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Permutation Pattern Matching for Doubly Partially Ordered Patterns

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Abstract

We study in this paper the DOUBLY PARTIALLY ORDERED PATTERN MATCHING (or DPOP MATCHING) problem, a natural extension of the PERMUTATION PATTERN MATCHING problem. PERMUTATION PATTERN MATCHING takes as input two permutations σ and π , and asks whether there exists an occurrence of σ in π ; whereas DPOP MATCHING takes *two partial orders* P_V and P_D defined on the same set X and a permutation π , and asks whether there exist $|X|$ elements in π whose values (resp., positions) are in accordance with P_V (resp., P_D). Posets P_V and P_D aim at relaxing the conditions formerly imposed by the permutation σ , since σ yields a total order on both positions and values. Our problem being NP-hard in general (as PERMUTATION PATTERN MATCHING is), we consider restrictions on several parameters/properties of the input, e.g., bounding the size of the pattern, assuming symmetry of the posets (i.e., P_V and P_D are identical), assuming that one partial order is a total (resp., weak) order, bounding the length of the longest chain/anti-chain in the posets, or forbidding specific patterns in π . For each such restriction, we provide results which together give a(n almost) complete landscape for the algorithmic complexity of the problem.

2012 ACM Subject Classification Theory of computation \rightarrow Design and analysis of algorithms

Keywords and phrases Partial orders, Permutations, Pattern Matching, Algorithmic Complexity, Parameterized Complexity

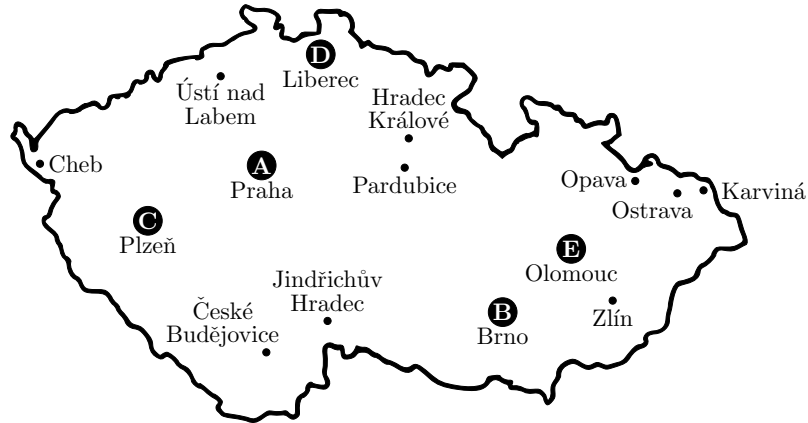
1 Preamble

Let us play the following little puzzle game. Among the selection of fifteen cities of the Czech Republic depicted in Figure 1 together with their geographic coordinates, find (if they exist) five cities, say **A**, **B**, **C**, **D** and **E**, such that:

- **A** and **C** are west of **D** and north of **B**,
- **E** is east of **B** and south of **A**,
- **D** is west of **B** and north of **A** and **C**.

It is assumed that no two cities have the same longitude or latitude. Notice that the game does not provide complete information as, for example, no information is provided about the relative positioning of **A** and **C** (and silence is tantamount to consent). We may assume that the information is minimal: requiring **C** is west of **B** is unnecessary since **C** is west of **D** and **D** is west of **B**. One solution is **A** = Praha, **B** = Brno, **C** = Plzeň, **D** = Liberec and **E** = Olomouc. Note that the solution is not unique, as **A** = Plzeň, **B** = Jindřichův Hradec, **C** = Cheb, **D** = Ústí nad Labem and **E** = Brno is another solution.

We show that this puzzle game can be modeled as a permutation pattern matching problem for doubly partially ordered patterns. Let us first associate a permutation $\pi \in \mathfrak{S}(15)$ with the problem (see Figure 2). We sort the fifteen cities of the Czech Republic depicted in Figure 1 both by increasing longitude (E) and by increasing latitude (N), so that $\pi(i) = j$ if



■ **Figure 1** A Czech Republic map showing 15 of its cities. Their names and GPS coordinates, in increasing order of their longitudes, are: Cheb (50°4′N 12°22′E), Plzeň (49°44′N 13°22′E), Ústí nad Labem (50°39′N 14°1′E), Praha (50°5′N 14°25′E), České Budějovice (48°58′N 14°28′E), Jindřichův Hradec (49°9′N 15°0′E), Liberec (50°46′N 15°3′E), Pardubice (50°2′N 15°46′E), Hradec Králové (50°12′N 15°49′E), Brno (49°11′N 16°36′E), Olomouc (49°35′N 17°15′E), Zlín (49°13′N 17°40′), Opava (49°56′N 17°54′E), Ostrava (49°50′N 18°17′E) and Karviná (49°51′N 18°32′E).
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46 the i -th city going west to east is also the j -th city going south to north. In our example,
 47 the “Czech Republic permutation” is $\pi = 11\ 6\ 14\ 12\ 1\ 2\ 15\ 10\ 13\ 3\ 5\ 4\ 9\ 7\ 8$. For example,
 48 $\pi(2) = 6$ since Plzeň is the second city going west to east, and the sixth city going south
 49 to north. What is left is to define our pattern P : P is composed of two partially ordered sets on
 50 the variables $\{A, B, C, D, E\}$ (see Figure 3): one partially ordered set (denoted P_V for *value*
 51 *poset* in the sequel) describes the south-to-north constraints and another partially ordered
 52 set (denoted P_P for *position poset* in the sequel) describes the west-to-east constraints.

53 2 Introduction

54 We say that a permutation σ *occurs* in another permutation π (or that π *contains* σ) if there
 55 exists a subsequence of elements of π that has the same relative order as σ . Otherwise, we
 56 say that π *avoids* σ . For example, π contains the permutation $\sigma = 123$ (resp., $\sigma = 321$) if
 57 it has an increasing (resp., a decreasing) subsequence of size 3. Similarly, $\sigma = 4312$ occurs
 58 in $\pi = 6152347$, as shown in $\textcircled{6}1\textcircled{5}2\textcircled{3}\textcircled{4}7$, but the same $\pi = 6152347$ avoids $\sigma' = 2341$.

59 Deciding whether a permutation $\sigma \in \mathfrak{S}(k)$ occurs in some permutation $\pi \in \mathfrak{S}(n)$
 60 is NP-complete [7], but is fixed-parameter tractable for the parameter k [15, 16]. Two
 61 exponential-time algorithms have been recently proposed [5], improving upon [1, 10]. A vast
 62 literature is devoted to the case where both the pattern σ and the target π are restricted to
 63 a proper permutation class, e.g., 321-avoiding permutations [17, 2, 20], (213, 231)-avoiding
 64 permutations [25], (2413, 3142)-avoiding (a.k.a. *separable*) permutations [18, 24], and $(k \dots 1)$ -
 65 avoiding permutations [11]. For more background on permutation patterns and pattern
 66 avoidance, we refer to [6] and [23].

67 In the last years, the notion of *pattern* has been generalized in several ways. A *vincular*
 68 *pattern* is a permutation in which some entries must occur consecutively [4]. *Consecutive*
 69 *patterns* are a special case of vincular patterns in which all entries need to be adjacent [13].
 70 *Bivincular patterns* generalize classical patterns even further than vincular patterns by
 71 requiring that not only positions but also values of elements involved in a matching may

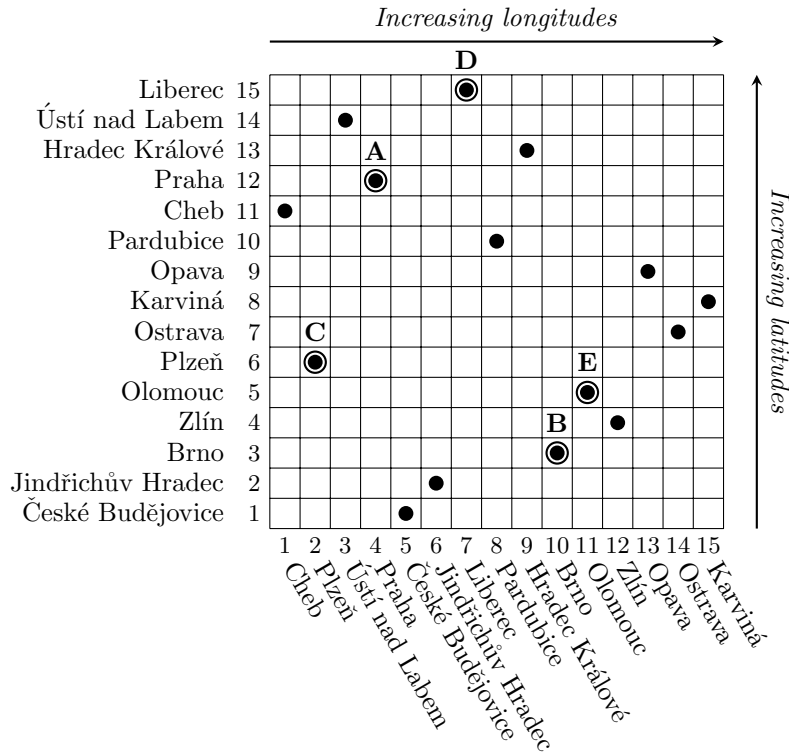


Figure 2 Permutation $\pi = 11\ 6\ 14\ 12\ 1\ 2\ 15\ 10\ 13\ 3\ 5\ 4\ 9\ 7\ 8$ corresponding to the map from Figure 1. The solution of our puzzle, depicted Figure 1, is also represented.

