



HAL
open science

Calabi-Yau structures on (quasi-)bisymplectic algebras

Tristan Bozec, Damien Calaque, Sarah Scherotzke

► **To cite this version:**

Tristan Bozec, Damien Calaque, Sarah Scherotzke. Calabi-Yau structures on (quasi-)bisymplectic algebras. Forum of Mathematics, Sigma, 2023, 11, pp.e87. 10.1017/fms.2023.88 . hal-03624186v1

HAL Id: hal-03624186

<https://hal.science/hal-03624186v1>

Submitted on 24 Feb 2024 (v1), last revised 26 Feb 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Calabi–Yau versus quasi-Hamiltonian structures

Tristan Bozec*, Damien Calaque†, Sarah Scherotzke‡

Abstract

We compare the Calabi–Yau structures defined and studied in [2, 3] with noncommutative analogs of quasi-Hamiltonian and Poisson geometries developed in [7, 8].

Contents

1	Link between Hochschild and de Rham homologies	1
1.1	Recollection of Ginzburg and Schedler [6]	1
1.2	Computations for $A = k[x^{\pm 1}]$	2
2	CY \Rightarrow quasi-bisymplectic in the sense of [8]	3
2.1	Relative setting	3
2.2	Non-degeneracy	4
2.3	Compatibility for the quiver A_2	4
2.4	Double Poisson structures and the 1-Calabi-Yau property	7
2.5	Calabi-Yau cospans and fusion	9
3	Comparison of Lagrangian structures	10
4	Simplicial complexes	13

1 Link between Hochschild and de Rham homologies

1.1 Recollection of Ginzburg and Schedler [6]

Fix a field k of characteristic zero. Consider a unital associative k -algebra A and fix a complementary subspace $\bar{A} = A/k$ of k . Denote by $d : A \rightarrow \bar{A}$ the associated quotient. The dg-algebra ΩA of noncommutative differential forms is defined by the quotient of $T_k(A \oplus \bar{A}[-1])$ by the relations

$$a \otimes b = ab \quad \text{and} \quad d(ab) = a \otimes d(b) + d(a) \otimes b$$

for every $a, b \in A$. The differential of ΩA is the derivation induced by d , satisfying $d^2 = 0$. We will systematically use the $\bar{}$ notation for the quotient by k .

*IMAG, Univ. Montpellier, CNRS, Montpellier, France
tristan.bozec@umontpellier.fr

†IMAG, Univ. Montpellier, CNRS, Montpellier, France
damien.calaque@umontpellier.fr

‡Mathematical Institute, University of Luxembourg, Luxembourg
sarah.scherotzke@uni.lu

We have a contraction $\iota : \Omega^\bullet A \rightarrow \Omega^{\bullet-1} A$ defined by

$$\iota(a_0 da_1 \dots da_n) = \sum_{l=1}^n (-1)^{(l-1)(n-1)+1} [a_l, da_{l+1} \dots da_n a_0 da_1 \dots da_{l-1}]$$

that factors through $\text{DRA} := \Omega A / [\Omega A, \Omega A]$, a Hochschild differential b on ΩA defined by

$$b(\alpha da) = (-1)^{|\alpha|} [\alpha, a]$$

and a Karoubi operator on ΩA given by

$$\kappa(\alpha da) = (-1)^{|\alpha|} da\alpha.$$

We have a harmonic decomposition $\bar{\Omega} A = P\bar{\Omega} A \oplus P^\perp \bar{\Omega}$ where

$$P\bar{\Omega} A = \ker(1 - \kappa)^2 \quad \text{and} \quad P^\perp \bar{\Omega} = \text{ima}(1 - \kappa)^2.$$

that gives

$$\iota = bN|_P \quad \text{and} \quad B = Nd|_P$$

where N is the grading operator and B the Connes differential.

Furthermore b and d are invariant under the harmonic decomposition and $(P^\perp \bar{\Omega}, b)$ is acyclic.

We observe that

$$\overline{DR} := \bar{\Omega} / [\bar{\Omega}, \bar{\Omega}] \simeq P\bar{\Omega} / P[\bar{\Omega}, \bar{\Omega}] \simeq P\bar{\Omega} / bP\bar{\Omega}$$

By [6], we have

$$(P\bar{\Omega}, b)^\bullet \simeq \overline{HH}_\bullet.$$

Remark 1.1. We call a k -algebra A 1-smooth if it has projective dimension one as an $A \otimes A$ -module. In particular a 1-smooth algebra is smooth and has projective dimension at most 1. This can be seen as follows: given a projective resolution $\Omega^1(A) \rightarrow A \otimes A \rightarrow A$ of A -bimodules, tensoring this resolution with $M \otimes_A -$ for $M \in \text{Mod}_A$ yields a projective resolution of M as A -module of length at most 1.

1.2 Computations for $A = k[x^{\pm 1}]$

We always mean $(dx)y$ if no brackets appear in $dx y$.

Set $\alpha_n = (x^{-1} dx)^{2n-1}$, $\beta_n = \kappa(\alpha_n) = (dx x^{-1})^{2n-1} \in \bar{\Omega}^{2n-1} A$. Then

$$\kappa(\beta_n) = \kappa(-\beta_{n-1} dx dx^{-1}) = -dx^{-1} \beta_{n-1} dx = \alpha_n.$$

Hence $\alpha_n + \beta_n \in P\bar{\Omega} A$ and $\alpha_n - \beta_n = \frac{1}{2}(1 - \kappa)^2(\alpha_n) \in P^\perp \bar{\Omega} A$. Then,

$$\begin{aligned} \iota \alpha_n &= \frac{1}{2}(2n-1)b(\alpha_n + \beta_n) \\ &= \frac{1}{2}(2n-1)([\alpha_{n-1} x^{-1} dx x^{-1}, x] - [\beta_{n-1} dx, x^{-1}]) \\ &= \frac{1}{2}(2n-1)(x^{-1} \beta_{n-1} dx + \alpha_{n-1} x^{-1} dx - \beta_{n-1} dx x^{-1} - x \alpha_{n-1} x^{-1} dx x^{-1}) \\ &= (2n-1)((x^{-1} dx)^{2n-2} - (dx x^{-1})^{2n-2}). \end{aligned}$$

On the other hand, $d\alpha_1 = -(x^{-1}dx)^2$, and if we assume $d\alpha_{n-1} = -(x^{-1}dx)^{2n-2}$, we get

$$\begin{aligned} d\alpha_n &= d(x^{-1}dx(x^{-1}dx)^{2n-2}) \\ &= d(x^{-1}dx)(x^{-1}dx)^{2n-2} - x^{-1}dxd((x^{-1}dx)^{2n-2}) \\ &= -x^{-1}dxx^{-1}dx(x^{-1}dx)^{2n-2} - x^{-1}dxd^2\alpha_{n-1} \\ &= -(x^{-1}dx)^{2n}. \end{aligned}$$

and similarly $d\beta_n = (dxx^{-1})^{2n}$ for all n . Thus, as $\iota\alpha_n = \iota\beta_n$,

$$\iota(\alpha_n + \beta_n) = 2\iota\alpha_n = -2(2n-1)d(\beta_{n-1} + \alpha_{n-1}).$$

