of taking limits on the  $\tau(T, w_N, f_N)$  defined by the operator (38) for the pairs  $(w_N, f_N)$  given by Definition 3.4, at some fixed time  $t \in \mathbb{R}_+$ . Since the time variable plays no role in this part, we omit it from the calculations. The first aim of this section is to analyze the following question already raised in the previous section: If  $\lim_{N \rightarrow \infty} \tau(T, w_N, f_N)$  exists for any  $T \in \mathbb{T}$ , can we find appropriate  $w$  and  $f$  such that we recover the representation

$$\tau(T, w, f) = \lim_{N \rightarrow \infty} \tau(T, w_N, f_N), \quad \forall T \in \mathbb{T}?$$

As we will see below, this can be rephrased as the question of how to take appropriate limits of  $w_N$  and  $f_N$  so that they allow passing to the limit in  $\tau(T, w_N, f_N)$ . The following result answers this question.

**Theorem 4.1.** *The definition of  $\tau(T, w, f)$  can be uniquely extended for any  $w \in L_\zeta^\infty \mathcal{M}_\xi \cap L_\xi^\infty \mathcal{M}_\zeta$  and  $f \in L_\xi^\infty L_x^\infty$ . Furthermore, consider any sequence  $\{w_N\}_{N \in \mathbb{N}}$  and  $\{f_N\}_{N \in \mathbb{N}}$  such that the following hypothesis hold true*

$$(i) \quad \sup_{N \in \mathbb{N}} \sup_{\xi \in [0, 1]} \int_0^1 |w_N(\xi, \zeta)| d\zeta < \infty, \quad \sup_{N \in \mathbb{N}} \sup_{\zeta \in [0, 1]} \int_0^1 |w_N(\xi, \zeta)| d\xi < \infty,$$

$$(ii) \quad \sup_{N \in \mathbb{N}} \|f_N\|_{L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})} < \infty.$$

Then, there exists an extracted subsequence of  $N$  (that we still denote as  $N$  for simplicity) and  $w \in L_\zeta^\infty \mathcal{M}_\xi \cap L_\xi^\infty \mathcal{M}_\zeta$  together with  $f \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$  such that

$$\tau(T, w_N, f_N) \rightarrow \tau(T, w, f) \quad \text{in } L_{loc}^p(\mathbb{R}^{d|T|}),$$

as  $N \rightarrow \infty$ , for each  $T \in \mathbb{T}$  and any  $1 \leq p < \infty$ .

We note that Theorem 4.1 only imposes that the sequences  $\{w_N\}_{N \in \mathbb{N}}$  and  $\{f_N\}_{N \in \mathbb{N}}$  fulfill the above uniform estimates (i) and (ii). Those sequences can, hence, be more general than the empirical graphons in Definition 3.4 associated to discrete objects  $(w_{ij})_{i,j=1,\dots,N}$  and  $(f_i)_{i=1,\dots,N} \subseteq \mathcal{P}(\mathbb{R}^d)$ . However, when restricting to empirical  $w_N$  and  $f_N$ , note that we have equivalently the following uniform bounds

$$(i') \quad \sup_{N \in \mathbb{N}} \max_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| < \infty, \quad \sup_{N \in \mathbb{N}} \max_{1 \leq j \leq N} \sum_{i=1}^N |w_{ij}| < \infty,$$

$$(ii') \quad \sup_{N \in \mathbb{N}} \max_{1 \leq i \leq N} \|f_i\|_{W_x^{1,\infty}} < \infty.$$

Let us remark some critical differences of our generalized graphons with standard graphons:

- Graphons were originally introduced in the context of random graphs where  $N w_{ij}$  is the probability of having an edge connecting  $i$  to  $j$ . In that case, it was assumed that  $w_{ij} \lesssim 1/N$  or  $0 \leq w_N(\xi, \zeta) \leq 1$ , which leads to a dense matrix  $w_{ij}$  on the probability of connections but a sparse resulting graphs (since the probability of having a connection between any two given nodes is low). Compactness is obtained by considering  $w_N$  as the kernel of an operator from  $L^1$  to  $L^\infty$ . This corresponds to the natural topology on graphons characterized by the *cut metric*, which can be reframed in terms of the homomorphism densities  $\tau(G, w)$  in Lemma 3.14 for general simple graphs  $G$ , see the pioneer works of Lovász and Lovász-Szegedy [56, 57].
- Instead, in our work  $w_N$  are not uniformly bounded in  $L_{\xi,\zeta}^\infty$ , but only in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ . This still allows considering  $w_N$  as the kernel of an operator, for example from  $L^\infty$  to  $L^\infty$  and from  $L^1$  to  $L^1$  as it has been depicted in Lemma 3.7. But the connection with the cut metric and the natural compactness that derives from it are lost. Instead Theorem 4.1 suggests an alternative natural topology on pairs  $(w, f)$  with  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $f \in L_\xi^\infty L_x^\infty$ .
- One straightforward result that highlights these differences is the following: As we have seen in the previous section (cf. Lemma 3.15), we have

$$\|\tau(T, w, f)\|_{L^\infty} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|f\|_{L_\xi^\infty L_x^\infty}^{|T|},$$

for any tree  $T$  and any  $w \in L_\xi^\infty \mathcal{M}_\zeta$ , while this estimate is obviously false if we consider  $\tau(G, w)$  for general simple graphs  $G$  as in done in the standard theory of graphons.

Of direct impact to our work here is the fact that we cannot easily define  $\tau(T, w, f)$  through formula (38) in Definition 3.13 when we only have  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and  $f \in L_\xi^\infty L_x^\infty$ . We explain how this can be done in the next subsection through the construction of an appropriate algebra. The rest of the section is devoted to the proof of Theorem 4.1 which is based on obtaining simultaneous compactness on all elements of the algebra.

**4.2. Defining  $\tau(T, w, f)$  for unbounded  $w$ .** The key to the definition of  $\tau(T, w, f)$  (and later of the compactness argument) for only  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , is an iterative way to construct it from the leaves of the tree. This leads us to the following definition.

**Definition 4.2** (A countable algebra). *We will denote by  $\mathcal{T}$  the countable algebra of transforms over spaces of arbitrarily large dimensions which is built as follows: For each transform  $F \in \mathcal{T}$  there exists  $n \in \mathbb{N}$  (the rank of  $F$ ) so that  $F$  maps each couple  $(w, f)$  into a scalar function  $F(w, f)$  on  $[0, 1] \times \mathbb{R}^{dn}$ . The full algebra  $\mathcal{T}$  is obtained in a recursive way according to the following rules:*

- (i) *The elementary 1-rank transform  $F_0 : (w, f) \mapsto f$  belongs to the algebra  $\mathcal{T}$ .*

(ii) Let  $F_1 \in \mathcal{T}$  and  $F_2 \in \mathcal{T}$  be  $n_1$ -rank and  $n_2$ -rank transforms respectively. Then, the following  $(n_1 + n_2)$ -rank transform also belongs to  $\mathcal{T}$ :

$$F_1 \otimes F_2 : (w, f) \mapsto F_1(w, f)(\xi, x_1, \dots, x_{n_1}) F_2(w, f)(\xi, x_{n_1+1}, \dots, x_{n_1+n_2}).$$

(iii) Let  $F \in \mathcal{T}$  be a  $n$ -rank transform. Then, the following  $n$ -rank transform also belongs to  $\mathcal{T}$ :

$$F^* : (w, f) \mapsto \int_0^1 F(w, f)(\zeta, x_1, \dots, x_n) w(\xi, d\zeta).$$

For a given choice of  $(w, f)$ , we also denote by  $M(w, f)$  the countable algebra consisting of functions over spaces of arbitrary large dimensions given by  $F(w, f)$  for any transform  $F \in \mathcal{T}$ . Of course  $M(w, f)$  can also be obtained directly by transposing the rules (i), (ii) and (iii) above:

(i) We have  $f(x_1, \xi) \in M(w, f)$ .

(ii) If  $\phi(\xi, x_1, \dots, x_{n_1}), \psi(\xi, x_1, \dots, x_{n_2}) \in M(w, f)$ , then we have:

$$\phi(\xi, x_1, \dots, x_{n_1}) \psi(\xi, x_{n_1+1}, \dots, x_{n_1+n_2}) \in M(w, f).$$

(iii) If  $\phi(\xi, x_1, \dots, x_n) \in M(w, f)$ , then we have:

$$\int_0^1 \phi(\zeta, x_1, \dots, x_n) w(\xi, d\zeta) \in M(w, f).$$

The first point is to observe that  $M(w, f)$  (and thus the algebra  $\mathcal{T}$ ) is well defined with the only hypotheses  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ .

**Lemma 4.3.** Consider any  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ . Then all functions of  $M(w, f)$  are well defined and belong to  $L^\infty([0, 1], (L^1 \cap L^\infty)(\mathbb{R}^{dn}))$  for some  $n \in \mathbb{N}$ . Moreover,

$$F(w_N, f_N) \rightarrow F(w, f) \quad \text{in } L^1([0, 1] \times \mathbb{R}^{dn}),$$

an  $N \rightarrow \infty$  for any fixed  $F \in \mathcal{T}$ , any sequence  $\{f_N\}_{N \in \mathbb{N}}$  uniformly bounded in  $L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and converging to some  $f$  in  $L_\xi^\infty(L_x^1 \cap L_x^\infty)$ , and any sequence of  $\{w_N\}_{N \in \mathbb{N}}$  uniformly bounded in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and converging to  $w$  in  $L_\xi^1 H_\zeta^{-1} \cap L_\zeta^1 H_\xi^{-1}$ .

*Proof.* We of course use an induction argument based on the recursive rules above defining the countable algebra  $M(w, f)$ .

◊ *Step 1: Good definition of elements of  $M(w, f)$ .*

Obviously, for the elementary transform  $F_0$  in rule (i) we have that  $F_0(w, f) = f$  is well defined and belongs to  $L^\infty([0, 1] \times \mathbb{R}^d)$  by hypothesis. The second rule (ii) also poses no issue. Indeed, let us set  $F = F_1 \otimes F_2$  and assume  $F_1, F_2 \in \mathcal{T}$  are transform with rank  $n_1, n_2 \in \mathbb{N}$  for which we already now that  $F_1(w, f)$  and  $F_2(w, f)$  are well defined and belong to  $L^\infty([0, 1] \times \mathbb{R}^{dn_1})$  and  $L^\infty([0, 1] \times \mathbb{R}^{dn_2})$  respectively. Then,  $F(w, f)$  is well defined and belongs to  $L^\infty([0, 1] \times \mathbb{R}^{d(n_1+n_2)})$  as the product of two bounded functions. Finally, it only remains to check the third rule (iii). Consider any  $F \in \mathcal{T}$  of rank  $n \in \mathbb{N}$  for which we already now that  $F(w, f)$  is well defined and belongs  $L^\infty([0, 1] \times \mathbb{R}^{dn})$ . Then,  $F^*(w, f)$  is also well defined and belongs to  $L^\infty([0, 1] \times \mathbb{R}^{dn})$  by the estimate (34)<sub>2</sub> in Lemma 3.8.

Moreover, since  $L^1$  norms are also stable under tensor products, by the same argument we also have that if  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , then all functions in  $M(w, f)$  are well defined and belong to  $L_\xi^\infty(L_{x_1, \dots, x_n}^1 \cap L_{x_1, \dots, x_n}^\infty)$  for some  $n \in \mathbb{N}$ .

◊ *Step 2: Convergence of elements of  $M(w_N, f_N)$ .*

First note that taking the elementary transform  $F_0 \in \mathcal{T}$  in rule (i) we have that

$$F_0(w_N, f_N) = f_N \rightarrow f = F_0(w, f) \quad \text{in } L^1([0, 1] \times \mathbb{R}^d),$$

by the convergence hypothesis on  $\{f_N\}_{N \in \mathbb{N}}$ . Let us set  $F = F_1 \otimes F_2$  as in the rule (ii) for a couple of transforms  $F_1, F_2 \in \mathcal{T}$  and assume that we already now that  $F_1(w_N, f_N)$  and  $F_2(w_N, f_N)$  are both convergent in  $L^1$  and uniformly bounded in  $L^\infty$ . Therefore, it is clear that

$$F(w_N, f_N) \rightarrow F(w, f) \quad \text{in } L^1([0, 1] \times \mathbb{R}^{d(n_1+n_2)}),$$

and it is uniformly bounded in  $L^\infty$  as the product of uniformly bounded sequences in  $L^\infty$ , each of them convergent in  $L^1$ . Finally, consider a  $n$ -rank transform  $F \in \mathcal{T}$  and assume that we already now that  $F(w_N, f_N)$  converges to  $F(w, f)$  in  $L^1$  and is uniformly bounded in  $L^\infty$ . Our last goal is to show that  $F^*(w_N, f_N)$  given by the rule (iii) is also convergent in  $L^1$  and uniformly bounded in  $L^\infty$ . Note that Lemma 3.7 yields a partial answer in the weak topology of  $L^\infty$  though. We shall improve it here

at the expense of the stronger convergence assumed on  $\{w_N\}_{N \in \mathbb{N}}$ . For simplicity of notation, we set  $\phi_N := F(w_N, f_N)$ , which by the induction hypothesis converges to  $\phi := F(w, f)$  in  $L^1$  and uniformly bounded in  $L^\infty$ . Then, we find that

$$F^*(w_N, f_N)(\xi, x_1, \dots, x_n) = \int_0^1 \phi_N(\zeta, x_1, \dots, x_n) w_N(\xi, d\zeta),$$

is uniformly bounded in  $L^\infty$  by estimate (34)<sub>2</sub> in Lemma 3.8 thanks to the uniform bound of  $\{w_N\}_{N \in \mathbb{N}}$  in  $L^\infty \mathcal{M}_\zeta$  and of  $\phi_N$  in  $L^\infty$ . Regarding convergence, we have

$$F^*(w_N, f_N) - F^*(w, f) = I_N^1 + I_N^2,$$

where each factor reads

$$\begin{aligned} I_N^1(\xi, x_1, \dots, x_n) &:= \int_0^1 (\phi_N - \phi)(\zeta, x_1, \dots, x_n) w_N(\xi, d\zeta), \\ I_N^2(\xi, x_1, \dots, x_n) &:= \int_0^1 \phi(\zeta, x_1, \dots, x_n) (w_N(\xi, d\zeta) - w(\xi, d\zeta)). \end{aligned}$$

for  $N \in \mathbb{N}$ . On the one hand, it is clear by estimate (34)<sub>1</sub> in Lemma 3.8 that we have

$$\|I_N^1\|_{L^1} \leq \|w_N\|_{L^\infty \mathcal{M}_\zeta} \|\phi_N - \phi\|_{L^1},$$

and, therefore,  $I_N^1 \rightarrow 0$  in  $L^1$  by the convergence of  $\phi_N$  to  $\phi$  in  $L^1$  and the uniform estimate of  $w_N$  in  $L^\infty \mathcal{M}_\zeta$ . On the other hand, consider any sequence  $\{\phi_\varepsilon\}_{\varepsilon > 0} \subset C_c^\infty([0, 1] \times \mathbb{R}^{dn})$  such that  $\phi_\varepsilon \rightarrow \phi$  in  $L^1$  as  $\varepsilon \rightarrow 0$  and define

$$I_{N,\varepsilon}^2(\xi, x_1, \dots, x_n) := \int_0^1 \phi_\varepsilon(\zeta, x_1, \dots, x_n) (w_N(\xi, d\zeta) - w(\xi, d\zeta)),$$

for  $N \in \mathbb{N}$  and  $\varepsilon > 0$ . Then, by (34)<sub>1</sub> in Lemma 3.8 we have

$$\begin{aligned} \|I_{N,\varepsilon}^2\|_{L^1} &\leq \|w_N\|_{L^\infty \mathcal{M}_\zeta} \|\phi - \phi_\varepsilon\|_{L^1} + \|I_{N,\varepsilon}^2\|_{L^1} \\ &\leq \|w_N\|_{L^\infty \mathcal{M}_\zeta} \|\phi - \phi_\varepsilon\|_{L^1} + \|\phi_\varepsilon\|_{L^1_{x_1, \dots, x_n} H_\zeta^1} \|w_N - w\|_{L^1_{\xi} H_\zeta^{-1}}, \end{aligned}$$

for every  $N \in \mathbb{N}$  and  $\varepsilon > 0$ . Taking limsup as  $N \rightarrow \infty$  we have that

$$\limsup_{N \rightarrow \infty} \|I_N^2\|_{L^1} \leq \sup_{N \in \mathbb{N}} \|w_N\|_{L^\infty \mathcal{M}_\zeta} \|\phi - \phi_\varepsilon\|_{L^1}.$$

Since  $\varepsilon > 0$  is arbitrary, taking  $\varepsilon \rightarrow 0$  and recalling that  $w_N$  is uniformly bounded in  $L^\infty \mathcal{M}_\zeta$  and  $\phi_\varepsilon \rightarrow \phi$  in  $L^1$  allow concluding.  $\square$

The critical reason for the introduction of the algebras  $M(w, f)$  and  $\mathcal{T}$  is that they allow easily recovering all  $\tau(T, w, f)$ , at least for bounded  $w$ . Namely, for any tree  $T \in \mathbb{T}$ , there exists a transform  $F \in \mathcal{T}$  so that  $\tau(T, w, f)$  can be described in terms of the transform  $F(w, f)$  of the pair  $(w, f)$ .

**Lemma 4.4.** *For any tree  $T \in \mathbb{T}$  there exists a transform  $F \in \mathcal{T}$  such that*

$$\tau(T, w, f)(x_1, \dots, x_{|T|}) = \int_0^1 F(w, f)(\zeta, x_1, \dots, x_{|T|}) d\zeta,$$

for any  $w \in L^\infty_{\xi, \zeta}$ , and any  $f \in L^\infty(L^1_x \cap L^\infty_x)$ .

*Proof.* Recall that any tree  $T_{n+1} \in \mathbb{T}_{n+1}$  can be written uniquely as  $T_{n+1} = T_n + i$  for a tree  $T_n \in \mathbb{T}_n$  and a vertex  $i \in \{1, \dots, n\}$ . We define the modified  $\tau$  operator as follows

$$\hat{\tau}(T_{n+1}, w, f)(\xi, x_1, \dots, x_{n+1}) := f(\xi, x_{n+1}) \int_{[0, 1]^n} w(\xi_i, \xi) \prod_{(k,l) \in T_n} w(\xi_k, \xi_l) \prod_{m=1}^n f(\xi_m, x_m) d\xi_1 \dots d\xi_n.$$

Note that since  $w \in L^\infty_{\xi, \zeta}$  and  $f \in L^\infty_x L^\infty_x$  then all the computations above make sense and this will also apply in the manipulations below. We also note that we readily obtain the following representation

$$\tau(T, w, f) = \int_0^1 \hat{\tau}(T, w, f)(\xi, \cdot) d\xi,$$

for every tree  $T \in \mathbb{T}$ . Then, we prove the more precise result that there must exist  $F \in \mathcal{T}$  such that  $\hat{\tau}(T, w, f) = F(w, f)$ . We proceed by induction on the size of the tree  $T$ .

Let  $T_2 \in \mathbb{T}_2$  be the only tree with two vertices. Then, we have

$$\hat{\tau}(T_2, w, f)(\xi, x_1, x_2) = f(\xi, x_2) \int_0^1 w(\zeta, \xi) f(x_1, \zeta) d\zeta.$$

In other words,  $\hat{\tau}(T_2, w, f) = F_2(w, f)$  with  $F_2 := F_0^* \otimes F_0$ . Of course,  $F_0 \in \mathcal{T}$  by item (i) in Definition 4.2. Hence  $F_0^* \in \mathcal{T}$  by item (iii), and finally  $F_2$  so does by item (ii).