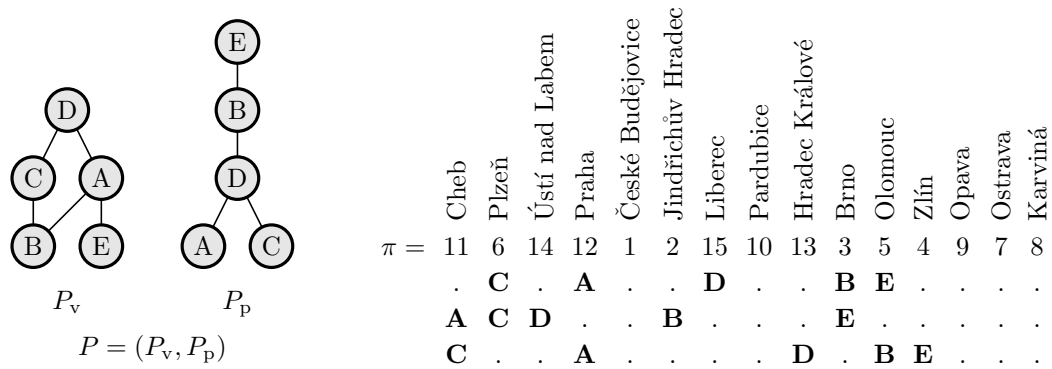


Figure 3 A dpop $P = (P_v, P_p)$ representing the pattern for our puzzle, together with three distinct occurrences. Note that partial orders are represented by Hasse diagrams, i.e., a bottom-up path in P_v (resp., P_p) implies a bottom-up (resp., left-right) relation in the occurrence of P in π .

be forced to be adjacent [8]. Mesh patterns (a further generalization of bivincular patterns) impose further restrictions on the relative positions of the entries in an occurrence of a pattern [9] and boxed mesh patterns are special cases of mesh patterns [3]. Strongly related to our approach are partially ordered patterns that are vincular patterns in which the relative order of some elements is not fixed [22]. The best general reference is [23].

In this paper, we consider a new generalization of classical patterns in which both the

78 relative order and the relative positioning of some elements are not fixed. The idea is to
 79 allow the possibility for some elements to be incomparable in value (i.e., their relative order
 80 is unknown) and to go one step further by allowing the possibility for some elements to be
 81 incomparable in position (i.e., their relative positioning in the occurrence is unknown). Since
 82 the problem is clearly NP-hard (as it contains PERMUTATION PATTERN MATCHING as a
 83 sub-problem), our goal is to identify tractable cases when restrictions apply to the pattern
 84 and/or to the permutation.

85 The restrictions we consider here apply to the following parameters of the problem: size
 86 of the pattern; symmetry (i.e., same partial order in positions and values); one partial order
 87 is a total (resp., weak) order; size of the longest chain (resp., anti-chain) in the partial orders
 88 (height and width); forbidden patterns in π . On the positive side, we show that the FPT
 89 algorithm for PERMUTATION PATTERN MATCHING can be generalized to our setting (with
 90 the pattern size as parameter). We further give polynomial-time algorithms when the pattern
 91 is a symmetric disjoint union of a constant number of weak orders. Finally, we also provide
 92 polynomial-time algorithms when the pattern is symmetric and the permutation belongs
 93 to some restricted classes, such as (123, 132)-avoiding permutations. We complement these
 94 positive results with NP- or W[1]-hardness proofs in most of the remaining cases.

95 3 Definitions

96 Permutations and Patterns

97 A permutation σ is said to be *contained* in (or is a *sub-permutation* of) another permutation π ,
 98 which we denote by $\sigma \preceq \pi$, if π has a (not necessarily contiguous) subsequence whose terms
 99 are order-isomorphic to σ . We also say that π *admits an occurrence* of the *pattern* σ . If no
 100 such subsequence exists, we say that π *avoids* σ (or is σ -*avoiding*). A permutation is *separable*
 101 if it avoids both 2413 and 3142. PERMUTATION PATTERN MATCHING is the problem of
 102 deciding whether a permutation is contained into another permutation.

103 For any non-negative integer n , we denote by $[n]$ the set $\{1, 2, \dots, n\}$. When $n \geq 1$, we also
 104 note $\text{ip}(n) = 1\,2\,\dots\,n$ the *increasing permutation* of length n and $\text{dp}(n) = n\,(n-1)\,\dots\,1$ the
 105 *decreasing permutation* of length n . Let $\pi \in \mathfrak{S}(n)$. The *reverse* (resp., *complement*) of π is the
 106 permutation $\pi^r = \pi(n)\pi(n-1)\,\dots\,\pi(1)$ (resp., $\pi^c = (n-\pi(1)+1)(n-\pi(2)+1)\,\dots\,(n-\pi(n)+1)$).
 107 The *inverse* of π is the permutation $\pi^{-1} \in \mathfrak{S}(n)$ defined by $\pi^{-1}(j) = i$ if and only if $\pi(i) = j$.
 108 Given a permutation π of size m and a permutation σ of size n , the *skew sum* of π and σ is
 109 the permutation of size $m+n$ defined by

$$110 \quad (\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + n & \text{for } 1 \leq i \leq m, \\ \sigma(i - m) & \text{for } m + 1 \leq i \leq m + n, \end{cases}$$

111 and the *direct sum* of π and σ is the permutation of size $m+n$ defined by

$$112 \quad (\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{for } 1 \leq i \leq m, \\ \sigma(i - m) + m & \text{for } m + 1 \leq i \leq m + n. \end{cases}$$

113 Orders

114 A relation \leq is a *partial order* on a set X if it has:

- 115 ■ *reflexivity*: for all $x \in X$, $x \leq x$ (i.e., every element is related to itself);
- 116 ■ *transitivity*: for all $x, x', x'' \in X$, if $x \leq x'$ and $x' \leq x''$, then $x \leq x''$;

117 ■ *antisymmetry*: for all $x, x' \in X$, if $x \leq x'$ and $x' \leq x$, then $x = x'$ (i.e., no two distinct
118 elements precede each other).

119 If \leq has the following additional property, we say that it is a *weak order* on X :

120 ■ *transitivity of incomparability*: for all $x, x', x'' \in X$, if x is incomparable with x' (i.e.,
121 neither $x \leq x'$ nor $x' \leq x$ is true) and if x' is incomparable with x'' , then x is incomparable
122 with x'' .

123 Two subsets X_1, X_2 are *independent* if there is no $x_1 \in X_1, x_2 \in X_2$ such that $x_1 \leq x_2$
124 or $x_2 \leq x_1$. We say that a partial order is *k-weak* if there exists a partition of X into k
125 pairwise independent sets X_1, \dots, X_k such that, for each i , the restriction of \leq to X_i is a
126 weak order (in other words, \leq is the disjoint union of k weak partial orders).