As a consequence $(\iota - ud)(\gamma) = 0$, where $\gamma_k = \frac{1}{2}(\alpha_k + \beta_k) \in P\bar{\Omega}^{2k-1}k[x^{\pm 1}]$ and

$$\gamma = \sum_{k \geq 0} \frac{k!}{(2k+1)!} (-u)^k \gamma_{k+1}.$$

Now, we have the following chain of quasi-isomorphisms, functorial in A

$$\left(\frac{\bar{\Omega}[[u]]}{[d\Omega, d\bar{\Omega}]}, \iota - ud \right) \xrightarrow{P} (P\bar{\Omega}[[u]], \iota - ud) \xrightarrow{N!} (P\bar{\Omega}[[u]], b - uB) \xrightarrow{C} (\bar{\Omega}[[u]], b - uB)$$

whose homology gives the reduced negative cyclic homology $\overline{\text{HC}}_{\bullet}^{-}$. Through this chain and the isomorphism $\Omega^n A \simeq A \otimes \bar{A}^{\otimes n}$, γ is mapped to

$$\sum_{k \geq 0} k! u^k \left((x^{-1} \otimes x)^{\otimes(k+1)} - (x \otimes x^{-1})^{\otimes(k+1)} \right)$$

as

$$\begin{aligned} \alpha_{k+1} &= (x^{-1}dx)^{2k+1} = (-1)^k x^{-1} (dxdx^{-1})^k dx, \\ \beta_{k+1} &= (dxx^{-1})^{2k+1} = (-1)^{k+1} x (dx^{-1}dx)^k dx^{-1}, \\ \text{and } \gamma_{k+1} &\in P\bar{\Omega}^{2k+1}, \end{aligned}$$

all of which being consistent with [3, 3.1.1].

α_n there is
 γ_n here!

2 CY \Rightarrow quasi-bisymplectic in the sense of [8]

2.1 Relative setting

Consider a morphism $\Phi : k[x^{\pm 1}] \rightarrow A$ where A is smooth and concentrated in degree 0. Assume that $\Phi[\gamma_1] = 0 \in \overline{\text{HH}}^1 A = H_1(\bar{\Omega}A, b)$. The space of lifts of $\Phi(\gamma_1)$ in relative homology is given by elements $\omega_1 \in \bar{\Omega}^2 A / b\bar{\Omega}^3 A$ such that $b\omega_1 = \Phi(\gamma_1)$. As we also have

$$\overline{\text{HH}} = \ker(\overline{\text{DR}} \xrightarrow{\iota} \bar{\Omega}),$$

this space of lifts is also described by elements $\omega_1 \in \overline{\text{DR}}^2 A$ such that $\iota\omega_1 = \Phi(\gamma_1)$, which is condition (B2) in [8].

Similarly, if $\Phi[\gamma] = 0 \in \overline{\text{HC}}_1^{-} A$, we have the existence of $\omega_k \in \overline{\text{DR}}^{2k} A$ for all k such that

$$(\iota - ud) \left(\sum_{k \geq 0} u^k \omega_{k+1} \right) = \Phi(\gamma)$$

or equivalently

$$\omega_1 = \Phi(\gamma_1) = \frac{1}{2}(\Phi^{-1}d\Phi + d\Phi\Phi^{-1}) \quad (\mathbb{B}2)$$

$$\omega_2 - d\omega_1 = -\frac{1}{6}\Phi(\gamma_2) \Rightarrow d\omega_1 = \frac{1}{6}(\Phi^{-1}d\Phi)^3 \pmod{[-, -]} \quad (\mathbb{B}1)$$

$$\omega_3 - d\omega_2 = \frac{2!}{5!}\Phi(\gamma_3)$$

\vdots

$$\omega_{k+1} - d\omega_k = (-1)^k \frac{k!}{(2k+1)!}\Phi(\gamma_{k+1}) \quad k \geq 1.$$

2.2 Non-degeneracy

Set $R = k[x^{\pm 1}]$ and write the relative 1-pre-Calabi–Yau structure

$$A^\vee[1] \rightarrow R^\vee[1] \otimes_{R^e} A^e \xrightarrow{\simeq} R \otimes_{R^e} A^e \rightarrow A$$

with short resolutions to get the homotopy commuting diagram

$$\begin{array}{ccccccc} A^e & \xrightarrow{\text{id}} & A^e & \xrightarrow{(\Phi^{-1} \otimes 1 + 1 \otimes \Phi^{-1})/2} & A^e & \xrightarrow{d\Phi} & \Omega^1 A \\ E \downarrow & & \downarrow & & \downarrow & & \downarrow \\ DA & \xrightarrow{\text{ev}_\Phi} & A^e & \xrightarrow{(\Phi^{-1} \otimes 1 + 1 \otimes \Phi^{-1})/2} & A^e & \xrightarrow{\text{id}} & A^e \end{array}$$

where $DA := \text{Der}(A, A^e) \simeq (\Omega^1 A)^\vee$. The homotopy $DA \rightarrow \Omega^1 A$ gives $\iota_E \omega_1 = (\Phi^{-1}d\Phi + d\Phi\Phi^{-1})/2$.

Remark 2.1. If $\omega \in \text{DR}^2 A$ and $\delta \in DA \simeq (\Omega^1 A)^\vee$, then $\iota_\delta(\omega) \in \Omega^1 A \hookrightarrow A^e$ matches $\langle \delta, \iota_E(\omega) \rangle \in A^e$.

Now assume that our Calabi–Yau structure is non-degenerate, that is

$$A^\vee[1] \simeq \text{fib}(R^\vee[1] \otimes_{R^e} A^e \xrightarrow{\simeq} R \otimes_{R^e} A^e \rightarrow A).$$

In short resolutions, this yields a quasi-isomorphism (between vertical complexes)

$$\begin{array}{ccc} A^e & \xrightarrow{(\Phi^{-1} \otimes 1 + 1 \otimes \Phi^{-1})/2} & A^e \\ E \downarrow & & \downarrow d\Phi \\ DA & \xrightarrow{\iota_E \omega_1} & \Omega^1 A \end{array}$$

which in particular gives a surjection $DA \rightarrow \Omega^1 A / \langle d\Phi \rangle$, that is (\mathbb{B}_3) in [8].

2.3 Compatibility for the quiver A_2 .

Consider the quiver $A_2 = (V = \{1, 2\}, E = \{e : 1 \rightarrow 2\})$, with orthogonal idempotents e_1 and e_2 satisfying $1 = e_1 + e_2$, and set

$$a_1 = e_1 + e^* e \text{ and } a_2 = e_2 + e e^*.$$

ι_E is the ι of previous sections

use Lemma 2.6.3 in C-BEG

and smoothness to have the short resolution!

