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Calabi–Yau versus quasi-Hamiltonian structures

Tristan Bozec*, Damien Calaque†, Sarah Scherotzke‡

Abstract

We compare the Calabi–Yau structures defined and studied in [2, 3] with noncommutative analogs of quasi-Hamiltonian and Poisson geometries developed in [7, 8].

Contents

1	Link between Hochschild and de Rham homologies	1
1.1	Recollection of Ginzburg and Schedler [6]	1
1.2	Computations for $A = k[x^{\pm 1}]$	2
2	CY \Rightarrow quasi-bisymplectic in the sense of [8]	3
2.1	Relative setting	3
2.2	Non-degeneracy	4
2.3	Compatibility for the quiver A_2	4
2.4	Double Poisson structures and the 1-Calabi-Yau property	7
2.5	Calabi-Yau cospans and fusion	9
3	Comparison of Lagrangian structures	10
4	Simplicial complexes	13

1 Link between Hochschild and de Rham homologies

1.1 Recollection of Ginzburg and Schedler [6]

Fix a field k of characteristic zero. Consider a unital associative k -algebra A and fix a complementary subspace $\bar{A} = A/k$ of k . Denote by $d : A \rightarrow \bar{A}$ the associated quotient. The dg-algebra ΩA of noncommutative differential forms is defined by the quotient of $T_k(A \oplus \bar{A}[-1])$ by the relations

$$a \otimes b = ab \quad \text{and} \quad d(ab) = a \otimes d(b) + d(a) \otimes b$$

for every $a, b \in A$. The differential of ΩA is the derivation induced by d , satisfying $d^2 = 0$. We will systematically use the $\bar{}$ notation for the quotient by k .

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We have a contraction $\iota : \Omega^\bullet A \rightarrow \Omega^{\bullet-1} A$ defined by

$$\iota(a_0 da_1 \dots da_n) = \sum_{l=1}^n (-1)^{(l-1)(n-1)+1} [a_l, da_{l+1} \dots da_n a_0 da_1 \dots da_{l-1}]$$

that factors through $\text{DRA} := \Omega A / [\Omega A, \Omega A]$, a Hochschild differential b on ΩA defined by

$$b(\alpha da) = (-1)^{|\alpha|} [\alpha, a]$$

and a Karoubi operator on ΩA given by

$$\kappa(\alpha da) = (-1)^{|\alpha|} da\alpha.$$

We have a harmonic decomposition $\bar{\Omega} A = P\bar{\Omega} A \oplus P^\perp \bar{\Omega}$ where

$$P\bar{\Omega} A = \ker(1 - \kappa)^2 \quad \text{and} \quad P^\perp \bar{\Omega} = \text{ima}(1 - \kappa)^2.$$

that gives

$$\iota = bN|_P \quad \text{and} \quad B = Nd|_P$$

where N is the grading operator and B the Connes differential.

Furthermore b and d are invariant under the harmonic decomposition and $(P^\perp \bar{\Omega}, b)$ is acyclic.

We observe that

$$\overline{DR} := \bar{\Omega} / [\bar{\Omega}, \bar{\Omega}] \simeq P\bar{\Omega} / P[\bar{\Omega}, \bar{\Omega}] \simeq P\bar{\Omega} / bP\bar{\Omega}$$

By [6], we have

$$(P\bar{\Omega}, b)^\bullet \simeq \overline{HH}_\bullet.$$

Remark 1.1. We call a k -algebra A 1-smooth if it has projective dimension one as an $A \otimes A$ -module. In particular a 1-smooth algebra is smooth and has projective dimension at most 1. This can be seen as follows: given a projective resolution $\Omega^1(A) \rightarrow A \otimes A \rightarrow A$ of A -bimodules, tensoring this resolution with $M \otimes_A -$ for $M \in \text{Mod}_A$ yields a projective resolution of M as A -module of length at most 1.

1.2 Computations for $A = k[x^{\pm 1}]$

We always mean $(dx)y$ if no brackets appear in $dx y$.

Set $\alpha_n = (x^{-1} dx)^{2n-1}$, $\beta_n = \kappa(\alpha_n) = (dx x^{-1})^{2n-1} \in \bar{\Omega}^{2n-1} A$. Then

$$\kappa(\beta_n) = \kappa(-\beta_{n-1} dx dx^{-1}) = -dx^{-1} \beta_{n-1} dx = \alpha_n.$$

Hence $\alpha_n + \beta_n \in P\bar{\Omega} A$ and $\alpha_n - \beta_n = \frac{1}{2}(1 - \kappa)^2(\alpha_n) \in P^\perp \bar{\Omega} A$. Then,

$$\begin{aligned} \iota \alpha_n &= \frac{1}{2}(2n-1)b(\alpha_n + \beta_n) \\ &= \frac{1}{2}(2n-1)([\alpha_{n-1} x^{-1} dx x^{-1}, x] - [\beta_{n-1} dx, x^{-1}]) \\ &= \frac{1}{2}(2n-1)(x^{-1} \beta_{n-1} dx + \alpha_{n-1} x^{-1} dx - \beta_{n-1} dx x^{-1} - x \alpha_{n-1} x^{-1} dx x^{-1}) \\ &= (2n-1)((x^{-1} dx)^{2n-2} - (dx x^{-1})^{2n-2}). \end{aligned}$$

On the other hand, $d\alpha_1 = -(x^{-1}dx)^2$, and if we assume $d\alpha_{n-1} = -(x^{-1}dx)^{2n-2}$, we get

$$\begin{aligned} d\alpha_n &= d(x^{-1}dx(x^{-1}dx)^{2n-2}) \\ &= d(x^{-1}dx)(x^{-1}dx)^{2n-2} - x^{-1}dxd((x^{-1}dx)^{2n-2}) \\ &= -x^{-1}dxx^{-1}dx(x^{-1}dx)^{2n-2} - x^{-1}dxd^2\alpha_{n-1} \\ &= -(x^{-1}dx)^{2n}. \end{aligned}$$

and similarly $d\beta_n = (dxx^{-1})^{2n}$ for all n . Thus, as $\iota\alpha_n = \iota\beta_n$,

$$\iota(\alpha_n + \beta_n) = 2\iota\alpha_n = -2(2n-1)d(\beta_{n-1} + \alpha_{n-1}).$$

