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Primitive elements of the Hopf algebra of free quasi-symmetric functions

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ABSTRACT: Using the dendriform and the bidendriform Cartier-Quillen-Milnor-Moore theorem, we construct a basis of the space of primitive elements of the Hopf algebra of free quasi-symmetric functions, indexed by a certain set of trees, and inductively computable.

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Introduction

The Hopf algebra of free quasi-symmetric functions \mathbf{FQSym} , also known as the Malvenuto-Reutenauer Hopf algebra, is introduced in [7]. It is a graded self-dual Hopf algebra, with the set of all permutations as a basis. Certain interesting properties are shown in [2]: in particular, it is shown that it is both free and cofree, and a basis of the space of its primitive elements is given, using the self-duality and a monomial basis. Note that computing the primitive elements of degree n by this method implies to inverse a certain $n! \times n!$ matrix.

The aim of this paper is to describe another basis of $Prim_{coAss}(\mathbf{FQSym})$, which can be inductively computed. We use for this the dendriform structure of \mathbf{FQSym} . Recall that a dendriform algebra is an associative algebra such that its product can be split into two non-associative products \prec and \succ , with good compatibilities [5, 6, 8]. It is known that \mathbf{FQSym} , or more precisely its augmentation ideal, is dendriform. More precisely, it is a dendriform Hopf algebra, in the sense of [8]. This implies, by the dendriform Cartier-Quillen-Milnor-Moore theorem, that $Prim_{coAss}(\mathbf{FQSym})$ is a brace algebra.

We introduced in [4] the notion of bidendriform bialgebra and showed that \mathbf{FQSym} is bidendriform. The bidendriform Cartier-Quillen-Milnor-Moore theorem implies that \mathbf{FQSym} is freely generated, as a dendriform algebra, by the space $Prim_{coDend}(\mathbf{FQSym})$ of primitive elements

in the codendriform sense. Combining this result with the dendriform Cartier-Quillen-Milnor-Moore theorem, we show that $Prim_{coAss}(\mathbf{FQSym})$ is, as a brace algebra, freely generated by $Prim_{coDend}(\mathbf{FQSym})$. We recall in section 2 a description of free brace algebras. If $(v_i)_{i \in I}$ is a basis of the vector space V , then the free brace algebra generated by V has a basis indexed by planar rooted trees decorated by I , and the brace structure is described in this basis by the help of graftings. Hence, for any basis of $Prim_{coDend}(\mathbf{FQSym})$, it is possible to recover a basis of $Prim_{coAss}(\mathbf{FQSym})$, indexed by a certain set of planar rooted trees.

Let, for all $n \in \mathbb{N}$:

$$\begin{cases} p_n &= \dim(Prim_{coAss}(\mathbf{FQSym})_n), \\ q_n &= \dim(Prim_{coDend}(\mathbf{FQSym})_n). \end{cases}$$

We then prove in section 3 that for $n \geq 2$, $q_n = (n-2)p_{n-1}$. We then give $n-2$ applications from $Prim_{coAss}(\mathbf{FQSym})_{n-1}$ to $Prim_{coDend}(\mathbf{FQSym})_n$, which give all elements of $Prim_{coDend}(\mathbf{FQSym})_n$. These applications are given by the insertion of $n+1$ at a given place in elements of the symmetric group S_n , seen as words in letters $1, \dots, n$.

Combining the results of the second and third sections, we define inductively in the fourth section a new basis of $Prim_{coAss}(\mathbf{FQSym})_n$, indexed by certain planar decorated rooted trees. The trees which are only a root give a basis of $Prim_{coDend}(\mathbf{FQSym})_n$.

Notations.

1. K is a commutative field of any characteristic.
2. If V is a K -vector field which is \mathbb{N} -graded, we shall denote by V_k the space of homogeneous elements of V of degree k .

