



**HAL**  
open science

# Permutation Pattern Matching for Doubly Partially Ordered Patterns

Laurent Bulteau, Guillaume Fertin, Vincent Jugé, Stéphane Vialette

► **To cite this version:**

Laurent Bulteau, Guillaume Fertin, Vincent Jugé, Stéphane Vialette. Permutation Pattern Matching for Doubly Partially Ordered Patterns. 33rd Annual Symposium on Combinatorial Pattern Matching, Hideo Bannai and Jan Holub, Jun 2022, Prague, Czech Republic. hal-03624311

**HAL Id: hal-03624311**

**<https://hal.science/hal-03624311v1>**

Submitted on 30 Mar 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Permutation Pattern Matching for Doubly Partially Ordered Patterns

Laurent Bulteau ✉ 

LIGM, CNRS, Univ Gustave Eiffel, F77454 Marne-la-Vallée, France

Guillaume Fertin ✉ 

Nantes Université, LS2N (UMR 6004), CNRS, Nantes, France

Vincent Jugé ✉ 

LIGM, CNRS, Univ Gustave Eiffel, F77454 Marne-la-Vallée, France

Stéphane Vialette ✉ 

LIGM, CNRS, Univ Gustave Eiffel, F77454 Marne-la-Vallée, France

---

## Abstract

We study in this paper the DOUBLY PARTIALLY ORDERED PATTERN MATCHING (or DPOP MATCHING) problem, a natural extension of the PERMUTATION PATTERN MATCHING problem. PERMUTATION PATTERN MATCHING takes as input two permutations  $\sigma$  and  $\pi$ , and asks whether there exists an occurrence of  $\sigma$  in  $\pi$ ; whereas DPOP MATCHING takes *two partial orders*  $P_V$  and  $P_D$  defined on the same set  $X$  and a permutation  $\pi$ , and asks whether there exist  $|X|$  elements in  $\pi$  whose values (resp., positions) are in accordance with  $P_V$  (resp.,  $P_D$ ). Posets  $P_V$  and  $P_D$  aim at relaxing the conditions formerly imposed by the permutation  $\sigma$ , since  $\sigma$  yields a total order on both positions and values. Our problem being NP-hard in general (as PERMUTATION PATTERN MATCHING is), we consider restrictions on several parameters/properties of the input, e.g., bounding the size of the pattern, assuming symmetry of the posets (i.e.,  $P_V$  and  $P_D$  are identical), assuming that one partial order is a total (resp., weak) order, bounding the length of the longest chain/anti-chain in the posets, or forbidding specific patterns in  $\pi$ . For each such restriction, we provide results which together give a(n almost) complete landscape for the algorithmic complexity of the problem.

**2012 ACM Subject Classification** Theory of computation  $\rightarrow$  Design and analysis of algorithms

**Keywords and phrases** Partial orders, Permutations, Pattern Matching, Algorithmic Complexity, Parameterized Complexity

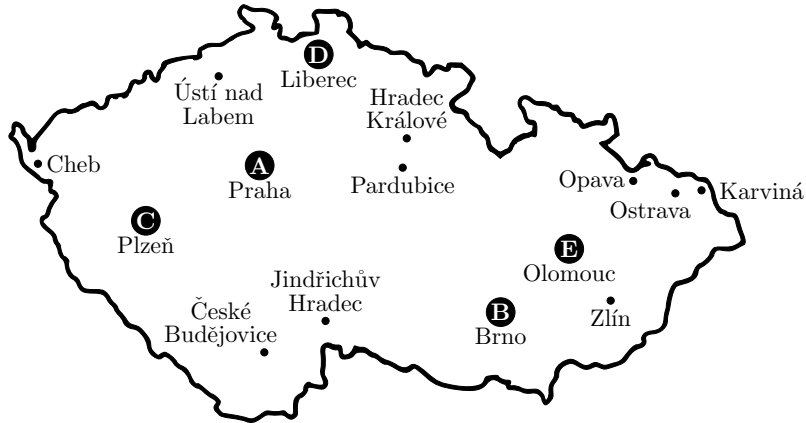
## 1 Preamble

Let us play the following little puzzle game. Among the selection of fifteen cities of the Czech Republic depicted in Figure 1 together with their geographic coordinates, find (if they exist) five cities, say **A**, **B**, **C**, **D** and **E**, such that:

- **A** and **C** are west of **D** and north of **B**,
- **E** is east of **B** and south of **A**,
- **D** is west of **B** and north of **A** and **C**.

It is assumed that no two cities have the same longitude or latitude. Notice that the game does not provide complete information as, for example, no information is provided about the relative positioning of **A** and **C** (and silence is tantamount to consent). We may assume that the information is minimal: requiring **C** is west of **B** is unnecessary since **C** is west of **D** and **D** is west of **B**. One solution is **A** = Praha, **B** = Brno, **C** = Plzeň, **D** = Liberec and **E** = Olomouc. Note that the solution is not unique, as **A** = Plzeň, **B** = Jindřichův Hradec, **C** = Cheb, **D** = Ústí nad Labem and **E** = Brno is another solution.

We show that this puzzle game can be modeled as a permutation pattern matching problem for doubly partially ordered patterns. Let us first associate a permutation  $\pi \in \mathfrak{S}(15)$  with the problem (see Figure 2). We sort the fifteen cities of the Czech Republic depicted in Figure 1 both by increasing longitude (E) and by increasing latitude (N), so that  $\pi(i) = j$  if



■ **Figure 1** A Czech Republic map showing 15 of its cities. Their names and GPS coordinates, in increasing order of their longitudes, are: Cheb (50°4′N 12°22′E), Plzeň (49°44′N 13°22′E), Ústí nad Labem (50°39′N 14°1′E), Praha (50°5′N 14°25′E), České Budějovice (48°58′N 14°28′E), Jindřichův Hradec (49°9′N 15°0′E), Liberec (50°46′N 15°3′E), Pardubice (50°2′N 15°46′E), Hradec Králové (50°12′N 15°49′E), Brno (49°11′N 16°36′E), Olomouc (49°35′N 17°15′E), Zlín (49°13′N 17°40′), Opava (49°56′N 17°54′E), Ostrava (49°50′N 18°17′E) and Karviná (49°51′N 18°32′E).  
© Creative Commons CC0 1.0 Universal Public Domain Dedication.

46 the  $i$ -th city going west to east is also the  $j$ -th city going south to north. In our example,  
 47 the “Czech Republic permutation” is  $\pi = 11\ 6\ 14\ 12\ 1\ 2\ 15\ 10\ 13\ 3\ 5\ 4\ 9\ 7\ 8$ . For example,  
 48  $\pi(2) = 6$  since Plzeň is the second city going west to east, and the sixth city going south  
 49 to north. What is left is to define our pattern  $P$ :  $P$  is composed of two partially ordered sets on  
 50 the variables  $\{A, B, C, D, E\}$  (see Figure 3): one partially ordered set (denoted  $P_V$  for *value*  
 51 *poset* in the sequel) describes the south-to-north constraints and another partially ordered  
 52 set (denoted  $P_P$  for *position poset* in the sequel) describes the west-to-east constraints.

## 53 2 Introduction

54 We say that a permutation  $\sigma$  *occurs* in another permutation  $\pi$  (or that  $\pi$  *contains*  $\sigma$ ) if there  
 55 exists a subsequence of elements of  $\pi$  that has the same relative order as  $\sigma$ . Otherwise, we  
 56 say that  $\pi$  *avoids*  $\sigma$ . For example,  $\pi$  contains the permutation  $\sigma = 123$  (resp.,  $\sigma = 321$ ) if  
 57 it has an increasing (resp., a decreasing) subsequence of size 3. Similarly,  $\sigma = 4312$  occurs  
 58 in  $\pi = 6152347$ , as shown in  $\textcircled{6}1\textcircled{5}2\textcircled{3}\textcircled{4}7$ , but the same  $\pi = 6152347$  avoids  $\sigma' = 2341$ .

59 Deciding whether a permutation  $\sigma \in \mathfrak{S}(k)$  occurs in some permutation  $\pi \in \mathfrak{S}(n)$   
 60 is NP-complete [7], but is fixed-parameter tractable for the parameter  $k$  [15, 16]. Two  
 61 exponential-time algorithms have been recently proposed [5], improving upon [1, 10]. A vast  
 62 literature is devoted to the case where both the pattern  $\sigma$  and the target  $\pi$  are restricted to  
 63 a proper permutation class, e.g., 321-avoiding permutations [17, 2, 20], (213, 231)-avoiding  
 64 permutations [25], (2413, 3142)-avoiding (a.k.a. *separable*) permutations [18, 24], and  $(k \dots 1)$ -  
 65 avoiding permutations [11]. For more background on permutation patterns and pattern  
 66 avoidance, we refer to [6] and [23].

67 In the last years, the notion of *pattern* has been generalized in several ways. A *vincular*  
 68 *pattern* is a permutation in which some entries must occur consecutively [4]. *Consecutive*  
 69 *patterns* are a special case of vincular patterns in which all entries need to be adjacent [13].  
 70 *Bivincular patterns* generalize classical patterns even further than vincular patterns by  
 71 requiring that not only positions but also values of elements involved in a matching may

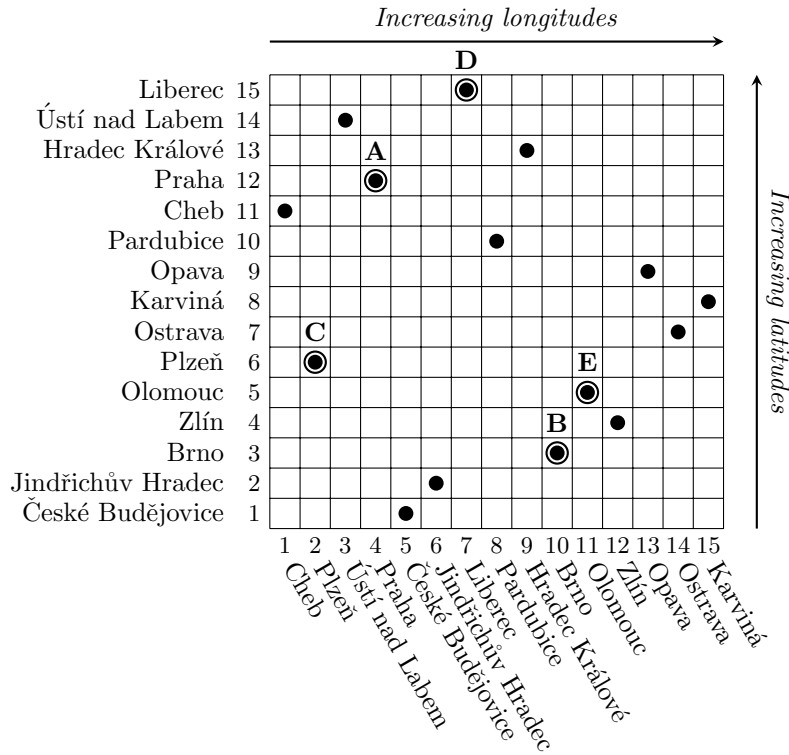


Figure 2 Permutation  $\pi = 11\ 6\ 14\ 12\ 1\ 2\ 15\ 10\ 13\ 3\ 5\ 4\ 9\ 7\ 8$  corresponding to the map from Figure 1. The solution of our puzzle, depicted Figure 1, is also represented.