Let us denote by A the localization $(\overline{kA_2})_{a_1, a_2}$. Recall that we have given in [3] a relative 1-Calabi–Yau structure on $\Phi : k[x^{\pm 1}] \rightarrow A$ defined by

$$\Phi_1(x_1) = a_1^{-1} \quad \text{and} \quad \Phi_2(x_2) = a_2.$$

In the previous section we proved that this Calabi–Yau structure induces a quasi-bisymplectic one $\omega_1 \in \overline{\text{DR}}^2 A$ on A . We want to prove that the double quasi-Poisson bracket compatible with ω_1 through [8, Theorem 7.1] is the one described in [8, §8.3]:

$$P = \frac{1}{2} \left((1 + ee^*) \frac{\partial}{\partial e^*} \frac{\partial}{\partial e} - (1 + e^*e) \frac{\partial}{\partial e} \frac{\partial}{\partial e^*} \right) \in (DA/[DA, DA])_2.$$

Note that we use the convention regarding concatenation of paths opposite to the one in [7], that is $e = e_2 e e_1$. In [2], one homotopy $\phi(\gamma_1) \sim 0$ is given by

$$\beta_1 = \frac{1}{2} (e^* \otimes e \otimes \Phi + \Phi \otimes e^* \otimes e - e^* \otimes \Phi^{-1} \otimes e - \Phi^{-1} \otimes e \otimes e^* + 1 \otimes e^* \otimes e\Phi - 1 \otimes e\Phi \otimes e^*) \quad (2.2)$$

1/2 missing in BCS2

where $\Phi = \Phi_1(x_1) + \Phi_2(x_2)$. It yields an element

$$\omega_1 = \frac{1}{4} (e^* ded\Phi + \Phi de^* de - e^* d\Phi^{-1} de - \Phi^{-1} dede^* + de^* d(e\Phi) - d(e\Phi) de^*)$$

1/4 because of the degree operator

in $\overline{\text{DR}}^2 A = (\overline{\Omega A}/[\overline{\Omega A}, \overline{\Omega A}])_2$. We can heavily simplify this expression working modulo $[\overline{\Omega A}, \overline{\Omega A}]$. First note that (again, dab stands for $(da)b$)

$$\begin{aligned} d\Phi &= -a_1^{-1} (de^* e + e^* de) a_1^{-1} + dee^* + ede^* = -\Phi (de^* e + e^* de) \Phi + dee^* + ede^* \\ d\Phi^{-1} &= de^* e + e^* de - a_2^{-1} (dee^* + ede^*) a_2^{-1} = de^* e + e^* de - \Phi^{-1} (dee^* + ede^*) \Phi^{-1}, \end{aligned}$$

thus, using $\Phi e \Phi = e$ and $\Phi e^* \Phi = e^*$ (cf [3, (4.3)]),

$$\begin{aligned} 4\omega_1 &= \Phi de^* de - \Phi^{-1} dede^* + e^* ded\Phi - e^* d\Phi^{-1} de + 2de^* d(e\Phi) \\ &= \Phi de^* de - \Phi^{-1} dede^* - e^* de\Phi (de^* e + e^* de) \Phi \\ &\quad + e^* \Phi^{-1} (dee^* + ede^*) \Phi^{-1} de + 2de^* de\Phi - 2de^* e\Phi (de^* e + e^* de) \Phi \\ &= \Phi de^* de - \Phi^{-1} dede^* - e^* de\Phi de^* e\Phi \\ &\quad - \underbrace{e^* de\Phi e^* de\Phi + e^* \Phi^{-1} dee^* \Phi^{-1} de}_{\equiv 0} + e^* \Phi^{-1} ede^* \Phi^{-1} de \\ &\quad + 2de^* de\Phi - 2 \underbrace{de^* e\Phi de^* e\Phi}_{\equiv 0} - 2de^* e\Phi e^* de\Phi \\ &\equiv 3\Phi de^* de - \Phi^{-1} dede^* - e\Phi e^* de\Phi de^* + e^* \Phi^{-1} ede^* \Phi^{-1} de + 2de^* e\Phi e^* de\Phi \\ &= 3\Phi de^* de - \Phi^{-1} dede^* - ee^* \Phi_2^{-1} de\Phi de^* + e^* e\Phi_1 de^* \Phi^{-1} de + 2de^* ee^* \Phi_2^{-1} de\Phi \\ &= 3\Phi de^* de - \Phi^{-1} dede^* - de\Phi de^* + \Phi^{-1} de\Phi de^* \\ &\quad + de^* \Phi^{-1} de - \Phi de^* \Phi^{-1} de - 2de^* de\Phi + 2de^* \Phi^{-1} de\Phi \\ &\equiv 2\Phi de^* de - 2\Phi^{-1} dede^* \end{aligned}$$

which matches the (unproven) formula in [9, Example 2.16].

Remark 2.3 (Sanity check).

$$\begin{aligned}
\iota_E(2\omega_1) &= -[e^*, de\Phi] + [e, \Phi de^*] + [e, de^*\Phi^{-1}] - [e^*, \Phi^{-1}de] \\
&= -e^*de\Phi + de\Phi e^* + e\Phi de^* - \Phi de^*e + ede^*\Phi^{-1} - de^*\Phi^{-1}e - e^*\Phi^{-1}de + \Phi^{-1}dee^* \\
&= -d(e^*e)\Phi + d(ee^*)\Phi^{-1} + \Phi^{-1}d(ee^*) - \Phi d(e^*e) \\
&= \Phi^{-1}(-\Phi d(e^*e)\Phi + d(ee^*)) + (-\Phi d(e^*e)\Phi + d(ee^*))\Phi^{-1} \\
&= \Phi^{-1}d\Phi + d\Phi\Phi^{-1}.
\end{aligned}$$

Let us prove Yamakawa's statement. Thanks to [8, Proposition 7.4], P and ω_1 are compatible if

$$\iota(\omega_1)\iota(P) = 1 - \frac{1}{4}T \quad (2.4)$$

with $T(dp) = [p, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}]$. For $p = e$, the LHS is

$$\begin{aligned}
\iota(\omega_1)\iota(P)(de) &= \frac{1}{2}\iota(\omega_1) \left(\frac{\partial}{\partial e^*}(1 + e^*e) + (1 + ee^*)\frac{\partial}{\partial e^*} \right) \\
&= \frac{1}{2}(\circ i_{\partial/\partial e^*}(\omega_1)(1 + e^*e) + (1 + ee^*)\circ i_{\partial/\partial e^*}(\omega_1))
\end{aligned}$$

where

$$i_\delta(pdqdr) = p\delta(q)' \otimes \delta(q)'' dr - pdq\delta(r)' \otimes \delta(r)'' \in A \otimes \Omega^1 + \Omega^1 \otimes A.$$