1 Bidendriform bialgebras and \mathbf{FQSym}

1.1 Bidendriform bialgebras

We introduced in [4] the following definition:

Definition 1 A bidendriform bialgebra is a family $(A, \prec, \succ, \Delta_\prec, \Delta_\succ)$ such that:

1. A is a K -vector space and:

$$\begin{array}{l} \prec: \left\{ \begin{array}{l} A \otimes A \longrightarrow A \\ a \otimes b \longrightarrow a \prec b, \end{array} \right. \quad \left| \quad \Delta_\prec: \left\{ \begin{array}{l} A \longrightarrow A \otimes A \\ a \longrightarrow \Delta_\prec(a) = a'_\prec \otimes a''_\prec, \end{array} \right. \\ \succ: \left\{ \begin{array}{l} A \otimes A \longrightarrow A \\ a \otimes b \longrightarrow a \succ b, \end{array} \right. \quad \left| \quad \Delta_\succ: \left\{ \begin{array}{l} A \longrightarrow A \otimes A \\ a \longrightarrow \Delta_\succ(a) = a'_\succ \otimes a''_\succ. \end{array} \right. \end{array}$$

2. (Dendriform axioms). (A, \prec, \succ) is a dendriform algebra: for all $a, b, c \in A$,

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \quad (1)$$

$$(a \succ b) \prec c = a \succ (b \prec c), \quad (2)$$

$$5a \prec b + a \succ b \succ c = a \succ (b \succ c). \quad (3)$$

3. (Codendriform axioms). $(A, \Delta_\prec, \Delta_\succ)$ is a codendriform coalgebra: for all $a \in A$,

$$(\Delta_\prec \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec + Id \otimes \Delta_\succ) \circ \Delta_\prec(a), \quad (4)$$

$$(\Delta_\succ \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec) \circ \Delta_\succ(a), \quad (5)$$

$$(\Delta_\prec \otimes Id + \Delta_\succ \otimes Id) \circ \Delta_\succ(a) = (Id \otimes \Delta_\succ) \circ \Delta_\succ(a). \quad (6)$$

4. (Bidendriform axioms). For all $a, b \in A$,

$$\Delta_{\succ}(a \succ b) = a'b'_{\prec} \otimes a'' \succ b''_{\prec} + a' \otimes a'' \succ b + b'_{\prec} \otimes a \succ b''_{\prec} + ab'_{\prec} \otimes b''_{\prec} + a \otimes b, \quad (7)$$

$$\Delta_{\succ}(a \prec b) = a'b'_{\prec} \otimes a'' \prec b''_{\prec} + a' \otimes a'' \prec b + b'_{\prec} \otimes a \prec b''_{\prec}, \quad (8)$$

$$\Delta_{\prec}(a \succ b) = a'b'_{\prec} \otimes a'' \succ b''_{\prec} + ab'_{\prec} \otimes b''_{\prec} + b'_{\prec} \otimes a \succ b''_{\prec}, \quad (9)$$

$$\Delta_{\prec}(a \prec b) = a'b'_{\prec} \otimes a'' \prec b''_{\prec} + a'b \otimes a'' + b'_{\prec} \otimes a \prec b''_{\prec} + b \otimes a. \quad (10)$$

Remarks.

1. If A is a bidendriform bialgebra, then $K \oplus A$ is naturally a Hopf algebra, extending $\prec + \succ$ and $\Delta_{\prec} + \Delta_{\succ}$ on $K \oplus A$.
2. If A is a bidendriform bialgebra, it is also a dendriform hopf algebra in the sense of [8, 9], with coassociative coproduct given by $\tilde{\Delta} = \Delta_{\prec} + \Delta_{\succ}$. The compatibilities of dendriform Hopf algebras are given by (7) + (9) and (8) + (10).