Assume by induction that the above holds for any tree in  $\mathbb{T}_n$ . We need to show that it is also verified for any tree in  $\mathbb{T}_{n+1}$ . Specifically, take any  $T_{n+1} = T_n + i$  as above and notice again that

$$\hat{\tau}(T_{n+1}, w, f)(\xi, x_1, \dots, x_{n+1}) = f(\xi, x_{n+1}) \int_0^1 w(\zeta, \xi) \tilde{\tau}(T_n, w, f)(\zeta, x_1, \dots, x_n) d\zeta.$$

By the induction hypothesis we have that there exists  $F_n \in \mathcal{T}$  such that  $\hat{\tau}(T_n, w, f) = F_n(w, f)$ . We finally conclude that  $\hat{\tau}(T_{n+1}, w, f) = F_{n+1}(w, f)$ , where  $F_{n+1} := F_n^* \otimes F_0 \in \mathcal{T}$  by the rules (i), (ii) and (iii) defining the countable algebra  $\mathcal{T}$ .  $\square$

The combination of Lemmas 4.3 and 4.4 naturally lead to the following extended definition of  $\tau(T, w, f)$

**Definition 4.5** (Extension of the  $\tau$  operator). *Consider a transform  $F \in \mathcal{T}$ , for any  $T \in \mathbb{T}$ , as provided by Lemma 4.4. We then define*

$$\tau(T, w, f)(x_1, \dots, x_{|T|}) = \int_0^1 F(w, f)(\zeta, x_1, \dots, x_{|T|}) d\zeta,$$

for any  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ .

In Lemma 4.4 we built a special transform  $F \in \mathcal{T}$  associated to each tree  $T \in \mathbb{T}$ . It relies on a recursive construction involving the operations (i), (ii) and (iii) in Definition 4.1, starting from the leaves of  $T$  and descending towards the root. As there could a priori be several transforms  $F$  that fit a given tree, the last point to check is that Definition 4.5 is independent of the particular choice of  $F$ .

**Lemma 4.6.** *Consider any  $F_1, F_2 \in \mathcal{T}$  such that*

$$\int_0^1 F_1(w, f)(\zeta, \cdot) d\zeta = \int_0^1 F_2(w, f)(\zeta, \cdot) d\zeta, \quad (50)$$

for all  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and all  $w \in L_{\xi, \zeta}^\infty$ . Then, the equality (50) also holds for all  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and all  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ .

*Proof.* Define  $w_N = K_N \star_{\xi, \zeta} w$  for some smooth convolution kernel  $K_N$  with  $K_N \rightarrow \delta_0$ , and for any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ .

First of all  $w_N \in L_{\xi, \zeta}^\infty$  since  $w \in L_\xi^\infty \mathcal{M}_\zeta$ . We also have that  $w_N$  converges to  $w$  in  $L_\xi^1 H_\zeta^{-1} \cap L_\zeta^1 H_\xi^{-1}$ .

Therefore, we first have that for  $w_N$

$$\int_0^1 F_1(w_N, f)(\zeta, \cdot) d\zeta = \int_0^1 F_2(w_N, f)(\zeta, \cdot) d\zeta.$$

But, furthermore, by applying Lemma 4.3, we have that  $F_1(w_N, f)$  and  $F_2(w_N, f)$  converge strongly in  $L^1$  to  $F_1(w, f)$  and  $F_2(w, f)$ , respectively. Hence, we obtain the desired equality.  $\square$

Being able to correctly define  $\tau(T, w, f)$  is of course only the first and simplest step in the proof of Theorem 4.1. Passing to the limit in  $\tau(T, w_N, f_N)$  is considerably more intricate and in particular we cannot apply Lemma 4.3 as of course we cannot have the required compactness on  $f_N$  and  $w_N$  from the assumptions of Theorem 4.1.

Instead we have to derive compactness through the clever use of measure-preserving transforms which are the object of the next subsections.

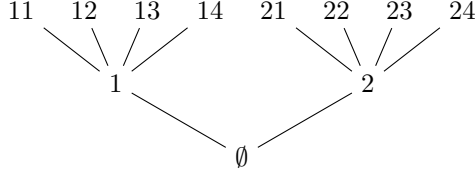
**4.3. The key compactness lemma.** A key tool to prove Theorem 4.1 is the following result that encompasses the inherent regularity on graphons.

**Lemma 4.7.** *Consider any sequence  $g_n$  in  $L^\infty([0, 1])$  with  $0 \leq g_n(x) \leq 2^{-n+1}$ . Then, there exists  $\Phi : [0, 1] \rightarrow [0, 1]$ , a.e. injective, measure-preserving, such that the following uniform regularity estimate is verified*

$$\sup_{n \in \mathbb{N}} \int_0^1 |(g_n \circ \Phi)(\xi) - (g_n \circ \Phi)(\xi + h)| d\xi \leq 2^{-C} \sqrt{\log \frac{1}{|h|}},$$

for any  $0 < |h| < 1$  and some universal constant  $C$ .

**Remark 4.8.** *By a.e. injective, we mean that there exists a full measure subset  $F' \subset [0, 1]$ ,  $|[0, 1] \setminus F'| = 0$ , such that  $\Phi : F' \rightarrow [0, 1]$  injective.*


 FIGURE 3. Words in  $W_0$ ,  $W_1$  and  $W_2$ 

We note that independently on the measure preserving map  $\Phi$ , by hypothesis we get that  $g_n \circ \Phi \rightarrow 0$  in  $L^1$  when  $n \rightarrow \infty$ . In particular,  $g_n \circ \Phi$  is compact in  $L^1$ . Therefore, by the Fréchet-Kolmogorov theorem we infer that the sequence  $g_n \circ \Phi$  must be uniformly equicontinuous so that the left hand side in the above inequality always converges to zero when  $h \rightarrow 0$ . The novelty of the previous result is that we can choose a special measure preserving map  $\Phi$  so that such an equicontinuity condition is made explicit in terms of a modulus of continuity. As usual, we extend by zero for evaluations outside  $[0, 1]$ , namely  $(g_n \circ \Phi)(\xi + h) = 0$ , if  $\xi + h \notin [0, 1]$ .

*Proof of Lemma 4.7.* Our proof is performed in several steps and relies on a hierarchical construction associated to a suitable hierarchical decomposition of the interval  $[0, 1]$ . Throughout the proof we shall assume that  $g_n > 0$  and that the  $g_n$  do not charge any point, *i.e.*,  $|\{x \in [0, 1] : g_n(x) = t\}| = 0$  for every  $t \in \mathbb{R}$  which will make decomposing  $[0, 1]$  according to the level sets of the  $g_n$  easier. For this, we observe that if  $g_n$  charges some points (at most in a countable way), then there exists  $g_{n,\varepsilon}$ , for every  $\varepsilon > 0$ , that does not charge any point and with  $\|g_n - g_{n,\varepsilon}\|_{L^\infty}$ . We may hence obtain the claimed regularity on each  $g_{n,\varepsilon}$  independently of  $\varepsilon$  which implies the desired result on  $g_n$ .

◇ *Step 1: The hierarchical decomposition.*

We built a sequence of covering of  $[0, 1]$  determined through a hierarchical decomposition. For simplicity of the construction, we use indices ranging over the following special sets of words with  $k$  letters

$$W_k := \{i_1 i_2 \dots i_k : i_m \in \{1, \dots, 2^m\}, m = 1, \dots, k\},$$

for any  $k \in \mathbb{Z}_+$  (see Figure 3). Note that  $W_0 = \{\emptyset\}$  and for  $k \geq 1$  we have that  $W_k$  contains all possible words with  $k$  letters, where the letter at position  $m \in \{1, \dots, k\}$  is allowed to take values in  $\{1, \dots, 2^m\}$ . For any  $i \in W_k$  we shall define  $j_k(i) := \{i1, \dots, i2^{k+1}\}$  the subset of  $W_{k+1}$  obtained by juxtaposition of a letter at the end of the word  $i$ . If we denote  $n_k := \#W_k$ , then the following induction follows

$$W_{k+1} = \bigcup_{i \in W_k} j_k(i), \quad n_{k+1} = 2^{k+1} n_k. \quad (51)$$

Hence, we actually obtain that  $n_k = \prod_{m=0}^k 2^m = 2^{\sum_{m=0}^k m} = 2^{\frac{k(k+1)}{2}}$ .

For every level  $k \in \mathbb{Z}_+$  we shall consider a covering of  $[0, 1]$  into  $n_k$  sub-intervals of identical size as follows. At level 0 we set  $I_0^\emptyset = [0, 1]$  and given a covering  $[0, 1] = \bigcup_{i \in W_k} I_k^i$  at level  $k$ , we define the covering at level  $k+1$  by decomposing each  $I_k^i$  with  $i \in W_k$  into  $2^{k+1}$  identical sub-intervals and labeling them in lexicographical order with indices in  $j_k(i)$ . For instance, at level 1 we obtain 2 sub-intervals  $I_1^1 = [0, 1/2]$ ,  $I_1^2 = [1/2, 1]$ , and at level 2 we obtain 8 sub-intervals: 4 associated with  $I_1^1$

$$I_2^{11} = [0, 1/8], \quad I_2^{12} = [1/8, 1/4], \quad I_2^{13} = [1/4, 3/8], \quad I_2^{14} = [3/8, 1/2],$$

and other 4 associated with  $I_1^2$

$$I_2^{21} = [1/2, 5/8], \quad I_2^{22} = [5/8, 3/4], \quad I_2^{23} = [3/4, 7/8], \quad I_2^{24} = [7/8, 1].$$

We proceed in an analogous way at higher levels. Note that  $I_{k+1}^j \subset I_k^i$  for every  $j \in j_k(i)$  and  $|I_k^i| = \frac{1}{n_k}$  since there are exactly  $n_k$  intervals  $I_k^i$  with  $i \in W_k$ .

Now, we define new coverings  $[0, 1] = \bigcup_{i \in W_k} O_k^i$  with measurable subsets  $O_k^i \subset [0, 1]$  such that  $O_{k+1}^j \subset O_k^i$  for every  $j \in j_k(i)$  and  $|O_k^i| = |I_k^i| = \frac{1}{n_k}$ . Specifically, they will take the form

$$O_k^i := \{x \in [0, 1] : s_{k,m}^i < g_m(x) \leq t_{k,m}^i, m = 1, \dots, k\}, \quad (52)$$

for some values  $0 \leq s_{k,m}^i < t_{k,m}^i \leq 2^{-m+1}$  with  $m = 1, \dots, k$  to be determined below by recursion on  $k$ . Again,  $O_0^\emptyset = [0, 1]$ . We define  $O_1^1$  and  $O_1^2$  at level 1 as follows. Since  $t \mapsto |\{x \in [0, 1] : g_1(x) \geq t\}|$  is continuous (because  $|\{x \in [0, 1] : g_1(x) = t\}| = 0$ ) then there must exist some  $\tilde{t} \in [0, 1]$  such that

$$|\{x \in [0, 1] : g_1(x) > \tilde{t}\}| = |\{x \in [0, 1] : g_1(x) \leq \tilde{t}\}| = \frac{1}{2}.$$



Therefore, we can set the values  $s_{1,1}^1 = 0$ ,  $t_{1,1}^1 = \tilde{t}$  for  $O_1^1$  and  $s_{1,1}^2 = \tilde{t}$ ,  $t_{1,1}^2 = 1$  for  $O_1^2$  so that

$$\begin{aligned} O_1^1 &= \{x \in [0, 1] : s_{1,1}^1 < g_1(x) \leq t_{1,1}^1\} = \{x \in [0, 1] : 0 < g_1(x) \leq \tilde{t}\}, \\ O_1^2 &= \{x \in [0, 1] : s_{1,1}^2 < g_1(x) \leq t_{1,1}^2\} = \{x \in [0, 1] : \tilde{t} < g_1(x) \leq 1\}. \end{aligned}$$

So defined note that  $[0, 1] = O_1^1 \cup O_1^2$  thanks to the hypothesis  $0 < g_1 \leq 1$ , and  $|O_1^1| = |O_1^2| = \frac{1}{2}$ . Let us now assume that at level  $k$  the values  $0 \leq s_{k,m}^i < t_{k,m}^i \leq 2^{-m+1}$  have been set so that the  $O_k^i$  given in (52) determine a covering  $[0, 1] = \bigcup_{i \in W_k} O_k^i$  and  $|O_k^i| = \frac{1}{n_k}$ . Let us now proceed in the recursion by defining at level  $k+1$  the values  $0 \leq s_{k+1,m}^j < t_{k+1,m}^j \leq 2^{-m+1}$  which will determine the  $O_{k+1}^j$ .

For any fixed  $i \in W_k$  we will built the  $O_{k+1}^j$  with  $j \in j_k(i)$  by appropriately decomposing  $O_k^i$  into  $2^{k+1}$  measurable pieces with identical measure. In this construction we will use words  $\tilde{W}_m$  containing  $m$  letters with only allowed values  $\{1, 2\}$ , *i.e.*,

$$\tilde{W}_m := \{i_1 i_2 \cdots i_m : i_l \in \{1, 2\}, l = 1, \dots, m\},$$

for any  $m = 0, \dots, k+1$ . By a new recursion on  $m$ , we shall proceed by defining a family of auxiliary subsets  $\tilde{O}_m^\alpha$  for  $\alpha \in \tilde{W}_m$  and  $m = 0, 1, \dots, k+1$  verifying  $\tilde{O}_m^\alpha = \tilde{O}_{m+1}^{\alpha 1} \cup \tilde{O}_{m+1}^{\alpha 2}$  and  $|\tilde{O}_{m+1}^{\alpha 1}| = |\tilde{O}_{m+1}^{\alpha 2}| = \frac{1}{2}|\tilde{O}_m^\alpha|$  starting at  $\tilde{O}_0^\emptyset = O_k^i$ . For  $m = 1$  we note that  $t \mapsto |\{x \in O_k^i : t < g_1(x) \leq t_{k,1}^i\}|$  is continuous. Then, there must exist some  $\tilde{t}_0^\emptyset \in (s_{k,1}^i, t_{k,1}^i)$  such that

$$|\{x \in O_k^i : s_{k,1}^i < g_1(x) \leq \tilde{t}_0^\emptyset\}| = |\{x \in O_k^i : \tilde{t}_0^\emptyset < g_1(x) \leq t_{k,1}^i\}| = \frac{1}{2}|O_k^i|.$$

Then, we set

$$\begin{aligned} \tilde{O}_1^1 &:= \{x \in O_k^i : s_{k,1}^i < g_1(x) \leq \tilde{t}_0^\emptyset\}, \\ \tilde{O}_1^2 &:= \{x \in O_k^i : \tilde{t}_0^\emptyset < g_1(x) \leq t_{k,1}^i\}. \end{aligned}$$

Note that in  $\tilde{O}_1^1$  and  $\tilde{O}_1^2$  we have simply split the interval  $[s_{k,1}^i, t_{k,1}^i]$  for the  $g_1$  into  $[s_{k,1}^i, \tilde{t}_0^\emptyset]$  and  $[\tilde{t}_0^\emptyset, t_{k,1}^i]$  respectively to divide  $O_k^i$  into two pieces with exactly half the mass. Assume that  $\tilde{O}_m^\alpha$  have been defined for  $\alpha \in \tilde{W}_m$  and  $m < k$  and let us define  $\tilde{O}_{m+1}^{\alpha 1}$  and  $\tilde{O}_{m+1}^{\alpha 2}$ . The argument is similar: since  $t \mapsto |\{x \in \tilde{O}_m^\alpha : t < g_{m+1}(x) \leq t_{k,m+1}^i\}|$  is continuous, there exists  $\tilde{t}_m^\alpha \in (s_{k,m+1}^i, t_{k,m+1}^i)$  with

$$|\{x \in \tilde{O}_m^\alpha : s_{k,m+1}^i < g_{m+1}(x) \leq \tilde{t}_m^\alpha\}| = |\{x \in \tilde{O}_m^\alpha : \tilde{t}_m^\alpha < g_{m+1}(x) \leq t_{k,m+1}^i\}| = \frac{1}{2}|\tilde{O}_m^\alpha|.$$

Then, we define

$$\begin{aligned} \tilde{O}_{m+1}^{\alpha 1} &= \{x \in \tilde{O}_m^\alpha : s_{k,m+1}^i < g_{m+1}(x) \leq \tilde{t}_m^\alpha\}, \\ \tilde{O}_{m+1}^{\alpha 2} &= \{x \in \tilde{O}_m^\alpha : \tilde{t}_m^\alpha < g_{m+1}(x) \leq t_{k,m+1}^i\}. \end{aligned}$$

Again note that in  $\tilde{O}_{m+1}^{\alpha 1}$  and  $\tilde{O}_{m+1}^{\alpha 2}$  we have simply split the interval  $[s_{k,m}^i, t_{k,m}^i]$  for the  $g_m$  into two pieces  $[s_{k,m}^i, \tilde{t}_m^\alpha]$  and  $[\tilde{t}_m^\alpha, t_{k,m}^i]$  respectively to divide  $\tilde{O}_m^\alpha$  into pieces with exactly half the mass. Once the auxiliary family has been set up to  $m = k$ , note that we can repeat the above argument once more for each  $\alpha \in \tilde{W}_k$  and find  $\tilde{t}_k^\alpha \in (0, 2^{-k})$  (by the bound  $0 < g_{k+1} \leq 2^{-k}$ ) such that the following subsets divide  $\tilde{O}_k^\alpha$  into two halves with exact mass

$$\begin{aligned} \tilde{O}_{k+1}^{\alpha 1} &:= \{x \in \tilde{O}_k^\alpha : 0 < g_{k+1}(x) \leq \tilde{t}_k^\alpha\}, \\ \tilde{O}_{k+1}^{\alpha 2} &:= \{x \in \tilde{O}_k^\alpha : \tilde{t}_k^\alpha < g_{k+1}(x) \leq 2^{-k}\}. \end{aligned}$$

By construction the last level  $\{\tilde{O}_{k+1}^\alpha : \alpha \in \tilde{W}_{k+1}\}$  contain exactly  $2^{k+1}$  pieces with same mass so that  $O_k^i = \bigcup_{\alpha \in \tilde{W}_{k+1}} \tilde{O}_{k+1}^\alpha$ . We then define  $O_{k+1}^j$  with  $j \in j_k(i)$  by labeling them in lexicographical order and we set the resulting values  $0 \leq s_{k+1,m}^j < t_{k+1,m}^j \leq 2^{-m+1}$  accordingly.