Note that above we have used, for $\pi, \nu \in A$ and $\delta \in D_A$,

$$\begin{aligned}
\circ i_{\pi\delta\nu}(pdqdr) &= \circ(p\delta(q)'\nu \otimes \pi\delta(q)'' dr - pdq\delta(r)'\nu \otimes \pi\delta(r)'') \\
&= \pi \circ i_\delta(pdqdr)\nu
\end{aligned}$$

stated in
the proof
of 2.8.6 in
CBEG...

since the bimodule structure on D_A is induced by the inner one on A^e . We have:

$$\circ i_{\partial/\partial e^*}(2\omega_1) = \circ(\Phi \otimes de + \Phi^{-1}de \otimes e_2) = de\Phi + \Phi^{-1}de$$

thus

$$\begin{aligned}
4\iota(\omega_1)\iota(P)(da) &= (de\Phi + \Phi^{-1}de)(1 + e^*e) + (1 + ee^*)(de\Phi + \Phi^{-1}de) \\
&= 2de + \Phi^{-1}de\Phi^{-1} + \Phi de\Phi
\end{aligned}$$

whereas 4 times the RHS of 2.4 is

$$\begin{aligned}
4de - [e, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}] &= 4de - e\Phi^{-1}(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*) \\
&\quad + e(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)\Phi^{-1} \\
&\quad + \Phi^{-1}(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)e \\
&\quad - (-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)\Phi^{-1}e \\
&= 4de + ede^*e\Phi + ee^*de\Phi - e\Phi de^*e - e\Phi e^*de \\
&\quad + \Phi^{-1}dee^*e + \Phi^{-1}ede^*e - dee^*\Phi^{-1}e - ede^*\Phi^{-1}e \\
&= 4de + ee^*de\Phi - \Phi^{-1}ee^*de + \Phi^{-1}dee^*e - dee^*e\Phi \\
&= 4de + \Phi de\Phi - de\Phi - de + \Phi^{-1}de \\
&\quad + \Phi^{-1}de\Phi^{-1} - \Phi^{-1}de - de + de\Phi \\
&= 2de + \Phi^{-1}de\Phi^{-1} + \Phi de\Phi.
\end{aligned}$$

Computations are similar to prove 2.4 evaluated at de^* .

Hourra.

2.4 Double Poisson structures and the 1-Calabi-Yau property

Double Poisson algebras have been introduced by Van den Berg in [8]. Recall

Definition 2.5. • A double Poisson algebra is an algebra A together with a double bracket $\{\{-, -\}\} : A \times A \rightarrow A \otimes A$ satisfying that $\{\{-, -\}\}$ is a derivation in the second argument, satisfies $\{\{u, v\}\} = \tau\{\{v, u\}\}$ for the twist function $\tau : A \otimes A \rightarrow A \otimes A, a \otimes b \mapsto b \otimes a$ and the Jacobi identity.

- We call an algebra A bi-symplectic, if there is a closed form $\omega \in \text{DR}(A)^2$ which is non-degenerate, that is $\iota_\omega : D_A \rightarrow \Omega_A$ is an isomorphism of A -bimodules.

A double Poisson algebra structure induces a classical Poisson structures on the moduli space of representations of A , while a bi-symplectic structure induces a symplectic structure on the moduli space of representations of A .

If A is 1-smooth, the existence of a double Poisson structure is equivalent to the existence of an element $P \in (DA/[DA, DA])_2$ such that $\{P, P\} = 0$, where $\{-, -\}$ denotes in this context the Schouten-Nijenhuis bracket on $DA/[DA, DA]$.

Definition 2.6. [?, 8] Let $P \in (DA/[DA, DA])_2$ and (A, P) be a double Poisson algebra. A moment map for the double Poisson algebra is an element $\mu = (\mu_i)_i \in \bigoplus e_i A e_i$ such that $\{P, \mu\} = -E$.

In [8] A3 it is shown that every bi-symplectic form gives rise to non-commutative Hamiltonian structure with Poisson bracket

$$P_\omega := (\iota(\omega)^{-1} \otimes \iota(\omega)^{-1})(\omega) \in (DA/[DA, DA])_2.$$

Definition 2.7. We say that a relative 1-Calabi-Yau structure

$$\mu : \bigoplus_{i=1}^n k[x_i] \rightarrow A, x_i \mapsto \mu_i$$

given by $\mu(\beta) \stackrel{\sim}{\sim} 0$ with $\beta \in \bigoplus_{i=1}^n HH_1(k[x_i])$ is compatible with a non-commutative Hamiltonian structure (P, μ) if $P_\omega = P$.

Let Q be a finite quiver with n vertices and \bar{Q} the double quiver. Note that $k\bar{Q}$ is 1-smooth. By Theorem 1.8 of [8], we have a non-commutative Hamiltonian structure on $A := k\bar{Q}$ given by the double Poisson structure

$$P = \sum_{a \in Q_1} \frac{\partial}{\partial a^*} \frac{\partial}{\partial a} \in (DA/[DA, DA])_2.$$

and corresponding moment map $\mu := \sum_{a \in Q_1} [a, a^*]$.

In [?], we have shown that the map $\bigoplus_{i=1}^n k[x_i] \rightarrow k\bar{Q}, x_i \mapsto \mu_i$ carries a relative 1-Calabi-Yau structure which is induced by the natural 1-Calabi-Yau structure $\sum_{i=1}^n 1 \otimes x_i \in \bigoplus_{i=1}^n HH_1(k[x_i])$.

Lemma 2.8. *The non-commutative Hamiltonian structure on $k\bar{Q}$ coincides with the relative 1-Calabi-Yau structure on $\bigoplus_{i=1}^n k[x_i] \rightarrow k\bar{Q}$.*

Proof. The homotopy of $\mu(1 \otimes x) \sim 0$ defining the relative Calabi-Yau structure on μ is given by $B(\sum_{\alpha \in Q_1} \alpha^* \otimes \alpha) = \sum_{\alpha \in Q_1} 1 \otimes \alpha^* \otimes \alpha$ which corresponds to $\omega := \sum_{\alpha \in Q_1} d\alpha^* d\alpha \in \overline{\text{DR}}^2 A$. By Proposition 8.1.1. of [4] ω is a bi-symplectic form on $k\bar{Q}$.

Indeed, it is easy to verify that

$$P_\omega = \sum_{a \in Q_1} \frac{\partial}{\partial a^*} \frac{\partial}{\partial a} \in (DA/[DA, DA])_2$$

which finishes the proof. \square

We next investigate the relationship between fusion of double Poisson and bi-symplectic structures and relate them to the compositions of Calabi-Yau cospans. By [8] Proposition 2.6.6 if (A, P) is a double Poisson algebra, then so is (A^f, P^f) , where A^f denotes the fusion algebra. If (A, P, μ) is a non-commutative Hamiltonian structure, then (A^f, P^f, μ^f) , where $\mu^f = \bigoplus_{i \geq 3} \mu_i \oplus \mu_1 + e_{21} \mu_2 e_{12}$ is also a non-commutative Hamiltonian structure.