If A is a bidendriform algebra, we define:

$$\text{Prim}_{\text{codend}}(A) = \text{Ker}(\Delta_{\prec}) \cap \text{Ker}(\Delta_{\succ}).$$

The following result is proved in [4] (theorem 35 and corollary 17):

Theorem 2 (Bidendriform Cartier-Quillen-Milnor-Moore theorem) *Let A be a \mathbb{N} -graded bidendriform bialgebra, such that $A_0 = (0)$. Then A is freely generated as a dendriform algebra by $\text{Prim}_{\text{coDend}}(A)$. Moreover, consider the following formal series:*

$$R(X) = \sum_{n=1}^{+\infty} \dim(A_n) X^n, \quad Q(X) = \sum_{n=1}^{+\infty} \dim(\text{Prim}_{\text{coDend}}(A_n)) X^n.$$

$$\text{Then } Q(X) = \frac{R(X)}{(R(X) + 1)^2}.$$

1.2 An example: the Hopf algebra \mathbf{FQSym}

See [1, 2, 7]. The algebra \mathbf{FQSym} is the vector space generated by the elements $(\mathbf{F}_u)_{u \in \mathbb{S}}$, where \mathbb{S} is the disjoint union of the symmetric groups S_n ($n \in \mathbb{N}$). Its product and its coproduct are given in the following way: for all $u \in S_n, v \in S_m$, putting $u = (u_1 \dots u_n)$,

$$\begin{aligned} \Delta(\mathbf{F}_u) &= \sum_{i=0}^n \mathbf{F}_{st(u_1 \dots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \dots u_n)}, \\ \mathbf{F}_u \cdot \mathbf{F}_v &= \sum_{\zeta \in sh(n, m)} \mathbf{F}_{(u \times v) \cdot \zeta^{-1}}, \end{aligned}$$

where $sh(n, m)$ is the set of (n, m) -shuffles, and st is the standardisation. Its unit is $1 = \mathbf{F}_{\emptyset}$, where \emptyset is the unique element of S_0 . Moreover, \mathbf{FQSym} is a \mathbb{N} -graded Hopf algebra, by putting $|\mathbf{F}_u| = n$ if $u \in S_n$.

Examples.

$$\begin{aligned} \mathbf{F}_{(12)} \mathbf{F}_{(123)} &= \mathbf{F}_{(12345)} + \mathbf{F}_{(13245)} + \mathbf{F}_{(13425)} + \mathbf{F}_{(13452)} + \mathbf{F}_{(31245)} \\ &\quad + \mathbf{F}_{(31425)} + \mathbf{F}_{(31452)} + \mathbf{F}_{(34125)} + \mathbf{F}_{(34152)} + \mathbf{F}_{(34512)}; \end{aligned}$$

$$\begin{aligned} \Delta(\mathbf{F}_{(12543)}) &= 1 \otimes \mathbf{F}_{(12543)} + \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1432)} + \mathbf{F}_{(12)} \otimes \mathbf{F}_{(321)} \\ &\quad + \mathbf{F}_{(123)} \otimes \mathbf{F}_{(21)} + \mathbf{F}_{(1243)} \otimes \mathbf{F}_{(1)} + \mathbf{F}_{(12543)} \otimes 1. \end{aligned}$$

Let $(\mathbf{FQSym})_+ = \text{Vect}(\mathbf{F}_u / u \in S_n, n \geq 1)$ be the augmentation ideal of \mathbf{FQSym} . We define $\prec, \succ, \Delta_\prec$ and Δ_\succ on $(\mathbf{FQSym})_+$ in the following way: for all $u \in S_n, v \in S_m$, by putting $u = (u_1 \dots u_n)$,

$$\begin{aligned} \mathbf{F}_u \prec \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m) \\ \zeta^{-1}(n+m)=n}} \mathbf{F}_{(u \times v) \cdot \zeta^{-1}}, \\ \mathbf{F}_u \succ \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m) \\ \zeta^{-1}(n+m)=n+m}} \mathbf{F}_{(u \times v) \cdot \zeta^{-1}}, \\ \Delta_\prec(\mathbf{F}_u) &= \sum_{i=u^{-1}(n)}^{n-1} \mathbf{F}_{st(u_1 \dots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \dots u_n)}, \\ \Delta_\succ(\mathbf{F}_u) &= \sum_{i=1}^{u^{-1}(n)-1} \mathbf{F}_{st(u_1 \dots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \dots u_n)}. \end{aligned}$$