As a consequence $(\iota - ud)(\gamma) = 0$, where $\gamma_k = \frac{1}{2}(\alpha_k + \beta_k) \in P\bar{\Omega}^{2k-1}k[x^{\pm 1}]$ and

$$\gamma = \sum_{k \geq 0} \frac{k!}{(2k+1)!} (-u)^k \gamma_{k+1}.$$

Now, we have the following chain of quasi-isomorphisms, functorial in A

$$\left(\frac{\bar{\Omega}[[u]]}{[d\Omega, d\bar{\Omega}]}, \iota - ud \right) \xrightarrow{P} (P\bar{\Omega}[[u]], \iota - ud) \xrightarrow{N!} (P\bar{\Omega}[[u]], b - uB) \xrightarrow{\hookrightarrow} (\bar{\Omega}[[u]], b - uB)$$

whose homology gives the reduced negative cyclic homology $\overline{\text{HC}}_{\bullet}^{-}$. Through this chain and the isomorphism $\Omega^n A \simeq A \otimes \bar{A}^{\otimes n}$, γ is mapped to

$$\sum_{k \geq 0} k! u^k \left((x^{-1} \otimes x)^{\otimes(k+1)} - (x \otimes x^{-1})^{\otimes(k+1)} \right)$$

as

$$\begin{aligned} \alpha_{k+1} &= (x^{-1}dx)^{2k+1} = (-1)^k x^{-1} (dxdx^{-1})^k dx, \\ \beta_{k+1} &= (dxx^{-1})^{2k+1} = (-1)^{k+1} x (dx^{-1}dx)^k dx^{-1}, \\ \text{and } \gamma_{k+1} &\in P\bar{\Omega}^{2k+1}, \end{aligned}$$

all of which being consistent with [3, 3.1.1].

α_n there is
 γ_n here!

2 CY \Rightarrow quasi-bisymplectic in the sense of [8]

2.1 Relative setting

Consider a morphism $\Phi : k[x^{\pm 1}] \rightarrow A$ where A is smooth and concentrated in degree 0. Assume that $\Phi[\gamma_1] = 0 \in \overline{\text{HH}}^1 A = H_1(\bar{\Omega}A, b)$. The space of lifts of $\Phi(\gamma_1)$ in relative homology is given by elements $\omega_1 \in \bar{\Omega}^2 A / b\bar{\Omega}^3 A$ such that $b\omega_1 = \Phi(\gamma_1)$. As we also have

$$\overline{\text{HH}} = \ker(\overline{\text{DR}} \xrightarrow{\iota} \bar{\Omega}),$$

this space of lifts is also described by elements $\omega_1 \in \overline{\text{DR}}^2 A$ such that $\iota\omega_1 = \Phi(\gamma_1)$, which is condition (B2) in [8].

Similarly, if $\Phi[\gamma] = 0 \in \overline{\text{HC}}_1^{-} A$, we have the existence of $\omega_k \in \overline{\text{DR}}^{2k} A$ for all k such that

$$(\iota - ud) \left(\sum_{k \geq 0} u^k \omega_{k+1} \right) = \Phi(\gamma)$$

or equivalently

$$\omega_1 = \Phi(\gamma_1) = \frac{1}{2}(\Phi^{-1}d\Phi + d\Phi\Phi^{-1}) \quad (\mathbb{B}2)$$

$$\omega_2 - d\omega_1 = -\frac{1}{6}\Phi(\gamma_2) \Rightarrow d\omega_1 = \frac{1}{6}(\Phi^{-1}d\Phi)^3 \pmod{[-, -]} \quad (\mathbb{B}1)$$

$$\omega_3 - d\omega_2 = \frac{2!}{5!}\Phi(\gamma_3)$$

\vdots

$$\omega_{k+1} - d\omega_k = (-1)^k \frac{k!}{(2k+1)!}\Phi(\gamma_{k+1}) \quad k \geq 1.$$

2.2 Non-degeneracy

Set $R = k[x^{\pm 1}]$ and write the relative 1-pre-Calabi–Yau structure

$$A^\vee[1] \rightarrow R^\vee[1] \otimes_{R^e} A^e \xrightarrow{\simeq} R \otimes_{R^e} A^e \rightarrow A$$

with short resolutions to get the homotopy commuting diagram

$$\begin{array}{ccccccc} A^e & \xrightarrow{\text{id}} & A^e & \xrightarrow{(\Phi^{-1} \otimes 1 + 1 \otimes \Phi^{-1})/2} & A^e & \xrightarrow{d\Phi} & \Omega^1 A \\ E \downarrow & & \downarrow & & \downarrow & & \downarrow \\ DA & \xrightarrow{\text{ev}_\Phi} & A^e & \xrightarrow{(\Phi^{-1} \otimes 1 + 1 \otimes \Phi^{-1})/2} & A^e & \xrightarrow{\text{id}} & A^e \end{array}$$

where $DA := \text{Der}(A, A^e) \simeq (\Omega^1 A)^\vee$. The homotopy $DA \rightarrow \Omega^1 A$ gives $\iota_E \omega_1 = (\Phi^{-1}d\Phi + d\Phi\Phi^{-1})/2$.

Remark 2.1. If $\omega \in \text{DR}^2 A$ and $\delta \in DA \simeq (\Omega^1 A)^\vee$, then $\iota_\delta(\omega) \in \Omega^1 A \hookrightarrow A^e$ matches $\langle \delta, \iota_E(\omega) \rangle \in A^e$.

Now assume that our Calabi–Yau structure is non-degenerate, that is

$$A^\vee[1] \simeq \text{fib}(R^\vee[1] \otimes_{R^e} A^e \xrightarrow{\simeq} R \otimes_{R^e} A^e \rightarrow A).$$

In short resolutions, this yields a quasi-isomorphism (between vertical complexes)

$$\begin{array}{ccc} A^e & \xrightarrow{(\Phi^{-1} \otimes 1 + 1 \otimes \Phi^{-1})/2} & A^e \\ E \downarrow & & \downarrow d\Phi \\ DA & \xrightarrow{\iota_E \omega_1} & \Omega^1 A \end{array}$$

which in particular gives a surjection $DA \rightarrow \Omega^1 A / \langle d\Phi \rangle$, that is (\mathbb{B}_3) in [8].

2.3 Compatibility for the quiver A_2 .

Consider the quiver $A_2 = (V = \{1, 2\}, E = \{e : 1 \rightarrow 2\})$, with orthogonal idempotents e_1 and e_2 satisfying $1 = e_1 + e_2$, and set

$$a_1 = e_1 + e^* e \text{ and } a_2 = e_2 + e e^*.$$

ι_E is the ι of previous sections

use Lemma 2.6.3 in C-BEG

and smoothness to have the short resolution!