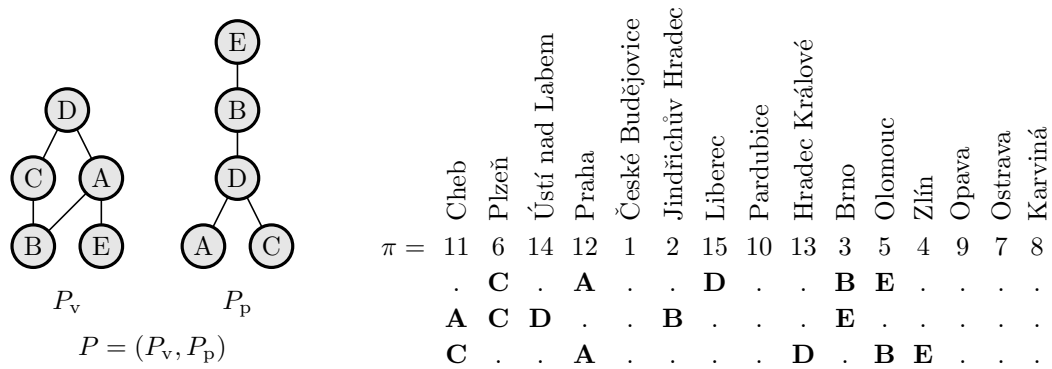


Figure 3 A dpop  $P = (P_v, P_p)$  representing the pattern for our puzzle, together with three distinct occurrences. Note that partial orders are represented by Hasse diagrams, i.e., a bottom-up path in  $P_v$  (resp.,  $P_p$ ) implies a bottom-up (resp., left-right) relation in the occurrence of  $P$  in  $\pi$ .

be forced to be adjacent [8]. Mesh patterns (a further generalization of bivincular patterns) impose further restrictions on the relative positions of the entries in an occurrence of a pattern [9] and boxed mesh patterns are special cases of mesh patterns [3]. Strongly related to our approach are partially ordered patterns that are vincular patterns in which the relative order of some elements is not fixed [22]. The best general reference is [23].

In this paper, we consider a new generalization of classical patterns in which both the

78 relative order and the relative positioning of some elements are not fixed. The idea is to  
 79 allow the possibility for some elements to be incomparable in value (i.e., their relative order  
 80 is unknown) and to go one step further by allowing the possibility for some elements to be  
 81 incomparable in position (i.e., their relative positioning in the occurrence is unknown). Since  
 82 the problem is clearly NP-hard (as it contains PERMUTATION PATTERN MATCHING as a  
 83 sub-problem), our goal is to identify tractable cases when restrictions apply to the pattern  
 84 and/or to the permutation.

85 The restrictions we consider here apply to the following parameters of the problem: size  
 86 of the pattern; symmetry (i.e., same partial order in positions and values); one partial order  
 87 is a total (resp., weak) order; size of the longest chain (resp., anti-chain) in the partial orders  
 88 (height and width); forbidden patterns in  $\pi$ . On the positive side, we show that the FPT  
 89 algorithm for PERMUTATION PATTERN MATCHING can be generalized to our setting (with  
 90 the pattern size as parameter). We further give polynomial-time algorithms when the pattern  
 91 is a symmetric disjoint union of a constant number of weak orders. Finally, we also provide  
 92 polynomial-time algorithms when the pattern is symmetric and the permutation belongs  
 93 to some restricted classes, such as (123, 132)-avoiding permutations. We complement these  
 94 positive results with NP- or W[1]-hardness proofs in most of the remaining cases.

### 95 3 Definitions

#### 96 Permutations and Patterns

97 A permutation  $\sigma$  is said to be *contained* in (or is a *sub-permutation* of) another permutation  $\pi$ ,  
 98 which we denote by  $\sigma \preceq \pi$ , if  $\pi$  has a (not necessarily contiguous) subsequence whose terms  
 99 are order-isomorphic to  $\sigma$ . We also say that  $\pi$  *admits an occurrence* of the *pattern*  $\sigma$ . If no  
 100 such subsequence exists, we say that  $\pi$  *avoids*  $\sigma$  (or is  $\sigma$ -*avoiding*). A permutation is *separable*  
 101 if it avoids both 2413 and 3142. PERMUTATION PATTERN MATCHING is the problem of  
 102 deciding whether a permutation is contained into another permutation.

103 For any non-negative integer  $n$ , we denote by  $[n]$  the set  $\{1, 2, \dots, n\}$ . When  $n \geq 1$ , we also  
 104 note  $\text{ip}(n) = 1\ 2\ \dots\ n$  the *increasing permutation* of length  $n$  and  $\text{dp}(n) = n\ (n-1)\ \dots\ 1$  the  
 105 *decreasing permutation* of length  $n$ . Let  $\pi \in \mathfrak{S}(n)$ . The *reverse* (resp., *complement*) of  $\pi$  is the  
 106 permutation  $\pi^r = \pi(n)\pi(n-1)\ \dots\ \pi(1)$  (resp.,  $\pi^c = (n-\pi(1)+1)(n-\pi(2)+1)\ \dots\ (n-\pi(n)+1)$ ).  
 107 The *inverse* of  $\pi$  is the permutation  $\pi^{-1} \in \mathfrak{S}(n)$  defined by  $\pi^{-1}(j) = i$  if and only if  $\pi(i) = j$ .  
 108 Given a permutation  $\pi$  of size  $m$  and a permutation  $\sigma$  of size  $n$ , the *skew sum* of  $\pi$  and  $\sigma$  is  
 109 the permutation of size  $m+n$  defined by

$$110 \quad (\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + n & \text{for } 1 \leq i \leq m, \\ \sigma(i - m) & \text{for } m + 1 \leq i \leq m + n, \end{cases}$$

111 and the *direct sum* of  $\pi$  and  $\sigma$  is the permutation of size  $m+n$  defined by

$$112 \quad (\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{for } 1 \leq i \leq m, \\ \sigma(i - m) + m & \text{for } m + 1 \leq i \leq m + n. \end{cases}$$

#### 113 Orders

114 A relation  $\leq$  is a *partial order* on a set  $X$  if it has:

- 115 ■ *reflexivity*: for all  $x \in X$ ,  $x \leq x$  (i.e., every element is related to itself);
- 116 ■ *transitivity*: for all  $x, x', x'' \in X$ , if  $x \leq x'$  and  $x' \leq x''$ , then  $x \leq x''$ ;

117 ■ *antisymmetry*: for all  $x, x' \in X$ , if  $x \leq x'$  and  $x' \leq x$ , then  $x = x'$  (i.e., no two distinct  
118 elements precede each other).

119 If  $\leq$  has the following additional property, we say that it is a *weak order* on  $X$ :

120 ■ *transitivity of incomparability*: for all  $x, x', x'' \in X$ , if  $x$  is incomparable with  $x'$  (i.e.,  
121 neither  $x \leq x'$  nor  $x' \leq x$  is true) and if  $x'$  is incomparable with  $x''$ , then  $x$  is incomparable  
122 with  $x''$ .

123 Two subsets  $X_1, X_2$  are *independent* if there is no  $x_1 \in X_1, x_2 \in X_2$  such that  $x_1 \leq x_2$   
124 or  $x_2 \leq x_1$ . We say that a partial order is *k-weak* if there exists a partition of  $X$  into  $k$   
125 pairwise independent sets  $X_1, \dots, X_k$  such that, for each  $i$ , the restriction of  $\leq$  to  $X_i$  is a  
126 weak order (in other words,  $\leq$  is the disjoint union of  $k$  weak partial orders).

127 Let  $\mathcal{P} = (X, \leq)$  be a finite partially ordered set. A *chain* in  $\mathcal{P}$  is a set of pairwise  
128 comparable elements (i.e., a totally ordered subset) and an *antichain* in  $\mathcal{P}$  is a set of pairwise  
129 incomparable elements. The *partial order height* of  $\mathcal{P}$ , denoted by  $\text{height}(\mathcal{P})$ , is defined as  
130 the maximum cardinality of a chain in  $\mathcal{P}$ , and the *partial order width* of  $\mathcal{P}$ , denoted by  
131  $\text{width}(\mathcal{P})$ , is defined as the maximum cardinality of an antichain in  $\mathcal{P}$ . By Dilworth Theorem,  
132  $\text{width}(\mathcal{P})$  is also the minimum number of chains in any partition of  $\mathcal{P}$  into chains. The *dual*  
133 of  $\mathcal{P}$  is the partial order  $\mathcal{P}^\partial = (X, \leq^\partial)$  defined by letting  $\leq^\partial$  be the converse relation of  $\leq$ ,  
134 i.e.,  $x \leq^\partial x'$  if and only if  $x' \leq x$ . The dual of a partial order is a partial order and the dual  
135 of the dual of a relation is the original relation. A *total order* is a partial order in which any  
136 two elements are comparable, and a set equipped with a total order is a *totally ordered set*.  
137 A *linear extension* of a partial order is a total order that is compatible with the partial order.  
138 It will be convenient to represent a linear extension of a poset  $\mathcal{P} = (X, \leq)$  as the mapping  
139  $\tau_{\mathcal{P}} : X \rightarrow [|X|]$  such that  $\tau_{\mathcal{P}}(i) < \tau_{\mathcal{P}}(j)$  if  $i < j$  in the linear extension.

140 A *doubly partially ordered pattern* (dpop)  $P$  is a pair, denoted by  $P = (P_v, P_p)$ , of posets  
141  $P_v = (X, \leq_v)$  and  $P_p = (X, \leq_p)$  defined over the same set  $X$ . We call  $P_v$  and  $P_p$  the *value*  
142 *poset* and the *position poset*, respectively. A dpop  $P = (P_v, P_p)$  is *symmetric* if  $P_v = P_p$ ,  
143 *dual* if  $P_v = P_p^\partial$ , and *semi-total* if one of  $P_p$  or  $P_v$  is a total order. We let  $\text{height}(P)$  and  
144  $\text{width}(P)$  stand for  $\max\{\text{height}(P_v), \text{height}(P_p)\}$  and  $\max\{\text{width}(P_v), \text{width}(P_p)\}$ , respect-  
145 ively. Finally, the *size* of  $P$  is defined as the cardinality  $|X|$  and is denoted by  $|P|$ .