Examples.

$$\begin{aligned} \mathbf{F}_{(12)} \prec \mathbf{F}_{(123)} &= \mathbf{F}_{(13452)} + \mathbf{F}_{(31452)} + \mathbf{F}_{(34152)} + \mathbf{F}_{(34512)}, \\ \mathbf{F}_{(12)} \succ \mathbf{F}_{(123)} &= \mathbf{F}_{(12345)} + \mathbf{F}_{(13245)} + \mathbf{F}_{(13425)} + \mathbf{F}_{(31245)} + \mathbf{F}_{(31425)} + \mathbf{F}_{(34125)}, \\ \Delta_\prec(\mathbf{F}_{(12543)}) &= \mathbf{F}_{(123)} \otimes \mathbf{F}_{(21)} + \mathbf{F}_{(1243)} \otimes \mathbf{F}_{(1)}, \\ \Delta_\succ(\mathbf{F}_{(12543)}) &= \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1432)} + \mathbf{F}_{(12)} \otimes \mathbf{F}_{(321)}. \end{aligned}$$

The following result is proved in [4], theorem 38:

Theorem 3 $((\mathbf{FQSym})_+, \prec, \succ, \Delta_\prec, \Delta_\succ)$ is a connected bidendriform bialgebra.

Moreover, $(\mathbf{FQSym})_+$ is \mathbb{N} -graded, by putting the elements of S_n homogeneous of degree n . By theorem 2, with $q_n = \dim(\text{Prim}_{coDend}(\mathbf{FQSym})_n)$, we obtain:

n	1	2	3	4	5	6	7	8	9	10	11	12
q_n	1	0	1	6	39	284	2 305	20 682	203 651	2 186 744	25 463 925	319 989 030

2 Recovering $\text{Prim}_{coAss}(\mathbf{FQSym})$ from $\text{Prim}_{coDend}(\mathbf{FQSym})$

2.1 Dendriform Cartier-Quillen-Milnor-Moore theorem and variations

Recall that a brace algebra is a K -vector space A together with a n -multilinear operation for all $n \geq 2$:

$$\langle \dots \rangle : \begin{cases} A^{\otimes n} & \longrightarrow A \\ a_1 \otimes \dots \otimes a_n & \longrightarrow \langle a_1, \dots, a_n \rangle, \end{cases}$$

satisfying certain relations; see [8, 9] for more details. For example:

$$\langle a_1, \langle a_2, a_3 \rangle \rangle = \langle a_1, a_2, a_3 \rangle + \langle \langle a_1, a_2 \rangle, a_3 \rangle + \langle a_2, a_1, a_3 \rangle.$$

The following theorem is proved in [8, 9]; more precisely, the first point of this theorem is proposition 2-8 and theorem 3-4 of [8] and the second point is theorem 4-6 of [9]:

Theorem 4 (Dendriform Cartier-Quillen-Milnor-Moore theorem) *Let A be a dendriform Hopf algebra. We denote $\text{Prim}_{coAss}(A) = \text{Ker}(\tilde{\Delta})$.*

1. $Prim_{coAss}(A)$ is a brace algebra, with brackets given by:

$$\langle p_1, \dots, p_n \rangle = \sum_{i=0}^{n-1} (-1)^{n-1-i} (p_1 \prec (p_2 \prec (\dots \prec p_i) \dots) \succ p_n \prec (\dots (p_{i+1} \succ p_{i+2}) \succ \dots) \succ p_{n-1}).$$

2. If A is freely generated as a dendriform algebra by a subvector space $V \subseteq Prim_{coAss}(A)$, then $Prim_{coAss}(A)$ is freely generated as a brace algebra by V .