Let us denote by A the localization $(\overline{kA_2})_{a_1, a_2}$. Recall that we have given in [3] a relative 1-Calabi–Yau structure on $\Phi : k[x^{\pm 1}] \rightarrow A$ defined by

$$\Phi_1(x_1) = a_1^{-1} \quad \text{and} \quad \Phi_2(x_2) = a_2.$$

In the previous section we proved that this Calabi–Yau structure induces a quasi-bisymplectic one $\omega_1 \in \overline{\text{DR}}^2 A$ on A . We want to prove that the double quasi-Poisson bracket compatible with ω_1 through [8, Theorem 7.1] is the one described in [8, §8.3]:

$$P = \frac{1}{2} \left((1 + ee^*) \frac{\partial}{\partial e^*} \frac{\partial}{\partial e} - (1 + e^*e) \frac{\partial}{\partial e} \frac{\partial}{\partial e^*} \right) \in (DA/[DA, DA])_2.$$

Note that we use the convention regarding concatenation of paths opposite to the one in [7], that is $e = e_2 e e_1$. In [2], one homotopy $\phi(\gamma_1) \sim 0$ is given by

$$\beta_1 = \frac{1}{2} (e^* \otimes e \otimes \Phi + \Phi \otimes e^* \otimes e - e^* \otimes \Phi^{-1} \otimes e - \Phi^{-1} \otimes e \otimes e^* + 1 \otimes e^* \otimes e\Phi - 1 \otimes e\Phi \otimes e^*) \quad (2.2)$$

1/2 missing in BCS2

where $\Phi = \Phi_1(x_1) + \Phi_2(x_2)$. It yields an element

$$\omega_1 = \frac{1}{4} (e^* ded\Phi + \Phi de^* de - e^* d\Phi^{-1} de - \Phi^{-1} dede^* + de^* d(e\Phi) - d(e\Phi) de^*)$$

1/4 because of the degree operator

in $\overline{\text{DR}}^2 A = (\overline{\Omega A}/[\overline{\Omega A}, \overline{\Omega A}])_2$. We can heavily simplify this expression working modulo $[\overline{\Omega A}, \overline{\Omega A}]$. First note that (again, dab stands for $(da)b$)

$$\begin{aligned} d\Phi &= -a_1^{-1} (de^* e + e^* de) a_1^{-1} + dee^* + ede^* = -\Phi (de^* e + e^* de) \Phi + dee^* + ede^* \\ d\Phi^{-1} &= de^* e + e^* de - a_2^{-1} (dee^* + ede^*) a_2^{-1} = de^* e + e^* de - \Phi^{-1} (dee^* + ede^*) \Phi^{-1}, \end{aligned}$$

thus, using $\Phi e \Phi = e$ and $\Phi e^* \Phi = e^*$ (cf [3, (4.3)]),

$$\begin{aligned} 4\omega_1 &= \Phi de^* de - \Phi^{-1} dede^* + e^* ded\Phi - e^* d\Phi^{-1} de + 2de^* d(e\Phi) \\ &= \Phi de^* de - \Phi^{-1} dede^* - e^* de\Phi (de^* e + e^* de) \Phi \\ &\quad + e^* \Phi^{-1} (dee^* + ede^*) \Phi^{-1} de + 2de^* de\Phi - 2de^* e\Phi (de^* e + e^* de) \Phi \\ &= \Phi de^* de - \Phi^{-1} dede^* - e^* de\Phi de^* e\Phi \\ &\quad - \underbrace{e^* de\Phi e^* de\Phi + e^* \Phi^{-1} dee^* \Phi^{-1} de}_{\equiv 0} + e^* \Phi^{-1} ede^* \Phi^{-1} de \\ &\quad + 2de^* de\Phi - 2 \underbrace{de^* e\Phi de^* e\Phi}_{\equiv 0} - 2de^* e\Phi e^* de\Phi \\ &\equiv 3\Phi de^* de - \Phi^{-1} dede^* - e\Phi e^* de\Phi de^* + e^* \Phi^{-1} ede^* \Phi^{-1} de + 2de^* e\Phi e^* de\Phi \\ &= 3\Phi de^* de - \Phi^{-1} dede^* - ee^* \Phi_2^{-1} de\Phi de^* + e^* e\Phi_1 de^* \Phi^{-1} de + 2de^* ee^* \Phi_2^{-1} de\Phi \\ &= 3\Phi de^* de - \Phi^{-1} dede^* - de\Phi de^* + \Phi^{-1} de\Phi de^* \\ &\quad + de^* \Phi^{-1} de - \Phi de^* \Phi^{-1} de - 2de^* de\Phi + 2de^* \Phi^{-1} de\Phi \\ &\equiv 2\Phi de^* de - 2\Phi^{-1} dede^* \end{aligned}$$

which matches the (unproven) formula in [9, Example 2.16].

Remark 2.3 (Sanity check).

$$\begin{aligned}
\iota_E(2\omega_1) &= -[e^*, de\Phi] + [e, \Phi de^*] + [e, de^*\Phi^{-1}] - [e^*, \Phi^{-1}de] \\
&= -e^*de\Phi + de\Phi e^* + e\Phi de^* - \Phi de^*e + ede^*\Phi^{-1} - de^*\Phi^{-1}e - e^*\Phi^{-1}de + \Phi^{-1}dee^* \\
&= -d(e^*e)\Phi + d(ee^*)\Phi^{-1} + \Phi^{-1}d(ee^*) - \Phi d(e^*e) \\
&= \Phi^{-1}(-\Phi d(e^*e)\Phi + d(ee^*)) + (-\Phi d(e^*e)\Phi + d(ee^*))\Phi^{-1} \\
&= \Phi^{-1}d\Phi + d\Phi\Phi^{-1}.
\end{aligned}$$

Let us prove Yamakawa's statement. Thanks to [8, Proposition 7.4], P and ω_1 are compatible if

$$\iota(\omega_1)\iota(P) = 1 - \frac{1}{4}T \quad (2.4)$$

with $T(dp) = [p, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}]$. For $p = e$, the LHS is

$$\begin{aligned}
\iota(\omega_1)\iota(P)(de) &= \frac{1}{2}\iota(\omega_1) \left(\frac{\partial}{\partial e^*}(1 + e^*e) + (1 + ee^*)\frac{\partial}{\partial e^*} \right) \\
&= \frac{1}{2}(\circ i_{\partial/\partial e^*}(\omega_1)(1 + e^*e) + (1 + ee^*)\circ i_{\partial/\partial e^*}(\omega_1))
\end{aligned}$$

where

$$i_\delta(pdqdr) = p\delta(q)' \otimes \delta(q)'' dr - pdq\delta(r)' \otimes \delta(r)'' \in A \otimes \Omega^1 + \Omega^1 \otimes A.$$