Lemma 2.9. *Let (A, ω) with $\omega \in \text{DR}(A)^2$ be a bi-symplectic structure and A^f the fusion algebra. Then (A^f, ω^f) is also a bi-symplectic structure and $P_\omega^f = P_{\omega^f}$, that is fusion commutes with the morphism that associates a non-commutative Hamiltonian structure to a bi-symplectic structure.*

Proof. By functoriality, $\omega^f \in \text{DR}(A^f)^2$ is a closed form and ι_{ω^f} fits into a commutative diagram

$$\begin{array}{ccc} D_A & \xrightarrow{\iota_\omega} & \Omega_A \\ \downarrow & & \downarrow \\ D_{A^f} & \xrightarrow{\iota_{\omega^f}} & \Omega_{A^f} \end{array}$$

As the vertical arrows are surjective and the top horizontal arrow ι_ω is an isomorphism by the non-degeneracy of ω , it follows that ι_{ω^f} is also an isomorphism. By the definition

$$P_{\omega^f} = (\iota(\omega^f)^{-1} \otimes \iota(\omega^f)^{-1})(\omega^f) = (\iota(\omega)^{-1} \otimes \iota(\omega)^{-1})(\omega)^f = P_\omega^f.$$

The moment map associated to ω is given by the element μ which satisfies $\{P, \mu\} = E$. Clearly $\{P_{\omega^f}, \mu^f\} = \{P_\omega^f, \mu^f\} = \{P_\omega, \mu\}^f = E^f$. \square

Lemma 2.10. *The cospan $k[x] \amalg k[y] \xrightarrow{f} k\langle x, y \rangle \xleftarrow{g} k[z]$ with $g(z) = x + y$ and $f(x) = x, f(y) = y$ is a 1-Calabi-Yau cospan.*

Proof. We equip $k[x] \amalg k[y]$ and $k[z]$ with their natural Calabi-Yau structure given by $1 \otimes x + 1 \otimes y = 1 \otimes (x + y) \in HH_1(k[x] \amalg k[y])$ and $1 \otimes z \in HH_1(k[z])$. Clearly, $f(1 \otimes (x + y)) = g(1 \otimes z)$. \square

Given a relative 1-Calabi-Yau structure $\mu : k[x] \amalg k[y] \rightarrow A$, we can consider the composition of cospans

$$\begin{array}{ccccc} & & A \amalg_{k[x] \amalg k[y]} k\langle x, y \rangle & & \\ & \nearrow & & \nwarrow & \\ k\langle x, y \rangle & & & & A \\ \nearrow & & \nwarrow & & \nwarrow \\ k[z] & & k[x] \amalg k[y] & & \emptyset \end{array}$$

where x, y is mapped to μ_1, μ_2 . This yields a relative Calabi-Yau structure $k[z] \rightarrow B$.

Theorem 2.11. *Let (A, P, μ) be a non-Hamiltonian structure which coincides with the relative 1-Calabi-Yau structure $\mu : k[x] \amalg k[y] \rightarrow A$. The fusion of the non-Hamiltonian structure (A, P, μ) coincides with the relative 1-Calabi-Yau structure $k[z] \rightarrow A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$ given by the composition of cospans above.*

Proof. As the non-commutative Hamiltonian structure is compatible with the relative 1-Calabi-Yau structure, we have $P = P_\omega$ and $\mu(x) = \mu_1$ and $\mu(y) = \mu_2$. We have a natural isomorphism $A^f \simeq A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$ and under this isomorphism $A \rightarrow A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$ is equivalent to the morphism $-^f : A \rightarrow A^f$ and $\mu^f : k[z] \rightarrow A^f$ coincides with $k[z] \rightarrow A \amalg k[x] \amalg k[y] \rightarrow k\langle x, y \rangle$. Furthermore, since the homotopy between the 1-forms in the cospan $k[x] \amalg k[y] \xrightarrow{f} k\langle x, y \rangle \xleftarrow{g} k[z]$ is trivial, the zero-homotopy of the composition is given by ω^f , where ω denotes the zero homotopy $\mu(1 \otimes (x + y)) \sim 0$. As $P_\omega^f = P_{\omega^f}$ the fusion non-Hamiltonian structure coincides with the relative 1-Calabi-Yau structure of the composition of the cospans. \square

2.5 Calabi-Yau cospans and fusion

The aim of this section is to compare the relative Calabi-Yau structure obtained by the cospan procedure with the double quasi-Poisson structure obtained by the fusion process as in [8] Section 7. Let A be an algebra or dg category with orthogonal idempotents e_1, \dots, e_n or respectively objects denoted $1, \dots, n$.

Given a 1-relative Calabi-Yau map $\Phi : \amalg_{i=1}^n k[x_i^{\pm 1}] \rightarrow A$ such that $x_i \mapsto \Phi_i := \Phi(x_i) \in (e_i A e_i)^*$, we can compose with the Calabi-Yau cospan from [3]

$$\amalg_{i=1}^n k[x_i^{\pm 1}] \rightarrow k\langle x_1^{\pm 1}, x_2^{\pm 1} \rangle \amalg_{i=3}^n k[x_i^{\pm 1}] \leftarrow k[z_{1,2}^{\pm 1}] \amalg_{i=3}^n k[x_i^{\pm 1}]$$

with $z_{1,2} \mapsto x_1 x_2$ and the identity map on the x_i for $i \geq 3$. The composition of cospans yields a relative 1-Calabi-Yau structure on

$$\Phi^f : \amalg_{i=3}^n k[x_i^{\pm 1}] \amalg k[z_{1,2}^{\pm 1}] \rightarrow A^f$$

where $A^f := A * \amalg_{i=1}^n k[x_i^{\pm 1}] \amalg k[z_{1,2}^{\pm 1}] \amalg_{i=3}^n k[x_i^{\pm 1}]$. This process fuses the objects respectively vertices denoted 1 and 2. As in [8] we denote by a^f the image of $a \in A$ in A^f and by $\Phi^f : k[z^{\pm 1}] \rightarrow A^f$ the new moment map. We have $\Phi^f(x_i) = \Phi_i^f \in A^f$ for $i \geq 3$ and $\Phi^f(z_{1,2}) = \Phi_1^f \Phi_2^f \in A^f$.