146 ► **Definition 1** (DPOP MATCHING). *Given a permutation  $\pi \in \mathfrak{S}(n)$  and a dpop  $P = (P_v, P_p)$ ,*  
147 *an occurrence (or mapping) of  $P$  in  $\pi$  is an injective function  $\varphi : X \rightarrow [n]$  such that:*

148 ■  *$\pi \circ \varphi$  is  $\leq_v$ -non-decreasing, i.e., for all  $x, y \in X$ , if  $x \leq_v y$  then  $\pi(\varphi(x)) \leq \pi(\varphi(y))$ , and*  
149 ■  *$\varphi$  is  $\leq_p$ -non-decreasing, i.e., for all  $x, y \in X$ , if  $x \leq_p y$  then  $\varphi(x) \leq \varphi(y)$ .*

150 *The DPOP MATCHING problem consists in deciding whether  $P$  occurs in  $\pi$ .*

## 151 First Observations

152 ► **Observation 2.** *PERMUTATION PATTERN MATCHING is the special case of DPOP MATCH-*  
153 *ING where both  $\leq_v$  and  $\leq_p$  are total orders.*

154 We note that applying a vertical and/or horizontal symmetry on both pattern and  
155 permutation does not alter the existence of an occurrence.

156 ► **Observation 3.** *Let  $P = (P_v, P_p)$  be a dpop and  $\pi$  be a permutation. The following*  
157 *statements are equivalent:*

- |  |  |
|--|--|
| 158 1. $(P_v, P_p)$ occurs in $\pi$ ;              |  |
| 159 2. $(P_v, (P_p)^\partial)$ occurs in $\pi^r$ ; | 4. $((P_v)^\partial, (P_p)^\partial)$ occurs in $\pi^{cr}$ ; |
| 160 3. $((P_v)^\partial, P_p)$ occurs in $\pi^c$ ; | 5. $(P_p, P_v)$ occurs in $\pi^{-1}$ .                       |

161 The following reformulation will prove useful.

162 ► **Observation 4.** *Let  $P = (P_V, P_P)$  be a dpop with  $P_V = (X, \leq_V)$ ,  $P_P = (X, \leq_P)$  and  $k = |X|$ ,  
163 and let  $\pi \in \mathfrak{S}(n)$  be a permutation. The following statements are equivalent:*

164 ■  *$P$  occurs in  $\pi$ .*

165 ■ *There exists a linear extension  $\tau_V : X \rightarrow [k]$  of  $P_V$  and a linear extension  $\tau_P : X \rightarrow [k]$   
166 of  $P_P$  such that the permutation  $\sigma \in \mathfrak{S}(k)$  defined by  $\sigma(i) = \tau_V(\tau_P^{-1}(i))$  for  $1 \leq i \leq k$  is  
167 contained in  $\pi$ .*

168 The rationale for the reformulation introduced in Observation 4 stems from the following  
169 corollary that sets the general context.

170 ► **Corollary 5 ([16]).** *DPOP MATCHING is FPT for the parameter  $|P|$ .*

171 Indeed, it is enough to guess two linear extensions  $\tau_V : X \rightarrow [k]$  of  $P_V$  and  $\tau_P : X \rightarrow [k]$   
172 of  $P_P$ , and to check if the permutation  $\sigma \in \mathfrak{S}(k)$  defined by  $\sigma(i) = \tau_V(\tau_P^{-1}(i))$  for  $1 \leq i \leq k$   
173 is contained in  $\pi$ . There are  $O(k!^2)$  pairs of such extensions and, for each of them, one can  
174 check whether  $\sigma$  occurs in  $\pi$  in  $n 2^{O(k^2 \log k)}$  time [16].

## 175 4 Semi-Total Patterns

176 In this section we focus on semi-total patterns, i.e., without loss of generality, on the case  
177 where  $P_P$  is a total order (up to symmetry by Observation 3). This case still contains  
178 PERMUTATION PATTERN MATCHING as a special case, and is thus NP-hard. We focus on  
179 small-height value partial orders, i.e., on dpop with constant  $\text{height}(P_V)$ , and give an XP  
180 algorithm for weak orders (Proposition 6) and paraNP-hardness in general (Proposition 7).

181 ► **Proposition 6.** *DPOP MATCHING is solvable in  $O(n^{\text{height}(P_V)})$  time if  $P_V$  is a weak order  
182 and  $P_P$  is a total order.*

183 **Proof.** Let  $\pi \in \mathfrak{S}(n)$  be a permutation and  $P = (P_V, P_P)$  be a dpop on some ground set  $X$ ,  
184 where  $P_V$  is a weak order and  $P_P$  is a total order. Without loss of generality, we assume  
185 that  $X$  is the set  $[k]$  and that  $P_P$  is the usual order on integers. For every  $x \in X$ , we abusively  
186 denote by  $\text{height}(x)$  the maximum cardinality of a chain with maximum element  $x$  in  $P_V$ .  
187 Finally, set  $\ell = \text{height}(P_V)$ .

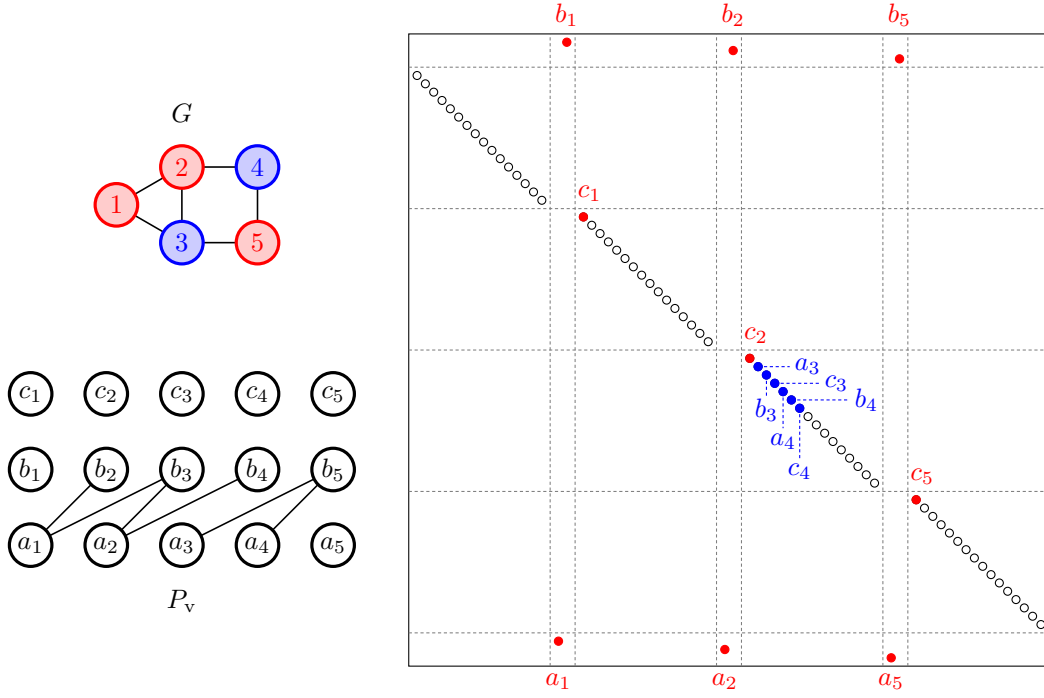
188 For any two distinct variables  $x, y \in X$ , we have  $x <_V y$  if and only if  $\text{height}(x) < \text{height}(y)$ .  
189 Thus,  $P$  occurs in  $\pi$  if and only there exists a sequence  $0 = a_0 < a_1 < a_2 < \dots < a_\ell = n$  such  
190 that  $w_\sigma$  is a subsequence of  $w_\pi$ , where  $w_\pi \in [\ell]^n$  and  $w_\sigma \in [\ell]^k$  are the two words defined  
191 by  $w_\pi[i] = \min\{j : a_{j-1} < \pi(i) \leq a_j\}$  and  $w_\sigma[i] = \text{height}(i)$ .

192 As for the running time, there exist  $\binom{n-1}{\ell-1}$  distinct sequences  $(a_i)_{0 \leq i \leq \ell}$  and deciding  
193 whether  $w_\sigma$  occurs in  $w_\pi$  as a subsequence is a linear-time procedure. ◀

194 ► **Proposition 7.** *DPOP MATCHING is NP-complete even if  $\text{height}(P_V) = 2$ ,  $P_P$  is a total  
195 order and  $\pi$  avoids 1234.*

196 **Proof.** We perform a reduction from VERTEX COVER, which is known to be NP-complete [21].  
197 Let  $G = (V, E)$  be a graph and let  $k$  be a positive integer. We identify  $V$  with the set  $[n]$ .

198 We construct a dpop  $P = (P_V, P_P)$ , where  $P_V = (X, \leq_V)$  is a height-2 partial order and  
199  $P_P = (X, \leq_P)$  a total order, as follows. We set  $X = \{a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_n, b_n, c_n\}$ , so  
200 that  $|X| = 3n$ . Then, we set  $a_1 \leq_P b_1 \leq_P c_1 \leq_P a_2 \leq_P b_2 \leq_P c_2 \leq_P \dots \leq_P a_n \leq_P b_n \leq_P c_n$ ,  
201 which defines the total order  $\leq_P$ . Finally, for each edge  $\{i, j\}$  in  $E$  with  $i < j$ , we set  $a_i \leq_V b_j$ ;



■ **Figure 4** Top left: vertex cover (in red) in a 5-vertex graph  $G$ . Bottom left: partially ordered set  $P_V$  constructed from  $G$ . Right: an occurrence of  $P$  in  $\pi$  obtained from our size-3 vertex cover.

202 all other elements of  $X$  are pairwise incomparable by  $\leq_v$ . This defines a partial order  $\leq_v$   
 203 such that  $\text{height}(\leq_v) = 2$ .

204 Write now  $N = 3n + 3$  and  $m = (k + 1)N - 2$ , and define a permutation  $\pi \in \mathfrak{S}(m)$  as  
 205 follows:

- 206 ■  $\pi(iN + j) = (m + 1) - (iN + j + k)$  whenever  $0 \leq i \leq k$  and  $1 \leq j \leq N - 2$ ;
- 207 ■  $\pi(iN - 1) = (k + 1) - i$  and  $\pi(iN) = (m + 1) - i$  whenever  $1 \leq i \leq k$ .