127 Let $\mathcal{P} = (X, \leq)$ be a finite partially ordered set. A *chain* in \mathcal{P} is a set of pairwise
128 comparable elements (i.e., a totally ordered subset) and an *antichain* in \mathcal{P} is a set of pairwise
129 incomparable elements. The *partial order height* of \mathcal{P} , denoted by $\text{height}(\mathcal{P})$, is defined as
130 the maximum cardinality of a chain in \mathcal{P} , and the *partial order width* of \mathcal{P} , denoted by
131 $\text{width}(\mathcal{P})$, is defined as the maximum cardinality of an antichain in \mathcal{P} . By Dilworth Theorem,
132 $\text{width}(\mathcal{P})$ is also the minimum number of chains in any partition of \mathcal{P} into chains. The *dual*
133 of \mathcal{P} is the partial order $\mathcal{P}^\partial = (X, \leq^\partial)$ defined by letting \leq^∂ be the converse relation of \leq ,
134 i.e., $x \leq^\partial x'$ if and only if $x' \leq x$. The dual of a partial order is a partial order and the dual
135 of the dual of a relation is the original relation. A *total order* is a partial order in which any
136 two elements are comparable, and a set equipped with a total order is a *totally ordered set*.
137 A *linear extension* of a partial order is a total order that is compatible with the partial order.
138 It will be convenient to represent a linear extension of a poset $\mathcal{P} = (X, \leq)$ as the mapping
139 $\tau_{\mathcal{P}} : X \rightarrow [|X|]$ such that $\tau_{\mathcal{P}}(i) < \tau_{\mathcal{P}}(j)$ if $i < j$ in the linear extension.

140 A *doubly partially ordered pattern* (dpop) P is a pair, denoted by $P = (P_v, P_p)$, of posets
141 $P_v = (X, \leq_v)$ and $P_p = (X, \leq_p)$ defined over the same set X . We call P_v and P_p the *value*
142 *poset* and the *position poset*, respectively. A dpop $P = (P_v, P_p)$ is *symmetric* if $P_v = P_p$,
143 *dual* if $P_v = P_p^\partial$, and *semi-total* if one of P_p or P_v is a total order. We let $\text{height}(P)$ and
144 $\text{width}(P)$ stand for $\max\{\text{height}(P_v), \text{height}(P_p)\}$ and $\max\{\text{width}(P_v), \text{width}(P_p)\}$, respect-
145 ively. Finally, the *size* of P is defined as the cardinality $|X|$ and is denoted by $|P|$.

146 ► **Definition 1** (DPOP MATCHING). *Given a permutation $\pi \in \mathfrak{S}(n)$ and a dpop $P = (P_v, P_p)$,*
147 *an occurrence (or mapping) of P in π is an injective function $\varphi : X \rightarrow [n]$ such that:*

148 ■ *$\pi \circ \varphi$ is \leq_v -non-decreasing, i.e., for all $x, y \in X$, if $x \leq_v y$ then $\pi(\varphi(x)) \leq \pi(\varphi(y))$, and*
149 ■ *φ is \leq_p -non-decreasing, i.e., for all $x, y \in X$, if $x \leq_p y$ then $\varphi(x) \leq \varphi(y)$.*

150 *The DPOP MATCHING problem consists in deciding whether P occurs in π .*

151 First Observations

152 ► **Observation 2.** *PERMUTATION PATTERN MATCHING is the special case of DPOP MATCH-*
153 *ING where both \leq_v and \leq_p are total orders.*

154 We note that applying a vertical and/or horizontal symmetry on both pattern and
155 permutation does not alter the existence of an occurrence.

156 ► **Observation 3.** *Let $P = (P_v, P_p)$ be a dpop and π be a permutation. The following*
157 *statements are equivalent:*

- | | |
|--|--|
| 158 1. (P_v, P_p) occurs in π ; | |
| 159 2. $(P_v, (P_p)^\partial)$ occurs in π^r ; | 4. $((P_v)^\partial, (P_p)^\partial)$ occurs in π^{cr} ; |
| 160 3. $((P_v)^\partial, P_p)$ occurs in π^c ; | 5. (P_p, P_v) occurs in π^{-1} . |

161 The following reformulation will prove useful.

162 ► **Observation 4.** *Let $P = (P_V, P_P)$ be a dpop with $P_V = (X, \leq_V)$, $P_P = (X, \leq_P)$ and $k = |X|$,
163 and let $\pi \in \mathfrak{S}(n)$ be a permutation. The following statements are equivalent:*

164 ■ *P occurs in π .*

165 ■ *There exists a linear extension $\tau_V : X \rightarrow [k]$ of P_V and a linear extension $\tau_P : X \rightarrow [k]$
166 of P_P such that the permutation $\sigma \in \mathfrak{S}(k)$ defined by $\sigma(i) = \tau_V(\tau_P^{-1}(i))$ for $1 \leq i \leq k$ is
167 contained in π .*

168 The rationale for the reformulation introduced in Observation 4 stems from the following
169 corollary that sets the general context.

170 ► **Corollary 5 ([16]).** *DPOP MATCHING is FPT for the parameter $|P|$.*

171 Indeed, it is enough to guess two linear extensions $\tau_V : X \rightarrow [k]$ of P_V and $\tau_P : X \rightarrow [k]$
172 of P_P , and to check if the permutation $\sigma \in \mathfrak{S}(k)$ defined by $\sigma(i) = \tau_V(\tau_P^{-1}(i))$ for $1 \leq i \leq k$
173 is contained in π . There are $O(k!^2)$ pairs of such extensions and, for each of them, one can
174 check whether σ occurs in π in $n 2^{O(k^2 \log k)}$ time [16].

175 4 Semi-Total Patterns

176 In this section we focus on semi-total patterns, i.e., without loss of generality, on the case
177 where P_P is a total order (up to symmetry by Observation 3). This case still contains
178 PERMUTATION PATTERN MATCHING as a special case, and is thus NP-hard. We focus on
179 small-height value partial orders, i.e., on dpop with constant $\text{height}(P_V)$, and give an XP
180 algorithm for weak orders (Proposition 6) and paraNP-hardness in general (Proposition 7).

181 ► **Proposition 6.** *DPOP MATCHING is solvable in $O(n^{\text{height}(P_V)})$ time if P_V is a weak order
182 and P_P is a total order.*

183 **Proof.** Let $\pi \in \mathfrak{S}(n)$ be a permutation and $P = (P_V, P_P)$ be a dpop on some ground set X ,
184 where P_V is a weak order and P_P is a total order. Without loss of generality, we assume
185 that X is the set $[k]$ and that P_P is the usual order on integers. For every $x \in X$, we abusively
186 denote by $\text{height}(x)$ the maximum cardinality of a chain with maximum element x in P_V .
187 Finally, set $\ell = \text{height}(P_V)$.

188 For any two distinct variables $x, y \in X$, we have $x <_V y$ if and only if $\text{height}(x) < \text{height}(y)$.
189 Thus, P occurs in π if and only there exists a sequence $0 = a_0 < a_1 < a_2 < \dots < a_\ell = n$ such
190 that w_σ is a subsequence of w_π , where $w_\pi \in [\ell]^n$ and $w_\sigma \in [\ell]^k$ are the two words defined
191 by $w_\pi[i] = \min\{j : a_{j-1} < \pi(i) \leq a_j\}$ and $w_\sigma[i] = \text{height}(i)$.

192 As for the running time, there exist $\binom{n-1}{\ell-1}$ distinct sequences $(a_i)_{0 \leq i \leq \ell}$ and deciding
193 whether w_σ occurs in w_π as a subsequence is a linear-time procedure. ◀

194 ► **Proposition 7.** *DPOP MATCHING is NP-complete even if $\text{height}(P_V) = 2$, P_P is a total
195 order and π avoids 1234.*

196 **Proof.** We perform a reduction from VERTEX COVER, which is known to be NP-complete [21].
197 Let $G = (V, E)$ be a graph and let k be a positive integer. We identify V with the set $[n]$.

198 We construct a dpop $P = (P_V, P_P)$, where $P_V = (X, \leq_V)$ is a height-2 partial order and
199 $P_P = (X, \leq_P)$ a total order, as follows. We set $X = \{a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_n, b_n, c_n\}$, so
200 that $|X| = 3n$. Then, we set $a_1 \leq_P b_1 \leq_P c_1 \leq_P a_2 \leq_P b_2 \leq_P c_2 \leq_P \dots \leq_P a_n \leq_P b_n \leq_P c_n$,
201 which defines the total order \leq_P . Finally, for each edge $\{i, j\}$ in E with $i < j$, we set $a_i \leq_V b_j$;