Let us precise the relation between $Prim_{coAss}(A)$ and $Prim_{coDend}(A)$ if A is a bidendriform bialgebra. Combining the dendriform and the bidendriform Cartier-Quillen-Milnor-Moore theorems:

Theorem 5 *Let A be a \mathbb{N} -graded bidendriform bialgebra, with $A_0 = (0)$. Then $Prim_{coAss}(A)$ is, as a brace algebra, freely generated by $Prim_{coDend}(A)$.*

Proof. By the bidendriform Cartier-Quillen-Milnor-Moore theorem, A is freely generated as a dendriform algebra by $Prim_{coDend}(A)$. By the second point of the dendriform Cartier-Quillen-Milnor-Moore theorem, $Prim_{coAss}(A)$ is freely generated as a brace algebra by the space $Prim_{coDend}(A)$. \square

Proposition 6 *If A is \mathbb{N} -graded dendriform Hopf algebra, such that $A_0 = (0)$, then A is generated as a dendriform algebra by $Prim_{coAss}(A)$. Moreover, consider the following formal series:*

$$R(X) = \sum_{n=1}^{\infty} \dim(A_n) X^n, \quad P(X) = \sum_{n=1}^{+\infty} \dim(Prim_{coDend}(A)_n) X^n.$$

Then $P(X) = \frac{R(X)}{1 + R(X)}$.

Proof. *First step.* Let $p_1, \dots, p_n \in Prim_{coAss}(A)$. We define by induction on n :

$$\omega(p_1, \dots, p_n) = \begin{cases} p_1 & \text{if } n = 1, \\ p_n \prec \omega(p_1, \dots, p_{n-1}) & \text{if } n \geq 2. \end{cases}$$

An easy induction on n allows to show the following result, using (8)+(10):

$$\tilde{\Delta}(\omega(p_1, \dots, p_n)) = \sum_{i=1}^{n-1} \omega(p_1, \dots, p_i) \otimes \omega(p_{i+1}, \dots, p_n).$$

We denote by $\tilde{\Delta}^n : A \rightarrow A^{n+1}$ the iterated coproducts of A . It comes by induction:

$$\tilde{\Delta}^m(\omega(p_1, \dots, p_n)) = \begin{cases} 0 & \text{if } m \geq n, \\ p_1 \otimes \dots \otimes p_n & \text{if } m = n - 1. \end{cases}$$

Second step. We consider the tensor (non counitary) coalgebra $C = T(Prim_{coAss}(A))$:

$$C = \bigoplus_{n=1}^{\infty} Prim_{coAss}(A)^{\otimes n}.$$

It is a coalgebra for the deconcatenation coproduct. As $Prim_{coAss}(A)$ is \mathbb{N} -graded, C is a graded coalgebra with formal series:

$$S(X) = \frac{1}{1 - P(X)} - 1 = \frac{P(X)}{1 - P(X)}.$$

By the first step, the following application is a morphism of graded coalgebras:

$$\Psi : \begin{cases} C & \longrightarrow A \\ p_1 \otimes \dots \otimes p_n & \longrightarrow \omega(p_1, \dots, p_n). \end{cases}$$

Third step. Suppose that $\text{Ker}(\Psi)$ is non zero. As it is a coideal of C , it contains primitive elements of C , that is to say elements of $\text{Prim}_{\text{coAss}}(A)$. As Ψ is obviously monic on $\text{Prim}_{\text{coAss}}(A)$, this is impossible. So $\text{Ker}(\Psi) = (0)$ and Ψ is monic.

Let $a \in A$. As $A_0 = (0)$, for a certain $N(a) \in \mathbb{N}^*$, $\tilde{\Delta}^{N(a)}(a) = 0$. We prove that $a \in \text{Im}(\Psi)$ by induction on $N(a)$. If $N(a) = 1$, then $a \in \text{Prim}_{\text{coAss}}(A)$ and the result is obvious. Suppose that the result is true for all $b \in A$ such that $N(b) < N(a)$. As $\tilde{\Delta}^{N(a)}(a) = 0$, necessarily $\tilde{\Delta}^{N(a)-1}(a) \in \text{Prim}_{\text{coAss}}(A)^{\otimes N(a)}$. We put:

$$\tilde{\Delta}^{N(a)-1}(a) = a_1 \otimes \dots \otimes a_n, \quad b = a - \omega(a_1, \dots, a_n).$$

By the first step, $\tilde{\Delta}^{N(a)-1}(b) = 0$, so $N(b) < N(a)$. By the induction hypothesis, $b \in \text{Im}(\Psi)$. As $\omega(a_1, \dots, a_n) \in \text{Im}(\Psi)$, $a \in \text{Im}(\Psi)$.