Note that above we have used, for $\pi, \nu \in A$ and $\delta \in D_A$,

$$\begin{aligned}
\circ i_{\pi\delta\nu}(pdqdr) &= \circ(p\delta(q)'\nu \otimes \pi\delta(q)'' dr - pdq\delta(r)'\nu \otimes \pi\delta(r)'') \\
&= \pi \circ i_\delta(pdqdr)\nu
\end{aligned}$$

stated in
the proof
of 2.8.6 in
CBEG...

since the bimodule structure on D_A is induced by the inner one on A^e . We have:

$$\circ i_{\partial/\partial e^*}(2\omega_1) = \circ(\Phi \otimes de + \Phi^{-1}de \otimes e_2) = de\Phi + \Phi^{-1}de$$

thus

$$\begin{aligned}
4\iota(\omega_1)\iota(P)(da) &= (de\Phi + \Phi^{-1}de)(1 + e^*e) + (1 + ee^*)(de\Phi + \Phi^{-1}de) \\
&= 2de + \Phi^{-1}de\Phi^{-1} + \Phi de\Phi
\end{aligned}$$

whereas 4 times the RHS of 2.4 is

$$\begin{aligned}
4de - [e, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}] &= 4de - e\Phi^{-1}(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*) \\
&\quad + e(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)\Phi^{-1} \\
&\quad + \Phi^{-1}(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)e \\
&\quad - (-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)\Phi^{-1}e \\
&= 4de + ede^*e\Phi + ee^*de\Phi - e\Phi de^*e - e\Phi e^*de \\
&\quad + \Phi^{-1}dee^*e + \Phi^{-1}ede^*e - dee^*\Phi^{-1}e - ede^*\Phi^{-1}e \\
&= 4de + ee^*de\Phi - \Phi^{-1}ee^*de + \Phi^{-1}dee^*e - dee^*e\Phi \\
&= 4de + \Phi de\Phi - de\Phi - de + \Phi^{-1}de \\
&\quad + \Phi^{-1}de\Phi^{-1} - \Phi^{-1}de - de + de\Phi \\
&= 2de + \Phi^{-1}de\Phi^{-1} + \Phi de\Phi.
\end{aligned}$$

Computations are similar to prove 2.4 evaluated at de^* .

Hourra.

2.4 Double Poisson structures and the 1-Calabi-Yau property

Double Poisson algebras have been introduced by Van den Berg in [8]. Recall

Definition 2.5. • A double Poisson algebra is an algebra A together with a double bracket $\{\{-, -\}\} : A \times A \rightarrow A \otimes A$ satisfying that $\{\{-, -\}\}$ is a derivation in the second argument, satisfies $\{\{u, v\}\} = \tau\{\{v, u\}\}$ for the twist function $\tau : A \otimes A \rightarrow A \otimes A, a \otimes b \mapsto b \otimes a$ and the Jacobi identity.

- We call an algebra A bi-symplectic, if there is a closed form $\omega \in \text{DR}(A)^2$ which is non-degenerate, that is $\iota_\omega : D_A \rightarrow \Omega_A$ is an isomorphism of A -bimodules.

A double Poisson algebra structure induces a classical Poisson structures on the moduli space of representations of A , while a bi-symplectic structure induces a symplectic structure on the moduli space of representations of A .

If A is 1-smooth, the existence of a double Poisson structure is equivalent to the existence of an element $P \in (DA/[DA, DA])_2$ such that $\{P, P\} = 0$, where $\{-, -\}$ denotes in this context the Schouten-Nijenhuis bracket on $DA/[DA, DA]$.

Definition 2.6. [?, 8] Let $P \in (DA/[DA, DA])_2$ and (A, P) be a double Poisson algebra. A moment map for the double Poisson algebra is an element $\mu = (\mu_i)_i \in \bigoplus e_i A e_i$ such that $\{P, \mu\} = -E$.

In [8] A3 it is shown that every bi-symplectic form gives rise to non-commutative Hamiltonian structure with Poisson bracket

$$P_\omega := (\iota(\omega)^{-1} \otimes \iota(\omega)^{-1})(\omega) \in (DA/[DA, DA])_2.$$

Definition 2.7. We say that a relative 1-Calabi-Yau structure

$$\mu : \bigoplus_{i=1}^n k[x_i] \rightarrow A, x_i \mapsto \mu_i$$

given by $\mu(\beta) \stackrel{\sim}{\sim} 0$ with $\beta \in \bigoplus_{i=1}^n HH_1(k[x_i])$ is compatible with a non-commutative Hamiltonian structure (P, μ) if $P_\omega = P$.

Let Q be a finite quiver with n vertices and \bar{Q} the double quiver. Note that $k\bar{Q}$ is 1-smooth. By Theorem 1.8 of [8], we have a non-commutative Hamiltonian structure on $A := k\bar{Q}$ given by the double Poisson structure

$$P = \sum_{a \in Q_1} \frac{\partial}{\partial a^*} \frac{\partial}{\partial a} \in (DA/[DA, DA])_2.$$

and corresponding moment map $\mu := \sum_{a \in Q_1} [a, a^*]$.

In [?], we have shown that the map $\bigoplus_{i=1}^n k[x_i] \rightarrow k\bar{Q}, x_i \mapsto \mu_i$ carries a relative 1-Calabi-Yau structure which is induced by the natural 1-Calabi-Yau structure $\sum_{i=1}^n 1 \otimes x_i \in \bigoplus_{i=1}^n HH_1(k[x_i])$.

Lemma 2.8. *The non-commutative Hamiltonian structure on $k\bar{Q}$ coincides with the relative 1-Calabi-Yau structure on $\bigoplus_{i=1}^n k[x_i] \rightarrow k\bar{Q}$.*

Proof. The homotopy of $\mu(1 \otimes x) \sim 0$ defining the relative Calabi-Yau structure on μ is given by $B(\sum_{\alpha \in Q_1} \alpha^* \otimes \alpha) = \sum_{\alpha \in Q_1} 1 \otimes \alpha^* \otimes \alpha$ which corresponds to $\omega := \sum_{\alpha \in Q_1} d\alpha^* d\alpha \in \overline{\text{DR}}^2 A$. By Proposition 8.1.1. of [4] ω is a bi-symplectic form on $k\bar{Q}$.