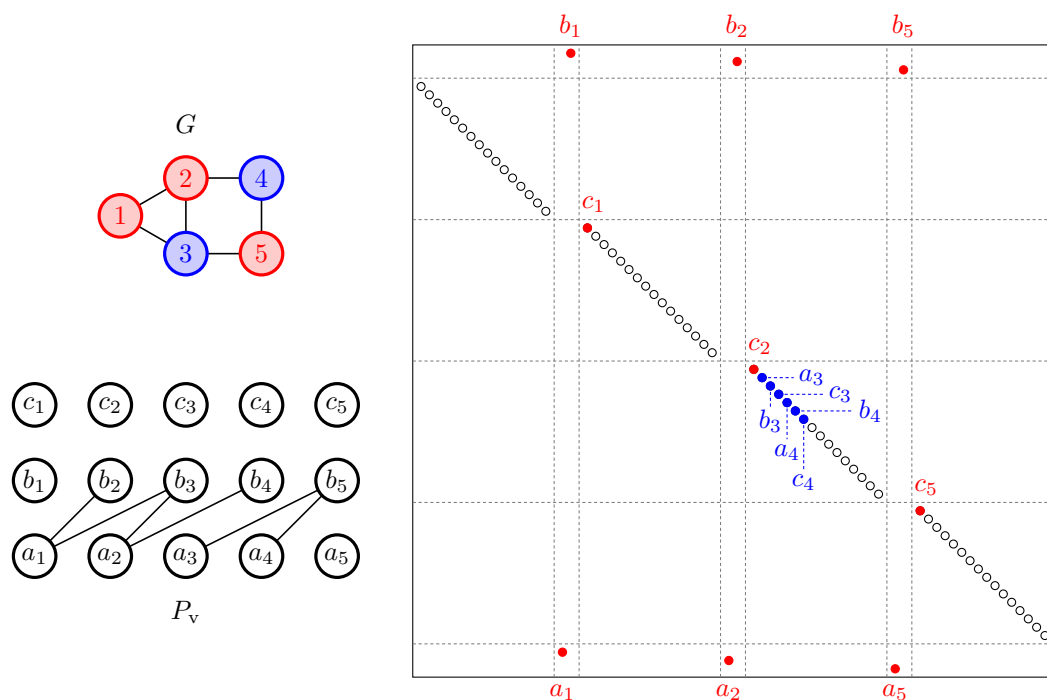


Figure 4 Top left: vertex cover (in red) in a 5-vertex graph G . Bottom left: partially ordered set P_V constructed from G . Right: an occurrence of P in π obtained from our size-3 vertex cover.

all other elements of X are pairwise incomparable by \leq_v . This defines a partial order \leq_v such that $\text{height}(\leq_v) = 2$.

Write now $N = 3n + 3$ and $m = (k + 1)N - 2$, and define a permutation $\pi \in \mathfrak{S}(m)$ as follows:

- $\pi(iN + j) = (m + 1) - (iN + j + k)$ whenever $0 \leq i \leq k$ and $1 \leq j \leq N - 2$;
- $\pi(iN - 1) = (k + 1) - i$ and $\pi(iN) = (m + 1) - i$ whenever $1 \leq i \leq k$.

It is straightforward to check that π is 1234-avoiding. It is also easy to see how the construction, illustrated in Figure 4, can be accomplished in polynomial-time.

Let us see under which conditions an injective \leq_p -non-decreasing function $\varphi: X \rightarrow [m]$ maps P into π . We say that a vertex i belongs to the j -th gadget if one of the integers $\varphi(a_i)$ or $\varphi(b_i)$ is equal to $jN - 1$ or to jN , i.e., if $\{\varphi(a_i), \varphi(b_i)\} \cap \{jN - 1, jN\} \neq \emptyset$. When two elements in the range of φ are consecutive, either they are integers $\varphi(a_i)$ and $\varphi(b_i)$ for a given i , or one of them is an integer $\varphi(c_i)$ for some i . Therefore, no two distinct vertices i and i' can belong to the same j -th gadget. Consequently, and since there are k gadgets, the set V' of vertices i that belong to some gadget is of size at most k .

Then, we define a notion of *height* as follows: for each element x of X , we set $\text{height}(x) = 0$ if N divides $\varphi(x) + 1$, $\text{height}(x) = 2$ if N divides $\varphi(x)$, and $\text{height}(x) = 1$ otherwise. By construction, for all $x, y \in X$ such that $x \leq_p y$, we have $\pi(\varphi(x)) \leq \pi(\varphi(y))$ if and only if x is of smaller height than y . Therefore, if φ maps P into π , and for each relation $a_i \leq_v b_j$, either a_i has height 0 or b_j has height 2. In particular, either i or j must belong to V' , and therefore V' is a vertex cover of size at most k .

Conversely, provided that there exist vertices $v(1) < v(2) < \dots < v(k)$ that form a vertex cover V' , we construct an occurrence of P in π as follows. First, we abusively set $v(0) = 0$. Then, for all $i \in [k]$, we set $f(i) = jN + 3(i - v(j))$, where j is the largest integer such

226 that $v(j) \leq i$. We set $\varphi(a_j) = f(j) - 1$, $\varphi(b_j) = f(j)$ and $\varphi(c_j) = f(j) + 1$.

227 By construction, we have $f(i) + 3 \leq f(i + 1)$ for all i , and therefore φ is an injective
 228 \leq_p -non-decreasing function. Moreover, for every $i \in [n]$, the elements a_i and b_i have heights 0
 229 and 2 if $i \in V'$, and they have height 1 if $i \notin V'$. It follows that $\pi(\varphi(a_i)) \leq \pi(\varphi(b_j))$ whenever
 230 $a_i \leq_v b_j$, i.e., that φ is an occurrence of P in π . ◀

231 5 Symmetric Patterns

232 This section is devoted to studying complexity issues of pattern matching for symmetric
 233 dpop (i.e., those dpop $P = (\mathcal{P}, \mathcal{P})$, whose value and position posets coincide). We further
 234 focus on two special cases, first when \mathcal{P} has a bounded width, then when π is restricted to
 235 constrained pattern-avoiding classes of permutations.

236 5.1 Symmetric Pattern with Bounded Width

237 We first observe that the problem is polynomial for width 1 (Observation 8). We further prove
 238 $W[1]$ -hardness for the parameter k when \mathcal{P} is a disjoint union of k chains (Proposition 10).
 239 We complement this result with an XP algorithm for the slightly more general case where \mathcal{P} is
 240 a disjoint union of weak orders (Proposition 11). Note that the existence of an XP algorithm
 241 for the width parameter remains open, and we conjecture that the problem is NP-hard even
 242 for constant width.

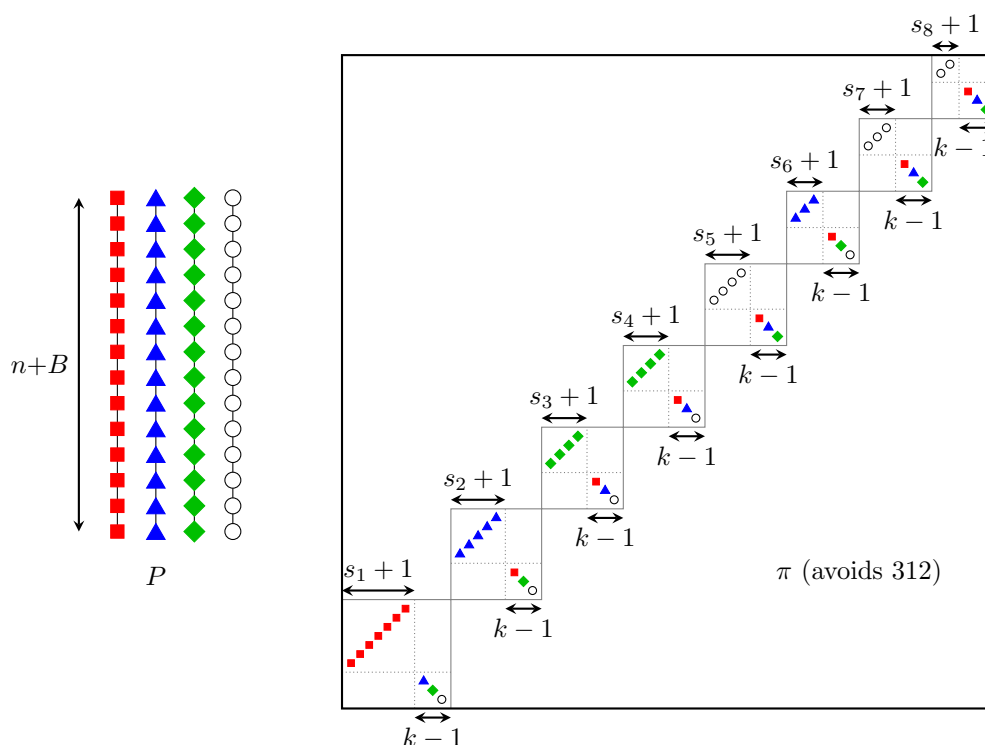
243 ▶ **Observation 8.** *DPOP MATCHING is solvable in $O(n \log \log |P|)$ time for a symmetric*
 244 *dpop P of width 1 (i.e., a total symmetric dpop P).*

245 **Proof.** If P has width 1, then $P = (\mathcal{P}, \mathcal{P})$ for some total order $\mathcal{P} = (X, \leq)$. In particular,
 246 we can write $X = \{x_1, \dots, x_{|X|}\}$ with $x_i \prec x_j$ for $i < j$, and in any mapping $\phi : X \rightarrow [n]$,
 247 the elements $\pi_{\phi(x_1)}, \dots, \pi_{\phi(x_{|X|})}$ must form an increasing subsequence of π . Conversely, any
 248 size- $|X|$ increasing subsequence of π can be used as an image for ϕ , so in this setting DPOP
 249 MATCHING corresponds to the longest increasing subsequence problem, which can be solved
 250 in $O(n \log \log |X|)$ time [12]. ◀

251 To simplify the exposition of our next result, we introduce a new problem that may be of
 252 independent interest. Given a positive integer k and a permutation $\pi \in \mathfrak{S}(kn)$, BALANCED
 253 k -INCREASING COLORING is the problem of deciding whether there exists a balanced k -
 254 coloring of π (i.e., a partition of $[kn]$ into k subsets of size exactly n) such that each color
 255 induces an increasing subsequence of π .