From this construction, we easily infer that fixing any  $m \leq k$  and any  $i \in W_k$ , then at least half of the new intervals  $[s_{k+1,m}^j, t_{k+1,m}^j]$  with  $j \in j_k(i)$  must have at most half the length of  $[s_{k,m}^i, t_{k,m}^i]$ , *i.e.*

$$\#\left\{j \in j_k(i) : \left(t_{k+1,m}^j - s_{k+1,m}^j\right) > \frac{1}{2} \left(t_{k,m}^i - s_{k,m}^i\right)\right\} < 2^k. \quad (53)$$

We can even be more precise. Note that going from  $[s_{k,m}^i, t_{k,m}^i]$  to the next level  $k+1$ , we introduced  $2^{k+1}$  new intervals  $[s_{k+1,m}^j, t_{k+1,m}^j]$ . They correspond to either of the two pieces  $[s_{k,m}^i, \tilde{t}_{m-1}^\alpha]$  or  $[\tilde{t}_{m-1}^\alpha, t_{k,m}^i]$

for  $\alpha \in \widetilde{W}_{m-1}$  in which we split the initial interval. By construction note that each of the pieces is repeated in exactly  $2^{k-m+1}$  of the  $O_{k+1}^j$  with  $j \in j_k(i)$ . Therefore, we obtain

$$\begin{aligned} & \frac{1}{2^{k+1}} \sum_{j \in j_k(i)} (t_{k+1,m}^j - s_{k+1,m}^j) \\ &= \frac{1}{2^{k+1}} 2^{k-m+1} \sum_{\alpha \in \widetilde{W}_{m-1}} ((t_{k,m}^i - \tilde{t}_{m-1}^\alpha) + (\tilde{t}_{m-1}^\alpha - s_{k,m}^i)) \\ &= \frac{1}{2^m} \sum_{\alpha \in \widetilde{W}_{m-1}} (t_{k,m}^i - s_{k,m}^i) = \frac{1}{2} (t_{k,m}^i - s_{k,m}^i). \end{aligned}$$

Summing over  $i \in W_k$ , dividing by  $n_k$  and recalling (51) yields

$$\frac{1}{n_{k+1}} \sum_{j \in W_{k+1}} (t_{k+1,m}^j - s_{k+1,m}^j) = \frac{1}{2n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i).$$

for any  $k \geq m$ . By induction we then infer that

$$\frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) \leq \frac{1}{2^{k-m}} \frac{1}{n_m} \sum_{i \in W_m} (t_{m,m}^i - s_{m,m}^i),$$

for any  $k \geq m$ . Notice that by construction and our assumption on the  $g_m$  we obtain that the intervals  $(s_{m,m}^i, t_{m,m}^i]$  they all reduce to  $(0, \frac{1}{2^{m-1}}]$ . Therefore, we obtain

$$\frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) \leq \frac{1}{2^k}, \quad (54)$$

for any  $k \in \mathbb{N}$  and every  $m \leq k$ . Note that the fact that  $0 < g_m \leq \frac{1}{2^m}$  has crucially used in the above cancellations, thus leading to a uniform in  $m$  bound.

◇ *Step 2: The measure preserving map  $\Psi$ .*

We begin by defining  $\Psi : [0, 1] \rightarrow [0, 1]$ , which will correspond to the inverse of  $\Phi : [0, 1] \rightarrow [0, 1]$ . Set any point  $x \in [0, 1]$  and use the previous covering by the disjoint  $\{O_k^i\}_{i \in W_k}$  to find a unique nested sequence  $i_{k+1}(x) \in j_k(i_k(x))$  such that  $x \in \bigcap_{k \in \mathbb{N}} O_k^{i_k(x)}$ . Since  $\{I_k^{i_k(x)}\}_{k \in \mathbb{N}}$  is a nested sequence of compact sets, by Cantor's intersection theorem it is evident that their intersection is a non-empty compact interval again. In fact, since  $|I_k^{i_k(x)}| \rightarrow 0$  as  $k \rightarrow \infty$  then  $\bigcap_{k \in \mathbb{N}} I_k^{i_k(x)}$  must consist in a singleton  $\{y\}$ , which we use to define  $\Psi(x) := y$ .

Now we prove that  $\Psi$  is a measure-preserving map. Specifically, we shall prove that for every Borel  $B \subset [0, 1]$  we have that  $\Psi^{-1}(B)$  is measurable and  $|\Psi^{-1}(B)| = |B|$ . First, we argue for  $B = I_k^i$  for any fixed  $k \in \mathbb{N}$  and  $i \in W_k$ . Note that the inclusion  $\Psi^{-1}(I_k^i) \supset O_k^i$  is clear by definition, but the converse does not necessarily hold. Indeed, we have that  $\Psi^{-1}(I_k^i) = O_k^i \cup N_k^i$ , where

$$N_k^i = \left( \bigcap_{m \in \mathbb{N}} O_m^{l_m} \right) \cup \left( \bigcap_{m \in \mathbb{N}} O_m^{r_m} \right).$$

Here,  $l_{m+1} \in j_m(l_m)$  and  $r_{m+1} \in j_m(r_m)$  describe the only two nested sequences of intervals  $\{I_m^{l_m}\}_{m \in \mathbb{N}}$  and  $\{I_m^{r_m}\}_{m \in \mathbb{N}}$  which stay always adjacent to (and respectively at the left or the right of)  $I_k^i$  for  $m \geq k$ . Obviously, if  $I_k^i$  is the first or last interval at level  $k$  in lexicographical order, then there is only one such sequence. In particular,  $\Psi^{-1}(I_k^i)$  is measurable because so are  $O_k^i$  and  $N_k^i$  (as countable union and intersection of measurable sets). In fact,  $N_k^i \subset [0, 1]$  is negligible (as the union of two negligible subsets), then  $|\Psi^{-1}(I_k^i)| = |O_k^i| = |I_k^i|$ . To extend the above property to general Borel sets  $B$ , we note that it is enough to verify the property for any semialgebra generating the Borel  $\sigma$ -algebra (cf. [82, Theorem 1.1]), e.g., the algebra of intervals with endpoints in the dyadic rationals  $J$ :

$$\mathcal{A} := \{[a, b) : a, b \in J, 0 \leq a \leq b \leq 1\} \cup \{[a, 1] : a \in J, 0 \leq a < 1\}.$$

Set any such  $B \in \mathcal{A}$ , for instance  $B = [a, b)$  for  $a, b \in J$  and  $0 \leq a < b \leq 1$  (the other case follows similarly). Then, there exists some (countable) subset  $\mathcal{I}$  of indices  $(k, l)$  so that  $[a, b) = \bigcup_{(k,l) \in \mathcal{I}} I_k^i$  because the  $I_k^i$  are a basis of neighborhoods of  $[0, 1]$ . By appropriately removing eventual nested elements in  $\mathcal{I}$  to avoid redundant information, we may assume that all the involved  $I_k^i$  are not contained in each other so that they can only intersect at points  $J$  of their boundaries. Therefore, we obtain

$$\Psi^{-1}(B) = \bigcup_{(k,i) \in \mathcal{I}} \Psi^{-1}(I_k^i) = \bigcup_{(k,i) \in \mathcal{I}} (O_k^i \cup N_k^i).$$

so that  $\Psi^{-1}(B)$  is measurable. Since  $N_k^i$  are negligible and all the involved  $O_k^i$  are disjoint then

$$|\Psi^{-1}(B)| = \sum_{(k,i) \in \mathcal{I}} |O_k^i| = \sum_{(k,i) \in \mathcal{I}} |I_k^i| = |B|,$$

where in the last equality we have used again that the  $I_k^i$  only intersect at most at points  $J$  of their boundaries and this has a null contribution to the sum.

Let us now study the injectivity and surjectivity *a.e.* of  $\Psi$ . Consider any point  $y \in [0, 1] \setminus J$  and set unique nested sequence  $i'_{k+1}(y) \in j_k(i'_k(y))$  so that  $\{y\} = \bigcap_{k \in \mathbb{N}} I_k^{i'_k(y)}$ . We can then consider the associated negligible measurable set  $\bigcap_{k \in \mathbb{N}} O_k^{i'_k(y)}$ . Notice that the *a.e.* surjectivity of  $\Psi$  amounts to proving that the latter intersection is non-empty for *a.e.*  $y$ , whilst the *a.e.* injectivity of  $\Psi$  amounts to proving that the intersection contains at most one point for *a.e.*  $y$ . For the injectivity, let us assume that  $g_1$  is strictly decreasing and continuous (by adding it to the sequence  $\{g_n\}_{n \in \mathbb{N}}$  if needed). Then note that

$$\bigcap_{k \in \mathbb{N}} O_k^{i'_k(y)} \subset \bigcap_{k \in \mathbb{N}} g_1^{-1}([s_{k,1}^{i'_k(y)}, t_{k,1}^{i'_k(y)}]) = g_1^{-1}\left(\bigcap_{k \in \mathbb{N}} [s_{k,1}^{i'_k(y)}, t_{k,1}^{i'_k(y)}]\right),$$

for each  $y \in [0, 1] \setminus J$ . Since the right hand side consists in an intersection of a nested sequence of compact intervals, then it reduces to a compact interval again. The condition for it to actually contain a single point is that  $t_{k,1}^{i'_k(y)} - s_{k,1}^{i'_k(y)} \rightarrow 0$  when  $k \rightarrow \infty$ . Although it does not necessarily happen for each  $y \in [0, 1] \setminus J$ , we shall prove that it does happen for *a.e.*  $y$ . Our argument relies heavily on (53) or, its more quantitative version (54) above. Namely, for any  $m \in \mathbb{N}$  let us define the sets

$$J_{k,m} := \left\{ i \in W_k : t_{k,m}^i - s_{k,m}^i \geq \frac{2^k}{3^k} \right\}. \quad (55)$$

for any level  $k \geq m$ . As it will be observed below, the choice of  $\frac{2}{3}$  in  $J_{k,m}$  is rather arbitrary and could be replaced by any fixed rate  $r \in (\frac{1}{2}, 1)$ . Therefore,

$$\frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) \geq \frac{1}{n_k} \sum_{i \in J_{k,m}} (t_{k,m}^i - s_{k,m}^i) \geq \frac{2^k |J_{k,m}|}{3^k n_k}.$$

Hence, using (54) implies

$$|J_{k,m}| \leq \frac{3^k}{4^k} n_k, \quad (56)$$

for any  $k \geq m \geq 1$ . In particular, if one defines the poor sets

$$P_m := \{y \in [0, 1] \setminus J : t_{k,m}^{i'_k(y)} - s_{k,m}^{i'_k(y)} \not\rightarrow 0 \text{ when } k \rightarrow \infty\}, \quad (57)$$

for any  $m \in \mathbb{N}$ , then one has that

$$P_m \subset \bigcap_{k_0 \in \mathbb{N}} \bigcup_{k \geq k_0} \bigcup_{j \in J_{k,m}} I_k^j. \quad (58)$$

Since  $|I_k^j| = \frac{1}{n_k}$ , using (56) and (58) we find that  $|P_m| = 0$  since

$$|P_m| \leq \sum_{k \geq k_0} \frac{3^k}{4^k},$$

for any  $k_0 \in \mathbb{N}$ . Define the associated global poor set

$$P := \bigcup_{m \in \mathbb{N}} P_m. \quad (59)$$

which is again negligible, and consider pair of full measure sets

$$F'_0 := [0, 1] \setminus (J \cup P), \quad F := \Psi^{-1}(F'_0).$$

Then, it is clear that  $\Psi : F \rightarrow [0, 1]$  becomes an injective (measurable) measure-preserving map. Since we are dealing with the (completed) Lebesgue  $\sigma$ -algebra, which determines a standard probability space, then  $\Psi(F)$  is measurable (*cf.* [25, Theorem 3-2]). Unfortunately, we only have the inclusion  $\Psi(F) \subset F'_0$ . Hence, we define the (eventually) smaller full measure set

$$F' := \Psi(F).$$

Therefore,  $\Psi : F \rightarrow F'$  becomes indeed a bijective (measurable) measure-preserving map. Therefore,  $\Psi^{-1} : F' \rightarrow F$  is also a measure-preserving map. Extending  $\Psi^{-1}$  to all  $[0, 1]$  we obtain our measure-preserving map  $\Phi$ . In particular, note that by construction  $\Phi$  and  $\Psi$  are characterized by the fact that

$$\Phi(I_k^i \cap F') = O_k^i \cap F, \quad \Psi(O_k^i \cap F) = I_k^i \cap F', \quad (60)$$

for each  $k \in \mathbb{N}$  and  $i \in W_k$ .

◇ *Step 3:  $L^1$  modulus of continuity of  $\{g_n \circ \Phi\}_{n \in \mathbb{N}}$ .*

For simplicity of notation, we define  $h_n := g_n \circ \Phi$  for any  $n \in \mathbb{N}$ . First, we notice that for any  $y \in F' \subset F'_0$  the value  $h_m(y)$  is determined by the sequence  $t_{k,m}^{i'_k(y)}$  (or  $s_{k,m}^{i'_k(y)}$ ), where  $i'_{k+1}(y) \in j_k(i'_k(y))$  is defined as before as the unique nested sequence with  $\{y\} = \bigcap_{k \in \mathbb{N}} I_k^{i'_k(y)}$ . Specifically, we shall prove

$$h_m(y) = \lim_{k \rightarrow \infty} t_{k,m}^{i'_k(y)} = \lim_{k \rightarrow \infty} s_{k,m}^{i'_k(y)}, \quad (61)$$

for each  $y \in F' \subset F'_0$ . Indeed, by (60) and (52) note that

$$s_{k,m}^{i'_k(y)} < h_m(y) \leq t_{k,m}^{i'_k(y)},$$

for any  $k \geq m$  and any  $y \in F' \subset F'_0$ . By the definitions (57) and (59) of  $P_m$  and  $P$  the nested intervals  $[s_{k,m}^{i'_k(y)}, t_{k,m}^{i'_k(y)}]$  verify  $t_{k,m}^{i'_k(y)} - s_{k,m}^{i'_k(y)} \rightarrow 0$  when  $k \rightarrow \infty$  so that we conclude (61).

To finish the proof, we need to show that

$$\int_0^1 |h_m(y) - h_m(y + \tau)| dy \leq 2^{-C} \sqrt{\log \frac{1}{|\tau|}},$$

for any  $0 < |\tau| < 1$  and some universal constant  $C$ . Let us fix any  $0 < \tau < 1$  (a similar argument yields  $-1 < \tau < 0$ ) and any level  $k \in \mathbb{N}$  so that  $\tau < \frac{1}{n_k}$ . That is, the size  $\tau$  of the shift is smaller than the length of the dyadic intervals  $I_k^i$  at the level  $k$ . Then, we can split each interval  $I_k^i$  as follows

$$I_k^i \cap F' = G_{k,\tau}^i \cup B_{k,\tau}^i,$$

for every  $i \in W_k$ , where  $G_{k,\tau}^i$  consist of the points in  $I_k^i$  which stay in  $I_k^i$  up to a shift of size  $\tau$  and  $B_{k,\tau}^i$  are all the other points in  $I_k^i$ . Specifically,

$$\begin{aligned} G_{k,\tau}^i &:= \{y \in I_k^i \cap F' : y + \tau \in I_k^i \cap F'\}, \\ B_{k,\tau}^i &:= \{y \in I_k^i \cap F' : y + \tau \notin I_k^i \cap F'\}. \end{aligned}$$

Note that a point  $y \in B_{k,\tau}^i$  must stay within  $\tau$  of the right boundary of  $I_k^i$ . Therefore, we infer

$$|G_{k,\tau}^i| = \frac{1}{n_k} - \tau \leq \frac{1}{n_k}, \quad |B_{k,\tau}^i| = \tau. \quad (62)$$

Using such a decomposition, we can split the above integrals as follows

$$\begin{aligned} \int_0^1 |h_m(y) - h_m(y + \tau)| dy &= \sum_{i \in W_k} \int_{G_{k,\tau}^i} |h_m(y) - h_m(y + \tau)| dy \\ &\quad + \sum_{i \in W_k} \int_{B_{k,\tau}^i} |h_m(y) - h_m(y + \tau)| dy. \end{aligned} \quad (63)$$

For points  $y \in G_{k,\tau}^i$ , since  $y, y + \tau \in I_k^i \cap F'$  then  $\Phi(y), \Phi(y + \tau) \in O_k^i$  so that

$$|h_m(y) - h_m(y + \tau)| \leq t_{k,m}^i - s_{k,m}^i.$$

For all other points  $y \in B_{k,\tau}^i$  we only have that

$$|h_m(y) - h_m(y + \tau)| \leq |h_m(y)| + |h_m(y + \tau)| \leq \frac{2}{2^{m-1}} \leq 2.$$

Using the above bounds along with (62) in (63), which exploit the decay of  $g_m$ , we obtain

$$\int_0^1 |h_m(y) - h_m(y + \tau)| dy \leq \frac{1}{n_k} \sum_{i \in W_k} (t_{k,m}^i - s_{k,m}^i) + \sum_{i \in W_k} 2\tau \leq \frac{1}{2^k} + 2\tau n_k. \quad (64)$$

Here,  $k \in \mathbb{N}$  has only been set under the restriction  $\tau < \frac{1}{n_k}$  but we could optimize our choice of  $k$ . Specifically, let us set  $k = k_\tau$  more precisely so that  $\frac{1}{n_{k+1}^2} < \tau \leq \frac{1}{n_k^2}$ . Then, we obtain that

$$\frac{1}{2^k} + 2\tau n_k \leq \frac{3}{2^k} \leq 2^{-C} \sqrt{\log \frac{1}{\tau}},$$

for universal  $C$ , which by (64) yields our result. For the last inequality, note that by our choice of  $k$  in terms of  $\tau$  we get  $\frac{1}{\tau} \leq n_{k+1}^2 = 2^{(k+1)(k+2)} \leq 2^{6k^2}$  so that  $\sqrt{\log \frac{1}{\tau}} \leq \sqrt{6 \log 2} k$ .  $\square$

We note that the dyadic character of the hierarchical construction in the proof of Lemma 4.7 required a specific geometric uniform decay  $0 \leq g_n \leq 2^{-n+1}$  in order for the  $L^1$  modulus of continuity of  $g_n \circ \Phi$  to be independent of  $k$ . However, applying the above to the rescaled sequence

$$\tilde{g}_n := \frac{g_n + \|g_n\|_{L^\infty}}{2^n \|g_n\|_{L^\infty}},$$

allows considering a generic sequence  $g_n$  in  $L^\infty([0, 1])$ , and obtaining a  $n$ -dependent  $L^1$  modulus of continuity (eventually growing with  $n$ ) as follows.

**Corollary 4.9.** *Consider any sequence  $g_n$  in  $L^\infty([0, 1])$ . Then, there exists  $\Phi : [0, 1] \rightarrow [0, 1]$ , a.e. injective, measure-preserving, such that the following estimate is verified*

$$\int_0^1 |(g_n \circ \Phi)(\xi) - (g_n \circ \Phi)(\xi + h)| d\xi \leq 2^n \|g_n\|_{L^\infty} 2^{-C} \sqrt{\log \frac{1}{|h|}},$$

for any  $0 < |h| < 1$ , each  $n \in \mathbb{N}$  and some universal constant  $C$ .

As a further remark, let us highlight the relationship of this lemma with the classical results of regularity on graphons. If  $w$  is a graphon, then the Lemma 4.7 implies the classical regularity lemma by Lovász-Szegedy [57], and in turn the famous regularity lemma on graphs by Szemerédi.

**Lemma 4.10** (Regularity lemma on graphons [57]). *If  $w \in L^\infty([0, 1]^2)$  is symmetric with  $0 \leq w \leq 1$ , then there exists  $\Phi : [0, 1] \rightarrow [0, 1]$ , measure-preserving, such that in the cut-distance*

$$\delta_\square(w(\Phi(\cdot), \Phi(\cdot)), w(\Phi(\cdot + h), \Phi(\cdot))) \leq \frac{C}{\sqrt{\log \frac{1}{h}}},$$

where the cut-distance is given by

$$\delta_\square(w, \tilde{w}) = \sup_{\Phi, \tilde{\Phi}} \sup_{S, T} \left| \int_{S \times T} (w(\Phi(\xi), \Phi(\zeta)) - \tilde{w}(\tilde{\Phi}(\xi), \tilde{\Phi}(\zeta))) d\xi d\zeta \right|,$$

where the supremum is taken over any pair  $\Phi, \tilde{\Phi} : [0, 1] \rightarrow [0, 1]$  of measure-preserving maps, and any pair  $S, T \subset [0, 1]$  of measurable subsets.

We once again emphasize that in our case  $w$  is not necessarily symmetric nor a bounded function.