Indeed, it is easy to verify that

$$P_\omega = \sum_{a \in Q_1} \frac{\partial}{\partial a^*} \frac{\partial}{\partial a} \in (DA/[DA, DA])_2$$

which finishes the proof. \square

We next investigate the relationship between fusion of double Poisson and bi-symplectic structures and relate them to the compositions of Calabi-Yau cospans. By [8] Proposition 2.6.6 if (A, P) is a double Poisson algebra, then so is (A^f, P^f) , where A^f denotes the fusion algebra. If (A, P, μ) is a non-commutative Hamiltonian structure, then (A^f, P^f, μ^f) , where $\mu^f = \bigoplus_{i \geq 3} \mu_i \oplus \mu_1 + e_{21} \mu_2 e_{12}$ is also a non-commutative Hamiltonian structure.

Lemma 2.9. *Let (A, ω) with $\omega \in \text{DR}(A)^2$ be a bi-symplectic structure and A^f the fusion algebra. Then (A^f, ω^f) is also a bi-symplectic structure and $P_\omega^f = P_{\omega^f}$, that is fusion commutes with the morphism that associates a non-commutative Hamiltonian structure to a bi-symplectic structure.*

Proof. By functoriality, $\omega^f \in \text{DR}(A^f)^2$ is a closed form and ι_{ω^f} fits into a commutative diagram

$$\begin{array}{ccc} D_A & \xrightarrow{\iota_\omega} & \Omega_A \\ \downarrow & & \downarrow \\ D_{A^f} & \xrightarrow{\iota_{\omega^f}} & \Omega_{A^f} \end{array}$$

As the vertical arrows are surjective and the top horizontal arrow ι_ω is an isomorphism by the non-degeneracy of ω , it follows that ι_{ω^f} is also an isomorphism. By the definition

$$P_{\omega^f} = (\iota(\omega^f)^{-1} \otimes \iota(\omega^f)^{-1})(\omega^f) = (\iota(\omega)^{-1} \otimes \iota(\omega)^{-1})(\omega)^f = P_\omega^f.$$

The moment map associated to ω is given by the the element μ which satisfies $\{P, \mu\} = E$. Clearly $\{P_{\omega^f}, \mu^f\} = \{P_\omega^f, \mu^f\} = \{P_\omega, \mu\}^f = E^f$. \square

Lemma 2.10. *The cospan $k[x] \amalg k[y] \xrightarrow{f} k\langle x, y \rangle \xleftarrow{g} k[z]$ with $g(z) = x + y$ and $f(x) = x, f(y) = y$ is a 1-Calabi-Yau cospan.*

Proof. We equip $k[x] \amalg k[y]$ and $k[z]$ with their natural Calabi-Yau structure given by $1 \otimes x + 1 \otimes y = 1 \otimes (x + y) \in HH_1(k[x] \amalg k[y])$ and $1 \otimes z \in HH_1(k[z])$. Clearly, $f(1 \otimes (x + y)) = g(1 \otimes z)$. \square

Given a relative 1-Calabi-Yau structure $\mu : k[x] \amalg k[y] \rightarrow A$, we can consider the composition of cospans

$$\begin{array}{ccccc} & & A \amalg_{k[x] \amalg k[y]} k\langle x, y \rangle & & \\ & \nearrow & & \nwarrow & \\ k\langle x, y \rangle & & & & A \\ \nearrow & & \nwarrow & & \nwarrow \\ k[z] & & k[x] \amalg k[y] & & \emptyset \end{array}$$

where x, y is mapped to μ_1, μ_2 . This yields a relative Calabi-Yau structure $k[z] \rightarrow B$.

Theorem 2.11. *Let (A, P, μ) be a non-Hamiltonian structure which coincides with the relative 1-Calabi-Yau structure $\mu : k[x] \amalg k[y] \rightarrow A$. The fusion of the non-Hamiltonian structure (A, P, μ) coincides with the relative 1-Calabi-Yau structure $k[z] \rightarrow A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$ given by the composition of cospans above.*

Proof. As the non-commutative Hamiltonian structure is compatible with the relative 1-Calabi-Yau structure, we have $P = P_\omega$ and $\mu(x) = \mu_1$ and $\mu(y) = \mu_2$. We have a natural isomorphism $A^f \simeq A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$ and under this isomorphism $A \rightarrow A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$ is equivalent to the morphism $-^f : A \rightarrow A^f$ and $\mu^f : k[z] \rightarrow A^f$ coincides with $k[z] \rightarrow A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$. Furthermore, since the homotopy between the 1-forms in the cospan $k[x] \amalg k[y] \xrightarrow{f} k\langle x, y \rangle \xleftarrow{g} k[z]$ is trivial, the zero-homotopy of the composition is given by ω^f , where ω denotes the zero homotopy $\mu(1 \otimes (x + y)) \sim 0$. As $P_\omega^f = P_{\omega^f}$ the fusion non-Hamiltonian structure coincides with the relative 1-Calabi-Yau structure of the composition of the cospans. \square

2.5 Calabi-Yau cospans and fusion

The aim of this section is to compare the relative Calabi-Yau structure obtained by the cospan procedure with the double quasi-Poisson structure obtained by the fusion process as in [8] Section 7. Let A be an algebra or dg category with orthogonal idempotents e_1, \dots, e_n or respectively objects denoted $1, \dots, n$.

Given a 1-relative Calabi-Yau map $\Phi : \amalg_{i=1}^n k[x_i^{\pm 1}] \rightarrow A$ such that $x_i \mapsto \Phi_i := \Phi(x_i) \in (e_i A e_i)^*$, we can compose with the Calabi-Yau cospan from [3]

$$\amalg_{i=1}^n k[x_i^{\pm 1}] \rightarrow k\langle x_1^{\pm 1}, x_2^{\pm 1} \rangle \amalg_{i=3}^n k[x_i^{\pm 1}] \leftarrow k[z_{1,2}^{\pm 1}] \amalg_{i=3}^n k[x_i^{\pm 1}]$$

with $z_{1,2} \mapsto x_1 x_2$ and the identity map on the x_i for $i \geq 3$. The composition of cospans yields a relative 1-Calabi-Yau structure on

$$\Phi^f : \amalg_{i=3}^n k[x_i^{\pm 1}] \amalg k[z_{1,2}^{\pm 1}] \rightarrow A^f$$

where $A^f := A * \amalg_{i=1}^n k[x_i^{\pm 1}] \amalg k[z_{1,2}^{\pm 1}] \amalg_{i=3}^n k[x_i^{\pm 1}]$. This process fuses the objects respectively vertices denoted 1 and 2. As in [8] we denote by a^f the image of $a \in A$ in A^f and by $\Phi^f : k[z^{\pm 1}] \rightarrow A^f$ the new moment map. We have $\Phi^f(x_i) = \Phi_i^f \in A^f$ for $i \geq 3$ and $\Phi^f(z_{1,2}) = \Phi_1^f \Phi_2^f \in A^f$.