Recall from [2] Section 3.3. that the homotopy on the cospan

$$k[x^{\pm 1}] \amalg k[y^{\pm 1}] \rightarrow k\langle x^{\pm 1}, y^{\pm 1} \rangle \leftarrow k[z^{\pm 1}]$$

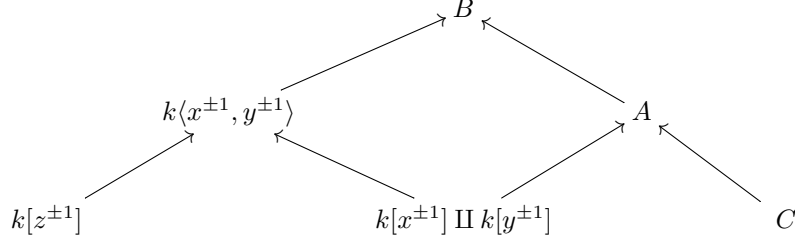
is given by $\gamma_1 := y^{-1} \otimes x^{-1} \otimes xy - y \otimes y^{-1} x^{-1} \otimes x$, which satisfies

$$\alpha_1(xy) - (\alpha_1(x) + \alpha_1(y)) = b(\gamma_1).$$

We can assume that $n = 2$. Let us assume that ω_1 and P are compatible in the sense of [8]. We want to show that ω_1^{ff} and $P^{ff} := P^f - E_1^f E_2^f$ are also compatible. The composition of cospans yields that $\omega_1^{ff} = \beta_1^f + \Phi^f(\zeta_1) = \beta_1^f + \Phi_1^{-1} d\Phi_1 d\Phi - \Phi_2 d\Phi^{-1} d\Phi_1$

let me try to reproduce below the strategy used in 2.3

We need to understand the following composition of Calabi–Yau cospans:



where x, y is mapped to Φ_2, Φ_1 . The homotopy

$$\zeta_1 = \frac{1}{2} \left(y^{-1} \otimes x^{-1} \otimes xy - y \otimes y^{-1} x^{-1} \otimes x \right)$$

is mapped in $\overline{\text{DR}}^2 k\langle x^{\pm 1}, y^{\pm 1} \rangle$ to

$$\begin{aligned}
 \omega &= \frac{1}{4} \left(y^{-1} dx^{-1} d(xy) - y d(y^{-1} x^{-1}) dx \right) \\
 &= \frac{1}{4} \left(-y^{-1} x^{-1} dx x^{-1} (xdy + dxy) + dyy^{-1} x^{-1} dx + x^{-1} dx x^{-1} dx \right) \\
 &= \frac{1}{4} \left(-y^{-1} x^{-1} dx dy - y^{-1} x^{-1} dx x^{-1} dxy + dyy^{-1} x^{-1} dx + x^{-1} dx x^{-1} dx \right) \\
 &\equiv -\frac{1}{2} x^{-1} dx dy y^{-1}
 \end{aligned}$$

which is mapped to

$$-\frac{1}{2} \Phi_1^{-1} d\Phi_1 d\Phi_2 \Phi_2^{-1} \in \overline{\text{DR}}^2 B.$$

which is precisely the opposite of the corrective term given in [9] following [1].

right now
I don't
see how
to correct
this sign

3 Comparison of Lagrangian structures

Assume that we have a finite set $\{e_i \mid i \in I\}$ of idempotents in A , and define the commutative algebra $R = \bigoplus_{i \in I} ke_i$. For any I -graded finite dimensional space V define A_V by

$$\text{Hom}_{\text{Alg}/R}(A, \text{End}(V)) = \text{Hom}_{\text{CommAlg}/k}(A_V, k).$$

Thanks to [4, (6.2.2)], setting $X_V = \text{Spec}(A_V)$, we have a map

$$\text{tr} : \text{DR}^\bullet A \longrightarrow \Omega^\bullet(X_V)^{\text{GL}_V}$$

given by $\alpha \mapsto \text{tr}(\hat{\alpha})$ where $\hat{\alpha}$ is induced by the evaluation

$$A \rightarrow (A_V \otimes \text{End}(V))^{\text{GL}_V} \quad ; \quad a \mapsto \hat{a}.$$

Thanks to [8, Proposition 6.1], there is a quasi-Hamiltonian structure on $(X_V, \text{tr}(\omega_1), \hat{\Phi})$. Now $\hat{\Phi} : X_V \rightarrow \text{GL}_V$ induces a lagrangian structure on $[X_V/\text{GL}_V] \rightarrow [\text{GL}_V/\text{GL}_V]$.

the ω_1 and
 $\hat{\Phi}$ from
section 2

On the other hand, thanks to [3], the 1-Calabi-Yau structure on Φ yields a lagrangian structure on $\text{Perf}_A \rightarrow \text{Perf}_{k[x^{\pm 1}]}$, and thus considering substacks on $[X_V/\text{GL}_V] \rightarrow [\text{GL}_V/\text{GL}_V]$ again.

Theorem 3.1. *These two lagrangian structures are identical.*

Recall that a relative 1-Calabi-Yau morphism of algebras $k[x^\pm] \rightarrow A$ induces a 1-shifted Lagrangian morphism of stacks $\mathbf{Perf}_A \rightarrow \mathbf{Perf}_{k[x^\pm]}$. Now by [?], a 1-shifted Lagrangian morphism induces a non-degenerate Poisson structure on \mathbf{Perf}_A via pullback of forms. In our setting, $[\mathbf{rep}_A/\mathrm{GL}_n]$ is an open substack of \mathbf{Perf}_A and therefore the non-degenerate Poisson form restricts to $[\mathbf{rep}_A/\mathrm{GL}_n]$. Equivalently, given a Lagrangian morphism $\mathbf{Perf}_A \rightarrow \mathbf{Perf}_{k[x^\pm]}$, we can consider the following commutative diagram

$$\begin{array}{ccc} \mathbf{Perf}_A & \longrightarrow & \mathbf{Perf}_{k[x^\pm]} \\ \uparrow & & \uparrow \\ [\mathbf{rep}_n A/\mathrm{GL}_n] & \longrightarrow & [\mathbf{rep}_n k[x^\pm]/\mathrm{GL}_n] \end{array}$$

where the vertical arrows are open embeddings and the bottom horizontal morphism inherits a 1-Lagrangian structure from the top morphism. As shown in [], the 1-Calabi-Yau structure on $k[x^\pm]$ given by $(x^{-1} \otimes 1 + 1 \otimes x^{-1})/2$ induces the classical symplectic structure on $[\mathbf{rep}_n k[x^\pm]/\mathrm{GL}_n] \simeq [\mathrm{GL}_n/\mathrm{GL}_n]$ via restriction from $\mathbf{Perf}_{k[x^\pm]}$ which we can pull back to $[\mathbf{rep}_n A/\mathrm{GL}_n]$.

Question: Does the non-degenerate quasi-Poisson form respectively quasi-Hamiltonian structure on $\mathbf{rep}_n A$ given in [7] yield the same Poisson structure on $[\mathbf{rep}_n A/\mathrm{GL}_n]$ than the two methods described above?

Here DR^\bullet denotes either the Karoubi-DeRham complex in the non-commutative case and the DeRham complex of derived stacks seen as a functor $\mathrm{DR}^\bullet : \mathrm{dSt} \rightarrow \mathrm{Mod}_k^{\epsilon-gr}$.