**4.4. Proof of Theorem 4.1.** We are now ready to prove the main theorem of this section. By hypothesis (i) and (ii) in Theorem 4.1 we have that there exist  $C_1, C_2 \in \mathbb{R}_+$  so that

$$\|w_N\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \leq C_1, \quad \|f_N\|_{L_\xi^\infty (W_x^{1,1} \cap W_x^{1,\infty})} \leq C_2,$$

for any  $N \in \mathbb{N}$ . For each fixed  $T \in \mathbb{T}$ , by Lemma 3.15, we have that  $\tau(T, w_N, f_N)$  is uniformly bounded in  $W^{1,1} \cap W^{1,\infty}$  and is hence locally compact in  $L_x^p$  for all  $p < \infty$ . By a standard diagonal extraction procedure, we may hence assume that for some subsequence of  $N$  (still denoted as  $N$  for simplicity)

$$\lim_{N \rightarrow \infty} \tau(T, w_N, f_N) \text{ exists in } L_{loc}^p, \quad \forall p < \infty, \quad \forall T \in \mathbb{T}.$$

Our goal is to identify the above limits for each  $T \in \mathbb{T}$ .

We also recall the above countable algebras  $M(w_N, f_N)$  in Definition 4.2 endowed with the family of transforms  $\mathcal{T}$  according to the rules (i), (ii) and (iii). From Lemma 4.4, we already know that this algebra represents the hierarchy accurately. The rest of the proof is handled in several steps.

$\diamond$  *Step 1: Stability by re-arrangements.*

In order to apply our compactness Lemma 4.7 note that we have to perform appropriate re-arrangements in the  $\xi$  variable by measure-preserving maps. The following result show that the representation through the algebra  $M(w_N, f_N)$  is stable under re-arrangements.

**Lemma 4.11.** Consider any  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ ,  $f \in L_\xi^\infty(L_x^1 \cap L_x^\infty)$  and any a.e. injective measure-preserving map  $\Phi : [0, 1] \rightarrow [0, 1]$ . Define the re-arranged objects

$$\tilde{w}(\xi, d\zeta) := \Phi_{\#}^{-1} w(\Phi(\xi), \cdot)(d\zeta), \quad \tilde{f}(x, \xi) := f(x, \Phi(\xi)). \quad (65)$$

Then we have that

$$F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) = F(w, f)(\Phi(\xi), x_1, \dots, x_m),$$

for any transform  $F \in \mathcal{T}$ , and for a.e.  $\xi \in [0, 1]$  and each  $x_1, \dots, x_m \in \mathbb{R}^d$ , where  $\Phi^{-1}$  is any a.e. defined left inverse of  $\Phi$ . Moreover,  $\tau(T, \tilde{w}, \tilde{f}) = \tau(T, w, f)$  for any tree  $T$ .

If  $w \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ , then the re-arranged object  $\tilde{w}$  in (65) clearly belongs to  $L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$  and we have the straightforward formula

$$\tilde{w}(\xi, \zeta) = w(\Phi(\xi), \Phi(\zeta)).$$

For general  $w \in L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$ , we have that  $\tilde{w}$  is defined as the pull-back of  $w$  through  $\Phi$ . Since  $\Phi$  is injective, the pull-back through  $\Phi$  agrees with the push-forward through  $\Phi^{-1}$ . We also emphasize that the definition is not sensitive to modifications of  $\Phi^{-1}$  over the atoms of  $w(\Phi(\xi), d\zeta)$  within a Lebesgue-negligible set, as we show in the following straightforward argument based on Lemma 3.7. Assume that  $\Phi_1^{-1}$  and  $\Phi_2^{-1}$  are two different choices of the a.e. left inverse of  $\Phi$  (i.e.,  $\Phi_1^{-1} \circ \Phi = Id = \Phi_2^{-1} \circ \Phi$  a.e.) and define

$$\begin{aligned} \tilde{w}_1(\xi, d\zeta) &:= \Phi_{1\#}^{-1} w(\Phi(\xi), \cdot)(d\zeta), & \tilde{w}_2(\xi, d\zeta) &:= \Phi_{2\#}^{-1} w(\Phi(\xi), \cdot)(d\zeta), \\ \phi_1(\zeta) &:= \phi(\Phi_1^{-1}(\zeta)), & \phi_2(\zeta) &:= \phi(\Phi_2^{-1}(\zeta)), \end{aligned}$$

for any  $\phi \in L^\infty([0, 1])$ . Then, by estimate (32)<sub>2</sub> we have

$$\begin{aligned} \left\| \int_0^1 \phi(\zeta) (\tilde{w}_1(\xi, d\zeta) - \tilde{w}_2(\xi, d\zeta)) \right\|_{L_\zeta^\infty} &= \left\| \int_0^1 (\phi(\Phi_1^{-1}(\zeta)) w(\Phi_1(\xi), d\zeta) - \phi(\Phi_2^{-1}(\zeta)) w(\Phi_2(\xi), d\zeta)) \right\|_{L_\zeta^\infty} \\ &\leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta} \|\phi_1 - \phi_2\|_{L^\infty} = 0, \end{aligned}$$

where we have used that  $\phi_1 = \phi_2$  a.e. by definition. Since  $\phi \in L^\infty([0, 1])$  is arbitrary, we can restrict to  $\phi \in C([0, 1])$  and by a straightforward separability argument we conclude that  $\tilde{w}_1 = \tilde{w}_2$ .

*Proof of Lemma 4.11.* The proof is done by induction on the number of operations performed by  $F$ . Of course, for the basic transformation  $F_0$  in item (i) of Definition 4.2, the property is obvious since

$$F_0(\tilde{w}, \tilde{f})(\xi, x) = \tilde{f}(\xi, x) = f(\Phi(\xi), x) = F_0(w, f)(\Phi(\xi), x),$$

by definition (65) of the re-arrangements. Now consider any  $F \in \mathcal{T}$  with at least one operation, either of the type (ii) or (iii) in Definition 4.2. On the one hand, let us first assume that  $F = F_1 \otimes F_2$  is obtained through operations of type (ii) by taking the tensor product of some  $F_1$  and  $F_2$ . Since both the transforms  $F_1$  and  $F_2$  must contain at least one less operation each, by the induction hypothesis we have

$$F_i(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_i}) = F_i(w, f)(\Phi(\xi), x_1, \dots, x_{m_i}),$$

for  $i = 1, 2$ . Then, we obviously have

$$\begin{aligned} F(\tilde{f}, \tilde{w})(\xi, x_1, \dots, x_{m_1+m_2}) &= F_1(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_1}) F_2(\tilde{w}, \tilde{f})(\xi, x_{m_1+1}, \dots, x_{m_1+m_2}) \\ &= F_1(w, f)(\Phi(\xi), x_1, \dots, x_{m_1}) F_2(w, f)(\Phi(\xi), x_{m_1+1}, \dots, x_{m_1+m_2}) \\ &= F(w, f)(\Phi(\xi), x_1, \dots, x_{m_1+m_2}). \end{aligned}$$

On the other hand, assume now that the identity holds for some  $F \in \mathcal{T}$  and let us consider  $F^*$  via an operation of (iii), i.e.,

$$F^*(w, f)(\xi, x_1, \dots, x_m) = \int_0^1 F(w, f)(\zeta, x_1, \dots, x_m) w(\xi, d\zeta).$$

Then, we have again that

$$\begin{aligned} F^*(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) &= \int_0^1 F(\tilde{w}, \tilde{f})(\zeta, x_1, \dots, x_m) \tilde{w}(\xi, d\zeta) \\ &= \int_0^1 F(w, f)(\Phi(\zeta), x_1, \dots, x_m) \tilde{w}(\xi, d\zeta) \\ &= \int_0^1 F(w, f)(\zeta, x_1, \dots, x_m) w(\Phi(\xi), d\zeta), \end{aligned}$$

where in the second line we have used that  $F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) = F(f, w)(\Phi(\xi), x_1, \dots, x_m)$  and in the third line we have used the definition of  $\tilde{w}$  as a push-forward measure. Hence, we find

$$F^*(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_m) = F^*(w, f)(\Phi(\xi), x_1, \dots, x_m).$$

Finally, by Lemma 4.4, for any  $T \in \mathbb{T}$  there exists  $F \in \mathcal{T}$  such that

$$\begin{aligned} \tau(T, w, f)(x_1, \dots, x_{|T|}) &= \int_0^1 F(w, f)(\xi, x_1, \dots, x_{|T|}) d\xi, \\ \tau(T, \tilde{w}, \tilde{f})(x_1, \dots, x_{|T|}) &= \int_0^1 F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{|T|}) d\xi. \end{aligned}$$

Since  $\Phi$  is measure-preserving, by the theorem of change of variables we obtain

$$\begin{aligned} \tau(T, \tilde{w}, \tilde{f}) &= \int_0^1 F(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{|T|}) d\xi = \int_0^1 F(w, f)(\Phi(\xi), x_1, \dots, x_{|T|}) d\xi \\ &= \int_0^1 F(w, f)(\xi, x_1, \dots, x_{|T|}) d\xi = \tau(T, w, f), \end{aligned}$$

which concludes the proof.  $\square$

$\diamond$  *Step 2: Obtaining compactness.*

Since  $\mathcal{T}$  is countable, we may label its elements with countable indices as follows  $\mathcal{T} = \{F_k : k \in \mathbb{N}\}$ , where we choose as the first element  $F_0$  the elementary transform in rule (i) of Definition 4.2. Assume that  $F_k$  involves  $m_k$  variables  $x_1, \dots, x_{m_k}$  and denote  $n_k \in \mathbb{N}$  the amount of operations of type (iii) involved in the transform  $F_k$ . Then, by Definition 4.2 of the algebra  $M(w_N, f_N)$  and arguing like in Lemma 3.15 by induction on the number of operations in  $F_k$  implies

$$\|F_k(w_N, f_N)\|_{L_\xi^\infty(W^{1,1} \cap W^{1,\infty})} \leq \|w_N\|_{L_\xi^\infty L_\xi^1}^{n_k} \|f_N\|_{L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})}^{m_k} \leq C_1^{m_k} C_2^{n_k}, \quad (66)$$

for any  $N \in \mathbb{N}$  and each  $k \in \mathbb{N}$ . We remark that by our hypothesis, the above estimate is independent on  $N$ . Our goal in this part is to use Lemma 4.7 (more specifically its Corollary 4.9) to obtain compactness with respect to  $N$  after suitable re-arrangements of the sequence  $F_k(w_N, f_N)$  for any  $k \in \mathbb{N}$ . Unfortunately,  $F_k(w_N, f_N)$  does not only depend on  $\xi$  but also on the extra variables  $x_1, \dots, x_{m_k}$ , so that it is not totally clear how compactness in the joint variables  $(\xi, x_1, \dots, x_{m_k})$  arises from Corollary 4.9. Note though that dependency on  $x_1, \dots, x_{m_k}$  is actually smooth. In the following result we derive rigorously the compactness supported by the aforementioned regularity and a density argument.

**Lemma 4.12.** *Under the assumptions in Theorem 4.1, there exist  $\Phi_N : [0, 1] \rightarrow [0, 1]$  measure-preserving maps for each  $N \in \mathbb{N}$  and  $\phi_k(\xi, x_1, \dots, x_{m_k}) \in L_\xi^\infty(W^{1,1} \cap W^{1,\infty})$  for each  $k \in \mathbb{N}$  so that we have*

$$F_k(w_N, f_N)(\Phi_N(\xi), x_1, \dots, x_{m_k}) \rightarrow \phi_k(\xi, x_1, \dots, x_{m_k}) \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^{d m_k}),$$

when  $N \rightarrow \infty$  (up to a subsequence on  $N$ ) for each  $k \in \mathbb{N}$  and any  $1 \leq p < \infty$ .

*Proof.* For every  $k \in \mathbb{N}$ , let  $\mathcal{D}_k = \{(x_1^l, \dots, x_{m_k}^l) : l \in \mathbb{N}\} \subset \mathbb{R}^{d m_k}$  be any countable dense subset (e.g.,  $\mathcal{D}_k = \mathbb{Q}^{d m_k}$ ), and define the functions  $g_{k,l}^N(\xi) := F_k(w_N, f_N)(\xi, x_1^l, \dots, x_{m_k}^l)$  for each  $\xi \in [0, 1]$  and any  $k, l, N \in \mathbb{N}$ . By Corollary 4.9 applied to the countable family  $\{g_{k,l}^N\}_{k,l \in \mathbb{N}} \subset L^\infty([0, 1])$ , we obtain that for each  $N \in \mathbb{N}$  there exists and a measure-preserving map  $\Phi_N : [0, 1] \rightarrow [0, 1]$  so that the re-arrangements  $\tilde{g}_{k,l}^N = g_{k,l}^N \circ \Phi_N$  fulfill the following estimate

$$\sup_{N \in \mathbb{N}} \int_0^1 |\tilde{g}_{k,l}^N(\xi + h) - \tilde{g}_{k,l}^N(\xi)| d\xi \leq C_{k,l} 2^{-C \sqrt{|\log \frac{1}{|h|}}},$$

for any  $0 < |h| < 1$ , an universal constant  $C \in \mathbb{R}_+$ , and some constants  $C_{k,l} \in \mathbb{R}_+$ . We remark that the constants  $C_{k,l}$  are indeed independent of  $N$  thanks to the above uniform bound (66). By the Fréchet-Kolmogorov theorem and using a diagonal extraction there exists some subsequence of  $N$ 's (which we still denote  $N$  for simplicity) and there exists  $\tilde{g}_{k,l} \in L^1([0, 1])$  so that  $\tilde{g}_{k,l}^N \rightarrow \tilde{g}_{k,l}$  in  $L^1([0, 1])$  as  $N \rightarrow \infty$  for any  $k, l \in \mathbb{N}$ . Our final step will be to lift the above convergence on  $\mathcal{D}_k$  to all  $\mathbb{R}^{d m_k}$ . To such an end, we define the functions  $\tilde{g}_k : [0, 1] \times \mathcal{D}_k \rightarrow \mathbb{R}$  by  $\tilde{g}_k(\xi, x_1^l, \dots, x_{m_k}^l) := \tilde{g}_{k,l}(\xi)$ . Since  $L^1$  convergence implies convergence *a.e.* of an appropriate subsequence, then the uniform Lipschitz bounds on (66) imply

$$\begin{aligned} |\tilde{g}_k(\xi, x_1^l, \dots, x_{m_k}^l)| &\leq C_1^{m_k} C_2^{n_k}, \\ |\tilde{g}_k(\xi, x_1^{l_1}, \dots, x_{m_k}^{l_1}) - \tilde{g}_k(\xi, x_1^{l_2}, \dots, x_{m_k}^{l_2})| &\leq C_1^{m_k} C_2^{n_k} |(x_1^{l_1}, \dots, x_{m_k}^{l_1}) - (x_1^{l_2}, \dots, x_{m_k}^{l_2})|, \end{aligned}$$

for any  $k, l, l_1, l_2 \in \mathbb{N}$  and *a.e.*  $\xi \in [0, 1]$ . Then,  $\tilde{g}_k : [0, 1] \times \mathcal{D}_k \rightarrow \mathbb{R}$  can be extended by continuity in a unique way into a function  $\phi_k \in L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1, \infty}$  with  $\|\phi_k\|_{L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1, \infty}} \leq C_1^{m_k} C_2^{m_k}$ . From the uniform Lipschitz bounds on the  $\tilde{g}_{k,l}^N$  and  $\phi_k$ , we obtain that

$$F_k(w_N, f_N)(\Phi_N(\xi), x_1, \dots, x_{m_k}) \rightarrow \phi_k(\xi, x_1, \dots, x_{m_k}),$$

as  $N \rightarrow \infty$ , for each  $k \in \mathbb{N}$ , *a.e.*  $\xi \in [0, 1]$  and any  $x_1, \dots, x_{m_k} \in \mathbb{R}^d$ .

Note we also know that  $F_k(w_N, f_N)(\Phi_N(\cdot), \cdot, \dots, \cdot)$  and  $\phi_k(\cdot, \cdot, \dots, \cdot)$  are bounded in  $L^\infty$  uniformly with respect to  $N$  thanks to the uniform bounds (66). Therefore, the above pointwise convergence can indeed be improved into  $L_{loc}^p$  convergence for any  $1 \leq p < \infty$ , on any compact set, through the dominated convergence theorem. Indeed, since  $F_k(w_N, f_N)(\Phi_N(\cdot), \cdot, \dots, \cdot)$  are also uniformly bounded in  $L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1,1} \subseteq L_\xi^\infty BV_{x_1, \dots, x_{m_k}}$  with respect to  $N$  by (66), then we can also take the subsequence of  $N$ 's so that it converges locally weakly-star in  $L_\xi^\infty BV_{x_1, \dots, x_{m_k}}$ , thus guaranteeing that  $\phi_k \in L_\xi^\infty W_{x_1, \dots, x_{m_k}}^{1,1}$ .  $\square$

Before entering into the last step in the proof of Theorem 4.1 (namely, the identification of the limits  $\phi_k$  in Lemma 4.12), let us first note that at this point we already have the necessary information to prove the following result.

**Theorem 4.13.** *Let  $(X_1, \dots, X_N)$  be solution to (1) with  $K \in W^{1,1} \cap W^{1,\infty}$  and consider the associated laws  $f_i(t, \cdot) := \text{Law}(X_i(t))$ . Assume that  $X_i^0$  are independent with  $\mathbb{E}[|X_i^0|^2] < \infty$  and*

$$\sup_{1 \leq i \leq N} \sqrt{\mathbb{E}[|X_i^0|^2]} \leq M, \quad \sup_{1 \leq i \leq N} \|f_i^0\|_{W^{1,1} \cap W^{1,\infty}} \leq L, \quad \sup_{1 \leq i \leq N} \sum_{j=1}^N |w_{ij}| \leq C, \quad \sup_{1 \leq j \leq N} \sum_{i=1}^N |w_{ij}| \leq C,$$

for every  $N \in \mathbb{N}$  and appropriate  $M, L, C \in \mathbb{R}_+$ . Then, there exists  $\tilde{w}_N \in L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$  and  $\tilde{f}_N \in L_\xi^\infty (W_x^{1,1} \cap W_x^{1,\infty})$  for every  $N \in \mathbb{N}$  satisfying the graphon-type Vlasov PDE (35), i.e.,

$$\partial_t \tilde{f}_N(t, x, \xi) + \text{div}_x \left( \tilde{f}_N(t, x, \xi) \int_0^1 \int_{\mathbb{R}^d} K(x-y) \tilde{w}_N(\xi, \zeta) \tilde{f}_N(t, y, \zeta) dy d\zeta \right) = 0, \quad (67)$$

in the distributional sense. Moreover, for any finite time interval  $[0, t_*]$  and each compact set  $\Omega \subset \mathbb{R}^d$  we have that  $\tilde{w}_N$  and  $\tilde{f}_N$  verify

$$\begin{aligned} \int_0^{t_*} \int_\Omega \int_0^1 \left| \int_0^1 (\tilde{w}_N(\xi, \zeta) - \tilde{w}_N(\xi + h, \zeta)) \tilde{f}_N(t, x, \zeta) d\zeta \right| d\xi dx dt &\leq \varepsilon(|h|), \\ \sup_{t \in [0, t_*]} W_1 \left( \mathbb{E} \mu_N(t, \cdot), \int_0^1 \tilde{f}_N(t, \cdot, \xi) d\xi \right) &\leq \tilde{C} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}, \end{aligned} \quad (68)$$

for any  $h \in \mathbb{R}$ , each  $N \in \mathbb{N}$ , some constant  $\tilde{C} \in \mathbb{R}_+$  and some continuous function  $\varepsilon(|h|)$  with  $\varepsilon(0) = 0$ . Both  $\tilde{C}$  and  $\varepsilon(h)$  are independent on  $N$ :  $\tilde{C}$  only depends on  $C, \|K\|_{W^{1,\infty}}$  and  $t_*$ , and  $\varepsilon(|h|)$  only depends on  $C, L, t_*$  and  $\Omega$ . Here,  $\mu_N$  is the empirical measure  $\mu_N(t, x) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}(x)$ .