208 It is straightforward to check that  $\pi$  is 1234-avoiding. It is also easy to see how the  
 209 construction, illustrated in Figure 4, can be accomplished in polynomial-time.

210 Let us see under which conditions an injective  $\leq_p$ -non-decreasing function  $\varphi: X \rightarrow [m]$   
 211 maps  $P$  into  $\pi$ . We say that a vertex  $i$  belongs to the  $j$ -th gadget if one of the integers  $\varphi(a_i)$   
 212 or  $\varphi(b_i)$  is equal to  $jN - 1$  or to  $jN$ , i.e., if  $\{\varphi(a_i), \varphi(b_i)\} \cap \{jN - 1, jN\} \neq \emptyset$ . When two  
 213 elements in the range of  $\varphi$  are consecutive, either they are integers  $\varphi(a_i)$  and  $\varphi(b_i)$  for a  
 214 given  $i$ , or one of them is an integer  $\varphi(c_i)$  for some  $i$ . Therefore, no two distinct vertices  $i$   
 215 and  $i'$  can belong to the same  $j$ -th gadget. Consequently, and since there are  $k$  gadgets, the  
 216 set  $V'$  of vertices  $i$  that belong to some gadget is of size at most  $k$ .

217 Then, we define a notion of *height* as follows: for each element  $x$  of  $X$ , we set  $\text{height}(x) = 0$   
 218 if  $N$  divides  $\varphi(x) + 1$ ,  $\text{height}(x) = 2$  if  $N$  divides  $\varphi(x)$ , and  $\text{height}(x) = 1$  otherwise. By  
 219 construction, for all  $x, y \in X$  such that  $x \leq_p y$ , we have  $\pi(\varphi(x)) \leq \pi(\varphi(y))$  if and only if  $x$   
 220 is of smaller height than  $y$ . Therefore, if  $\varphi$  maps  $P$  into  $\pi$ , and for each relation  $a_i \leq_v b_j$ ,  
 221 either  $a_i$  has height 0 or  $b_j$  has height 2. In particular, either  $i$  or  $j$  must belong to  $V'$ , and  
 222 therefore  $V'$  is a vertex cover of size at most  $k$ .

223 Conversely, provided that there exist vertices  $v(1) < v(2) < \dots < v(k)$  that form a vertex  
 224 cover  $V'$ , we construct an occurrence of  $P$  in  $\pi$  as follows. First, we abusively set  $v(0) = 0$ .  
 225 Then, for all  $i \in [k]$ , we set  $f(i) = jN + 3(i - v(j))$ , where  $j$  is the largest integer such



226 that  $v(j) \leq i$ . We set  $\varphi(a_j) = f(j) - 1$ ,  $\varphi(b_j) = f(j)$  and  $\varphi(c_j) = f(j) + 1$ .

227 By construction, we have  $f(i) + 3 \leq f(i + 1)$  for all  $i$ , and therefore  $\varphi$  is an injective  
 228  $\leq_p$ -non-decreasing function. Moreover, for every  $i \in [n]$ , the elements  $a_i$  and  $b_i$  have heights 0  
 229 and 2 if  $i \in V'$ , and they have height 1 if  $i \notin V'$ . It follows that  $\pi(\varphi(a_i)) \leq \pi(\varphi(b_j))$  whenever  
 230  $a_i \leq_v b_j$ , i.e., that  $\varphi$  is an occurrence of  $P$  in  $\pi$ . ◀

## 231 5 Symmetric Patterns

232 This section is devoted to studying complexity issues of pattern matching for symmetric  
 233 dpop (i.e., those dpop  $P = (\mathcal{P}, \mathcal{P})$ , whose value and position posets coincide). We further  
 234 focus on two special cases, first when  $\mathcal{P}$  has a bounded width, then when  $\pi$  is restricted to  
 235 constrained pattern-avoiding classes of permutations.

### 236 5.1 Symmetric Pattern with Bounded Width

237 We first observe that the problem is polynomial for width 1 (Observation 8). We further prove  
 238  $W[1]$ -hardness for the parameter  $k$  when  $\mathcal{P}$  is a disjoint union of  $k$  chains (Proposition 10).  
 239 We complement this result with an XP algorithm for the slightly more general case where  $\mathcal{P}$  is  
 240 a disjoint union of weak orders (Proposition 11). Note that the existence of an XP algorithm  
 241 for the width parameter remains open, and we conjecture that the problem is NP-hard even  
 242 for constant width.

243 ▶ **Observation 8.** *DPOP MATCHING is solvable in  $O(n \log \log |P|)$  time for a symmetric*  
 244 *dpop  $P$  of width 1 (i.e., a total symmetric dpop  $P$ ).*

245 **Proof.** If  $P$  has width 1, then  $P = (\mathcal{P}, \mathcal{P})$  for some total order  $\mathcal{P} = (X, \leq)$ . In particular,  
 246 we can write  $X = \{x_1, \dots, x_{|X|}\}$  with  $x_i \prec x_j$  for  $i < j$ , and in any mapping  $\phi : X \rightarrow [n]$ ,  
 247 the elements  $\pi_{\phi(x_1)}, \dots, \pi_{\phi(x_{|X|})}$  must form an increasing subsequence of  $\pi$ . Conversely, any  
 248 size- $|X|$  increasing subsequence of  $\pi$  can be used as an image for  $\phi$ , so in this setting DPOP  
 249 MATCHING corresponds to the longest increasing subsequence problem, which can be solved  
 250 in  $O(n \log \log |X|)$  time [12]. ◀

251 To simplify the exposition of our next result, we introduce a new problem that may be of  
 252 independent interest. Given a positive integer  $k$  and a permutation  $\pi \in \mathfrak{S}(kn)$ , BALANCED  
 253  $k$ -INCREASING COLORING is the problem of deciding whether there exists a balanced  $k$ -  
 254 coloring of  $\pi$  (i.e., a partition of  $[kn]$  into  $k$  subsets of size exactly  $n$ ) such that each color  
 255 induces an increasing subsequence of  $\pi$ .

256 ▶ **Proposition 9.** *BALANCED  $k$ -INCREASING COLORING for 312-avoiding permutations is*  
 257  *$W[1]$ -hard for the parameter  $k$ .*

258 **Proof.** We perform a reduction from UNARY BIN PACKING parameterized by the number of  
 259 bins, which is known to be  $W[1]$ -hard [19]. In this version of BIN PACKING, we are given  
 260 a list of integers  $s_1, s_2, \dots, s_n$  encoded in unary, and two integers  $B$  and  $k$ . These integers  
 261 are interpreted as item sizes, and the task is to decide whether the items can be partitioned  
 262 into  $k$  subsets, each of total size  $B$ . We show that there is a reduction from UNARY BIN  
 263 PACKING, parameterized by the number of bins, to BALANCED  $k$ -INCREASING COLORING,  
 264 parameterized by the number of colors.

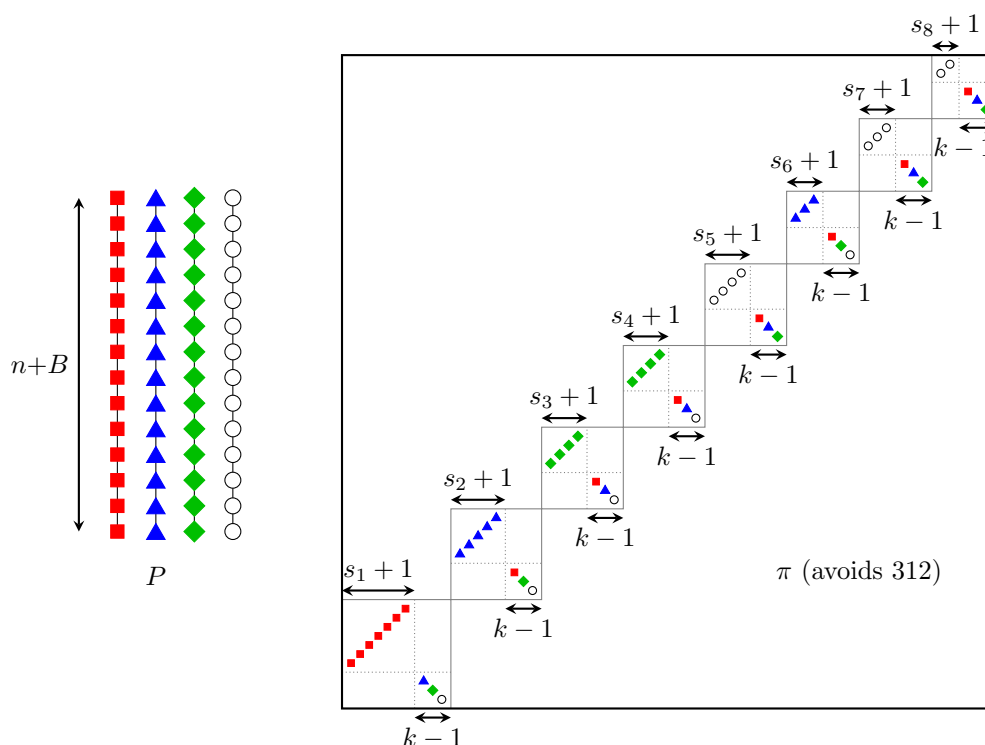


Figure 5 Reduction from UNARY BIN PACKING to BALANCED  $k$ -INCREASING COLORING for the list  $6, 4, 3, 3, 3, 2, 2, 1$ , which admits the partition  $(\{6\}, \{4, 2\}, \{3, 3\}, \{3, 2, 1\})$ , and the integers  $B = 6, k = 4$ . Left: dpop  $P$  that consists of  $k$  chains, each of length  $n + B$ . Right: 312-avoiding permutation  $\pi$  that consists of  $n$  blocks. Each color/shape induces an increasing subsequence of  $\pi$ .

265 Consider an arbitrary instance of UNARY BIN PACKING containing  $n$  items with item  
 266 sizes  $S = \{s_1, s_2, \dots, s_n\}$ , and two integers  $B$  and  $k$ . Define  $\pi \in \mathfrak{S}(kB + kn)$  by

$$267 \quad \pi = \bigoplus_{i=1}^n (\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)).$$

268 Each pattern  $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$  is called the  $i$ -th block of  $\pi$ . See Figure 5 for an illustration.  
 269 It is straightforward to check that  $\pi$  is 312-avoiding.

270 We claim that the  $n$  items  $s_1, s_2, \dots, s_n$  can be partitioned into  $k$  subsets, each of total  
 271 size  $B$ , if and only if there exists a  $k$ -coloring of  $\pi$  such that each color induces an increasing  
 272 pattern of length  $B + n$ .