Last step. As Ψ is an isomorphism of graded coalgebras, $S(X) = R(X)$. Hence:

$$R(X) = \frac{P(X)}{1 - P(X)},$$

$$\text{so } R(X) - R(X)P(X) = P(X) \text{ and } P(X) = \frac{R(X)}{1 + R(X)}. \quad \square$$

2.2 Free brace algebras

Using a description of the free dendriform algebra generated by a set \mathcal{D} with planar decorated forests, we gave a description of the free brace algebra $\text{Brace}(\mathcal{D})$ in [3]. A basis of this brace algebra is given by the set $T^{\mathcal{D}}$ of planar rooted trees decorated by \mathcal{D} . For example:

$$\begin{aligned} \text{Brace}(\mathcal{D})_1 &= \text{Vect}(\bullet_a, a \in \mathcal{D}), \\ \text{Brace}(\mathcal{D})_2 &= \text{Vect}(\uparrow_a^b, a, b \in \mathcal{D}), \\ \text{Brace}(\mathcal{D})_3 &= \text{Vect}({}^c\mathbb{V}_a^b, \downarrow_a^c, a, b, c \in \mathcal{D}), \\ \text{Brace}(\mathcal{D})_4 &= \text{Vect}({}^d\mathbb{V}_a^c, {}^d\mathbb{V}_a^b, {}^d\mathbb{V}_a^c, {}^d\mathbb{V}_a^c, \downarrow_a^d, a, b, c, d \in \mathcal{D}), \dots \end{aligned}$$

The brace bracket satisfies, for all $t_1, \dots, t_{n-1} \in T^{\mathcal{D}}$, $d \in \mathcal{D}$:

$$\langle t_1, \dots, t_{n-1}, \bullet_d \rangle = B_d(t_{n-1} \dots t_1),$$

where $B_d(t_{n-1} \dots t_1)$ is the tree obtained by grafting the trees t_{n-1}, \dots, t_1 (in this order) on a common root decorated by d . For example, if $a, b, c, d \in \mathcal{D}$,

$$\langle \bullet_a, \downarrow_b^c, \bullet_d \rangle = {}^c\downarrow_b^a.$$

As a consequence, if A is a connected bidendriform bialgebra and if $(q_d)_{d \in \mathcal{D}}$ is a basis of $\text{Prim}_{\text{coDend}}(A)$, then a basis of $\text{Prim}_{\text{coAss}}(A)$ is given by $(p_t)_{t \in T^{\mathcal{D}}}$ defined inductively by:

$$\begin{cases} p_{\bullet_d} &= q_d, \\ p_{B_d^+(t_1 \dots t_n)} &= \langle p_{t_n}, \dots, p_{t_1}, q_d \rangle. \end{cases}$$

3 Recovering $Prim_{coDend}(\mathbf{FQSym})$ from $Prim_{coAss}(\mathbf{FQSym})$

For all $n \in \mathbb{N}^*$, we put:

$$\begin{cases} p_n &= \dim(Prim_{coAss}(\mathbf{FQSym})_n), \\ q_n &= \dim(Prim_{coDend}(\mathbf{FQSym})_n). \end{cases}$$

Proposition 7 For all $n \geq 2$, $q_n = (n-2)p_{n-1}$.