Recall from [2] Section 3.3. that the homotopy on the cospan

$$k[x^{\pm 1}] \amalg k[y^{\pm 1}] \rightarrow k\langle x^{\pm 1}, y^{\pm 1} \rangle \leftarrow k[z^{\pm 1}]$$

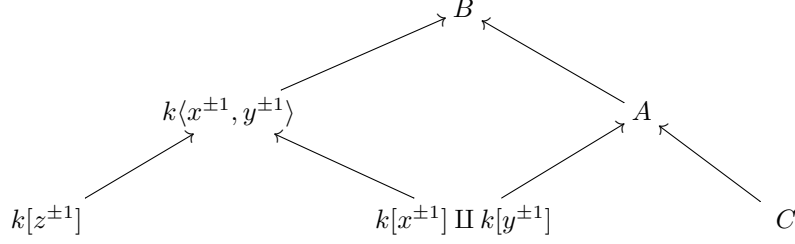
is given by $\gamma_1 := y^{-1} \otimes x^{-1} \otimes xy - y \otimes y^{-1} x^{-1} \otimes x$, which satisfies

$$\alpha_1(xy) - (\alpha_1(x) + \alpha_1(y)) = b(\gamma_1).$$

We can assume that $n = 2$. Let us assume that ω_1 and P are compatible in the sense of [8]. We want to show that ω_1^{ff} and $P^{ff} := P^f - E_1^f E_2^f$ are also compatible. The composition of cospans yields that $\omega_1^{ff} = \beta_1^f + \Phi^f(\zeta_1) = \beta_1^f + \Phi_1^{-1} d\Phi_1 d\Phi - \Phi_2 d\Phi^{-1} d\Phi_1$

let me try to reproduce below the strategy used in 2.3

We need to understand the following composition of Calabi–Yau cospans:



where x, y is mapped to Φ_2, Φ_1 . The homotopy

$$\zeta_1 = \frac{1}{2} \left(y^{-1} \otimes x^{-1} \otimes xy - y \otimes y^{-1} x^{-1} \otimes x \right)$$

is mapped in $\overline{\text{DR}}^2 k\langle x^{\pm 1}, y^{\pm 1} \rangle$ to

$$\begin{aligned}
 \omega &= \frac{1}{4} \left(y^{-1} dx^{-1} d(xy) - y d(y^{-1} x^{-1}) dx \right) \\
 &= \frac{1}{4} \left(-y^{-1} x^{-1} dx x^{-1} (xdy + dxy) + dyy^{-1} x^{-1} dx + x^{-1} dx x^{-1} dx \right) \\
 &= \frac{1}{4} \left(-y^{-1} x^{-1} dx dy - y^{-1} x^{-1} dx x^{-1} dxy + dyy^{-1} x^{-1} dx + x^{-1} dx x^{-1} dx \right) \\
 &\equiv -\frac{1}{2} x^{-1} dx dyy^{-1}
 \end{aligned}$$

which is mapped to

$$-\frac{1}{2} \Phi_1^{-1} d\Phi_1 d\Phi_2 \Phi_2^{-1} \in \overline{\text{DR}}^2 B.$$

which is precisely the opposite of the corrective term given in [9] following [1].

right now
I don't
see how
to correct
this sign

3 Comparison of Lagrangian structures

Assume that we have a finite set $\{e_i \mid i \in I\}$ of idempotents in A , and define the commutative algebra $R = \bigoplus_{i \in I} ke_i$. For any I -graded finite dimensional space V define A_V by

$$\text{Hom}_{\text{Alg}/R}(A, \text{End}(V)) = \text{Hom}_{\text{CommAlg}/k}(A_V, k).$$

Thanks to [4, (6.2.2)], setting $X_V = \text{Spec}(A_V)$, we have a map

$$\text{tr} : \text{DR}^\bullet A \longrightarrow \Omega^\bullet(X_V)^{\text{GL}_V}$$

given by $\alpha \mapsto \text{tr}(\hat{\alpha})$ where $\hat{\alpha}$ is induced by the evaluation

$$A \rightarrow (A_V \otimes \text{End}(V))^{\text{GL}_V} \quad ; \quad a \mapsto \hat{a}.$$

Thanks to [8, Proposition 6.1], there is a quasi-Hamiltonian structure on $(X_V, \text{tr}(\omega_1), \hat{\Phi})$. Now $\hat{\Phi} : X_V \rightarrow \text{GL}_V$ induces a lagrangian structure on $[X_V/\text{GL}_V] \rightarrow [\text{GL}_V/\text{GL}_V]$.

the ω_1 and
 Φ from
section 2

On the other hand, thanks to [3], the 1-Calabi-Yau structure on Φ yields a lagrangian structure on $\text{Perf}_A \rightarrow \text{Perf}_{k[x^{\pm 1}]}$, and thus considering substacks on $[X_V/\text{GL}_V] \rightarrow [\text{GL}_V/\text{GL}_V]$ again.

Theorem 3.1. *These two lagrangian structures are identical.*

Recall that a relative 1-Calabi-Yau morphism of algebras $k[x^\pm] \rightarrow A$ induces a 1-shifted Lagrangian morphism of stacks $\mathbf{Perf}_A \rightarrow \mathbf{Perf}_{k[x^\pm]}$. Now by [?], a 1-shifted Lagrangian morphism induces a non-degenerate Poisson structure on \mathbf{Perf}_A via pullback of forms. In our setting, $[\mathbf{rep}_A/\mathrm{GL}_n]$ is an open substack of \mathbf{Perf}_A and therefore the non-degenerate Poisson form restricts to $[\mathbf{rep}_A/\mathrm{GL}_n]$. Equivalently, given a Lagrangian morphism $\mathbf{Perf}_A \rightarrow \mathbf{Perf}_{k[x^\pm]}$, we can consider the following commutative diagram

$$\begin{array}{ccc} \mathbf{Perf}_A & \longrightarrow & \mathbf{Perf}_{k[x^\pm]} \\ \uparrow & & \uparrow \\ [\mathbf{rep}_n A/\mathrm{GL}_n] & \longrightarrow & [\mathbf{rep}_n k[x^\pm]/\mathrm{GL}_n] \end{array}$$

where the vertical arrows are open embeddings and the bottom horizontal morphism inherits a 1-Lagrangian structure from the top morphism. As shown in [], the 1-Calabi-Yau structure on $k[x^\pm]$ given by $(x^{-1} \otimes 1 + 1 \otimes x^{-1})/2$ induces the classical symplectic structure on $[\mathbf{rep}_n k[x^\pm]/\mathrm{GL}_n] \simeq [\mathrm{GL}_n/\mathrm{GL}_n]$ via restriction from $\mathbf{Perf}_{k[x^\pm]}$ which we can pull back to $[\mathbf{rep}_n A/\mathrm{GL}_n]$.