273 Suppose first that the  $n$  items  $s_1, s_2, \dots, s_n$  can be partitioned into  $k$  subsets, each of  
 274 total size  $B$ . Write  $S = S_1 \cup S_2 \cup \dots \cup S_k$  such a partition. Define a  $k$ -coloring of  $\pi$  as follows.  
 275 Consider the  $i$ -th block  $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$  of  $\pi$ , and suppose that  $s_i \in S_j$ . Color the  
 276 whole ascending pattern  $\text{ip}(s_i + 1)$  with color  $c_j$  and arbitrarily color the elements of the  
 277 descending pattern  $\text{dp}(k - 1)$  with the remaining  $k - 1$  colors (each element of  $\text{dp}(k - 1)$  is  
 278 assigned to a distinct color). We claim that every color  $c_j$  induces an increasing pattern of  
 279 length  $B + n$  in  $\pi$ . First, it is clear that the above  $k$ -coloring induces increasing patterns only.  
 280 As for the length of each induced increasing pattern, focus on any color  $c_j$ . We note that,  
 281 in every block  $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$  of  $\pi$ , either the whole subpattern  $\text{ip}(s_i + 1)$  is colored  
 282 with color  $c_j$  (if  $s_i \in S_j$ ) or exactly one element of the subpattern  $\text{dp}(k - 1)$  is colored with

283 color  $c_j$  (if  $s_i \notin S_j$ ). It follows that the increasing pattern induced by color  $c_j$  in  $\pi$  has length  
 284  $\sum_{s_i \in S_j} (s_i + 1) + n - |S_j| = \sum_{s_i \in S_j} s_i + |S_j| + n - |S_j| = B + n$ .

285 For the reverse direction, suppose now that there exists a  $k$ -coloring of  $\pi$  such that each  
 286 color induces an increasing pattern of length  $B + n$ . Every block  $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$   
 287 requires at least  $k$  colors, as it contains a decreasing subpattern of length  $k$ . Therefore, the  
 288 whole subpattern  $\text{ip}(s_i + 1)$  is colored with the same color. For every  $j \leq k$ , let  $S_j$  be the set  
 289 of all  $s_i$  such that, in the  $i$ -th block  $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$ , the subpattern  $\text{ip}(s_i + 1)$  is colored  
 290 with color  $c_j$ . We have  $B + n = \sum_{s_i \in S_j} (s_i + 1) + n - |S_j| = \sum_{s_i \in S_j} s_i + |S_j| + n - |S_j|$ , and  
 291 hence  $\sum_{s_i \in S_j} s_i = B$ . Therefore, the  $n$  items  $s_1, s_2, \dots, s_n$  can be packed into  $k$  bins, each  
 292 of capacity  $B$ . ◀

293 Most of the interest in Proposition 9 stems from the following proposition.

294 ▶ **Proposition 10.** *DPOP MATCHING for symmetric dpop and 312-avoiding permutations is*  
 295 *W[1]-hard for the parameter width( $P$ ).*

296 **Proof.** We perform a reduction from BALANCED  $k$ -INCREASING COLORING, which is W[1]-  
 297 hard for the parameter  $k$ . Let  $\pi \in \mathfrak{S}(kn)$  for some positive integers  $k$  and  $n$ . We construct a  
 298 symmetric dpop  $P = (\mathcal{P}, \mathcal{P})$ , where  $\mathcal{P} = (X, \preceq)$ , as follows:  $X = [k] \times [n]$  and  $(i, j) \preceq (i', j')$   
 299 if and only if  $i = i'$  and  $j \leq j'$ . We claim that  $P$  occurs in  $\pi$  if and only if  $\pi$  admits a  $k$ -coloring  
 300 for which every color induces an increasing pattern of length  $n$ .

301 If  $\pi$  admits such a  $k$ -coloring into colors  $c_1, c_2, \dots, c_k$ , the function  $\varphi: X \rightarrow [kn]$  that  
 302 maps each pair  $(i, j)$  to the  $j$ -th smallest position with color  $c_i$  is an occurrence of  $P$  in  $\pi$ .

303 Conversely, suppose that some injective function  $\varphi: X \rightarrow [kn]$  is an occurrence of  $P$  in  $\pi$ .  
 304 For each  $i \leq k$ , the set  $\{i\} \times [n]$  forms a chain of  $\preceq$ , and therefore it is mapped to an increasing  
 305 pattern of size  $n$ . Coloring this pattern in color  $c_i$  produces the desired  $k$ -coloring. ◀

306 We show now that the problem where  $P$  consists of  $k$  independent chains is XP for the  
 307 parameter  $k$ . In fact, we generalize this result to  $k$ -weak partial orders (i.e., if  $P$  consists  
 308 of  $k$  independent weak orders).

309 ▶ **Proposition 11.** *DPOP MATCHING for  $k$ -weak symmetric dpop is XP with parameter  $k$ .*

310 **Proof.** Let  $P$  be a disjoint union of  $k$  weak symmetric dpop  $P_1, P_2, \dots, P_k$ . For each dpop  $P_i$ ,  
 311 let  $\preceq_i$  be a linear extension of  $P_i$ , and let  $P_{i,1}, P_{i,2}, \dots, P_{i,p_i}$  be the maximal antichains of  $P_i$ ,  
 312 ordered by  $\preceq_i$ . Finally, for each  $k$ -tuple  $\mathbf{a} = (a_1, a_2, \dots, a_k)$  of integers such that  $a_i \leq |P_i|$ ,  
 313 we denote by  $P_{\mathbf{a}}$  the dpop obtained from  $P$  by removing the  $a_i$   $\preceq_i$ -least elements of each  
 314 dpop  $P_i$ , and by  $P_{i,\mathbf{a}}^{\min}$  the antichain  $P_{i,j}$  that contains elements from  $P_{\mathbf{a}}$  and for which  $j$  is  
 315 minimal, or  $\emptyset$  if no such antichain exists.

316 Then, given a permutation  $\pi \in \mathfrak{S}(n)$ , a  $k$ -tuple  $\mathbf{I} = (I_1, \dots, I_k)$  of intervals of  $[n]$ , a  
 317  $k$ -tuple  $\mathbf{a}$  and an integer  $\ell$ , a function  $\varphi: P_{\mathbf{a}} \rightarrow \{\ell, \ell + 1, \dots, n\}$  is called a *partial matching*  
 318 for  $(\pi, \mathbf{I}, \mathbf{a}, \ell)$  if:

- 319 ■  $\varphi$  and  $\pi \circ \varphi$  are  $\preceq_i$ -non-decreasing for each  $i$ , and
- 320 ■ for each  $i$ , and each element  $x$  of  $P_{\mathbf{a}}$ ,  $\pi(x) \in I_i$  if and only if  $x \in P_{i,\mathbf{a}}^{\min}$ .

321 Before going further, we denote by  $\mathbf{1}_i$  the  $k$ -tuple with one element 1 (in position  $i$ ) and  
 322  $k - 1$  elements 0. We also denote by  $<_i$  the partial order on tuples  $\mathbf{I}$  of intervals, where  
 323  $\mathbf{I} <_i \mathbf{I}'$  if  $I_j = I'_j$  whenever  $j \neq i$  and  $x < x'$  whenever  $x \in I_i$  and  $x' \in I'_i$ .

324 When  $a_i = |P_i|$  for all  $i$ , such a partial matching exists for all permutations  $\pi$ , tuples  
 325 of intervals  $\mathbf{I}$  and integers  $\ell$ . When  $a_i = 0$  for all  $i$  and  $\ell = 1$ , and once  $\pi$  is fixed, such  
 326 partial matchings coincide with (standard) matchings, and thus we are interested in checking

327 whether a partial matching exists. Finally, for all tuples  $\mathbf{I}$  and  $\mathbf{a}$  and for all  $\ell \leq n$ , a partial  
 328 matching  $\varphi$  for  $(\pi, \mathbf{I}, \mathbf{a}, \ell)$  exists precisely when one of the following cases occur:

- 329 1.  $\varphi$  is a partial matching for  $(\pi, \mathbf{I}, \mathbf{a}, \ell + 1)$ , i.e.,  $\ell \notin \varphi(P_{\mathbf{a}})$ ;
- 330 2. there exists an integer  $i \leq k$  for which the  $\preceq_i$ -least element of  $P_{i, \mathbf{a}}^{\min}$ , say  $x$ , is such that  
 331  $\varphi(x) = \ell$  and  $\pi(\ell) \in I_i$ , and either  
 332  $\text{-- } x$  is not the only element of  $P_{i, \mathbf{a}}^{\min}$ , and  $\varphi$  is a partial matching for  $(\pi, \mathbf{I}, \mathbf{a} + \mathbf{1}_i, \ell + 1)$ , or  
 333  $\text{-- } x$  is the only element of  $P_{i, \mathbf{a}}^{\min}$  and there exists a tuple  $\mathbf{I}' \succ_i \mathbf{I}$  such that  $\varphi$  is a partial  
 334 matching for  $(\pi, \mathbf{I}', \mathbf{a} + \mathbf{1}_i, \ell + 1)$ .

335 Consequently, we can compute by dynamic programming the list of triples  $(\mathbf{I}, \mathbf{a}, \ell)$  such  
 336 that there exists a partial matching for  $(\pi, \mathbf{I}, \mathbf{a}, \ell)$ : deciding whether adding a triple  $(\mathbf{I}, \mathbf{a}, \ell)$  to  
 337 the list simply requires to check which triples of the form  $(\mathbf{I}', \mathbf{a}', \ell + 1)$  already belong to the  
 338 list. Since there are less than  $n^{3k+1}$  triples, this provides us with a  $\tilde{O}(n^{6k+2})$  algorithm.  $\blacktriangleleft$

## 339 5.2 Symmetric Pattern and Pattern-Avoiding $\pi$

340 In this final section, we consider restrictions on the *shape* of  $\pi$ , via pattern-avoiding restrictions.  
 341 Our goal here is to identify tractable cases among classes of permutations avoiding one or  
 342 more size-3 patterns. We give an almost complete dichotomy of polynomial/NP-hard cases,  
 343 as shown in Table 1. Hardness results are proven in Proposition 12, and also apply to height-2  
 344 dpops. Polynomial cases are proven in Proposition 13 and apply to dpops of any height.