**Remark 4.14.** *By interpolation with Lemma 4.7, the function  $\varepsilon(h)$  can be calculated explicitly by considering an explicit choice of a dense sequence of points  $x_1^1, \dots, x_{m_k}^1$  in the argument above (by constructing a dyadic grid for example), as we briefly explain in the proof.*

*Proof of Theorem 4.13.* First, by Proposition 2.2 we know that the laws  $f_i$  and the solution  $\tilde{f}_i$  to the coupled system (19) with the same initial data  $\tilde{f}_i^0 = f_i^0$  satisfy the estimate (20), i.e.,

$$W_1(f_i(t, \cdot), \tilde{f}_i(t, \cdot)) \leq \tilde{C} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2},$$

for each  $t \in [0, t_*]$ , any  $i = 1, \dots, N$ , and the constant  $\tilde{C} := \sqrt{2/C} (e^{2C t_* \|K\|_{W^{1,\infty}}} - 1)$ . Let us set  $f_N$  and  $w_N$  according to the graphon-type representation in Definition 3.4 associated with  $\tilde{f}_i$  and  $w_{ij}$ . Therefore, noting that  $\mathbb{E} \mu_N(t, \cdot) = \frac{1}{N} \sum_{i=1}^N f_i(t, \cdot)$  and  $\int_0^1 f_N(t, x, \xi) d\xi = \frac{1}{N} \sum_{i=1}^N \tilde{f}_i(t, x, \xi)$  we infer

$$\sup_{t \in [0, t_*]} W_1 \left( \mathbb{E} \mu_N(t, \cdot), \int_0^1 f_N(t, \cdot, \xi) d\xi \right) \leq \tilde{C} \sup_{1 \leq i, j \leq N} |w_{ij}|^{1/2}. \quad (69)$$

Second, as already studied in Section 3, we recall that  $f_N$  solves the same transport equations as in (67) in the sense of distribution. Arguing like in the proof of Proposition 3.11 we can propagate the initial  $L_\xi^\infty (W_x^{1,1} \cap W_x^{1,\infty})$  norms of the solution. This implies that on any bounded time interval  $[0, t_*]$ , we have that  $f_N \in L_\xi^\infty W_{t,x}^{1,1} \cap W_{t,x}^{1,\infty}$  and, in addition,  $\|f_N\|_{L_\xi^\infty W_{t,x}^{1,1} \cap W_{t,x}^{1,\infty}}$  is bounded uniformly with respect to  $N$  in terms of  $\|K\|_{W^{1,\infty}}$ , the constants  $L, C$  and  $t_*$ .



We note that Lemma 4.12 was initially proved for time-independent objects. However, we remark that since  $f_N$  have appropriate Lipschitz dependence in the joint variable  $(t, x)$  and the norms are bounded uniformly on  $N$ , then a similar result holds true for the time dependent objects. Specifically, there exists a measure-preserving map  $\Phi_N : [0, 1] \rightarrow [0, 1]$  for every  $N \in \mathbb{N}$  such that the sequence  $\{F_k(w_N, f_N)(\Phi_N(\cdot), \cdot, \dots, \cdot)\}_{N \in \mathbb{N}}$  is compact in  $L^1_{loc}([0, t_*] \times [0, 1] \times \mathbb{R}^{d m_k})$  for each  $k \in \mathbb{N}$ . For simplicity of notation, we denote again the re-arranged objects

$$\tilde{w}_N(\xi, \zeta) := w_N(\Phi_N(\xi), \Phi_N(\zeta)), \quad \tilde{f}_N(t, x, \xi) := f_N(t, x, \Phi_N(\xi)),$$

as given in (65). In the sequel, we shall restrict to the special transform  $F_0^* \in \mathcal{T}$  consisting of only one operation of the type (iii) on the elementary transform  $F_0$  of rule (i) in Definition 4.2. Specifically, we choose  $F_0^*(w_N, f_N)(t, \xi, x) = \int_0^1 w_N(\xi, \zeta) f_N(t, x, \zeta) d\zeta$ . Indeed, by Lemma 4.11 we can reformulate it as

$$F_0^*(w_N, f_N)(t, \Phi_N(\xi), x) = F_0^*(\tilde{w}_N, \tilde{f}_N)(t, \xi, x) = \int_0^1 \tilde{w}_N(\xi, \zeta) \tilde{f}_N(t, x, \zeta) d\zeta,$$

which is compact in  $L^1_{loc}([0, t_*] \times [0, 1] \times \mathbb{R}^d)$  in particular. Then, by the Fréchet-Kolmogorov theorem, for every compact set  $\Omega \subset \mathbb{R}^d$  there must exist then some  $N$ -independent continuous function  $\varepsilon_\Omega(|h|)$  with  $\varepsilon_\Omega(|0|) = 0$  such that

$$\int_0^1 \int_0^{t_*} \int_\Omega \left| \int_0^1 (\tilde{w}_N(\xi, \zeta) \tilde{f}_N(t, x, \zeta) - \tilde{w}_N(\xi + h, \zeta) \tilde{f}_N(t + h, x + h v, \zeta)) d\zeta \right| d\xi dx dt \leq \varepsilon_\Omega(|h|),$$

for every  $h \in \mathbb{R}$ , each  $v \in \mathbb{S}^{d-1}$  and each  $N \in \mathbb{N}$ .

Using the Lipschitz-continuity of  $f_N$  (thus  $\tilde{f}_N$ ) on the joint variables  $(t, x)$  implies

$$\int_0^1 \int_0^{t_*} \int_\Omega \left| \int_0^1 (\tilde{w}_N(\xi, \zeta) - \tilde{w}_N(\xi + h, \zeta)) \tilde{f}_N(t, x, \zeta) d\zeta \right| d\xi dx dt \leq \varepsilon_\Omega(|h|) + C t_* |\Omega| \|\tilde{f}_N\|_{L^\infty W_{t,x}^{1,\infty}} |h|,$$

for every  $h \in \mathbb{R}$  and any  $N \in \mathbb{N}$ . Since  $f_N$  (thus  $\tilde{f}_N$ ) are bounded in  $L^\infty W_{t,x}^{1,\infty}$  uniformly in  $N$ , then the above estimate implies (68)<sub>1</sub>. Finally, note that (67) and the above estimate (69) for  $f_N$  are stable under re-arrangements on  $f_N$ . Hence, we conclude that  $\tilde{f}_N$  also satisfies (67) and (68)<sub>2</sub> by a simple change of variables.

We observe here that a more intricate interpolation argument can show that  $\varepsilon$  depend only on the uniform bounds on  $w_N$  and  $f_N$ , and on  $\Omega$  and could even be made explicit. As in the proof of Lemma 4.12, we construct a dense sequence  $(t_l, x_l) \in [0, t_*] \times \Omega$ , for example through dyadic grid such that for some  $\theta > 0$ , we have that there exists  $l \leq C_\Omega \varepsilon^{-\theta}$  with  $|(t, x) - (t_l, x_l)| \leq \varepsilon$ , for any  $(t, x)$  and any  $\varepsilon > 0$ . Now Corollary 4.9 implies that

$$\sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t_l, \Phi_N(\xi + h), x_l) - F_0^*(w_N, f_N)(t_l, \Phi_N(\xi), x_l)| d\xi \leq C_\Omega 2^l 2^{-C} \sqrt{\log \frac{1}{|h|}},$$

for some constant  $C_\Omega$ . Thanks to the Lipschitz bound in  $t$  and  $x$ , we also have

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t, \Phi_N(\xi + h), x) - F_0^*(w_N, f_N)(t, \Phi_N(\xi), x)| d\xi \\ & \leq \sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t_l, \Phi_N(\xi + h), x_l) - F_0^*(w_N, f_N)(t_l, \Phi_N(\xi), x_l)| d\xi \\ & \quad + C t_* |\Omega| \|\tilde{f}_N\|_{L^\infty W_{t,x}^{1,\infty}} |(t, x) - (t_l, x_l)|, \end{aligned}$$

for any  $(t, x)$  and any  $(t_l, x_l)$ . Recalling the property on the grid  $(t_l, x_l)$ , we find that

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \int_0^1 |F_0^*(w_N, f_N)(t, \Phi_N(\xi + h), x) - F_0^*(w_N, f_N)(t, \Phi_N(\xi), x)| d\xi \\ & \leq \inf_l \left( C_\Omega 2^l 2^{-C} \sqrt{\log \frac{1}{|h|}} + C t_* |\Omega| \|\tilde{f}_N\|_{L^\infty W_{t,x}^{1,\infty}} l^{1/\theta} \right), \end{aligned}$$

which yields an explicit  $\varepsilon_\Omega(|h|)$ , indeed depending only on  $\Omega$ ,  $t_*$  and  $\|\tilde{f}_N\|_{L^\infty W_{t,x}^{1,\infty}}$ .  $\square$

$\diamond$  *Step 3: Identifying limits.*

We are finally ready to identify the limits obtained by the compactness result in Lemma 4.12. We recall that, according to such a result, there exists a measure-preserving map  $\Phi_N : [0, 1] \rightarrow [0, 1]$  for every

$N \in \mathbb{N}$  and  $\phi_k(\xi, x_1, \dots, x_{m_k}) \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$  for every  $k \in \mathbb{N}$  so that for an appropriate subsequence in  $N$  (still denoted by  $N$  for simplicity) we obtain

$$F_k(w_N, f_N)(\Phi_N(\xi), x_1, \dots, x_{m_k}) \rightarrow \phi_k(\xi, x_1, \dots, x_{m_k}) \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^{d m_k}),$$

as  $N \rightarrow \infty$  for all  $k \in \mathbb{N}$  and each  $1 \leq p < \infty$ . For simplicity of notation, we shall denote again the re-arranged objects by  $\tilde{f}_N$  and  $\tilde{w}_N$  as given in (65). Using the stability under re-arrangements in Lemma 4.11, the above convergence can then be simply reformulated as follows:

$$F_k(\tilde{w}_N, \tilde{f}_N) \rightarrow \phi_k \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^{d m_k}), \quad (70)$$

as  $N \rightarrow \infty$  for all  $k \in \mathbb{N}$  and each  $1 \leq p < \infty$ . As mentioned above, our goal here reduces to identifying  $\phi_k$  as  $F(\tilde{w}, \tilde{f})$  for some appropriate limiting objects  $\tilde{f}$  and  $\tilde{w}$  of the re-arranged objects  $\tilde{f}_N$  and  $\tilde{w}_N$ .

On the one hand, since  $\tilde{f}_N$  is the first element in the algebra  $M(\tilde{w}_N, \tilde{f}_N)$ , the above implies that there exists  $\tilde{f} \in L_\xi^\infty(W_x^{1,1} \cap W_x^{1,\infty})$  (which is indeed the first of the above  $\phi_k$ ) so that we have

$$\tilde{f}_N \rightarrow \tilde{f} \quad \text{in } L_{loc}^p([0, 1] \times \mathbb{R}^d), \quad (71)$$

when  $N \rightarrow \infty$  for each  $1 \leq p < \infty$ . On the other hand, by the uniform bound of  $w_N$  in item (i) of Theorem 4.1, we readily obtain that  $\|\tilde{w}_N\|_{L_\xi^\infty L_\zeta^1} \leq C$  and  $\|\tilde{w}_N\|_{L_\zeta^\infty L_\xi^1} \leq C$  for every  $N \in \mathbb{N}$ . Since we have (see [43])

$$L^\infty([0, 1], L^1(0, 1)) \subset L^\infty([0, 1], \mathcal{M}([0, 1])) = (L^1([0, 1], C([0, 1])))^*,$$

then the Alaoglu-Bourbaki theorem claims that there exist a subsequence on  $N$  (again denoted by  $N$  for simplicity) and some  $\tilde{w} \in L_\xi^\infty \mathcal{M}_\zeta$  (which is in fact exchangeable because so are the  $\tilde{w}_N$ ) so that

$$\tilde{w}_N \xrightarrow{*} \tilde{w} \quad \text{in } L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi, \quad (72)$$

when  $N \rightarrow \infty$ . Our last step is to prove that indeed  $\phi_k = F_k(\tilde{w}, \tilde{f})$  for every  $k \in \mathbb{N}$ . Again, we proceed by induction on the number of operations in  $F_k$ .

First, let us assume that  $F_k$  is built via an operation of type (ii) in Definition 4.2. Specifically, assume that  $F_k = F_{k_1} \otimes F_{k_2}$ . Since each  $F_{k_1}$  and  $F_{k_2}$  must contain at least one less operation than  $F_k$ , then the induction hypothesis implies

$$F_{k_i}(\tilde{w}, \tilde{f}) = \phi_{k_i} = \lim_{N \rightarrow \infty} F_{k_i}(\tilde{w}_N, \tilde{f}_N),$$

for  $i = 1, 2$ . We recall that the above is a limit in  $L_{loc}^p$  for any  $1 \leq p < \infty$ . Therefore,

$$\begin{aligned} \phi_k(\xi, x_1, \dots, x_{m_k}) &= \lim_{N \rightarrow \infty} F_k(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_k}) \\ &= \lim_{N \rightarrow \infty} F_{k_1}(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_{k_1}}) \lim_{N \rightarrow \infty} F_{k_2}(\tilde{w}_N, \tilde{f}_N)(\xi, x_{m_{k_1}+1}, \dots, x_{m_k}) \\ &= \phi_{k_1}(\xi, x_1, \dots, x_{m_{k_1}}) \phi_{k_2}(\xi, x_{m_{k_1}+1}, \dots, x_{m_k}) \\ &= F_{k_1}(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_{k_1}}) F_{k_2}(\tilde{w}, \tilde{f})(\xi, x_{m_{k_1}+1}, \dots, x_{m_k}) \\ &= F_k(\tilde{w}, \tilde{f})(\xi, x_1, \dots, x_{m_k}). \end{aligned}$$

Second, let us assume that  $F_k$  is obtained through an operation of type (iii) in Definition 4.2. Specifically, we have there exists some transform  $F_{k'} \in \mathcal{T}$  with  $m_k = m_{k'}$  such that  $F_k = F_{k'}^*$ , that is

$$F_k(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_k}) = \int_0^1 \tilde{w}_N(\xi, \zeta) F_{k'}(\tilde{w}_N, \tilde{f}_N)(\zeta, x_1, \dots, x_{m_k}) d\zeta, \quad (73)$$

holds. Since  $F_{k'}$  contains at least one operation less than  $F_k$ , then the induction hypothesis shows that

$$F_{k'}(\tilde{w}, \tilde{f}) = \phi_{k'} = \lim_{N \rightarrow \infty} F_{k'}(\tilde{w}_N, \tilde{f}_N), \quad (74)$$

again in  $L_{loc}^p$  for any  $1 \leq p < \infty$ . Take any  $\varphi \in C_c([0, 1] \times \mathbb{R}^{d m_k})$  and define

$$\begin{aligned} \psi_N(\xi, \zeta) &:= \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_{k'}(\tilde{w}_N, \tilde{f}_N)(\zeta, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k}, \\ \psi(\xi, \zeta) &:= \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_{k'}(\tilde{w}, \tilde{f})(\zeta, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k}. \end{aligned}$$

Then, by (74) it is clear that  $\psi_N(\xi, \zeta) \rightarrow \psi(\xi, \zeta)$  in  $L_\zeta^1 C_\xi$ . Multiplying (73) by  $\varphi$  and integrating note that we can write

$$\int_0^1 \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_k(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k} d\xi$$

$$= \int_0^1 \int_0^1 \tilde{w}_N(\xi, \zeta) \psi_N(\xi, \zeta) d\xi d\zeta.$$

Using that  $\psi_N(\xi, \zeta) \rightarrow \psi(\xi, \zeta)$  in  $L^1_\zeta C_\xi$  and the weak-star convergence of  $\tilde{w}_N$  in  $L^\infty_\zeta \mathcal{M}_\xi$  as given in (72) imply that

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_0^1 \int_{\mathbb{R}^{d m_k}} \varphi(\xi, x_1, \dots, x_{m_k}) F_k(\tilde{f}_N, \tilde{w}_N)(\xi, x_1, \dots, x_{m_k}) dx_1 \dots dx_{m_k} d\xi \\ = \int_0^1 \int_0^1 \tilde{w}(\xi, d\zeta) \psi(\xi, \zeta) d\xi, \end{aligned}$$

where in the last step we have used Remark 2.3. Undoing the definition of  $\psi$  we have have proven (by arbitrariness of  $\varphi \in C_c([0, 1] \times \mathbb{R}^{d m_k})$ ) that

$$F_k(\tilde{w}_N, \tilde{f}_N) \xrightarrow{*} F_k(\tilde{w}, \tilde{f}) \quad \text{in } \mathcal{M}([0, 1] \times \mathbb{R}^{d m_k}),$$

as  $N \rightarrow \infty$ . Since  $F_k(\tilde{w}_N, \tilde{f}_N) \rightarrow \phi_k$  in  $L^p_{loc}$  by (70), then we conclude that  $\phi_k = F_k(\tilde{w}, \tilde{f})$ .

Finally, we conclude the identifications of the limits of  $\tau(T, w_N, f_N)$  by using Lemmas 4.4 4.11. Specifically, for any  $T \in \mathbb{T}$ , there exists some  $F \in \mathcal{T}$  such that

$$\tau(T, w_N, f_N)(x_1, \dots, x_{|T|}) = \tau(T, \tilde{w}_N, \tilde{f}_N)(x_1, \dots, x_{|T|}) = \int_0^1 F(\tilde{w}_N, \tilde{f}_N)(\xi, x_1, \dots, x_{|T|}) d\xi.$$

We recall the above convergence of the  $F(\tilde{f}_N, \tilde{w}_N)$  to  $F(\tilde{f}, \tilde{w})$  in  $L^p_{loc}$  as  $N \rightarrow \infty$ . Then, we can pass to the limit in the right hand side of the above relation to achieve

$$\tau(T, w_N, f_N)(x_1, \dots, x_{|T|}) \rightarrow \int_0^1 F(\tilde{f}, \tilde{w})(\xi, x_1, \dots, x_{|T|}) d\xi = \tau(T, \tilde{w}, \tilde{f}),$$

as  $N \rightarrow \infty$  in  $L^p_{loc}(\mathbb{R}^{d|T|})$ , for any  $1 \leq p < \infty$  and each  $T \in \mathbb{T}$ , where in the last step we have used Definition 4.5 for the extension of the operator  $\tau$ .

## 5. PROOF OF THE MAIN RESULT: THEOREM 1.1

We are now ready to perform the full proof of Theorem 1.1.

First of all, denote by  $f_i^0$  the density of each  $X_i^0$ . Then, we define  $w_N(\xi, \zeta)$  and  $f_N^0(x, \xi)$  according to equation (29) in Definition 3.4, which we recall

$$\begin{aligned} w_N(\xi, \zeta) &= \sum_{i,j=1}^N N w_{ij} \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi) \mathbb{I}_{[\frac{j-1}{N}, \frac{j}{N})}(\zeta), \quad \xi, \zeta \in [0, 1], \\ f_N^0(x, \xi) &= \sum_{i=1}^N f_i^0(x) \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi), \quad x \in \mathbb{R}^d, \xi \in [0, 1]. \end{aligned}$$

From the uniform bounds on  $f_i^0$  and  $w_{ij}$ , it is immediate to check that  $w_N$  and  $f_N^0$  satisfy the assumptions (i) – (ii) of Theorem 4.1. We note as well that  $\tau(T, w_N, f_N)$  is uniformly bounded in  $W^{1,\infty}$ , for any fixed  $T$ . Since trees are countable, by the usual diagonal extraction technique, we can obtain a subsequence (still denoted by  $N$ ) such that  $\tau(T, w_N, f_N)$  converges strongly in  $L^\infty$ , for every  $T$ . Hence, this subsequence also satisfies assumption (iii) of Theorem 4.1.