Recall that every B -point $x : \mathrm{Spec} B \rightarrow \mathbf{Perf}_A$ is given by a dg functor $A \rightarrow \mathrm{Mod}_B^{\mathrm{perf}}$. Applying HC^- yields a map $\mathrm{HC}^-(A) \rightarrow \mathrm{HC}^-(\mathrm{Mod}_B^{\mathrm{perf}}) \simeq \mathrm{HC}^-(B)$. The DeRham complex of \mathbf{Perf}_A can be computed as the limit $\lim_{x: \mathrm{Spec} B \rightarrow \mathbf{Perf}_A} \mathrm{HC}^-(B)$ and we therefore obtain a natural map $\mathrm{DR}(A) \simeq \mathrm{HC}^-(A) \rightarrow \mathrm{DR}(\mathbf{Perf}_A)$.

We have a diagram

$$\begin{array}{ccc} \mathrm{DR}A & \longrightarrow & \mathrm{DR}^\bullet(\mathbf{Perf}_A) \\ & \searrow & \downarrow \\ & & \mathrm{DR}([\mathbf{rep}_n A/\mathrm{GL}_n]) \\ & \searrow \scriptstyle{tr} & \downarrow \\ & & \mathrm{DR}(\mathbf{rep}_n A). \end{array}$$

The vertical arrows on the right are given by the functoriality of DR^\bullet with respect to the natural maps $\mathbf{rep}_n A \rightarrow [\mathbf{rep}_n A/\mathrm{GL}_n] \rightarrow \mathbf{Perf}_A$ of derived stacks. The maps $tr : \mathrm{HC}^- A \simeq \mathrm{DR}^\bullet(A) \rightarrow \mathrm{DR}^\bullet(\mathbf{rep}_n A)$ and $\mathrm{HC}^- A \simeq \mathrm{DR}^\bullet(A) \rightarrow \mathrm{DR}^\bullet([\mathbf{rep}_n A/\mathrm{GL}_n])$ are given in [7]. To answer our question, we have to show that the diagram above commutes.

We will first show that the outside triangle commutes.

Remark 3.3 (pre-quotient case). Thanks to [4, (6.2.2)], setting $X_V = \mathrm{Spec}(A_V)$, we have a map

$$\mathrm{DR}^\bullet A \longrightarrow \Omega^\bullet(X_V)^{\mathrm{GL}_V}$$

given by $\alpha \mapsto \mathrm{tr}(\hat{\alpha})$ where $\hat{\alpha}$ is induced by the evaluation

$$A \rightarrow (A_V \otimes \mathrm{End}(V))^{\mathrm{GL}_V}; \quad a \mapsto \hat{a}.$$

On the other hand, we have a chain

$$\mathrm{HC}^- A \longrightarrow \mathrm{HC}^-(\mathrm{Mod}_{A_V}^{\mathrm{perf}}) \xrightarrow{\sim} \mathrm{HC}^- A_V \xrightarrow[\mathrm{HKR}]{\sim} \mathrm{DR}^\bullet A_V$$

given by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \mathrm{tr}(\hat{a}_0) \mathrm{dtr}(\hat{a}_1) \dots \mathrm{dtr}(\hat{a}_n).$$

Let us clarify the map $\mathrm{DR}^\bullet A \simeq \mathrm{HC}^- A \longrightarrow \mathrm{DR}(\mathrm{Perf}_A)$. By the above description, we know that for every B -point $x : A \rightarrow \mathrm{Mod}_B^{\mathrm{perf}}$, we have a map $\mathrm{HC}^-(A) \rightarrow \mathrm{HC}^-(\mathrm{Mod}_B^{\mathrm{perf}}) \simeq \mathrm{HC}^-(B)$ given by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \mathrm{tr}(xa_0) \mathrm{dtr}(xa_1) \dots \mathrm{dtr}(xa_n)$. Now the map $\mathbf{rep}_V(A) \rightarrow \mathbf{Perf}_A$ corresponds to the A_V -point $A \longrightarrow A_V \otimes \mathrm{End}(V)$.

We get that the square

$$\begin{array}{ccc} \mathrm{DR}^\bullet A \simeq \mathrm{HC}^- A & \longrightarrow & \mathrm{DR}(\mathrm{Perf}_A) \\ \downarrow & & \downarrow \\ \Omega^\bullet(X_V)^{\mathrm{GL}_V} & \longrightarrow & \mathrm{DR}(X_V) \simeq \mathrm{DR}(\mathbf{rep}_V A) \end{array}$$

commutes.

4 Simplicial complexes

Now denote by $A(n)$ the category with objects (a, i) , $a \in \mathrm{Ob}(A)$, $i = 1, \dots, n$ and morphisms

$$A(n)((a, i), (b, j)) = \begin{cases} A(a, b), & \text{if } i = j \\ A(a, a)^\sim, & \text{if } j = i + 1 \\ 0, & \text{otherwise} \end{cases} \quad (4.1)$$

This gives rise to a simplicial object

$$A(1) \rightleftarrows A(2) \rightleftarrows A(3) \dots$$

Let $V = (V_a)_{a \in \mathrm{Ob} A}$ be a vector-space indexed by the objects of A . We denote $\mathbf{rep}_V A(m)$ the moduli space which sends every (a, i) to V_a . Then

$$\mathbf{rep}_V A(m) \simeq \mathbf{rep}_V A \times \mathrm{GL}_V^{m-1} = X_V \times (\mathrm{GL}_V)^{m-1}.$$

Applying the functor \mathbf{rep}_V to the simplicial complex yields therefore the simplicial complex

$$[X_V/\mathrm{GL}_V] = \mathrm{colim} \left(X_V \rightleftarrows X_V \times \mathrm{GL}_V \rightleftarrows X_V \times \mathrm{GL}_V \times \mathrm{GL}_V \dots \right).$$

Hence, we morphisms of derived stacks

$$X_V \times (\mathrm{GL}_V)^{n-1} \rightarrow \mathrm{Perf}_{A(n)}$$

that induce a morphism

$$[X_V/\mathrm{GL}_V] \rightarrow \mathrm{colim} \mathrm{Perf}_{A(n)} \simeq \mathrm{Perf}_A.$$

We have natural maps $A(n) \rightarrow A$, $(a, i) \mapsto a$. This induces a map between the simplicial complex above and the constant simplicial complex in A . Applying DR yields a map $\lim_n \mathrm{DR}(A(n)) \longrightarrow \mathrm{DR}(A)$ which is an isomorphism. Let us spell out this map $\mathrm{tr}(a_0) \mathrm{dtr}(a_1) \dots \mathrm{dtr}(a_n) \mapsto$.