345 **► Proposition 12.** *DPOP MATCHING for height-2 symmetric dpop  $P$  and permutation  $\pi$  is*  
 346 *NP-hard even if  $\pi$  is separable (it avoids 2413 and 3142) and one of the following restrictions*  
 347 *occurs:*

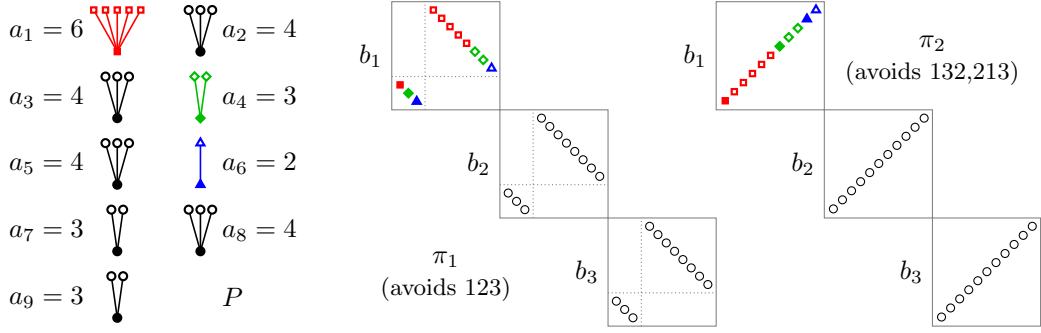
- 348 1.  $\pi$  is 123-avoiding;
- 349 2.  $\pi$  is (132, 213)-avoiding;
- 350 3.  $\pi$  is (132, 321)-avoiding;
4.  $\pi$  is (231, 312)-avoiding;
5.  $\pi$  is (132, 312)-avoiding;
6.  $\pi$  is (213, 321)-avoiding;
7.  $\pi$  is (213, 312)-avoiding;
8.  $\pi$  is (132, 231)-avoiding;
9.  $\pi$  is (213, 231)-avoiding.

351 **Proof.** In each of the cases presented below, we define a symmetric dpop  $P = (\mathcal{P}, \mathcal{P})$  for  
 352 some partially ordered set  $\mathcal{P} = (X, \preceq)$ . Each time, we identify  $P$  with the partial order  $\preceq$ .

353 **Case 1:**  $\pi$  is 123-avoiding and separable. We use a reduction from 3-PARTITION, as illustrated  
 354 in Figure 6 with permutation  $\pi_1$ . Let  $(A, B)$  be an instance of 3-PARTITION, where  $A$  is a  
 355 list of integers  $a_1, a_2, \dots, a_{3n}$  with sum  $nB$ , all being larger than 1.

$\pi$ avoids	123	132	213	231	312	321
... and ...						
123	12.1 3P					
132	13.2					
213	13.5	12.2 BIN				
231	13.1	12.8 BIC	12.9 BIC			
312	13.4	12.5 BIC	12.7 BIC	12.4 BIC		
321	13.3	12.3 BIS	12.6 BIS	(open)		

**■ Table 1** Polynomial (green/light) and NP-hard (red/dark) cases for DPOP MATCHING with symmetric dpop and pattern-avoiding permutation  $\pi$ , for combinations of size-3 avoided patterns. For each case, see the referenced proposition and case for more details. Diagonal cases follow from any other hard case in the same row or column. For hard cases, the problem used for reduction is indicated as follows: BIC: Biclique, 3P: 3-PARTITION, BIN: UNARY BIN PACKING, BIS: BISECTION.



■ **Figure 6** Reductions from 3-PARTITION and UNARY BIN PACKING to DPOP MATCHING on 123-avoiding and (132, 213)-avoiding permutations. Left: the height-2 dpop  $P$  used in both reductions. Right: the 123-avoiding and (132, 213)-avoiding permutations used in each reduction. The mapping of three subsets of  $P$  corresponding to a first bin gadget is highlighted in each figure.

356 For all  $p \leq n$ , we define a *bin gadget*  $b_p$  as the permutation  $\text{dp}(3) \oplus \text{dp}(B-3)$ : we see  
 357 this gadget as consisting of two parts. Our permutation  $\pi$  is now defined by  $\pi = \bigoplus_{p=1}^n b_p$ .  
 358 Then, our partial order  $\preceq$  is defined on a set  $X$  of  $nB$  elements, noted  $x_i, y_{i,2}, \dots, y_{i,a_i}$  for  
 359 each  $i \leq 3n$ , so that  $x_i \preceq y_{i,j}$  for all  $i$  and  $j$ .

360 If  $P$  has an occurrence  $\varphi: X \rightarrow [nB]$  in  $\pi$ , this occurrence is bijective. Moreover, each  
 361 element  $x_i$  is sent to the bottom-left of  $y_{i,2}$ , and thus it must be mapped to the left part of  
 362 some gadget, say  $b_{f(i)}$ . Each element  $y_{i,j}$  must then be mapped to the right part of the same  
 363 gadget. Now, for each  $p \leq n$ , the set  $S_p = \{i: f(i) = p\}$  is of size 3, and exactly  $B$  elements  
 364 of  $X$  are mapped to the gadget  $b_p$ , which means that  $\sum_{i \in S_p} a_i = B$ . Moreover,  $S_1 \cup \dots \cup S_p$   
 365 forms a partition of  $[3n]$ , hence it yields a 3-partition of  $A$ .

366 Conversely, given a partition  $S_1 \cup \dots \cup S_n$  of  $[3n]$  such that  $|S_p| = 3$  and  $\sum_{i \in S_p} a_i = B$   
 367 for each  $p$ , we build an occurrence of  $P$  in  $\pi$  by mapping the three elements  $x_i$  (for  $i \in S_p$ )  
 368 to the left part of  $b_p$ , and the  $B-3$  elements  $y_{i,j}$  (for  $i \in X_p$ ) to the right part of  $b_p$ .

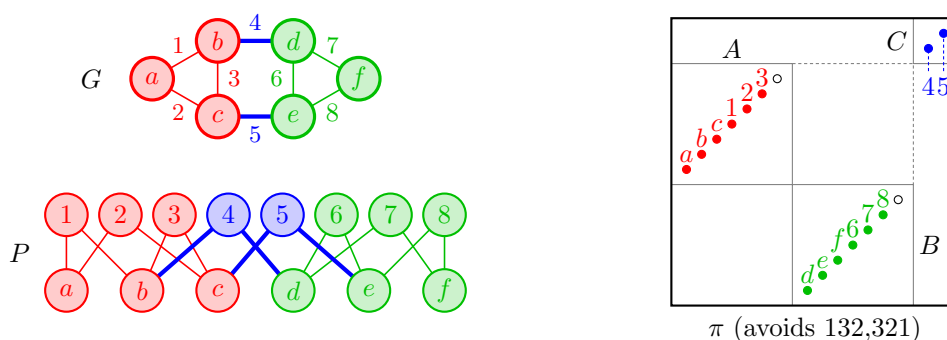
369 **Case 2:**  $\pi$  is (132, 213)-avoiding. We use a reduction from UNARY BIN PACKING, as  
 370 illustrated in Figure 6 with permutation  $\pi_2$ . Given an instance  $(A, B, k)$  of UNARY BIN  
 371 PACKING, where  $A$  is a list of integers  $a_1, a_2, \dots, a_n$  larger than 1, we use the same dpop as  
 372 in Case 1: our partial order  $\preceq$  is defined on a set  $X$  of  $nB$  elements, noted  $x_i, y_{i,2}, \dots, y_{i,a_i}$   
 373 for each  $i \leq n$ , so that  $x_i \preceq y_{i,j}$  for all  $i$  and  $j$ . However, this time, our gadget  $b_p$  is the  
 374 permutation  $\text{ip}(B)$ , and our permutation  $\pi$  is again defined by  $\pi = \bigoplus_{p=1}^n b_p$ .

375 If  $P$  has an occurrence  $\varphi: X \rightarrow [nB]$  in  $\pi$ , this occurrence is bijective. Each element  $x_i$   
 376 is sent to some gadget, say  $b_{f(i)}$ , and the elements  $y_{i,j}$  must then be mapped to the same  
 377 gadget. Now, for each  $p \leq k$ , let  $S_p = \{i: f(i) = p\}$ . Exactly  $B$  elements of  $X$  are mapped  
 378 to the gadget  $b_p$ , which means that  $\sum_{i \in S_p} a_i = B$ . This means that  $(S, B, k)$  is a positive  
 379 instance of the UNARY BIN PACKING problem.

380 Conversely, given a partition  $S_1 \cup \dots \cup S_k$  of  $[n]$  such that  $\sum_{i \in S_p} a_i = B$  for each  $i$ , we  
 381 build an occurrence of  $P$  in  $\pi$  by mapping the  $B$  elements  $x_i$  and  $y_{i,j}$  (for  $i \in S_p$ ) to  $b_p$ .

382 **Case 3:**  $\pi$  is (132, 321)-avoiding. We use a reduction from BISECTION, as illustrated in  
 383 Figure 7. Given a graph  $G = (V, E)$  and an integer  $k$ , the BISECTION problem consists in  
 384 deciding whether  $V$  admits a partition  $V_1 \cup V_2$  such that  $|V_1| = |V_2|$  and that *splits* at most  
 385  $k$  edges (i.e., at most  $k$  edges have one endpoint in  $V_1$  and one endpoint in  $V_2$ ).

386 Our reduction is as follows. Let  $n = |V|/2$ ,  $m = |E|$ ,  $W = m + k + 1$ , and  $L = nW + m$ .  
 387 Our partition  $\pi$  is defined by  $\pi = (\text{ip}(L) \oplus \text{ip}(L)) \oplus \text{ip}(k)$ . These three parts of  $\pi$  are noted  $A$ ,  
 388  $B$  and  $C$ , from left to right. Then, our partial order  $\preceq$  is defined on a set  $X$  of  $2nW + m$



■ **Figure 7** Simplified version (for  $W = 1$  and  $L = 7$ ) of the reduction from BISECTION to DPOP MATCHING on  $(132, 321)$ -avoiding permutations. Top left: a size-6 graph with a bisection  $(\{a, b, c\}, \{d, e, f\})$  that splits  $k = 2$  edges. Bottom left: height-2 dpop  $P$  (note that, in general, each element from the bottom line should appear  $W$  times and not just once). Right: permutation  $\pi$ ; elements of  $P$  are mapped to  $A$ ,  $B$  or  $C$  depending on whether they are colored in red, green or blue.

389 elements:  $2nW$  elements, noted  $x_{v,i}$  for each  $v \in V$  and  $i \leq W$ , and  $m$  elements, noted  $y_e$   
 390 for each  $e \in E$ . This order contains the relations  $x_{v,i} \preceq y_e$  for which  $v$  is an endpoint of  $e$ .