256 ▶ **Proposition 9.** *BALANCED k -INCREASING COLORING for 312-avoiding permutations is*
 257 *$W[1]$ -hard for the parameter k .*

258 **Proof.** We perform a reduction from UNARY BIN PACKING parameterized by the number of
 259 bins, which is known to be $W[1]$ -hard [19]. In this version of BIN PACKING, we are given
 260 a list of integers s_1, s_2, \dots, s_n encoded in unary, and two integers B and k . These integers
 261 are interpreted as item sizes, and the task is to decide whether the items can be partitioned
 262 into k subsets, each of total size B . We show that there is a reduction from UNARY BIN
 263 PACKING, parameterized by the number of bins, to BALANCED k -INCREASING COLORING,
 264 parameterized by the number of colors.



■ **Figure 5** Reduction from UNARY BIN PACKING to BALANCED k -INCREASING COLORING for the list $6, 4, 3, 3, 3, 2, 2, 1$, which admits the partition $(\{6\}, \{4, 2\}, \{3, 3\}, \{3, 2, 1\})$, and the integers $B = 6, k = 4$. Left: dpop P that consists of k chains, each of length $n + B$. Right: 312-avoiding permutation π that consists of n blocks. Each color/shape induces an increasing subsequence of π .

265 Consider an arbitrary instance of UNARY BIN PACKING containing n items with item
 266 sizes $S = \{s_1, s_2, \dots, s_n\}$, and two integers B and k . Define $\pi \in \mathfrak{S}(kB + kn)$ by

$$267 \quad \pi = \bigoplus_{i=1}^n (\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)).$$

268 Each pattern $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$ is called the i -th block of π . See Figure 5 for an illustration.
 269 It is straightforward to check that π is 312-avoiding.

270 We claim that the n items s_1, s_2, \dots, s_n can be partitioned into k subsets, each of total
 271 size B , if and only if there exists a k -coloring of π such that each color induces an increasing
 272 pattern of length $B + n$.

273 Suppose first that the n items s_1, s_2, \dots, s_n can be partitioned into k subsets, each of
 274 total size B . Write $S = S_1 \cup S_2 \cup \dots \cup S_k$ such a partition. Define a k -coloring of π as follows.
 275 Consider the i -th block $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$ of π , and suppose that $s_i \in S_j$. Color the
 276 whole ascending pattern $\text{ip}(s_i + 1)$ with color c_j and arbitrarily color the elements of the
 277 descending pattern $\text{dp}(k - 1)$ with the remaining $k - 1$ colors (each element of $\text{dp}(k - 1)$ is
 278 assigned to a distinct color). We claim that every color c_j induces an increasing pattern of
 279 length $B + n$ in π . First, it is clear that the above k -coloring induces increasing patterns only.
 280 As for the length of each induced increasing pattern, focus on any color c_j . We note that,
 281 in every block $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$ of π , either the whole subpattern $\text{ip}(s_i + 1)$ is colored
 282 with color c_j (if $s_i \in S_j$) or exactly one element of the subpattern $\text{dp}(k - 1)$ is colored with

283 color c_j (if $s_i \notin S_j$). It follows that the increasing pattern induced by color c_j in π has length
 284 $\sum_{s_i \in S_j} (s_i + 1) + n - |S_j| = \sum_{s_i \in S_j} s_i + |S_j| + n - |S_j| = B + n$.

285 For the reverse direction, suppose now that there exists a k -coloring of π such that each
 286 color induces an increasing pattern of length $B + n$. Every block $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$
 287 requires at least k colors, as it contains a decreasing subpattern of length k . Therefore, the
 288 whole subpattern $\text{ip}(s_i + 1)$ is colored with the same color. For every $j \leq k$, let S_j be the set
 289 of all s_i such that, in the i -th block $\text{ip}(s_i + 1) \ominus \text{dp}(k - 1)$, the subpattern $\text{ip}(s_i + 1)$ is colored
 290 with color c_j . We have $B + n = \sum_{s_i \in S_j} (s_i + 1) + n - |S_j| = \sum_{s_i \in S_j} s_i + |S_j| + n - |S_j|$, and
 291 hence $\sum_{s_i \in S_j} s_i = B$. Therefore, the n items s_1, s_2, \dots, s_n can be packed into k bins, each
 292 of capacity B . \blacktriangleleft

293 Most of the interest in Proposition 9 stems from the following proposition.

294 **► Proposition 10.** *DPOP MATCHING for symmetric dpop and 312-avoiding permutations is*
 295 *W[1]-hard for the parameter width(P).*

296 **Proof.** We perform a reduction from BALANCED k -INCREASING COLORING, which is W[1]-
 297 hard for the parameter k . Let $\pi \in \mathfrak{S}(kn)$ for some positive integers k and n . We construct a
 298 symmetric dpop $P = (\mathcal{P}, \mathcal{P})$, where $\mathcal{P} = (X, \preceq)$, as follows: $X = [k] \times [n]$ and $(i, j) \preceq (i', j')$
 299 if and only if $i = i'$ and $j \leq j'$. We claim that P occurs in π if and only if π admits a k -coloring
 300 for which every color induces an increasing pattern of length n .

301 If π admits such a k -coloring into colors c_1, c_2, \dots, c_k , the function $\varphi: X \rightarrow [kn]$ that
 302 maps each pair (i, j) to the j -th smallest position with color c_i is an occurrence of P in π .

303 Conversely, suppose that some injective function $\varphi: X \rightarrow [kn]$ is an occurrence of P in π .
 304 For each $i \leq k$, the set $\{i\} \times [n]$ forms a chain of \preceq , and therefore it is mapped to an increasing
 305 pattern of size n . Coloring this pattern in color c_i produces the desired k -coloring. \blacktriangleleft

306 We show now that the problem where P consists of k independent chains is XP for the
 307 parameter k . In fact, we generalize this result to k -weak partial orders (i.e., if P consists
 308 of k independent weak orders).

309 **► Proposition 11.** *DPOP MATCHING for k -weak symmetric dpop is XP with parameter k .*

310 **Proof.** Let P be a disjoint union of k weak symmetric dpop P_1, P_2, \dots, P_k . For each dpop P_i ,
 311 let \preceq_i be a linear extension of P_i , and let $P_{i,1}, P_{i,2}, \dots, P_{i,p_i}$ be the maximal antichains of P_i ,
 312 ordered by \preceq_i . Finally, for each k -tuple $\mathbf{a} = (a_1, a_2, \dots, a_k)$ of integers such that $a_i \leq |P_i|$,
 313 we denote by $P_{\mathbf{a}}$ the dpop obtained from P by removing the a_i \preceq_i -least elements of each
 314 dpop P_i , and by $P_{i,\mathbf{a}}^{\min}$ the antichain $P_{i,j}$ that contains elements from $P_{\mathbf{a}}$ and for which j is
 315 minimal, or \emptyset if no such antichain exists.

316 Then, given a permutation $\pi \in \mathfrak{S}(n)$, a k -tuple $\mathbf{I} = (I_1, \dots, I_k)$ of intervals of $[n]$, a
 317 k -tuple \mathbf{a} and an integer ℓ , a function $\varphi: P_{\mathbf{a}} \rightarrow \{\ell, \ell + 1, \dots, n\}$ is called a *partial matching*
 318 for $(\pi, \mathbf{I}, \mathbf{a}, \ell)$ if:

- 319 ■ φ and $\pi \circ \varphi$ are \preceq_i -non-decreasing for each i , and
- 320 ■ for each i , and each element x of $P_{\mathbf{a}}$, $\pi(x) \in I_i$ if and only if $x \in P_{i,\mathbf{a}}^{\min}$.