Applying now Theorem 4.1, we find  $w \in L^\infty_\xi \mathcal{M}_\zeta \cap L^\infty_\zeta \mathcal{M}_\xi$  and  $f^0 \in L^\infty_\xi (W_x^{1,1} \cap W_x^{1,\infty})$  such that

$$\tau(T, w, f^0) = \lim_{N \rightarrow \infty} \tau(T, w_N, f_N^0), \quad \forall T,$$

in any  $L^p_{loc}$  with  $p < \infty$ . Furthermore, we note that, for each  $T$ ,  $\tau(T, w, f^0)$  is uniformly bounded in  $L^1_x \cap L^\infty_x$ . Fixing now any  $R > 0$ , a straightforward extension of Lemma 3.15 leads to

$$\|\tau(T, w, f_N^0)\|_{L^1(\mathbb{R}^{d|T|}) \setminus B(0,R)^{|T|}} \leq |T| \|w\|_{L^\infty_\xi \mathcal{M}_\zeta}^{|T|-1} \|f_N^0\|_{L^\infty_\xi L^1(\mathbb{R}^d)}^{|T|-1} \|f_N^0\|_{L^\infty_\xi L^1(\mathbb{R}^d \setminus B(0,R))}.$$

On the other hand, from the assumptions the main theorem, we have that

$$\int_{\mathbb{R}^d} |x|^2 f_i^0(x) dx \leq C,$$

for some constant  $C$  independent of  $N$  and  $i$ . By the definition of  $f_N^0$ , this implies as well that

$$\int_{\mathbb{R}^d} |x|^2 f_N^0(x) dx \leq C.$$

This shows that

$$\|\tau(T, w, f_N^0)\|_{L^1(\mathbb{R}^d)^{|T|} \setminus B(0, R)^{|T|}} \leq \frac{C}{R^2} |T| \|w\|_{L_\xi^\infty M_\zeta}^{|T|-1} \|f_N^0\|_{L_\xi^\infty L^1(\mathbb{R}^d)}^{|T|-1}.$$

Combined with the local convergence in  $L^p$ , it implies that  $\tau(T, w_N, f_N^0)$  converges strongly to  $\tau(T, w_N, f^0)$  in every  $L^p(\mathbb{R}^d)^{|T|}$  and in particular in  $L^2(\mathbb{R}^d)^{|T|}$ .

We further note that also by Corollary 3.16, we have that

$$\|\tau(T, w, f^0)\|_{L_x^2} \leq \|w\|_{L_\xi^\infty \mathcal{M}_\zeta}^{|T|-1} \|f^0\|_{L_\xi^\infty L_x^2}^{|T|}.$$

Consequently, there exists some  $\lambda > 0$  small enough such that

$$\|\tau(\cdot, w, f^0) - \tau(\cdot, w_N, f_N^0)\|_\lambda \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (75)$$

By Proposition 3.11, we have existence of weak solution  $f \in L_{t,\xi}^\infty(W_x^{1,1} \cap W_x^{1,\infty})$  to (35) associated with  $w$  in the sense of Definition 3.10, namely

$$\begin{cases} \partial_t f(t, x, \xi) + \operatorname{div}_x \left( f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) = 0, \\ f(0, \cdot, \cdot) = f^0. \end{cases}$$

Of course,  $f$  is the function that we will use to compare to the empirical measure, meaning that we have to prove that

$$\mathbb{E} W_1 \left( \int_0^1 f(t, \cdot, \xi) d\xi, \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)} \right) \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (76)$$

First of all, we apply Proposition 2.2: Since  $\sup_{1 \leq i, j \leq N} |w_{ij}| \rightarrow 0$  as  $N \rightarrow \infty$ , then this shows that the solutions  $\bar{f}_i$  to (19) satisfy that

$$\mathbb{E} W_1 \left( \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, \cdot), \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)} \right) \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (77)$$

Then, we define  $f_N$  as per (29), namely

$$f_N(t, x, \xi) = \sum_{i=1}^N \bar{f}_i(t, x) \mathbb{I}_{[\frac{i-1}{N}, \frac{i}{N})}(\xi), \quad t \in \mathbb{R}_+, x \in \mathbb{R}^d, \xi \in [0, 1].$$

We recall that  $(w_N, f_N)$  is also a weak solution to (35), with  $f_N \in L_{t,\xi}^\infty(W_x^{1,1} \cap W_x^{1,\infty})$ . Therefore, both  $f_N$  and  $f$  satisfy the assumptions of Theorem 3.23. Consequently, using (75), we find

$$\left\| \int_0^1 (f - f_N)(t, \cdot, \xi) d\xi \right\|_{L_x^2} \leq \frac{C(t)}{(\log |\log \|\tau(\cdot, w_N, f_N^0) - \tau(\cdot, w, f^0)\|_\lambda|)^{1/2}} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Since the  $L^2$  norm locally dominates the Wasserstein distance and we have propagation of any moments, this implies that

$$W_1 \left( \int_0^1 f_N(t, \cdot, \xi) d\xi, \int_0^1 f(t, \cdot, \xi) d\xi \right) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Recalling that  $\int_0^1 f_N(t, x, \xi) d\xi = \frac{1}{N} \sum_{i=1}^N \bar{f}_i(t, x)$ , and combining this convergence with (77), finally we obtain (76), which therefore concludes the proof of Theorem 1.1.

## APPENDIX A. MODELS OF BIOLOGICAL NEURONS

**A.1. Smooth neuron dynamics of the Fitzhugh-Nagumo or Hodgkin-Huxley type.** Models of biological neurons offer a natural example which requires the more complex structure of the connectivities that we handle in this article. The purpose of this appendix is to give a very brief overview of some of the models, how they motivate our current study and what are the remaining open questions. There exists an extensive literature on the subject that we cannot do full justice to in this appendix. We first briefly refer to [36, 78], for a comprehensive point of view and to [22, 23] for the cortical network.

Typical models of neuron dynamics include the now famous Fitzhugh-Nagumo [32, 62], or Hodgkin-Huxley [41], which naturally fit in the framework of (1). In the case of Hodgkin-Huxley for example, one would take  $X_i(t) = (V_i(t), n_i(t), m_i(t), h_i(t))$  where  $V_i$  is the membrane potential,  $(n_i, m_i)$  are connected to the activation for the potassium and sodium channels and  $h_i$  is connected to the inactivation for the

Sodium channel.  $(n_i(t), m_i(t), h_i(t))$  solve uncoupled ODE's that can be represented by adding some self-interaction in our model, for example

$$\begin{cases} \frac{d}{dt}n_i(t) = \alpha_n(V_i(t))(1 - n_i(t)) - \beta_n(V_i(t))n_i(t), \\ \frac{d}{dt}m_i(t) = \alpha_m(V_i(t))(1 - m_i(t)) - \beta_m(V_i(t))m_i(t), \\ \frac{d}{dt}h_i(t) = \alpha_h(V_i(t))(1 - h_i(t)) - \beta_h(V_i(t))h_i(t), \end{cases} \quad (78)$$

for some given functions  $\alpha_{n,m,h}$  and  $\beta_{n,m,h}$ .

The neurons are coupled through their membrane potentials solving

$$C_m \frac{dV_i}{dt} + g_K n_i^4 (V_i - V_K) + g_{Na} m_i^3 h_i (V_i - V_{Na}) + g_l (V_i - V_l) + \sum_{j \neq i} w_{ij} (V_i - V_j) = 0, \quad (79)$$

for given constants  $C_m, g_K, g_{Na}, g_l, V_K, V_{Na}, V_l$ .

The coupled system composed of (78)-(79) obviously fits within the general framework of systems like (1). Provided all coefficients are smooth, it also satisfies the assumptions of Theorem 1.1 so that all our results are immediately applicable.

The simulation of such large systems is challenging, and mean-field limits are again an attractive alternative; see [59, 65, 68, 76]. For identical weights  $w_{ij} = 1/N$ , the mean-field limit can be rigorously derived by classical methods and has proved useful in understanding some of the large scale behavior of such systems, see for example [27, 26, 29, 33, 35] on the stochastic side or [15, 16, 17, 18, 33, 66, 69] on the deterministic side.

Those studies do not account for the structure of connectivity in the neural network, whose effect has been shown to be critical, as in [71, 72, 73, 83, 84, 85]. Of course, a first practical modeling issue, when considering more complex networks, simply is that the connectivity map  $w_{ij}$  has long remain mysterious. Random weights  $w_{ij}$  are popular for this reason and naturally lead to random networks, see [4]. The corresponding graph of interactions is typically dense and some notion of mean-field limit, based on graphons, has been derived for example in [21, 19, 20] for smooth interaction kernels  $K$ , and actually for the so-called Kuramoto [51, 52] model without learning (see (84) below with  $\eta = 0$ ).

Those follow an interesting approach to the problem when the interacting network is described as a random graph, usually based on symmetric Erdős-Rényi type graphs. The family of Erdős-Rényi graphs is one example of a convergent family of random graphs. The limiting behavior of such families is determined by a symmetric measurable function on the unit square: a classical graphon. In the case of the Erdős-Rényi graphs, the limiting graphon is even a constant function. This also leads to investigating bifurcations in the Kuramoto models on a variety of graphs ranging from symmetric and random small-world and power law graphs. The graph structure plays a key role in the transition to synchronization in the model. In [21], Kuramoto models on Erdős-Rényi and Paley graphs have been proven to have the same critical value that defines synchronization, and the authors explain the onset of synchronization in the Kuramoto model on a broad class of dense and sparse random graphs by establishing an explicit link between the network structure and the onset of synchronization. Furthermore, the mean-field limits of the Kuramoto models on these graphs can be proved to coincide with that for the Kuramoto models on weighted complete graphs [19, 20].

However, from a scaling point of view, it is expected that the *graph of connection maps*  $w_{ij}$  should be *sparse* for applications in neurosciences. Mammalian brains contain between  $10^8 - 10^{11}$  neurons ( $86 \cdot 10^9$  actually in humans). Although, of course, the models generally apply only to subdomains or certain neural clusters in charge of some specific mission, the dimensions of these clusters is very large  $N \gg 1$  and even these clusters tend to be correlated with others, which also implies very large dimensions. For example, in the human brain each neuron has on average 7000 synaptic connections to other neurons. This implies that synaptic weights must have a scale according to this number of the order of  $\sup_{i,j} w_{ij} \sim 10^{-3} - 10^{-4} \ll 1$ .

We also emphasize the importance of not imposing symmetry conditions to keep  $w_{ij} \neq w_{ji}$  in general. In the neuronal framework, the breakdown in symmetry of network connectivity can be correlated with a time scale hierarchy in some special situations. Small deviations from symmetric connectivity will cause small deviations from the invariant dynamics and introduce a temporal hierarchy, ensuring the consequence of an emergent separation on the time scale. Exploring network mechanisms due to asymmetric couplings to create more complex processes suggests that asymmetric coupling is not always a variation that needs to be averaged, since complex dynamics can be formed by simpler dynamics through a combination of asymmetric couplings. For example, stimulating a local part of neural tissue can trigger

temporarily extended paths of motion by inciting the spread and sustained activation of motor networks. Asymmetries in connectivity can reshape the emerging dynamics of a neural network, see [72, 85].

The derivation of mean-field limits for neuron models with sparse, non-symmetric connection maps had remained fully open and is critical for biological applications. The question was made even more pressing by recent progress that have lead to the first fully detailed representation of the connections between neurons, for some simple animals such as the *Drosophila*, see for example [42]. The present article is the first to allow handling of arbitrary connection maps. However our investigations still leave a certain number of open questions which we briefly discuss below.

**A.2. Open problem: Integrate and fire models.** Many variants of models (78)-(79) have been studied to make the individual dynamics of a neuron more accurate. Those include adding noise to the system in various forms that often still fully compatible with our results.

However, so-called integrate and fire models are considered to better capture the dynamics of large networks; see for example [13, 14] or [1] for a historical perspective. Among many elaborate variants, a basic model is described here for simplicity. The state of each neuron is again represented by its membrane potential  $V_i(t)$  which follows a simple SDE at almost all time,

$$dV_i(t) = b(V_i(t)) dt + \sigma_N dW_i, \quad (80)$$

but with a probability to spike

$$\mathbb{P}(\text{spike between } t \text{ and } t + dt) \approx f(V(t)) dt. \quad (81)$$

In the case of a spike at time  $t$ ,  $V_i$  is reset at 0 while all connected neurons are activated or inhibited

$$V_i(t_+) = 0, \quad V_j(t_+) = V_j(t_-) + \alpha w_{ij} \quad \forall j \neq i. \quad (82)$$

There exist many variations around those models, for example to include more complex equations instead of (80), or deterministic spiking when  $V_i$  reaches a given threshold. While (80)-(82) would formally fit in a modified version of (1), it includes *singular interactions* that prevent our results to apply. The nature of this singularity is however different from many of the other applications in this proposal, as it comes from the jumps in the  $V_i$  at spiking times, instead of unbounded interactions at  $X_i = X_j$ . The derivation of integrate and fire models for general connectivities hence remains a major, challenging open question

**A.3. Open problem: Learning mechanisms in neuron dynamics.** Long-term perspectives also include understanding how to incorporate learning mechanisms into system (1). This is a key question for biological and artificial neuron dynamics as learning is of course a critical function of the network.

Learning can simply translate in having time dependent connection weights  $w_{ij}(t)$  that evolve according to the dynamics of the agents themselves, see [2] for instance; this corresponds for example to the well-known Hebb rule that neurons wire together if they fire together. Considering again  $N$  neurons with activation  $X_i$  and time-dependent connectivity or synaptic weights  $w_{ij}(t)$  between neuron  $i$  and  $j$ , a straightforward extension of (1) simply reads

$$\left\{ \begin{array}{l} \frac{dX_i}{dt} = \sum_{j=1}^N w_{ij}(t) K(X_i, X_j) \quad (+\text{noise}), \\ \frac{dw_{ij}}{dt} = \eta (\Gamma(X_i, X_j) - w_{ij}) \quad (+\text{noise}), \\ X_i(0) = X_i^0, \quad w_{ij}(0) = w_{ij}^0, \end{array} \right. \quad (83)$$

for some plasticity function  $\Gamma$ , where  $\eta$  is regarded as the learning rate parameter.

How the mechanisms of learning can be reinforced or relaxed was specified in the neuronal context by Hebb [40]. Hebb proposed a theory of the adaptation of neurons in the brain during the learning process. The basic idea of his rule is based on the fact that neurons wire together if they fire together. Such a mechanism is based on the hypothesis that synchronous activation of cells (firing of neurons) leads to selectively pronounced increases in synaptic strength between these cells. From this process emerges the self-organized patterns. These postulates provide the neural basis for unsupervised complex learning of cognitive function in neural networks and may explain some processes that occur in the development of the nervous system.

The modeling is however made especially difficult in the present setting: changes in the  $w_{ij}(t)$  need to be able to altering the structure of sparse networks while preserving bounds such as (3). If all  $w_{ij}$  are of the same order, which is typical of dense graphs, then system (83) likely has a similar behavior to (1). However for sparse graphs, we expect the values of the  $w_{ij}$  to vary wildly and a straightforward system like (83) just cannot correctly capture very different orders of magnitude in connectivities.

On the analytical side, weights  $w_{ij}$  that adapt in time can lead to stronger synchronization between agents, strengthening correlations between them and putting in doubt the propagation of independence. Useful hints can be derived from studies on the Kuramoto model [51, 52] of coupled oscillators that was briefly mentioned above. The system is posed on the frequencies  $\theta_i$  and can read, with learning,

$$\begin{cases} \frac{d\theta_i}{dt} = \Omega_i + \frac{1}{N} \sum_{j=1}^N w_{i,j} \sin(\theta_j - \theta_i), \\ \frac{dw_{ij}}{dt} = \eta (\Gamma(\theta_i, \theta_j) - w_{ij}) \text{ (+noise)}, \\ \theta_i(0) = \theta_i^0, \quad w_{ij}(0) = w_{ij}^0, \end{cases} \quad (84)$$

where the  $\Omega_i$  are the natural frequencies of oscillators.

For large  $\eta$ , the other variable in the model that is the choice of  $\Gamma$  takes on great relevance since, at the limit  $\eta \rightarrow \infty$ , we have that  $w_{ij}(t) = \Gamma(\theta_i, \theta_j)$ . If for example  $\Gamma$  is assumed to be  $\Gamma(\theta_i, \theta_j) = \cos(\theta_i - \theta_j)$ , the attraction between near oscillators is reinforced whereas repulsive interaction arises between apart phases. On the other hand if one takes  $\Gamma(\theta_i, \theta_j) = C|\sin(\theta_i - \theta_j)|$ , synchronization only slowly emerges due to the reduction in connectivities for nearby oscillators.

Hebbian-type dynamics can be considered for the Kuramoto model. This learning law ensures that the weight of the adaptive coupling increases if the phases of the oscillators are close. One possibility discussed in [67] is to assume that the Hebbian-like plasticity function  $\Gamma$  is given by

$$\Gamma(\theta, \theta') := \frac{1}{c_{\alpha, \zeta}^\alpha |\theta - \theta'|_{\sigma}^{2\alpha}}, \quad (85)$$

where  $|\theta - \theta'|_{\sigma}$  is the orthodromic distance (to zero) over the unit circle and the parameter  $c_{\alpha, \zeta} := 1 - \zeta^{-1/\alpha}$ ,  $\zeta \in (0, 1]$ , has been chosen so that whenever two phases  $\theta_i$  and  $\theta_j$  stay at orthodromic distance  $\sigma$  or larger, then the adaptive function  $\Gamma$  predicts a maximum degree of connectedness not larger than  $\zeta$  between such oscillators. This is another way to create and amplify correlations between neurons.

#### APPENDIX B. TECHNICAL PROOFS FROM SUBSECTION 3.2

For the sake of completeness, we collect here the more technical proofs from subsection 3.2, starting with the proof of Lemma 3.7.