391 Assume that there exists a mapping of  $P$  into  $\pi$ . For each  $v \in V$ , and since  $C$  has  
 392 size  $k < W$ , at least one of the elements  $x_{v,i}$  is mapped to  $A$ , in which case we say that  $v$   
 393 has type  $A$ , or to  $B$ , in which case  $v$  has type  $B$ . Then, each vertex has at least one type,  
 394 and possibly both. We partition  $V$  into three sets  $V_A, V_B, V_{AB}$  containing the vertices of  
 395 type  $A$ ,  $B$  and both  $A$  and  $B$ , respectively. Moreover, for each  $v \in V_A$ , each element  $x_{v,i}$   
 396 must be mapped either to  $A$  or to  $C$ : these two parts together contain  $L + k$  elements,  
 397 so  $|V_A| \leq (L + k)/W = n + 1 - 1/W$ , and  $|V_A| \leq n$ . Similarly,  $|V_B| \leq n$ .

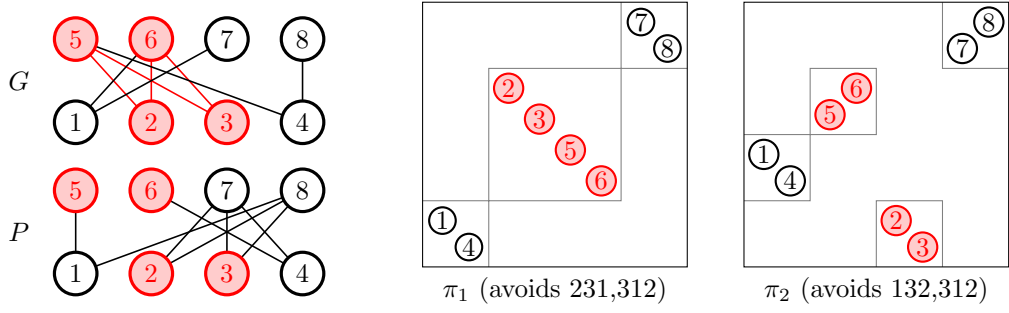
398 We build a set  $V_1$  as the union of  $V_A$  with  $n - |V_A|$  vertices of  $V_{AB}$ , and  $V_2$  as  $V \setminus V_1$ , so  
 399 that  $|V_1| = |V_2| = n$ . Moreover, for every  $v \in V_1$  (resp.,  $v \in V_2$ ), some element  $x_{v,i}$ , say  $x_{v,1}$ ,  
 400 is mapped to  $A$  (resp., to  $B$ ). Then, each edge  $e = (u, v)$  that is split by  $(V_1, V_2)$  must be  
 401 mapped to a point above some point of  $A$  and to the right of some point of  $B$ . This means  
 402 that  $y_e$  is mapped to  $C$ , and that  $(V_1, V_2)$  splits at most  $k$  edges, i.e., is a valid bisection.

403 Conversely, given a bisection  $(V_1, V_2)$  splitting at most  $k$  edges, we map  $P$  into  $\pi$  as  
 404 follows: map elements  $x_{v,i}$  for  $v \in V_1$  (resp.,  $V_2$ ) to the first  $nW$  elements of  $A$  (resp.,  $B$ ),  
 405 map elements  $y_e$  for which  $e$  is induced by  $V_1$  (resp.,  $V_2$ ) to the following elements of  $A$   
 406 (resp.  $B$ ), and finally map all elements  $y_e$  such that  $e$  is split by  $(V_1, V_2)$  into  $C$ . This mapping  
 407 is an occurrence of  $P$  in  $\pi$ .

408 **Case 4:**  $\pi$  is  $(231, 312)$ -avoiding. We use a reduction from BICLIQUE, as illustrated in  
 409 Figure 8 with permutation  $\pi_1$ . Given a bipartite graph  $G = (V, E)$  and an integer  $k$ , the  
 410 BICLIQUE problem consists in deciding whether  $V$  admits a complete bipartite subgraph  $K_{k,k}$ .  
 411 If  $V = A \cup B$  is a partition of  $V$  into two independent sets, adding independent vertices if  
 412 needed allows us to assume that  $A$  and  $B$  have the same size  $n$ , and that no vertex in either  
 413 side is fully connected to the other side.

414 Our permutation  $\pi$  is defined by  $\pi = \text{dp}(n - k) \oplus \text{dp}(2k) \oplus \text{dp}(n - k)$ . These three parts  
 415 of  $\pi$  are noted  $b_1, b_2$  and  $b_3$ . Our partial order  $\preceq$  is the order on  $V$  such that  $x \preceq y$  whenever  
 416  $x \in A, y \in B$  and  $\{x, y\} \notin E$ .

417 Consider a mapping of  $P$  into  $\pi$ . For each element  $x \in A$ , there exists  $y \in B$  such  
 418 that  $x \preceq y$ , and therefore  $x$  cannot be mapped into  $b_3$ . Symmetrically, no element  $y \in B$   
 419 may be mapped into  $b_1$ . Overall, since  $|\pi| = |V| = 2n$ ,  $b_1$  contains  $n - k$  elements from  $A$ ,  $b_3$



■ **Figure 8** Reduction from BICLIQUE to DPOP MATCHING on  $(231, 312)$ -avoiding and  $(213, 231)$ -avoiding permutations. Left: a bipartite graph  $G$  with a  $(2, 2)$  biclique and the corresponding height-2 dpop  $P$ , built from the complement of  $G$ . Right: permutations  $\pi_1$  and  $\pi_2$  with a mapping of the vertices 1 to 8, including the biclique vertices mapped into the central  $2k$  positions.

420 contains  $n - k$  elements from  $B$ , and  $b_2$  contains a size- $k$  subset  $A'$  of  $A$  and a size- $k$  subset  
 421  $B'$  of  $B$ . No two elements  $x \in A'$  and  $y \in B'$  that are mapped into  $b_2$  are comparable for  $\preceq$ ,  
 422 which means that  $\{x, y\} \in E$  for each such pair, i.e., that  $(A', B')$  is a biclique.

423 Conversely, if  $G$  has a biclique  $(A', B')$ , we map all elements of  $A \setminus A'$  into  $b_1$ , all elements  
 424 of  $A' \cup B'$  into  $b_2$ , and all elements of  $B \setminus B'$  into  $b_3$ . This mapping satisfies all relations  
 425  $x \preceq y$  with  $x \in A$  and  $y \in B$ , except for  $x \in A'$  and  $y \in B'$ , but indeed there is no such  
 426 relation since  $(A', B')$  is a biclique.

427 **Case 5:**  $\pi$  is  $(132, 312)$ -avoiding. We also use a reduction from BICLIQUE, as illustrated  
 428 in Figure 6 with permutation  $\pi_2$ . Our partial order  $\preceq$  is the same as in Case 4, and our  
 429 permutation  $\pi$  is defined by  $\pi = ((dp(n - k) \oplus ip(k)) \ominus dp(k)) \oplus ip(n - k)$ . These three parts  
 430 of  $\pi$  are noted  $b_1, b_2, b_3$  and  $b_4$ .

431 Consider a mapping of  $P$  into  $\pi$ . For each element  $y \in B$ , there exists  $x \in A$  such  
 432 that  $x \preceq y$ , and therefore  $y$  cannot be mapped into  $b_1$  or  $b_3$ . Thus, and since  $|B| = n$ ,  
 433 the elements of  $B$  are mapped to  $b_2$  or  $b_4$ , and the elements of  $A$  are mapped to  $b_1$  or  $b_3$ .  
 434 Hence,  $b_2$  contains a size- $k$  subset  $B'$  of  $B$  and  $b_3$  contains a size- $k$  subset  $A'$  of  $A$ . No  
 435 element of  $b_2$  is comparable to any element of  $b_3$ , and therefore  $(A', B')$  is a biclique.

436 Conversely, if  $G$  has a biclique  $(A', B')$ , we map all elements of  $A \setminus A'$  into  $b_1$ , all elements  
 437 of  $A'$  into  $b_3$ , all elements of  $B'$  into  $b_2$  and all elements of  $B \setminus B'$  into  $b_4$ . This mapping  
 438 satisfies all relations  $x \preceq y$  with  $x \in A$  and  $y \in B$ , except for  $x \in A'$  and  $y \in B'$ , but indeed  
 439 there is no such relation since  $(A', B')$  is a biclique.

440 **Cases 6–9:** These cases are symmetric to Cases 3, 5, 5 and 7, respectively. Indeed, if  $(P, \pi)$  is  
 441 an instance of DPOP MATCHING with  $P$  a height-2 symmetric dpop,  $(P^\partial, \pi^{cr})$  and  $(P, \pi^{-1})$   
 442 are equivalent instances of DPOP MATCHING with height-2 symmetric dpop, and

443 **Case 6:** if  $\pi$  avoids 132 and 321 (Case 3),  $\pi^{cr}$  avoids  $132^{cr} = 213$  and  $321^{cr} = 321$ ;

444 **Case 7:** if  $\pi$  avoids 132 and 312 (Case 5),  $\pi^{cr}$  avoids  $132^{cr} = 213$  and  $312^{cr} = 312$ ;

445 **Case 8:** if  $\pi$  avoids 132 and 312 (Case 5),  $\pi^{-1}$  avoids  $132^{-1} = 132$  and  $312^{-1} = 231$ ;

446 **Case 9:** if  $\pi$  avoids 213 and 312 (Case 7),  $\pi^{-1}$  avoids  $213^{-1} = 213$  and  $312^{-1} = 231$ . ◀

447 ▶ **Proposition 13.** DPOP MATCHING is in P for symmetric dpop  $P$  if one of the following  
 448 restrictions on  $\pi$  occurs:

- 449 1.  $\pi$  is  $(123, 231)$ -avoiding;    3.  $\pi$  is  $(123, 321)$ -avoiding;  
 450 2.  $\pi$  is  $(123, 132)$ -avoiding;    4.  $\pi$  is  $(123, 312)$ -avoiding;    5.  $\pi$  is  $(123, 213)$ -avoiding.

451 **Proof.** In each of the cases presented below, we are given a permutation  $\pi$  and a symmetric  
 452 dpop  $P = (\mathcal{P}, \mathcal{P})$  for some partially ordered set  $\mathcal{P} = (X, \preceq)$ . Each time, we identify  $P$  with  
 453 the partial order  $\preceq$ .

454 **Case 1:**  $\pi$  is (123, 231)-avoiding. There exist integers  $k, \ell$  and  $m$ , with sum  $n$ , such that  
 455  $\pi = \text{dp}(k) \ominus (\text{dp}(\ell) \oplus \text{dp}(m))$ . These three parts of  $\pi$  are noted  $b_1, b_2$  and  $b_3$ . Then, for every  
 456 pair  $(u, v)$  such that  $u \prec v$ , we must map  $u$  into  $b_2$  and  $v$  into  $b_3$ . Such values can be mapped  
 457 greedily, since elements in  $b_2$  are pairwise incompatible, as well as those in  $b_3$ . Thus,  $P$  can  
 458 be mapped into  $\pi$  if and only if it has height at most 2, there are at most  $a$  elements that  
 459 are lower bounds, and at most  $b$  elements that are upper bounds.