321 Before going further, we denote by $\mathbf{1}_i$ the k -tuple with one element 1 (in position i) and
 322 $k - 1$ elements 0. We also denote by $<_i$ the partial order on tuples \mathbf{I} of intervals, where
 323 $\mathbf{I} <_i \mathbf{I}'$ if $I_j = I'_j$ whenever $j \neq i$ and $x < x'$ whenever $x \in I_i$ and $x' \in I'_i$.

324 When $a_i = |P_i|$ for all i , such a partial matching exists for all permutations π , tuples
 325 of intervals \mathbf{I} and integers ℓ . When $a_i = 0$ for all i and $\ell = 1$, and once π is fixed, such
 326 partial matchings coincide with (standard) matchings, and thus we are interested in checking

327 whether a partial matching exists. Finally, for all tuples \mathbf{I} and \mathbf{a} and for all $\ell \leq n$, a partial
 328 matching φ for $(\pi, \mathbf{I}, \mathbf{a}, \ell)$ exists precisely when one of the following cases occur:

- 329 1. φ is a partial matching for $(\pi, \mathbf{I}, \mathbf{a}, \ell + 1)$, i.e., $\ell \notin \varphi(P_{\mathbf{a}})$;
- 330 2. there exists an integer $i \leq k$ for which the \preceq_i -least element of $P_{i, \mathbf{a}}^{\min}$, say x , is such that
 331 $\varphi(x) = \ell$ and $\pi(\ell) \in I_i$, and either
 332 \quad – x is not the only element of $P_{i, \mathbf{a}}^{\min}$, and φ is a partial matching for $(\pi, \mathbf{I}, \mathbf{a} + \mathbf{1}_i, \ell + 1)$, or
 333 \quad – x is the only element of $P_{i, \mathbf{a}}^{\min}$ and there exists a tuple $\mathbf{I}' \succ_i \mathbf{I}$ such that φ is a partial
 334 matching for $(\pi, \mathbf{I}', \mathbf{a} + \mathbf{1}_i, \ell + 1)$.

335 Consequently, we can compute by dynamic programming the list of triples $(\mathbf{I}, \mathbf{a}, \ell)$ such
 336 that there exists a partial matching for $(\pi, \mathbf{I}, \mathbf{a}, \ell)$: deciding whether adding a triple $(\mathbf{I}, \mathbf{a}, \ell)$ to
 337 the list simply requires to check which triples of the form $(\mathbf{I}', \mathbf{a}', \ell + 1)$ already belong to the
 338 list. Since there are less than n^{3k+1} triples, this provides us with a $\tilde{O}(n^{6k+2})$ algorithm. \blacktriangleleft

339 5.2 Symmetric Pattern and Pattern-Avoiding π

340 In this final section, we consider restrictions on the *shape* of π , via pattern-avoiding restrictions.
 341 Our goal here is to identify tractable cases among classes of permutations avoiding one or
 342 more size-3 patterns. We give an almost complete dichotomy of polynomial/NP-hard cases,
 343 as shown in Table 1. Hardness results are proven in Proposition 12, and also apply to height-2
 344 dpops. Polynomial cases are proven in Proposition 13 and apply to dpops of any height.

345 **► Proposition 12.** *DPOP MATCHING for height-2 symmetric dpop P and permutation π is*
 346 *NP-hard even if π is separable (it avoids 2413 and 3142) and one of the following restrictions*
 347 *occurs:*

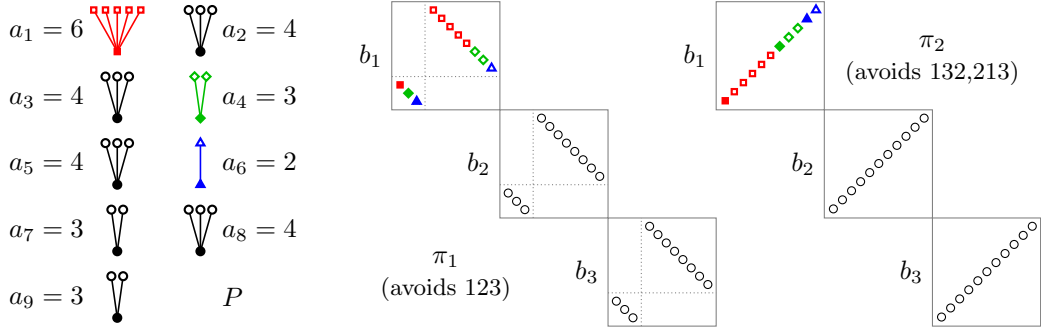
- 348 1. π is 123-avoiding;
- 349 2. π is (132, 213)-avoiding;
- 350 3. π is (132, 321)-avoiding;
4. π is (231, 312)-avoiding;
5. π is (132, 312)-avoiding;
6. π is (213, 321)-avoiding;
7. π is (213, 312)-avoiding;
8. π is (132, 231)-avoiding;
9. π is (213, 231)-avoiding.

351 **Proof.** In each of the cases presented below, we define a symmetric dpop $P = (\mathcal{P}, \mathcal{P})$ for
 352 some partially ordered set $\mathcal{P} = (X, \preceq)$. Each time, we identify P with the partial order \preceq .

353 **Case 1:** π is 123-avoiding and separable. We use a reduction from 3-PARTITION, as illustrated
 354 in Figure 6 with permutation π_1 . Let (A, B) be an instance of 3-PARTITION, where A is a
 355 list of integers a_1, a_2, \dots, a_{3n} with sum nB , all being larger than 1.

π avoids	123	132	213	231	312	321
... and ...						
123	12.1 3P					
132	13.2					
213	13.5	12.2 BIN				
231	13.1	12.8 BIC	12.9 BIC			
312	13.4	12.5 BIC	12.7 BIC	12.4 BIC		
321	13.3	12.3 BIS	12.6 BIS	(open)		

■ Table 1 Polynomial (green/light) and NP-hard (red/dark) cases for DPOP MATCHING with symmetric dpop and pattern-avoiding permutation π , for combinations of size-3 avoided patterns. For each case, see the referenced proposition and case for more details. Diagonal cases follow from any other hard case in the same row or column. For hard cases, the problem used for reduction is indicated as follows: BIC: Biclique, 3P: 3-PARTITION, BIN: UNARY BIN PACKING, BIS: BISECTION.



■ **Figure 6** Reductions from 3-PARTITION and UNARY BIN PACKING to DPOP MATCHING on 123-avoiding and (132, 213)-avoiding permutations. Left: the height-2 dpop P used in both reductions. Right: the 123-avoiding and (132, 213)-avoiding permutations used in each reduction. The mapping of three subsets of P corresponding to a first bin gadget is highlighted in each figure.

356 For all $p \leq n$, we define a *bin gadget* b_p as the permutation $\text{dp}(3) \oplus \text{dp}(B-3)$: we see
 357 this gadget as consisting of two parts. Our permutation π is now defined by $\pi = \bigoplus_{p=1}^n b_p$.
 358 Then, our partial order \preceq is defined on a set X of nB elements, noted $x_i, y_{i,2}, \dots, y_{i,a_i}$ for
 359 each $i \leq 3n$, so that $x_i \preceq y_{i,j}$ for all i and j .

360 If P has an occurrence $\varphi: X \rightarrow [nB]$ in π , this occurrence is bijective. Moreover, each
 361 element x_i is sent to the bottom-left of $y_{i,2}$, and thus it must be mapped to the left part of
 362 some gadget, say $b_{f(i)}$. Each element $y_{i,j}$ must then be mapped to the right part of the same
 363 gadget. Now, for each $p \leq n$, the set $S_p = \{i: f(i) = p\}$ is of size 3, and exactly B elements
 364 of X are mapped to the gadget b_p , which means that $\sum_{i \in S_p} a_i = B$. Moreover, $S_1 \cup \dots \cup S_p$
 365 forms a partition of $[3n]$, hence it yields a 3-partition of A .

366 Conversely, given a partition $S_1 \cup \dots \cup S_n$ of $[3n]$ such that $|S_p| = 3$ and $\sum_{i \in S_p} a_i = B$
 367 for each p , we build an occurrence of P in π by mapping the three elements x_i (for $i \in S_p$)
 368 to the left part of b_p , and the $B-3$ elements $y_{i,j}$ (for $i \in X_p$) to the right part of b_p .