We obtain the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{DR}(A) & \xleftarrow{\sim} & \lim_n \mathrm{DR}(A(n)) & \xrightarrow{\mathrm{tr}} & \lim_n \mathrm{DR}(\mathbf{rep}_V(A(n))) \\
\downarrow & & \downarrow & & \uparrow \sim \\
\mathrm{DR}(\mathbf{Perf}_A) & \xleftarrow{\sim} & \lim_n \mathrm{DR}(\mathbf{Perf}_{A(n)}) & \longrightarrow & \mathrm{DR}([X_V/\mathrm{GL}_V])
\end{array}$$

Assume first that our categories in

$$A = A(1) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} A(2) =: B$$

only have one object, so that an isomorphism $f \cong g$ corresponds to $z \in B^\times$ satisfying $zf = gz$. On simplicial complexes we have morphisms

$$\begin{array}{ccccc}
\dots & & A^{\otimes n} & \xleftarrow{(n+1)} & A^{\otimes n+1} & \dots \\
& & \downarrow f^{\otimes n} & & \downarrow f^{\otimes n+1} & \\
& & \downarrow g^{\otimes n} & & \downarrow g^{\otimes n+1} & \\
\dots & & B^{\otimes n} & \xleftarrow{(n+1)} & B^{\otimes n+1} & \dots
\end{array}$$

induced by f and g . Horizontal maps are given by m_i , $i = 1 \dots n+1$, the multiplication of the i -th and $i+1$ -th factors, with the convention $n+2 = 1$.

Proposition 4.2. *The homotopy between f and g induced by z*

$$(h : A \times \Delta^1 \rightarrow B) = ((h_{n,j})_{1 \leq j \leq n} : (A^{\otimes n})^n \rightarrow B^{\otimes n+1})_{n \geq 1}$$

reads

$$h_{n,j} = fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-j}$$

meaning

$$h_{n,j}(a_1 \otimes \dots \otimes a_n) = f(a_1)z^{-1} \otimes g^{\otimes j-1}(a_2 \otimes \dots \otimes a_j) \otimes z \otimes f^{\otimes n-j}(a_{j+1} \otimes \dots \otimes a_n).$$

Proof. First note that $m_1 h_{n,1} = f^{\otimes n}$ and $m_{n+1} h_{n,n} = g^{\otimes n}$.

Next consider $1 \leq i < j \leq n$, then

$$\begin{aligned}
m_i h_{n,j} &= m_i(fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-j}) \\
&= (fz^{-1} \otimes g^{\otimes j-2} \otimes z \otimes f^{\otimes n-j})m_i \\
&= h_{n-1,j-1}m_i
\end{aligned}$$

the only nontrivial case being $i = 1$ where we use $m_1(fz^{-1} \otimes g) = fz^{-1}m_1$.

If $1 < i = j$,

$$\begin{aligned}
m_i h_{n,i} &= m_i(fz^{-1} \otimes g^{\otimes i-1} \otimes z \otimes f^{\otimes n-i}) \\
&= fz^{-1} \otimes g^{\otimes i-2} \otimes gz \otimes f^{\otimes n-i} \\
&= fz^{-1} \otimes g^{\otimes i-2} \otimes zf \otimes f^{\otimes n-i} \\
&= m_i h_{n,i-1}.
\end{aligned}$$

Finally, when $i > j+1$ we have

$$\begin{aligned}
m_i h_{n,j} &= m_i(fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-j}) \\
&= (fz^{-1} \otimes g^{\otimes j-1} \otimes z \otimes f^{\otimes n-1-j})m_{i-1} \\
&= h_{n-1,j}m_{i-1}
\end{aligned}$$

even if $i = n+1$. □

Proposition 4.3. *Assume that $zf = gz$ and $wg = hw$ for $f, g, h : A \rightarrow B$ and $z, w \in B^\times$. Denote by h^z, h^w and h^{wz} the induced homotopies. The homotopy $H = (H_n : A^{\otimes n} \rightarrow B^{\otimes n+2})$ between $h^z + h^w$ and h^{wz} reads*

$$H_n = \sum_{\substack{r,s,t \geq 0 \\ r+s+t=n-1}} \pm fz^{-1}w^{-1} \otimes h^{\otimes r} \otimes w \otimes g^{\otimes s} \otimes z \otimes f^{\otimes t}.$$

Proof. Write $H_{r,s,t} = fz^{-1}w^{-1} \otimes h^{\otimes r} \otimes w \otimes g^{\otimes s} \otimes z \otimes f^{\otimes t}$, and $h_{p,q}^z = fz^{-1} \otimes g^{\otimes p} \otimes z \otimes f^{\otimes q}$, same with h^w, h^{wz} . Then note that

$$\begin{aligned} m_1 H_{0,s,t} &= h_{s,t}^z \\ m_{n+2} H_{r,s,0} &= h_{r,s}^w \\ m_{r+2} H_{r,0,t} &= h_{r,t}^{wz}. \end{aligned}$$

If $i \leq r$,

$$m_i H_{r,s,t} = H_{r-1,s,t} m_i,$$

thanks to $m_1(fz^{-1}w^{-1} \otimes h) = fz^{-1}w^{-1}m_1$ when $i = 1$.

If $i = r + 1$,

$$m_i H_{r,s,t} = m_i H_{r-1,s+1,t}$$

since $hw = wg$.

If $r + 3 \leq i \leq r + s + 1$,

$$m_i H_{r,s,t} = H_{r,s-1,t} m_{i-1}.$$

If $i = r + s + 2$,

$$m_i H_{r,s,t} = m_i H_{r,s-1,t+1}$$

since $gz = zf$.

If $r + s + 4 \leq i \leq n + 2$,

$$m_i H_{r,s,t} = H_{r,s,t-1} m_{i-2}.$$

□

References

- [1] A. Alekseev, A. Malkin & E. Meinrenken, Lie group valued moment maps, J. Differential Geom. **48** (1998), no. 3, 445–495.
- [2] T. Bozec, D. Calaque & S. Scherotzke, Relative critical loci and quiver moduli, preprint available at <https://arxiv.org/abs/2006.01069>.
- [3] T. Bozec, D. Calaque & S. Scherotzke, Calabi-Yau structures for multiplicative preprojective algebras, preprint available at <https://arxiv.org/abs/2102.12336>.
- [4] W. Crawley-Boevey, P. Etingof & V. Ginzburg, Noncommutative geometry and quiver algebras, Adv. Math. **209** (2007), no. 1, 274–336.
- [5] V. Ginzburg & T. Schedler, Free products, cyclic homology, and the Gauss-Manin connection, Adv. Math. **231** (2012), no. 3-4, 2352–2389.
- [6] V. Ginzburg & T. Schedler, A new construction of cyclic homology, Proc. Lond. Math. Soc. **112** no. 3 (2016), 549–587.

- [7] M. Van den Bergh, Double Poisson algebras, *Trans. Amer. Math. Soc.* **360** (2008), no. 11, 5711–5769.
- [8] M. Van den Bergh, Non-commutative quasi-Hamiltonian spaces, in *Poisson geometry in mathematics and physics*, *Contemp. Math.* **450** (2008), 273–300.
- [9] D. Yamakawa, Geometry of multiplicative preprojective algebra, *International Mathematics Research Papers IMRP*, (2008).