460 Note that, if  $P$  is not symmetric, the problem becomes NP-hard, since reversing the  
 461 horizontal order of  $P_{\text{p}}$  and  $\pi$  transforms  $\pi$  into the (132, 321)-avoiding permutation of the  
 462 NP-hard Case 3 in Proposition 12.

463 **Case 2:**  $\pi$  is (123, 132)-avoiding. The permutation  $\pi$  is a skew sum  $\pi = \ominus_{p=1}^k d_p$  of patterns of  
 464 the form  $d_p = \text{dp}(a_p) \oplus \text{dp}(1)$  for some integer  $a_p \geq 0$ . Then, no two elements in  $X$  can share  
 465 a strict lower bound, i.e., if  $u \prec v$  and  $u \prec w$  then  $v = w$ . Thus,  $P$  is of height at most 2,  
 466 and there exists a partition  $S_1 \cup \dots \cup S_\ell$  of  $X$  in which each set  $S_i$  contains a distinguished  
 467 element  $s_i$ , such that  $x \preceq s_i$  if and only if  $x \in S_i$ . Up to reordering the patterns  $d_p$  and  
 468 the sets  $S_i$ , which are pairwise incomparable, we assume that  $a_1 \geq a_2 \geq \dots \geq a_k$  and that  
 469  $|S_1| \geq |S_2| \geq \dots \geq |S_\ell|$ . Let also  $m$  be the number of sets  $S_i$  with size at least 2.

470 Each set  $S_i$  must be mapped into a single pattern, say  $d_{p(i)}$ , and if  $i \leq m$ , i.e., if  $|S_i| \geq 2$ ,  
 471 the element  $s_i$  must be mapped to the unique top-right element of  $d_{p(i)}$ . Such a mapping  
 472 exists if and only if  $k \geq m$  and  $a_i \geq |S_i| - 1$  for all  $i \leq m$ : we shall choose  $p(i) = i$  and  
 473 map greedily the elements of  $S_i \setminus \{s_i\}$  to the bottom-left part of  $d_i$ . Finally, the elements of  
 474 singleton sets  $S_i$  can be mapped to the remaining places in  $\pi$ .

475 **Case 3:**  $\pi$  is (123, 321)-avoiding. Erdős-Szekeres theorem [14] proves that  $n \leq 4$ .

476 **Cases 4–5:** These cases are symmetric to Cases 1 and 2, respectively. Indeed, if  $(P, \pi)$  is an  
 477 instance of DPOP MATCHING with  $P$  a height-2 symmetric dpop,  $(P^\partial, \pi^{\text{cr}})$  and  $(P, \pi^{-1})$   
 478 are equivalent instances of DPOP MATCHING with height-2 symmetric dpos, and

479 **Case 4:** if  $\pi$  avoids 123 and 231 (Case 1),  $\pi^{-1}$  avoids  $123^{-1} = 123$  and  $231^{-1} = 312$ ;

480 **Case 5:** if  $\pi$  avoids 123 and 132 (Case 2),  $\pi^{\text{cr}}$  avoids  $123^{\text{cr}} = 123$  and  $132^{\text{cr}} = 213$ . ◀

## 481 6 Concluding Remarks

482 Some open complexity questions remain among the parameters we identified for DPOP  
 483 MATCHING. For semi-total dpos, the complexity is open for constant width, and for most  
 484 classes of pattern-avoiding permutations (although, according to Propositions 7 and 10, the  
 485 problem is NP-hard when  $\pi$  avoids 1234 or 312, respectively). For symmetric dpos, it  
 486 would be interesting to settle the complexity status of deciding whether a dpop occurs in  
 487 a (231, 321)-avoiding or (312, 321)-avoiding permutation. In particular, for these cases, we  
 488 conjecture that the problem becomes polynomial when  $\text{height}(P)$  is constant.

489 Regarding the original puzzle formulation of the problem, an interesting question is to  
 490 generate instances that yield a unique solution, i.e., given a permutation  $\pi$ , find a dpop  
 491 with a unique occurrence in  $\pi$ . This can be done by using a semi-total dpop (e.g., take  $X$   
 492 with  $|X| = |\pi|$ , let  $P_{\text{p}}$  be a total order and  $P_{\text{v}}$  be an empty order), but one could try to  
 493 minimize  $|X|$  or the number of pairs of comparable elements in  $P$  (i.e., the number of clues)  
 494 in order to have a unique solution.



## References

- 495 —
- 496 1 Shlomo Ahal and Yuri Rabinovich. On complexity of the subpattern problem. *SIAM J. Discret.*  
497 *Math.*, 22(2):629–649, 2008.
- 498 2 Michael H. Albert, Marie-Louise Lackner, Martin Lackner, and Vincent Vatter. The complexity  
499 of pattern matching for 321-avoiding and skew-merged permutations. *Discret. Math. Theor.*  
500 *Comput. Sci.*, 18(2), 2016.
- 501 3 Sergey V. Avgustinovich, Sergey Kitaev, and Alexandr Valyuzhenich. Avoidance of boxed  
502 mesh patterns on permutations. *Discret. Appl. Math.*, 161(1-2):43–51, 2013.
- 503 4 Eric Babson and Einar Steingrímsson. Generalized permutation patterns and a classification  
504 of the mahonian statistics. *Séminaire Lotharingien de Combinatoire [electronic only]*, 44:B44b,  
505 18 p.–B44b, 18 p., 2000. URL: <http://eudml.org/doc/120841>.
- 506 5 Benjamin Aram Berendsohn, László Kozma, and Dániel Marx. Finding and counting permuta-  
507 tions via csps. *Algorithmica*, 83(8):2552–2577, 2021.
- 508 6 Miklós Bóna. *Combinatorics of Permutations, Second Edition*. Discrete mathematics and its  
509 applications. CRC Press, 2012.
- 510 7 Prosenjit Bose, Jonathan F. Buss, and Anna Lubiw. Pattern matching for permutations. *Inf.*  
511 *Process. Lett.*, 65(5):277–283, 1998.
- 512 8 Mireille Bousquet-Mélou, Anders Claesson, Mark Dukes, and Sergey Kitaev. (2+2)-free posets,  
513 ascent sequences and pattern avoiding permutations. *J. Comb. Theory, Ser. A*, 117(7):884–909,  
514 2010.
- 515 9 Petter Brändén and Anders Claesson. Mesh patterns and the expansion of permutation  
516 statistics as sums of permutation patterns. *Electron. J. Comb.*, 18(2), 2011.
- 517 10 Marie-Louise Bruner and Martin Lackner. A fast algorithm for permutation pattern matching  
518 based on alternating runs. *Algorithmica*, 75(1):84–117, 2016.
- 519 11 Laurent Bulteau, Romeo Rizzi, and Stéphane Vialette. Pattern matching for k-track permuta-  
520 tions. In Costas S. Iliopoulos, Hon Wai Leong, and Wing-Kin Sung, editors, *Combinatorial*  
521 *Algorithms - 29th International Workshop, IWOCA 2018, Singapore, July 16-19, 2018, Pro-*  
522 *ceedings*, volume 10979 of *Lecture Notes in Computer Science*, pages 102–114. Springer, 2018.
- 523 12 Maxime Crochemore and Ely Porat. Fast computation of a longest increasing subsequence  
524 and application. *Inf. Comput.*, 208(9):1054–1059, 2010.
- 525 13 Sergi Elizalde and Marc Noy. Consecutive patterns in permutations. *Adv. Appl. Math.*,  
526 30(1-2):110–125, 2003.
- 527 14 Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio mathem-*  
528 *atica*, 2:463–470, 1935.
- 529 15 Jacob Fox. Stanley-wilf limits are typically exponential. *CoRR*, abs/1310.8378, 2013. URL:  
530 <http://arxiv.org/abs/1310.8378>, arXiv:1310.8378.
- 531 16 Sylvain Guillemot and Dániel Marx. Finding small patterns in permutations in linear time. In  
532 Chandra Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on*  
533 *Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 82–101.  
534 SIAM, 2014.
- 535 17 Sylvain Guillemot and Stéphane Vialette. Pattern matching for 321-avoiding permutations.  
536 In Yingfei Dong, Ding-Zhu Du, and Oscar H. Ibarra, editors, *Algorithms and Computation,*  
537 *20th International Symposium, ISAAC 2009, Honolulu, Hawaii, USA, December 16-18, 2009.*  
538 *Proceedings*, volume 5878 of *Lecture Notes in Computer Science*, pages 1064–1073. Springer,  
539 2009.
- 540 18 Louis Ibarra. Finding pattern matchings for permutations. *Inf. Process. Lett.*, 61(6):293–295,  
541 1997.
- 542 19 Klaus Jansen, Stefan Kratsch, Dániel Marx, and Ildikó Schlotter. Bin packing with fixed  
543 number of bins revisited. *J. Comput. Syst. Sci.*, 79(1):39–49, 2013.
- 544 20 Vít Jelínek and Jan Kyncl. Hardness of permutation pattern matching. In Philip N. Klein, ed-  
545 itor, *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms,*  
546 *SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*, pages 378–396. SIAM, 2017.

- 547 21 Richard M. Karp. *Reducibility among Combinatorial Problems*, pages 85–103. Springer US,  
548 Boston, MA, 1972.
- 549 22 Sergey Kitaev. Introduction to partially ordered patterns. *Discret. Appl. Math.*, 155(8):929–944,  
550 2007.
- 551 23 Sergey Kitaev. *Patterns in Permutations and Words*. Monographs in Theoretical Computer  
552 Science. An EATCS Series. Springer, 2011.
- 553 24 Both Emerite Neou, Romeo Rizzi, and Stéphane Vialette. Pattern matching for separable  
554 permutations. In Shunsuke Inenaga, Kunihiko Sadakane, and Tetsuya Sakai, editors, *String  
555 Processing and Information Retrieval - 23rd International Symposium, SPIRE 2016, Beppu,  
556 Japan, October 18-20, 2016, Proceedings*, volume 9954 of *Lecture Notes in Computer Science*,  
557 pages 260–272, 2016.
- 558 25 Both Emerite Neou, Romeo Rizzi, and Stéphane Vialette. Permutation pattern matching in  
559 (213, 231)-avoiding permutations. *Discret. Math. Theor. Comput. Sci.*, 18(2), 2016.