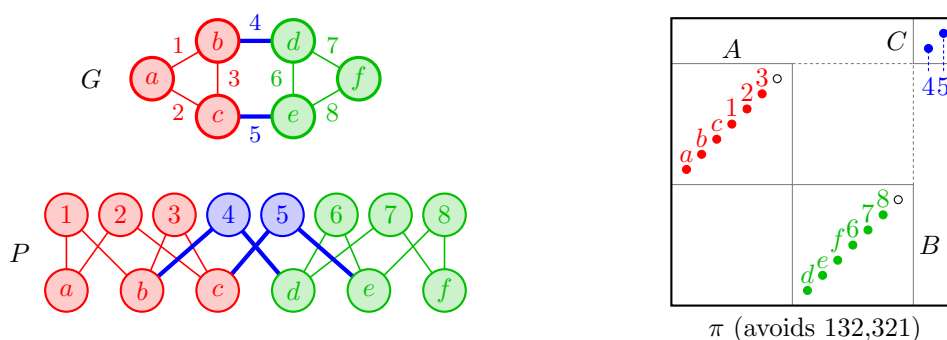
369 **Case 2:** π is (132, 213)-avoiding. We use a reduction from UNARY BIN PACKING, as
 370 illustrated in Figure 6 with permutation π_2 . Given an instance (A, B, k) of UNARY BIN
 371 PACKING, where A is a list of integers a_1, a_2, \dots, a_n larger than 1, we use the same dpop as
 372 in Case 1: our partial order \preceq is defined on a set X of nB elements, noted $x_i, y_{i,2}, \dots, y_{i,a_i}$
 373 for each $i \leq n$, so that $x_i \preceq y_{i,j}$ for all i and j . However, this time, our gadget b_p is the
 374 permutation $\text{ip}(B)$, and our permutation π is again defined by $\pi = \bigoplus_{p=1}^n b_p$.

375 If P has an occurrence $\varphi: X \rightarrow [nB]$ in π , this occurrence is bijective. Each element x_i
 376 is sent to some gadget, say $b_{f(i)}$, and the elements $y_{i,j}$ must then be mapped to the same
 377 gadget. Now, for each $p \leq k$, let $S_p = \{i: f(i) = p\}$. Exactly B elements of X are mapped
 378 to the gadget b_p , which means that $\sum_{i \in S_p} a_i = B$. This means that (S, B, k) is a positive
 379 instance of the UNARY BIN PACKING problem.

380 Conversely, given a partition $S_1 \cup \dots \cup S_k$ of $[n]$ such that $\sum_{i \in S_p} a_i = B$ for each i , we
 381 build an occurrence of P in π by mapping the B elements x_i and $y_{i,j}$ (for $i \in S_p$) to b_p .

382 **Case 3:** π is (132, 321)-avoiding. We use a reduction from BISECTION, as illustrated in
 383 Figure 7. Given a graph $G = (V, E)$ and an integer k , the BISECTION problem consists in
 384 deciding whether V admits a partition $V_1 \cup V_2$ such that $|V_1| = |V_2|$ and that *splits* at most
 385 k edges (i.e., at most k edges have one endpoint in V_1 and one endpoint in V_2).

386 Our reduction is as follows. Let $n = |V|/2$, $m = |E|$, $W = m + k + 1$, and $L = nW + m$.
 387 Our partition π is defined by $\pi = (\text{ip}(L) \oplus \text{ip}(L)) \oplus \text{ip}(k)$. These three parts of π are noted A ,
 388 B and C , from left to right. Then, our partial order \preceq is defined on a set X of $2nW + m$



■ **Figure 7** Simplified version (for $W = 1$ and $L = 7$) of the reduction from BISECTION to DPOP MATCHING on (132, 321)-avoiding permutations. Top left: a size-6 graph with a bisection $(\{a, b, c\}, \{d, e, f\})$ that splits $k = 2$ edges. Bottom left: height-2 dpop P (note that, in general, each element from the bottom line should appear W times and not just once). Right: permutation π ; elements of P are mapped to A, B or C depending on whether they are colored in red, green or blue.

389 elements: $2nW$ elements, noted $x_{v,i}$ for each $v \in V$ and $i \leq W$, and m elements, noted y_e
 390 for each $e \in E$. This order contains the relations $x_{v,i} \preceq y_e$ for which v is an endpoint of e .

391 Assume that there exists a mapping of P into π . For each $v \in V$, and since C has
 392 size $k < W$, at least one of the elements $x_{v,i}$ is mapped to A , in which case we say that v
 393 has type A , or to B , in which case v has type B . Then, each vertex has at least one type,
 394 and possibly both. We partition V into three sets V_A, V_B, V_{AB} containing the vertices of
 395 type A, B and both A and B , respectively. Moreover, for each $v \in V_A$, each element $x_{v,i}$
 396 must be mapped either to A or to C : these two parts together contain $L + k$ elements,
 397 so $|V_A| \leq (L + k)/W = n + 1 - 1/W$, and $|V_A| \leq n$. Similarly, $|V_B| \leq n$.

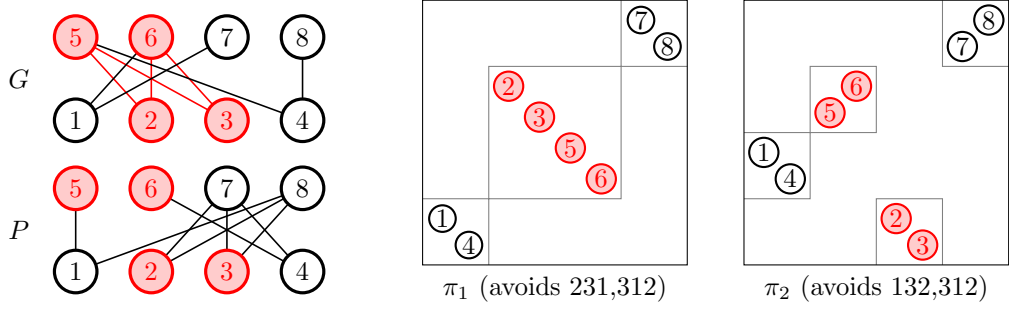
398 We build a set V_1 as the union of V_A with $n - |V_A|$ vertices of V_{AB} , and V_2 as $V \setminus V_1$, so
 399 that $|V_1| = |V_2| = n$. Moreover, for every $v \in V_1$ (resp., $v \in V_2$), some element $x_{v,i}$, say $x_{v,1}$,
 400 is mapped to A (resp., to B). Then, each edge $e = (u, v)$ that is split by (V_1, V_2) must be
 401 mapped to a point above some point of A and to the right of some point of B . This means
 402 that y_e is mapped to C , and that (V_1, V_2) splits at most k edges, i.e., is a valid bisection.

403 Conversely, given a bisection (V_1, V_2) splitting at most k edges, we map P into π as
 404 follows: map elements $x_{v,i}$ for $v \in V_1$ (resp., V_2) to the first nW elements of A (resp., B),
 405 map elements y_e for which e is induced by V_1 (resp., V_2) to the following elements of A
 406 (resp. B), and finally map all elements y_e such that e is split by (V_1, V_2) into C . This mapping
 407 is an occurrence of P in π .

408 **Case 4:** π is (231, 312)-avoiding. We use a reduction from BICLIQUE, as illustrated in
 409 Figure 8 with permutation π_1 . Given a bipartite graph $G = (V, E)$ and an integer k , the
 410 BICLIQUE problem consists in deciding whether V admits a complete bipartite subgraph $K_{k,k}$.
 411 If $V = A \cup B$ is a partition of V into two independent sets, adding independent vertices if
 412 needed allows us to assume that A and B have the same size n , and that no vertex in either
 413 side is fully connected to the other side.

414 Our permutation π is defined by $\pi = \text{dp}(n - k) \oplus \text{dp}(2k) \oplus \text{dp}(n - k)$. These three parts
 415 of π are noted b_1, b_2 and b_3 . Our partial order \preceq is the order on V such that $x \preceq y$ whenever
 416 $x \in A, y \in B$ and $\{x, y\} \notin E$.

417 Consider a mapping of P into π . For each element $x \in A$, there exists $y \in B$ such
 418 that $x \preceq y$, and therefore x cannot be mapped into b_3 . Symmetrically, no element $y \in B$
 419 may be mapped into b_1 . Overall, since $|\pi| = |V| = 2n$, b_1 contains $n - k$ elements from A, b_3



■ **Figure 8** Reduction from BICLIQUE to DPOP MATCHING on $(231, 312)$ -avoiding and $(213, 231)$ -avoiding permutations. Left: a bipartite graph G with a $(2, 2)$ biclique and the corresponding height-2 dpop P , built from the complement of G . Right: permutations π_1 and π_2 with a mapping of the vertices 1 to 8, including the biclique vertices mapped into the central $2k$ positions.