Question: Does the non-degenerate quasi-Poisson form respectively quasi-Hamiltonian structure on $\mathbf{rep}_n A$ given in [7] yield the same Poisson structure on $[\mathbf{rep}_n A/\mathrm{GL}_n]$ than the two methods described above?

Here DR^\bullet denotes either the Karoubi-DeRham complex in the non-commutative case and the DeRham complex of derived stacks seen as a functor $\mathrm{DR}^\bullet : \mathrm{dSt} \rightarrow \mathrm{Mod}_k^{\epsilon-gr}$.

Recall that every B -point $x : \mathrm{Spec} B \rightarrow \mathbf{Perf}_A$ is given by a dg functor $A \rightarrow \mathrm{Mod}_B^{\mathrm{perf}}$. Applying HC^- yields a map $\mathrm{HC}^-(A) \rightarrow \mathrm{HC}^-(\mathrm{Mod}_B^{\mathrm{perf}}) \simeq \mathrm{HC}^-(B)$. The DeRham complex of \mathbf{Perf}_A can be computed as the limit $\lim_{x: \mathrm{Spec} B \rightarrow \mathbf{Perf}_A} \mathrm{HC}^-(B)$ and we therefore obtain a natural map $\mathrm{DR}(A) \simeq \mathrm{HC}^-(A) \rightarrow \mathrm{DR}(\mathbf{Perf}_A)$.

We have a diagram

$$\begin{array}{ccc} \mathrm{DR}A & \longrightarrow & \mathrm{DR}^\bullet(\mathbf{Perf}_A) \\ & \searrow & \downarrow \\ & & \mathrm{DR}([\mathbf{rep}_n A/\mathrm{GL}_n]) \\ & \searrow \scriptstyle{tr} & \downarrow \\ & & \mathrm{DR}(\mathbf{rep}_n A). \end{array}$$

The vertical arrows on the right are given by the functoriality of DR^\bullet with respect to the natural maps $\mathbf{rep}_n A \rightarrow [\mathbf{rep}_n A/\mathrm{GL}_n] \rightarrow \mathbf{Perf}_A$ of derived stacks. The maps $tr : \mathrm{HC}^- A \simeq \mathrm{DR}^\bullet(A) \rightarrow \mathrm{DR}^\bullet(\mathbf{rep}_n A)$ and $\mathrm{HC}^- A \simeq \mathrm{DR}^\bullet(A) \rightarrow \mathrm{DR}^\bullet([\mathbf{rep}_n A/\mathrm{GL}_n])$ are given in [7]. To answer our question, we have to show that the diagram above commutes.

We will first show that the outside triangle commutes.

Remark 3.3 (pre-quotient case). Thanks to [4, (6.2.2)], setting $X_V = \mathrm{Spec}(A_V)$, we have a map

$$\mathrm{DR}^\bullet A \longrightarrow \Omega^\bullet(X_V)^{\mathrm{GL}_V}$$

given by $\alpha \mapsto \mathrm{tr}(\hat{\alpha})$ where $\hat{\alpha}$ is induced by the evaluation

$$A \rightarrow (A_V \otimes \mathrm{End}(V))^{\mathrm{GL}_V}; \quad a \mapsto \hat{a}.$$

On the other hand, we have a chain

$$\mathrm{HC}^- A \longrightarrow \mathrm{HC}^-(\mathrm{Mod}_{A_V}^{\mathrm{perf}}) \xrightarrow{\sim} \mathrm{HC}^- A_V \xrightarrow[\mathrm{HKR}]{\sim} \mathrm{DR}^\bullet A_V$$

given by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \mathrm{tr}(\hat{a}_0) \mathrm{dtr}(\hat{a}_1) \dots \mathrm{dtr}(\hat{a}_n).$$

Let us clarify the map $\mathrm{DR}^\bullet A \simeq \mathrm{HC}^- A \longrightarrow \mathrm{DR}(\mathrm{Perf}_A)$. By the above description, we know that for every B -point $x : A \rightarrow \mathrm{Mod}_B^{\mathrm{perf}}$, we have a map $\mathrm{HC}^-(A) \rightarrow \mathrm{HC}^-(\mathrm{Mod}_B^{\mathrm{perf}}) \simeq \mathrm{HC}^-(B)$ given by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \mathrm{tr}(xa_0) \mathrm{dtr}(xa_1) \dots \mathrm{dtr}(xa_n)$. Now the map $\mathbf{rep}_V(A) \rightarrow \mathbf{Perf}_A$ corresponds to the A_V -point $A \longrightarrow A_V \otimes \mathrm{End}(V)$.

We get that the square

$$\begin{array}{ccc} \mathrm{DR}^\bullet A \simeq \mathrm{HC}^- A & \longrightarrow & \mathrm{DR}(\mathrm{Perf}_A) \\ \downarrow & & \downarrow \\ \Omega^\bullet(X_V)^{\mathrm{GL}_V} & \longrightarrow & \mathrm{DR}(X_V) \simeq \mathrm{DR}(\mathbf{rep}_V A) \end{array}$$

commutes.

4 Simplicial complexes

Now denote by $A(n)$ the category with objects (a, i) , $a \in \mathrm{Ob}(A)$, $i = 1, \dots, n$ and morphisms

$$A(n)((a, i), (b, j)) = \begin{cases} A(a, b), & \text{if } i = j \\ A(a, a)^\sim, & \text{if } j = i + 1 \\ 0, & \text{otherwise} \end{cases} \quad (4.1)$$

This gives rise to a simplicial object

$$A(1) \rightleftarrows A(2) \rightleftarrows A(3) \dots$$

Let $V = (V_a)_{a \in \mathrm{Ob} A}$ be a vector-space indexed by the objects of A . We denote $\mathbf{rep}_V A(m)$ the moduli space which sends every (a, i) to V_a . Then

$$\mathbf{rep}_V A(m) \simeq \mathbf{rep}_V A \times \mathrm{GL}_V^{m-1} = X_V \times (\mathrm{GL}_V)^{m-1}.$$

Applying the functor \mathbf{rep}_V to the simplicial complex yields therefore the simplicial complex

$$[X_V/\mathrm{GL}_V] = \mathrm{colim} \left(X_V \rightleftarrows X_V \times \mathrm{GL}_V \rightleftarrows X_V \times \mathrm{GL}_V \times \mathrm{GL}_V \dots \right).$$

Hence, we morphisms of derived stacks

$$X_V \times (\mathrm{GL}_V)^{n-1} \rightarrow \mathrm{Perf}_{A(n)}$$

that induce a morphism

$$[X_V/\mathrm{GL}_V] \rightarrow \mathrm{colim} \mathrm{Perf}_{A(n)} \simeq \mathrm{Perf}_A.$$

We have natural maps $A(n) \rightarrow A$, $(a, i) \mapsto a$. This induces a map between the simplicial complex above and the constant simplicial complex in A . Applying DR yields a map $\lim_n \mathrm{DR}(A(n)) \longrightarrow \mathrm{DR}(A)$ which is an isomorphism. Let us spell out this map $\mathrm{tr}(a_0) \mathrm{dtr}(a_1) \dots \mathrm{dtr}(a_n) \mapsto$.