*Proof of Lemma 3.7.* Set any  $w \in L_{\xi}^{\infty} \mathcal{M}_{\zeta} \cap L_{\zeta}^{\infty} \mathcal{M}_{\xi}$ , any  $\phi \in L^{\infty}([0, 1])$  and any couple of sequences  $\{w_n\}_{n \in \mathbb{N}}$  and  $\{\phi_n\}_{n \in \mathbb{N}}$  as in the statement of the lemma. On the one hand, since  $\phi \in L^1([0, 1])$ , then

$$\int_0^1 \psi(\xi) \left( \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi = \int_0^1 \phi(\zeta) \left( \int_0^1 \psi(\xi) w(d\xi, \zeta) \right) d\zeta \leq \|w\|_{L_{\xi}^{\infty} \mathcal{M}_{\xi}} \|\phi\|_{L^1} \|\psi\|_{L^{\infty}},$$

for any  $\psi \in C([0, 1])$ . This implies that  $\int_0^1 \phi(\zeta) w(\cdot, d\zeta) \in \mathcal{M}([0, 1])$  and

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{\mathcal{M}_{\xi}} \leq \|w\|_{L_{\xi}^{\infty} \mathcal{M}_{\xi}} \|\phi\|_{L^1}. \quad (86)$$

Moreover, given any  $\psi \in C([0, 1])$  we have that  $\psi \otimes \phi_n \rightarrow \psi \otimes \phi$  strongly in  $L_{\zeta}^1 C_{\xi}$ . Since in addition  $w_n \xrightarrow{*} w$  weakly-star in  $L_{\zeta}^{\infty} \mathcal{M}_{\xi} = (L_{\zeta}^1 C_{\xi})^*$ , then we also obtain

$$\int_0^1 \psi(\xi) \left( \int_0^1 \phi_n(\zeta) w_n(\xi, d\zeta) \right) d\xi \rightarrow \int_0^1 \psi(\xi) \left( \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi.$$

Due to the arbitrariness of  $\psi$ , the above amounts to

$$\int_0^1 \phi_n(\zeta) w_n(\cdot, d\zeta) \xrightarrow{*} \int_0^1 \phi(\zeta) w(\cdot, d\zeta), \quad (87)$$

weakly-star in  $\mathcal{M}([0, 1])$ . On the other hand, set any  $\psi \in C([0, 1])$  and another sequence  $\{\tilde{\phi}_n\}_{n \in \mathbb{N}}$  as above consisting of continuous functions. It clearly exists by density of  $C([0, 1])$  in  $L^1([0, 1])$  and we can indeed assume that  $\{\tilde{\phi}_n\}_{n \in \mathbb{N}} \in C([0, 1])$ ,  $\tilde{\phi}_n \rightarrow \phi$  strongly in  $L^1$  and  $\|\tilde{\phi}_n\|_{L^{\infty}} \leq \|\phi\|_{L^{\infty}}$ . Then,

$$\int_0^1 \psi(\xi) \left( \int_0^1 \tilde{\phi}_n(\zeta) w(\xi, d\zeta) \right) d\xi \leq \|w\|_{L_{\xi}^{\infty} \mathcal{M}_{\zeta}} \|\psi\|_{L^1} \|\tilde{\phi}_n\|_{L^{\infty}} \leq \|w\|_{L_{\xi}^{\infty} \mathcal{M}_{\zeta}} \|\psi\|_{L^1} \|\phi\|_{L^{\infty}}.$$

By the above weak-star convergence, we obtain

$$\int_0^1 \psi(\xi) \left( \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi \leq \|w\|_{L_{\xi}^{\infty} \mathcal{M}_{\zeta}} \|\psi\|_{L^1} \|\phi\|_{L^{\infty}},$$

for every  $\psi \in C([0, 1])$ . Noting that  $C([0, 1])$  is dense in  $L^1([0, 1])$ , we find that the above Radon measure  $\int_0^1 \phi(\zeta) w(\cdot, d\zeta)$  actually belongs to  $L^1([0, 1])^* = L^\infty([0, 1])$  and

$$\left\| \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_{\xi}^\infty} \leq \|w\|_{L_{\xi}^\infty \mathcal{M}_\xi} \|\phi\|_{L^\infty},$$

which implies (32)<sub>2</sub>. Using (86) we also infer (32)<sub>1</sub> again because the Radon measure  $\int_0^1 \phi(\zeta) w(\cdot, d\zeta)$  is absolutely continuous with respect to the Lebesgue measure and we already had an estimate in the total variation norm. To end the proof of (33) note that  $\int_0^1 \phi_n(\zeta) w_n(\cdot, d\zeta)$  are uniformly bounded in  $L^1$  and  $L^\infty$  thanks to the already proven estimates (32). Since we already had that they converge weakly-star in  $\mathcal{M}([0, 1])$  to  $\int_0^1 \phi(\zeta) w(\cdot, d\zeta)$  by (87), we conclude that they also converge weakly-star in  $L^\infty$ .  $\square$

We now turn to the proof of Proposition 3.11.

*Proof of Proposition 3.11.* The argument relies on a direct Picard fixed point. Specifically, let us define

$$E := L^\infty([0, t_*] \times [0, 1], W^{1,1} \cap W^{1,\infty}(\mathbb{R}^d)), \quad E_R := \left\{ f \in E : \|f\|_{L_{t,\xi}^\infty W_x^{1,1} \cap W_x^{1,\infty}} < R \right\},$$

for any  $R > 0$  to be determined later. It is straightforward to check that  $E_R$  is a closed subset of  $L^1([0, t_*] \times [0, 1] \times \mathbb{R}^d)$  under the  $L^1$  norm in all variables  $t, \xi$  and  $x$ . Then, endowed with the  $L^1$  norm,  $E_R$  becomes a complete and convex metric space, which we shall use in the fixed point argument below.

Notice that the nonlinear PDE (35) can be reformulated as a fixed point equation in the usual way. Specifically, for any  $f \in E$  let us set  $\mathcal{L}f \in E$  to be the unique solution of the linear Cauchy problem

$$\begin{cases} \partial_t \mathcal{L}f(t, x, \xi) + \operatorname{div}_x \left( \mathcal{L}f(t, x, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy \right) = \nu \Delta_x \mathcal{L}f(t, x, \xi), \\ \mathcal{L}f(0, \cdot, \cdot) = f^0. \end{cases} \quad (88)$$

Hence, the fixed points of the operator  $\mathcal{L}$  amount to weak solutions of (35). The rest of the proof will focus on proving that the operator  $\mathcal{L} : E \rightarrow E$  is well defined, and we have that  $\mathcal{L}(E_R) \subset E_R$  and it is contractive in the  $L^1$  norm for appropriate  $R > 0$ .

$\diamond$  *Step 1: Well-posedness of (88).*

We note that (88) is a linear transport equation in  $\mathcal{L}f$  parametrized by  $\xi$  and whose velocity field  $\mathcal{V}_f$  is given in Definition 3.9. Notice that integrating by parts we have

$$\nabla_x \mathcal{V}_f(t, x, \xi) = \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) \otimes \nabla_y f(t, y, \zeta) dy,$$

which is bounded uniformly in all variables by the regularity of  $f \in E$ . Therefore, the velocity field verifies  $\mathcal{V}_f \in L^\infty([0, t_*] \times [0, 1], W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d))$ . For  $\nu = 0$  and for *a.e.* value  $\xi \in [0, 1]$  we can associate a unique and well-defined flow solving the characteristic system

$$\begin{cases} \frac{dX_f}{dt}(t, s, x; \xi) = \mathcal{V}_f(t, X_f(t, s, x; \xi); \xi), \\ X_f(s, s, x; \xi) = x, \end{cases} \quad (89)$$

in the sense of Caratheodory. Hence, the classical results ensure that the unique solution of (88) can be obtained for example by the method of characteristics, namely,  $\mathcal{L}f(t, \cdot, \xi) = X_f(t, 0, \cdot; \xi)_{\#} f^0(\cdot, \xi)$  for  $t \in [0, t_*]$  and *a.e.*  $\xi \in [0, 1]$ . In other words, we have that

$$\mathcal{L}f(t, x, \xi) = f_0(X_f(0, t, x; \xi)) J_f(0, t, x; \xi), \quad (90)$$

where  $J_f(t, s, x; \xi) := \det(\nabla_x X_f(t, s, x; \xi))$  denotes the Jacobian associated to the flow.

For  $\nu > 0$ , we may instead use the heat kernel  $G_\nu(t, x)$  and we can again classically obtain a solution through the mild formulation

$$\begin{aligned} \mathcal{L}f(t, x, \xi) &= G_\nu(t, \cdot) \star_x f^0(x, \xi) \\ &+ \int_0^t \int_{\mathbb{R}^d} G_\nu(t-s, x-y) \operatorname{div}_y \left( \mathcal{L}f(s, y, \xi) \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(y-z) f(s, z, \zeta) dz \right) ds dy. \end{aligned} \quad (91)$$

$\diamond$  *Step 2: A priori estimates of (88).*

We first recall classical estimates for advection and advection-diffusion equations. Consider any  $v(t, x) \in L^\infty([0, t_*], W^{1,\infty}(\mathbb{R}^d))$ , any weak solution  $u$  in  $L^1_{loc}$  to

$$\begin{cases} \partial_t u + \operatorname{div}_x(uv) = \nu \Delta_x u + R(t, x), \\ u(0, x) = u^0(x). \end{cases} \quad (92)$$



Then if  $R \in L^1_{t,x}$ , we have that  $u \in L^\infty L^1_x$  with

$$\|u(t, \cdot)\|_{L^1_x} \leq \|u^0\|_{L^1_x} + \int_0^t \|R(s, \cdot)\|_{L^1_x} ds. \quad (93)$$

If  $R \in L^1_t L^\infty_x$ , then  $u \in L^\infty_{t,x}$  with

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty_x} &\leq \|u^0\|_{L^\infty_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) \\ &\quad + \int_0^t \|R(s, \cdot)\|_{L^\infty_x} \exp\left((t-s) \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) ds. \end{aligned} \quad (94)$$

If additional derivatives are available on  $u^0$ ,  $v$  and  $R$ , Sobolev bounds are also propagated. Provided that  $v$  is  $C^1$ ,  $u$  is automatically a classical solution to (92) so that

$$\partial_t \nabla_x u + \operatorname{div}(v \nabla_x u) + \operatorname{div}(\nabla_x v u) = \nu \Delta_x \nabla_x u + \nabla_x R.$$

In particular, we may apply estimates similar to the previous ones as  $\nabla_x u$  solves (92) with  $\tilde{R} = \nabla_x R - \operatorname{div}(\nabla_x v u)$ . We consequently find that, if  $u^0 \in W_x^{1,1}$  and  $R = 0$  for simplicity, then

$$\begin{aligned} \|\tilde{R}(t, \cdot)\|_{L^1_x} &\leq \|\operatorname{div} v(t, \cdot)\|_{W_x^{1,\infty}} \|u(t, \cdot)\|_{L^1_x} + \|v(t, \cdot)\|_{W_x^{1,\infty}} \|\nabla_x u(t, \cdot)\|_{L^1_x} \\ &\leq \|\operatorname{div} v(t, \cdot)\|_{W_x^{1,\infty}} \|u^0\|_{L^1_x} + \|v(t, \cdot)\|_{W_x^{1,\infty}} \|\nabla_x u(t, \cdot)\|_{L^1_x}. \end{aligned}$$

Hence we obtain by Gronwall's lemma that

$$\begin{aligned} \|\nabla_x u(t, \cdot)\|_{L^1_x} &\leq \|\nabla_x u^0\|_{L^1_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) \\ &\quad + \|u^0\|_{L^1} \int_0^t \|\operatorname{div} v(s, \cdot)\|_{W_x^{1,\infty}} \exp\left((t-s) \|v\|_{L^\infty_t W_x^{1,\infty}}\right) ds. \end{aligned} \quad (95)$$

Similarly if  $u^0 \in W_x^{1,\infty}$  and  $R = 0$ , then

$$\begin{aligned} \|\tilde{R}(t, \cdot)\|_{L^\infty_x} &\leq \|\operatorname{div} v(t, \cdot)\|_{W_x^{1,\infty}} \|u(t, \cdot)\|_{L^\infty_x} + \|v(t, \cdot)\|_{W_x^{1,\infty}} \|\nabla_x u(t, \cdot)\|_{L^\infty_x} \\ &\leq \|\operatorname{div} v(t, \cdot)\|_{W_x^{1,\infty}} \|u^0\|_{L^\infty_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) + \|v(t, \cdot)\|_{W_x^{1,\infty}} \|\nabla_x u(t, \cdot)\|_{L^\infty_x}, \end{aligned}$$

and one has that, still by Gronwall's lemma,

$$\begin{aligned} \|\nabla_x u(t, \cdot)\|_{L^\infty_x} &\leq \|\nabla_x u^0\|_{L^\infty_x} \exp\left(t \|\operatorname{div}_x v\|_{L^\infty_{t,x}}\right) \\ &\quad + \|u^0\|_{L^\infty} \int_0^t \|\operatorname{div} v(s, \cdot)\|_{W_x^{1,\infty}} \exp\left((t-s) (\|v\|_{L^\infty_t W_x^{1,\infty}} + \|\operatorname{div}_x v\|_{L^\infty_{t,x}})\right) ds. \end{aligned} \quad (96)$$

We may now propagate estimates to show that the solution  $\mathcal{L}f$  lies in  $E$ , and indeed  $\mathcal{L}(E_R) \subset E_R$  for an appropriate choice of  $R > 0$ . We note that Equation (88) is immediately under the form (92) where  $\xi$  is only a parameter and

$$v_\xi(t, x) = V_f(t, x, \xi) = \int_0^1 w(\xi, d\zeta) \int_{\mathbb{R}^d} K(x-y) f(t, y, \zeta) dy.$$

From here, since  $f^0 \in L^\infty_\xi L^1_x$  we obtain that

$$\begin{aligned} \|\mathcal{L}f(t, \cdot, \xi)\|_{L^1_x} &= \|f^0(\cdot, \xi)\|_{L^1_x}, \\ \|\mathcal{L}f(t, \cdot, \xi)\|_{L^\infty_x} &\leq \|f^0(\cdot, \xi)\|_{L^\infty} \exp\left(t_* \|\operatorname{div}_x \mathcal{V}_f\|_{L^\infty_{t,x,\xi}}\right), \end{aligned} \quad (97)$$

for  $t \in [0, t_*]$  and *a.e.*  $\xi \in [0, 1]$ .

From the definition of  $V_f$ , we also directly obtain that

$$\begin{aligned} \|\operatorname{div}_x V_f\|_{L^\infty_{t,x,\xi}} &\leq \|K\|_{L^\infty} \|w\|_{L^\infty_{\xi} \mathcal{M}_\zeta \cap L^\infty_{\zeta} \mathcal{M}_\xi} \|\nabla_x f\|_{L^\infty_{t,\xi} L^1_x} \\ \|V_f\|_{L^\infty_{t,\xi} W_x^{1,\infty}} &\leq \|K\|_{L^\infty} \|w\|_{L^\infty_{\xi} \mathcal{M}_\zeta \cap L^\infty_{\zeta} \mathcal{M}_\xi} \|\nabla_x f\|_{L^\infty_{t,\xi} L^1_x} \\ \|\operatorname{div} V_f\|_{L^\infty_{t,\xi} W_x^{1,\infty}} &\leq \|\operatorname{div} K\|_{L^1} \|w\|_{L^\infty_{\xi} \mathcal{M}_\zeta \cap L^\infty_{\zeta} \mathcal{M}_\xi} \|\nabla_x f\|_{L^\infty_{t,\xi} L^\infty_x}. \end{aligned} \quad (98)$$

Putting (97), and (98) together into (95) and (96) implies that

$$\begin{aligned} \|\mathcal{L}f\|_{L^\infty_{t,\xi} W_x^{1,1} \cap W_x^{1,\infty}} &\leq C_1 \|f^0\|_{L^\infty_{\xi} W_x^{1,1} \cap W_x^{1,\infty}} \\ &\quad \times \exp\left(C_2 t_* (\|K\|_{L^\infty} + \|\operatorname{div} K\|_{L^1}) \|w\|_{L^\infty_{\xi} \mathcal{M}_\zeta \cap L^\infty_{\zeta} \mathcal{M}_\xi} \|f\|_{L^\infty_{t,\xi} W_x^{1,1} \cap W_x^{1,\infty}}\right), \end{aligned} \quad (99)$$

for some universal  $C_1, C_2 \geq 1$ . In particular, the above estimate show that  $\mathcal{L}f \in E$  for every  $f \in E$ .

◇ *Step 3: Invariance of  $E_R$  and contractivity of  $\mathcal{L}$ .*

Set now any  $R > C_1 \|f^0\|_{L_\xi^\infty W_x^{1,1} W_x^{1,\infty}}$  and note that the above estimates also show that  $\mathcal{L}(E_R) \subset E_R$  if the maximal time  $t_*$  is taken sufficiently small, namely,

$$0 < t_* \leq t_1(R) := \frac{1}{C_2(\|K\|_{L^\infty} + \|\operatorname{div} K\|_{L^1})\|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} R} \log \left( \frac{R}{C_1 \|f^0\|_{L_\xi^\infty W_x^{1,1} \cap W_x^{1,\infty}}} \right).$$

Our next goal is to show that  $\mathcal{L}$  is contractive under the  $L^1$  norm. To do so, let us consider  $f, g \in E_R$  and note that  $f - g$  follows the following linear transport equation with a source term

$$\begin{cases} \partial_t(\mathcal{L}f - \mathcal{L}g) + \operatorname{div}_x(\mathcal{V}_g(\mathcal{L}f - \mathcal{L}g)) + \operatorname{div}_x((\mathcal{V}_f - \mathcal{V}_g)\mathcal{L}f) = \nu \Delta_x(\mathcal{L}f - \mathcal{L}g), \\ (\mathcal{L}f - \mathcal{L}g)(0, \cdot, \cdot) = 0. \end{cases}$$

Taking  $L^1$  norms with respect to  $x$  and  $\xi$  yields

$$\begin{aligned} \|(\mathcal{L}f - \mathcal{L}g)(t, \cdot, \cdot)\|_{L_{x,\xi}^1} &\leq \|\operatorname{div}_x((\mathcal{V}_f - \mathcal{V}_g)\mathcal{L}f)\|_{L_{t,x,\xi}^1} \\ &\leq \|\operatorname{div}_x(\mathcal{V}_f - \mathcal{V}_g)\|_{L_{t,\xi}^\infty L_x^1} \|\mathcal{L}f\|_{L_{t,x,\xi}^\infty} + \|\mathcal{V}_f - \mathcal{V}_g\|_{L_{t,\xi}^\infty L_x^1} \|\nabla_x \mathcal{L}f\|_{L_{t,x,\xi}^\infty}, \end{aligned} \quad (100)$$

for  $t \in [0, t_*]$ . By the estimate (34)<sub>1</sub> in Lemma 3.8 we have

$$\begin{aligned} \|\mathcal{V}_f - \mathcal{V}_g\|_{L_{t,x,\xi}^1} &\leq \|K\|_{L^1} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|f - g\|_{L_{t,x,\xi}^1}, \\ \|\operatorname{div}_x(\mathcal{V}_f - \mathcal{V}_g)\|_{L_{t,x,\xi}^1} &\leq \|\operatorname{div} K\|_{L^1} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|f - g\|_{L_{t,x,\xi}^1}. \end{aligned} \quad (101)$$

Then, putting (100) and (101) together, and using that  $\mathcal{L}f \in E_R$  by the above step, we have that

$$\|\mathcal{L}f - \mathcal{L}g\|_{L_{t,x,\xi}^1} \leq R t_* (\|K\|_{L^1} + \|\operatorname{div} K\|_{L^1}) \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|f - g\|_{L_{t,x,\xi}^1}. \quad (102)$$

Take  $t_*$  sufficiently small so that  $t_* \leq t_1(R)$  as above and, in addition

$$0 < t_* < t_2(R) := \frac{1}{R(\|K\|_{L^1} + \|\operatorname{div} K\|_{L^1})\|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi}}.$$

Hence  $\mathcal{L}$  is contractive on  $E_R$  under the  $L^1$  norm by the stability estimate (102). By the Banach fixed point theorem there must exist a (unique) fixed point  $f \in E_R$  of  $\mathcal{L}$ , leading to a weak solution of (35) in  $[0, t_*]$ . Note that  $t_*$  must be taken small enough so that  $0 < t_* < \min\{t_1(R), t_2(R)\}$ . However, restarting the argument from  $t_*$  allows extending this weak solution to any finite time interval as usual, since the above a priori estimate (99) above does not blow up in time.  $\square$

We finish this technical appendix with the proof of Lemma 3.12.

*Proof of Lemma 3.12.* As usual, one uses an approximation by convolution. Because  $w$  is defined on  $[0, 1]^2$ , we first extend  $w$  as a periodic function on with period 1 in both variables. Similarly we naturally consider any function on  $[0, 1]$  as periodic, allowing to naturally use convolutions. Then, choose any convolution kernel  $L_n \geq 0$  with  $L_n \in C_c^\infty(\mathbb{R})$  for any fixed  $n$  and define  $w_n = L_n \otimes L_n \star_{\xi,\zeta} w$ . Since  $w \in \mathcal{M}_{\xi,\zeta}$ , we immediately note that  $w_n \in L_{\xi,\zeta}^\infty$ .