420 contains $n - k$ elements from B , and b_2 contains a size- k subset A' of A and a size- k subset
 421 B' of B . No two elements $x \in A'$ and $y \in B'$ that are mapped into b_2 are comparable for \preceq ,
 422 which means that $\{x, y\} \in E$ for each such pair, i.e., that (A', B') is a biclique.

423 Conversely, if G has a biclique (A', B') , we map all elements of $A \setminus A'$ into b_1 , all elements
 424 of $A' \cup B'$ into b_2 , and all elements of $B \setminus B'$ into b_3 . This mapping satisfies all relations
 425 $x \preceq y$ with $x \in A$ and $y \in B$, except for $x \in A'$ and $y \in B'$, but indeed there is no such
 426 relation since (A', B') is a biclique.

427 **Case 5:** π is $(132, 312)$ -avoiding. We also use a reduction from BICLIQUE, as illustrated
 428 in Figure 6 with permutation π_2 . Our partial order \preceq is the same as in Case 4, and our
 429 permutation π is defined by $\pi = ((dp(n - k) \oplus ip(k)) \ominus dp(k)) \oplus ip(n - k)$. These three parts
 430 of π are noted b_1 , b_2 , b_3 and b_4 .

431 Consider a mapping of P into π . For each element $y \in B$, there exists $x \in A$ such
 432 that $x \preceq y$, and therefore y cannot be mapped into b_1 or b_3 . Thus, and since $|B| = n$,
 433 the elements of B are mapped to b_2 or b_4 , and the elements of A are mapped to b_1 or b_3 .
 434 Hence, b_2 contains a size- k subset B' of B and b_3 contains a size- k subset A' of A . No
 435 element of b_2 is comparable to any element of b_3 , and therefore (A', B') is a biclique.

436 Conversely, if G has a biclique (A', B') , we map all elements of $A \setminus A'$ into b_1 , all elements
 437 of A' into b_3 , all elements of B' into b_2 and all elements of $B \setminus B'$ into b_4 . This mapping
 438 satisfies all relations $x \preceq y$ with $x \in A$ and $y \in B$, except for $x \in A'$ and $y \in B'$, but indeed
 439 there is no such relation since (A', B') is a biclique.

440 **Cases 6–9:** These cases are symmetric to Cases 3, 5, 5 and 7, respectively. Indeed, if (P, π) is
 441 an instance of DPOP MATCHING with P a height-2 symmetric dpop, (P^∂, π^{cr}) and (P, π^{-1})
 442 are equivalent instances of DPOP MATCHING with height-2 symmetric dpop, and

443 **Case 6:** if π avoids 132 and 321 (Case 3), π^{cr} avoids $132^{cr} = 213$ and $321^{cr} = 321$;

444 **Case 7:** if π avoids 132 and 312 (Case 5), π^{cr} avoids $132^{cr} = 213$ and $312^{cr} = 312$;

445 **Case 8:** if π avoids 132 and 312 (Case 5), π^{-1} avoids $132^{-1} = 132$ and $312^{-1} = 231$;

446 **Case 9:** if π avoids 213 and 312 (Case 7), π^{-1} avoids $213^{-1} = 213$ and $312^{-1} = 231$. ◀

447 ▶ **Proposition 13.** DPOP MATCHING is in P for symmetric dpop P if one of the following
 448 restrictions on π occurs:

- 449 1. π is $(123, 231)$ -avoiding; 3. π is $(123, 321)$ -avoiding;
 450 2. π is $(123, 132)$ -avoiding; 4. π is $(123, 312)$ -avoiding; 5. π is $(123, 213)$ -avoiding.

451 **Proof.** In each of the cases presented below, we are given a permutation π and a symmetric
 452 dpop $P = (\mathcal{P}, \mathcal{P})$ for some partially ordered set $\mathcal{P} = (X, \preceq)$. Each time, we identify P with
 453 the partial order \preceq .

454 **Case 1:** π is (123, 231)-avoiding. There exist integers k, ℓ and m , with sum n , such that
 455 $\pi = \text{dp}(k) \ominus (\text{dp}(\ell) \oplus \text{dp}(m))$. These three parts of π are noted b_1, b_2 and b_3 . Then, for every
 456 pair (u, v) such that $u \prec v$, we must map u into b_2 and v into b_3 . Such values can be mapped
 457 greedily, since elements in b_2 are pairwise incompatible, as well as those in b_3 . Thus, P can
 458 be mapped into π if and only if it has height at most 2, there are at most a elements that
 459 are lower bounds, and at most b elements that are upper bounds.

460 Note that, if P is not symmetric, the problem becomes NP-hard, since reversing the
 461 horizontal order of P_{p} and π transforms π into the (132, 321)-avoiding permutation of the
 462 NP-hard Case 3 in Proposition 12.

463 **Case 2:** π is (123, 132)-avoiding. The permutation π is a skew sum $\pi = \ominus_{p=1}^k d_p$ of patterns of
 464 the form $d_p = \text{dp}(a_p) \oplus \text{dp}(1)$ for some integer $a_p \geq 0$. Then, no two elements in X can share
 465 a strict lower bound, i.e., if $u \prec v$ and $u \prec w$ then $v = w$. Thus, P is of height at most 2,
 466 and there exists a partition $S_1 \cup \dots \cup S_\ell$ of X in which each set S_i contains a distinguished
 467 element s_i , such that $x \preceq s_i$ if and only if $x \in S_i$. Up to reordering the patterns d_p and
 468 the sets S_i , which are pairwise incomparable, we assume that $a_1 \geq a_2 \geq \dots \geq a_k$ and that
 469 $|S_1| \geq |S_2| \geq \dots \geq |S_\ell|$. Let also m be the number of sets S_i with size at least 2.

470 Each set S_i must be mapped into a single pattern, say $d_{p(i)}$, and if $i \leq m$, i.e., if $|S_i| \geq 2$,
 471 the element s_i must be mapped to the unique top-right element of $d_{p(i)}$. Such a mapping
 472 exists if and only if $k \geq m$ and $a_i \geq |S_i| - 1$ for all $i \leq m$: we shall choose $p(i) = i$ and
 473 map greedily the elements of $S_i \setminus \{s_i\}$ to the bottom-left part of d_i . Finally, the elements of
 474 singleton sets S_i can be mapped to the remaining places in π .

475 **Case 3:** π is (123, 321)-avoiding. Erdős-Szekeres theorem [14] proves that $n \leq 4$.

476 **Cases 4–5:** These cases are symmetric to Cases 1 and 2, respectively. Indeed, if (P, π) is an
 477 instance of DPOP MATCHING with P a height-2 symmetric dpop, $(P^\partial, \pi^{\text{cr}})$ and (P, π^{-1})
 478 are equivalent instances of DPOP MATCHING with height-2 symmetric dpos, and

479 **Case 4:** if π avoids 123 and 231 (Case 1), π^{-1} avoids $123^{-1} = 123$ and $231^{-1} = 312$;

480 **Case 5:** if π avoids 123 and 132 (Case 2), π^{cr} avoids $123^{\text{cr}} = 123$ and $132^{\text{cr}} = 213$. ◀

481 6 Concluding Remarks

482 Some open complexity questions remain among the parameters we identified for DPOP
 483 MATCHING. For semi-total dpos, the complexity is open for constant width, and for most
 484 classes of pattern-avoiding permutations (although, according to Propositions 7 and 10, the
 485 problem is NP-hard when π avoids 1234 or 312, respectively). For symmetric dpos, it
 486 would be interesting to settle the complexity status of deciding whether a dpop occurs in
 487 a (231, 321)-avoiding or (312, 321)-avoiding permutation. In particular, for these cases, we
 488 conjecture that the problem becomes polynomial when $\text{height}(P)$ is constant.

489 Regarding the original puzzle formulation of the problem, an interesting question is to
 490 generate instances that yield a unique solution, i.e., given a permutation π , find a dpop
 491 with a unique occurrence in π . This can be done by using a semi-total dpop (e.g., take X
 492 with $|X| = |\pi|$, let P_{p} be a total order and P_{v} be an empty order), but one could try to
 493 minimize $|X|$ or the number of pairs of comparable elements in P (i.e., the number of clues)
 494 in order to have a unique solution.

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