We obtain the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{DR}(A) & \xleftarrow{\sim} & \lim_n \mathrm{DR}(A(n)) & \xrightarrow{\mathrm{tr}} & \lim_n \mathrm{DR}(\mathbf{rep}_V(A(n))) \\
\downarrow & & \downarrow & & \uparrow \sim \\
\mathrm{DR}(\mathbf{Perf}_A) & \xleftarrow{\sim} & \lim_n \mathrm{DR}(\mathbf{Perf}_{A(n)}) & \longrightarrow & \mathrm{DR}([X_V/\mathrm{GL}_V])
\end{array}$$

Assume first that our categories in

$$A = A(1) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} A(2) =: B$$

only have one object, so that an isomorphism $f \cong g$ corresponds to $z \in B^\times$ satisfying $zf = gz$. On simplicial complexes we have morphisms

$$\begin{array}{ccccc}
\dots & & A^{\otimes n} & \xleftarrow{(n+1)} & A^{\otimes n+1} & & \dots \\
& & \downarrow f^{\otimes n} & & \downarrow f^{\otimes n+1} & & \\
& & \downarrow g^{\otimes n} & & \downarrow g^{\otimes n+1} & & \\
\dots & & B^{\otimes n} & \xleftarrow{(n+1)} & B^{\otimes n+1} & & \dots
\end{array}$$

induced by f and g . Horizontal maps are given by m_i , $i = 1 \dots n+1$, the multiplication of the i -th and $i+1$ -th factors, with the convention $n+2 = 1$.

Proposition 4.2. *The homotopy between f and g induced by z*

$$(h : A \times \Delta^1 \rightarrow B) = ((h_{n,j})_{1 \leq j \leq n} : (A^{\otimes n})^n \rightarrow B^{\otimes n+1})_{n \geq 1}$$

reads

$$h_{n,j} = fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-j}$$

meaning

$$h_{n,j}(a_1 \otimes \dots \otimes a_n) = f(a_1)z^{-1} \otimes g^{\otimes j-1}(a_2 \otimes \dots \otimes a_j) \otimes z \otimes f^{\otimes n-j}(a_{j+1} \otimes \dots \otimes a_n).$$

Proof. First note that $m_1 h_{n,1} = f^{\otimes n}$ and $m_{n+1} h_{n,n} = g^{\otimes n}$.

Next consider $1 \leq i < j \leq n$, then

$$\begin{aligned}
m_i h_{n,j} &= m_i(fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-j}) \\
&= (fz^{-1} \otimes g^{\otimes j-2} \otimes z \otimes f^{\otimes n-j})m_i \\
&= h_{n-1,j-1}m_i
\end{aligned}$$

the only nontrivial case being $i = 1$ where we use $m_1(fz^{-1} \otimes g) = fz^{-1}m_1$.

If $1 < i = j$,

$$\begin{aligned}
m_i h_{n,i} &= m_i(fz^{-1} \otimes g^{\otimes i-1} \otimes z \otimes f^{\otimes n-i}) \\
&= fz^{-1} \otimes g^{\otimes i-2} \otimes gz \otimes f^{\otimes n-i} \\
&= fz^{-1} \otimes g^{\otimes i-2} \otimes zf \otimes f^{\otimes n-i} \\
&= m_i h_{n,i-1}.
\end{aligned}$$

Finally, when $i > j+1$ we have

$$\begin{aligned}
m_i h_{n,j} &= m_i(fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-j}) \\
&= (fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-1-j})m_{i-1} \\
&= h_{n-1,j}m_{i-1}
\end{aligned}$$

even if $i = n+1$. □

Proposition 4.3. *Assume that $zf = gz$ and $wg = hw$ for $f, g, h : A \rightarrow B$ and $z, w \in B^\times$. Denote by h^z, h^w and h^{wz} the induced homotopies. The homotopy $H = (H_n : A^{\otimes n} \rightarrow B^{\otimes n+2})$ between $h^z + h^w$ and h^{wz} reads*

$$H_n = \sum_{\substack{r, s, t \geq 0 \\ r+s+t=n-1}} \pm fz^{-1}w^{-1} \otimes h^{\otimes r} \otimes w \otimes g^{\otimes s} \otimes z \otimes f^{\otimes t}.$$

Proof. Write $H_{r,s,t} = fz^{-1}w^{-1} \otimes h^{\otimes r} \otimes w \otimes g^{\otimes s} \otimes z \otimes f^{\otimes t}$, and $h_{p,q}^z = fz^{-1} \otimes g^{\otimes p} \otimes z \otimes f^{\otimes q}$, same with h^w, h^{wz} . Then note that

$$\begin{aligned} m_1 H_{0,s,t} &= h_{s,t}^z \\ m_{n+2} H_{r,s,0} &= h_{r,s}^w \\ m_{r+2} H_{r,0,t} &= h_{r,t}^{wz}. \end{aligned}$$

If $i \leq r$,

$$m_i H_{r,s,t} = H_{r-1,s,t} m_i,$$

thanks to $m_1(fz^{-1}w^{-1} \otimes h) = fz^{-1}w^{-1}m_1$ when $i = 1$.

If $i = r + 1$,

$$m_i H_{r,s,t} = m_i H_{r-1,s+1,t}$$

since $hw = wg$.

If $r + 3 \leq i \leq r + s + 1$,

$$m_i H_{r,s,t} = H_{r,s-1,t} m_{i-1}.$$

If $i = r + s + 2$,

$$m_i H_{r,s,t} = m_i H_{r,s-1,t+1}$$

since $gz = zf$.

If $r + s + 4 \leq i \leq n + 2$,

$$m_i H_{r,s,t} = H_{r,s,t-1} m_{i-2}.$$

□

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