Next, for any  $f \in L_\xi^1 C_\zeta$ , we trivially have that  $\|L_n \otimes L_n \star f\|_{L_\xi^1 L_\zeta^\infty} \leq \|f\|_{L_\xi^1 L_\zeta^\infty}$ . By duality, we deduce that  $w_n$  is uniformly bounded in  $L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi$  and, hence, in  $L_\xi^\infty L_\zeta^1 \cap L_\zeta^\infty L_\xi^1$ , since  $w_n \in L_{\xi,\zeta}^\infty$ .

Consider now any  $\phi_n$  converging weakly to  $\phi$  in  $H^1([0, 1])$ , we have that

$$\int_0^1 \phi_n(\zeta) L_n \star_\zeta w(\xi, \zeta) d\zeta = \int_0^1 \tilde{L}_n \star_\zeta \phi_n(\zeta) w(\xi, \zeta) d\zeta,$$

with  $\tilde{L}_n(\xi) = L_n(-\xi)$ . By Lemma 3.7, for any  $\psi \in L^\infty([0, 1])$ , we deduce that

$$\int_0^1 \psi(\xi) \left( \int_0^1 \phi_n(\zeta) L_n \star_\zeta w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) d\xi \leq \|\psi\|_{L^\infty} \|w\|_{L_\xi^\infty \mathcal{M}_\zeta \cap L_\zeta^\infty \mathcal{M}_\xi} \|\phi - \tilde{L}_n \star \phi_n\|_{L^1}.$$

Since  $H^1$  is compactly embedded in  $L^1$ , this shows that

$$\left\| \int_0^1 \phi_n(\zeta) L_n \star_\zeta w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_\xi^1} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Still by Lemma 3.7 since  $H^1 \subset L^\infty$ ,  $\int_0^1 \phi(\zeta) w(\xi, d\zeta)$  is a fixed function in  $L^\infty$  and consequently its convolution by  $L_n$  converges in  $L^1$  (and in all  $L^p$ ,  $p < \infty$ ). Therefore

$$\begin{aligned} & \left\| \int_0^1 \phi_n(\zeta) w_n(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_\xi^1} \\ & \leq \left\| L_n \star_\xi \left( \int_0^1 \phi_n(\zeta) L_n \star_\zeta w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right) \right\|_{L_\xi^1} \\ & \quad + \left\| L_n \star_\xi \int_0^1 \phi_n(\zeta) w(\xi, \zeta) d\zeta - \int_0^1 \phi(\zeta) w(\xi, d\zeta) \right\|_{L_\xi^1} \rightarrow 0, \end{aligned}$$

proving the convergence of  $w_n$  in  $L_\xi^1 H_\zeta^{-1}$ .  $\square$

## REFERENCES

- [1] L. F. Abbott, *Lapicque's introduction of the integrate-and-fire model neuron (1907)*, Brain Res. Bull. **50(5-6)** (1999), 303–4.
- [2] T. Aoki, T. Aoyagi, *Co-evolution of Phases and Connection Strengths in a Network of Phase Oscillators*, Phys. Rev. Lett. **102** (2009), 034101.
- [3] A. Backhausz, B. Szegedy, *Action convergence of operators and graphs*, Canad. J. Math. (2020), 1–50.
- [4] A. L. Barabási, R. Albert, *Emergence of scaling in random networks*, Science **286** (1999), 509–512.
- [5] E. Boissard, *Problèmes d'interaction discret-continu et distances de Wasserstein*. PhD thesis, Université de Toulouse III, 2011.
- [6] C. Borgs, J. T. Chayes, H. Cohn, Y. Zhao, *An  $L^p$  theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions*, Trans. Am. Math. Soc. **372(5)** (2019), 3019–3062.
- [7] C. Borgs, J. T. Chayes, H. Cohn, Y. Zhao, *An  $L^p$  theory of sparse graph convergence II: LD convergence, quotients, and right convergence*, Ann. Probab. **46(1)** (2018), 337–396.
- [8] C. Borgs, J. T. Chayes, J. Kahn, L. Lovász, *Left and right convergence of graphs with bounded degree*, Random Struct. Algorithms **42** (2013), 1–28.
- [9] C. Borgs, J. T. Chayes, L. Lovász, V. Sós, K. Vesztegombi, *Counting graph homomorphisms*. In Topics in discrete mathematics (eds. M. Klazar, J. Kratochvíl, M. Loeb, J. Matoušek, R. Thomas, and P. Valtr), Springer, 2006, 315–371.
- [10] C. Borgs, J. T. Chayes, L. Lovász, V. Sós, K. Vesztegombi, *Convergent graph sequences I: subgraph frequencies, metric properties, and testing*, Adv. Math. **219** (2008), 1801–1851.
- [11] C. Borgs, J. T. Chayes, L. Lovász, V. Sós, K. Vesztegombi, *Convergent graph sequences II: multiway cuts and statistical physics*, Ann. Math. **176** (2012), 151–219.
- [12] W. Braun, K. Hepp, *The Vlasov dynamics and its fluctuations in the  $1/N$  limit of interacting classical particles*, Comm. Math. Phys. **56(2)** (1977), 101–113.
- [13] N. Brunel, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking neurons*, J. Comput. Neurosci. **8(3)** (2000), 183–208.
- [14] A. N. Burkitt, *A review of the integrate-and-fire neuron model: I. Homogeneous synaptic input*, Biol. Cybern. **95(1)** (2006), 1–19.
- [15] M. J. Cáceres, J. A. Carrillo, B. Perthame, *Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states*, J. Math. Neurosci. **1** (2011), Art. 7, 33 pp.
- [16] M. J. Cáceres, B. Perthame, *Beyond blow-up in excitatory integrate and fire neuronal networks: refractory period and spontaneous activity*, J. Theor. Biol. **350** (2014), 81–89.
- [17] J. A. Carrillo, M. d. M. Gonzalez, M. P. Gualdani, M. E. Schonbek, *Classical solutions for a nonlinear Fokker-Planck equation arising in computational neuroscience*, Comm. Part. Differ. Equat. **38(3)** (2013), 385–409.
- [18] J. A. Carrillo, B. Perthame, D. Salort, D. Smets, *Qualitative Properties of Solutions for the Noisy Integrate & Fire model in Computational Neuroscience*, Nonlinearity **28(9)** (2015), 3365.
- [19] H. Chiba, G. S. Medvedev, *The mean field analysis for the Kuramoto model on graphs I. The mean field equation and transition point formulas*, Discrete Contin. Dyn. Syst. Ser. A **39(1)** (2019), 131–155.
- [20] H. Chiba, G. S. Medvedev, *The mean field analysis of the Kuramoto model on graphs II. Asymptotic stability of the incoherent state, center manifold reduction, and bifurcations*, Discrete Contin. Dyn. Syst. Ser. A **39(7)** (2019), 3897–3921.
- [21] H. Chiba, G. S. Medvedev, M. S. Muzuhara, *Bifurcations in the Kuramoto model on graphs*, Chaos **28** (2018), 073109.
- [22] A. Compte, N. Brunel, P. S. Goldman-Rakic, X.-J. Wang, *Synaptic mechanisms and network dynamics underlying spatial working memory in a cortical network model*, Cereb. Cortex **10** (2000), 910–923.
- [23] A. Compte, M. V. Sanchez-Vives, D. A. McCormick, X.-J. Wang, *Cellular and network mechanisms of slow oscillatory activity (<1 Hz) and wave propagations in a cortical network model*, J. Neurophysiol. **89** (2003), 2707–2725.
- [24] F. Coppini, H. Dietert, G. Giacomin, *A law of large numbers and large deviations for interacting diffusions on Erdős-Rényi graphs*, Stoch. Dyn. **20** (2020), 2050010.
- [25] T. de La Rue, *Espaces de Lebesgue*, Séminaire de probabilités de Strasbourg, Tome 27 (1993), pp. 15–21.
- [26] F. Delarue, J. Inglis, S. Rubenthaler, E. Tanré, *Particle systems with a singular mean-field self-excitation. Application to neuronal networks*, Stoch. Process. Their Appl. **125** (2015) 2451–2492.
- [27] F. Delarue, J. Inglis, S. Rubenthaler, E. Tanré, et al, *Global solvability of a networked integrate-and-fire model of McKean-Vlasov type*, Ann. Appl. Probab. **25(4)** (2015), 2096–2133.
- [28] S. Delattre, G. Giacomin, E. Luçon, *A note on dynamical models on random graphs and Fokker-Planck equations*, J. Stat. Phys. **165(4)** (2016), 785–798.

- [29] A. De Masi, A. Galves, E. Löcherbach, E. Presutti, *Hydrodynamic limit for interacting neurons*, J. Stat. Phys. **158** (2015), 866–902.
- [30] V. Dobrić, J. E. Yukich, *Asymptotics for transportation cost in high dimensions*, J. Theoret. Probab. **8**(1) (1995), 97–118.
- [31] R. L. Dobrushin, *Vlasov equations*, Funct. Anal. its Appl. **13**(2) (1979), 115–123.
- [32] R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical J. **1** (1961), 445–466.
- [33] F. Flandoli, E. Priola, G. Zanco, *A mean-field model with discontinuous coefficients for neurons with spatial interaction*, Discrete Contin. Dyn. Syst. Ser. A **39** (2019), 3037–3067.
- [34] N. Fournier, A. Guillin, *On the rate of convergence in Wasserstein distance of the empirical measure*, Probab. Theory Relat. Fields **162** (2015), 707–738.
- [35] N. Fournier, E. Löcherbach, *On a toy model of interacting neurons*, Ann. Inst. H. Poincaré Probab. Statist. **52**(4) (2016), 1844–1876.
- [36] W. Gerstner, W. M. Kistler, *Spiking neuron models: Single neurons, populations, plasticity*. Cambridge university press, 2002.
- [37] M. A. Gkogkas, C. Kuehn, *Graphop Mean-Field Limits for Kuramoto-Type Models*, SIAM J. Appl. Dyn. Syst. **21**(1) (2022), 248–283.
- [38] F. Golse, *On the dynamics of large particle systems in the mean field limit Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity*, Springer, Berlin 2016, 1–144.
- [39] M. Hauray, P. E. Jabin, *Particles approximations of Vlasov equations with singular forces: Propagation of chaos*, Ann. Sci. Ec. Norm. Super. **48**(4), 891–940 (2015).
- [40] D.O. Hebb, *The Organization of Behavior*, Wiley, New York, 1949.
- [41] A. L. Hodgkin, A. F. Huxley, *A quantitative description of membrane current and its application to conduction and excitation in nerve*, J. Physiol. **117** (1952), 500–544.
- [42] B. K. Hulse, H. Haberkern, R. Franconville, D. B. Turner-Evans, S. Y. Takemura, T. Wolff, M. Noorman, M. Dreher, C. Dan, R. Parekh, A. M. Hermundstad, G. M. Rubin, V. Jayaraman, *A connectome of the Drosophila central complex reveals network motifs suitable for flexible navigation and context-dependent action selection*, eLife **10** (2021), e66039.
- [43] A. Ionescu Tulcea, C. Ionescu Tulcea, *Topics in the theory of lifting*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge, vol. 48, Springer-Verlag, Berlin, Heidelberg, 1977.
- [44] P. E. Jabin, *A review of the mean field limits for Vlasov equations*, Kinet. Relat. Mod. **7** (2014), 661–711.
- [45] P. E. Jabin, Z. Wang, *Mean field limit and propagation of chaos for Vlasov systems with bounded forces*, J. Funct. Anal. **271**(12) (2016), 3588–3627.
- [46] P.-E. Jabin, Z. Wang, *Mean field limit for stochastic particle systems*. In: Active Particles, Volume 1, Theory, Models, Applications, pp 379–402. Birkhauser-Springer, Boston (2017).
- [47] P. E. Jabin, Z. Wang, *Quantitative estimates of propagation of chaos for stochastic systems with  $W^{-1,\infty}$  kernels*, Invent. Math. **214**(1) (2018), 523–591.
- [48] C. Kuehn, C. Xu, *Vlasov equations on digraph measures*, **339** (2022), J. Differ. Equ. 261–349.
- [49] D. Kaliuzhnyi-Verbovetskyi, G.S. Medvedev, *The Mean Field Equation for the Kuramoto Model on Graph Sequences with Non-Lipschitz Limit*, SIAM J. Math. Anal., 50(3), 2441–2465.
- [50] D. Kunszenti-Kovács, L. Lovász, B. Szegedy, *Measures on the square as sparse graph limits*, J. Comb. Theory. Ser. B **138** (2019), 1–40.
- [51] Y. Kuramoto, *Chemical Oscillations, waves and turbulence*. Springer-Verlag, Berlin, 1984.
- [52] Y. Kuramoto, *International symposium on mathematical problems in mathematical physics*, Lecture Notes in Theoretical Physics. **30** (1975), 420
- [53] D. Lacker, *Hierarchies, entropy and quantitative propagation of chaos for mean field diffusions*. Preprint arXiv:2105.02983.
- [54] C. Lancellotti, *On the Vlasov limit for systems of nonlinearly coupled oscillators without noise*. Transport Theor. Stat. Phys. **34** (2005) 523–535.
- [55] J. M. Levine, J. Bascompte, P. B. Adler, S. Allesina, *Beyond pairwise coexistence: biodiversity maintenance in complex communities*, Nature **456** (2017), 56–64.
- [56] L. Lovász, *Large Networks and Graph Limits*, AMS Colloquium Publications Volume: 60, 475 pp, 2012.
- [57] L. Lovász, B. Szegedy, *Limits of dense graph sequences*, J. Comb. Theory. Ser. B **96** (2006), 933–957.
- [58] E. Luçon, *Quenched asymptotics for interacting diffusions on inhomogeneous random graphs*, Stoch. Process. Their Appl. **130**(11) (2020), 6783–6842.
- [59] M. Mattia, P. Del Giudice, *Efficient event-driven simulation of large networks of spiking neurons and dynamical synapses*, Neural Comput. **12**(10) (2000), 2305–2329.
- [60] G. Medvedev, *The Nonlinear Heat Equation on Dense Graphs and Graph Limits*, SIAM J. Math. Anal. **46**(4) (2014), 883–898.
- [61] G. Medvedev, *The continuum limit of the Kuramoto model on sparse random graphs*, Commun. Math. Sci. **17**(4) (2019), 883–898.
- [62] J. Nagumo, S. Arimoto, S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. IRE. **50** (1962), 2061–2070.
- [63] H. Neunzert, *An introduction to the nonlinear Boltzmann–Vlasov equation*. In Kinetic Theories and the Boltzmann Equation, Lecture Notes in Mathematics vol. **1048**. Springer-Verlag, 1984.
- [64] H. Neunzert, J. Wick, *Die Approximation der Lösung von Integro-Differentialgleichungen durch endliche Punktmengen*, In R. Ansorge and W. Törnig, editors, Numerische Behandlung nichtlinearer Integrodifferential- und Differentialgleichungen, volume 395 of Lecture Notes in Mathematics, pages 275–290. Springer, Berlin, Heidelberg, 1974.

- [65] A. Omurtag, B.W. Knight, L. Sirovich, *On the simulation of large populations of neurons*, J. Comput. Neurosci. **8**(1) (2000), 51–63.
- [66] K. Pakdaman, B. Perthame, D. Salort, *Dynamics of a structured neuron population*, Nonlinearity, **23**(1) (2010), 55–75.
- [67] J. Park, D. Poyato, J. Soler, *Filippov trajectories and clustering in the Kuramoto model with singular couplings*, J. Eur. Math. Soc. **23** (2021), 3193–3278.
- [68] E. Pastorelli, C. Capone, F. Simula, M. V. Sanchez-Vives, P. Del Giudice, M. Mattia, P. S. Paolucci, *Scaling of a large-scale simulation of synchronous slow-wave and asynchronous awake-like activity of a cortical model with long-range interconnections*, Front. Syst. Neurosci. **13** (2019), 33.
- [69] B. Perthame, D. Salort, *On a voltage-conductance kinetic system for integrate & fire neural networks*, Kinet. Relat. Models **6** (2013), 841–864.
- [70] D. Pham, D. Karaboga, *Intelligent optimisation techniques: genetic algorithms, tabu search, simulated annealing and neural networks*, Springer Science & Business Media, 2012.
- [71] J. Pham, K. Pakdaman, J. Champagnat, J.-F. Vibert, *Activity in sparsely connected excitatory neural networks: effect of connectivity*, Neural Netw. **11**(3) (1998), 415–434.
- [72] A. S. Pillai, V. K. Jirsa, *Symmetry Breaking in Space-Time Hierarchies Shapes Brain Dynamics and Behavior*, Neuron **94**(5) (2017), 1010–1026.
- [73] J. W. Pillow, J. Shlens, L. Paninski, A. Sher, A. M. Litke, E. J. Chichilnisky, E. P. Simoncelli, *Spatio-temporal correlations and visual signalling in a complete neuronal population*, Nature **454** (2008), 995–999.
- [74] D. Poyato, *Filippov flows and mean-field limits in the kinetic singular Kuramoto model*. Preprint arXiv:1903.01305.
- [75] D. Poyato, J. Soler, *Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models*, Math. Models Methods Appl. Sci. **27**(6) (2017), 1089–1152.
- [76] A. Renart, N. Brunel, X. J. Wang, *Mean-field theory of irregularly spiking neuronal populations and working memory in recurrent cortical networks*, Computational neuroscience: A comprehensive approach, 431–490, J Feng Editor, CRC Press, 2003.
- [77] H. Spohn, *Dynamics of Charged Particles and their Radiation Field*, Cambridge University Press, 2004.
- [78] O. Sporns, *Networks of the Brain*, Cambridge, MA: MIT Press, 2010.
- [79] E. Szemerédi, *On sets of integers containing no  $k$  elements in arithmetic progression*, Acta Arith. **27** (1975), 299–345.
- [80] A.-S. Sznitman, *Topics in propagation of chaos*, Ecole d’Été de Probabilités de Saint-Flour XIX -1989 (P. L. Hennequin, ed.), Lecture Notes in Mathematics, vol. 1464, Springer, Berlin, Heidelberg, 1991, pp. 165–251.
- [81] V. S. Varadarajan, *On the convergence of sample probability distributions*, Sankhya **19** (1958), 23–26.
- [82] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, vol. 79, Springer, New-York, 1982.
- [83] D. J. Watts, S. H. Strogatz, *Collective dynamics of small-world networks*, Nature **393** (1998), 440–442.
- [84] H. R. Wilson, J. D. Cowan, *Excitatory and inhibitory interactions in localized populations of model neurons*, Biophys. J. **12** (1972), 1–24.
- [85] M. M. Woodman, V. K. Jirsa, *Emergent Dynamics from Spiking Neuron Networks through Symmetry Breaking of Connectivity*, PLoS ONE **8**(5) (2013), e64339.

(Pierre-Emmanuel Jabin)

DEPT. OF MATHEMATICS, HUCK INSTITUTES, AND EXCELLENCE RESEARCH UNIT “MODELING NATURE”, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16803, USA

*Email address:* pejabin@psu.edu

(David Poyato)

INSTITUT CAMILLE JORDAN (ICJ), UMR 5208 CNRS & UNIVERSITÉ CLAUDE BERNARD LYON 1, 69100 VILLEURBANNE, FRANCE, AND RESEARCH UNIT “MODELING NATURE” (MNAT), UNIVERSIDAD DE GRANADA, GRANADA, 18071, SPAIN

*Email address:* poyato@math.univ-lyon1.fr davidpoyato@ugr.es

(Juan Soler)

DEPARTAMENTO DE MATEMÁTICA APLICADA AND RESEARCH UNIT “MODELING NATURE” (MNAT), FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA, 18071 GRANADA, SPAIN

*Email address:* jsoler@ugr.es