Proof. We put:

$$R(X) = \sum_{n=1}^{\infty} n!X^n, \quad P(X) = \sum_{n=1}^{\infty} p_n X^n, \quad Q(X) = \sum_{n=1}^{\infty} q_n X^n.$$

By theorem 2 and proposition 6:

$$P(X) = \frac{R(X)}{1+R(X)}, \quad Q(X) = \frac{R(X)}{(1+R(X))^2}.$$

Hence:

$$P'(X) = \frac{R'(X)}{(1+R(X))^2}.$$

Moreover:

$$\begin{aligned} R'(X) &= \sum_{n=1}^{\infty} nn!X^{n-1} \\ &= \sum_{n=1}^{\infty} (n+1)!X^{n-1} - \sum_{n=1}^{\infty} n!X^{n-1} \\ &= \frac{R(X) - X}{X^2} - \frac{R(X)}{X} \\ &= \frac{R(X) - X(1+R(X))}{X^2}. \end{aligned}$$

We deduce:

$$X^2 P'(X) = \frac{R(X) - X(1+R(X))}{(1+R(X))^2} = Q(X) - \frac{X}{1+R(X)} = Q(X) - X + XP(X).$$

So:

$$X^2 P'(X) + XP(X) = \sum_{n=1}^{\infty} (n-1)p_n X^{n+1} = Q(X) - X = \sum_{n=2}^{\infty} q_n X^n.$$

As a conclusion, for all $n \geq 2$, $q_n = (n-2)p_{n-2}$. □

Definition 8 Let $i \in \mathbb{N}^*$. We define $\Phi_i : \mathbf{FQSym} \rightarrow \mathbf{FQSym}$ in the following way: for all $n \in \mathbb{N}$, for all $\sigma = (\sigma_1, \dots, \sigma_n) \in S_n$,

$$\Phi_i(\mathbf{F}_\sigma) = \begin{cases} 0 & \text{if } i \geq n, \\ \mathbf{F}_{(\sigma_1, \dots, \sigma_i, n+1, \sigma_{i+1}, \dots, \sigma_n)} & \text{if } i < n. \end{cases}$$

Theorem 9 Let $n \geq 2$. The following application is bijective:

$$\Phi : \begin{cases} (Prim_{coAss}(\mathbf{FQSym})_{n-1})^{n-2} & \longrightarrow Prim_{coDend}(\mathbf{FQSym})_n \\ (p_1, \dots, p_{n-2}) & \longrightarrow \Phi_1(p_1) + \dots + \Phi_{n-2}(p_{n-2}). \end{cases}$$

Proof. *First step.* Let us first prove that Φ takes its values in $Prim_{coDend}(\mathbf{FQSym})$. Let $p \in Prim_{coAss}(\mathbf{FQSym})$ and $1 \leq i \leq n-2$. For all $k \in \mathbb{N}$, let π_k be the projection on \mathbf{FQSym}_k . By definition of Δ_{\prec} and Δ_{\succ} , for all $\sigma \in S_{n-1}$:

$$\begin{aligned}\Delta_{\prec}(\Phi_i(\mathbf{F}_\sigma)) &= \left(\sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(\mathbf{F}_\sigma), \\ \Delta_{\succ}(\Phi_i(\mathbf{F}_\sigma)) &= \left(\sum_{j=1}^i \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(\mathbf{F}_\sigma).\end{aligned}$$

By linearity, we obtain:

$$\begin{aligned}\Delta_{\prec}(p) &= \left(\sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(p) = 0, \\ \Delta_{\succ}(p) &= \left(\sum_{j=1}^i \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(p) = 0.\end{aligned}$$

This proves the first step.

Second step. We now prove that Φ is monic. Let $(p_1, \dots, p_{n-2}) \in Ker(\Phi)$. Let be $1 \leq i \leq n-2$. We define:

$$\varpi_i : \begin{cases} \mathbf{FQSym}_n & \longrightarrow \mathbf{FQSym}_n \\ \mathbf{F}_\sigma & \longrightarrow \begin{cases} 0 & \text{if } \sigma^{-1}(n) \neq i+1, \\ \mathbf{F}_\sigma & \text{if } \sigma^{-1}(n) = i+1. \end{cases} \end{cases}$$

Then, in an obvious way, $\varpi_i(\Phi(p_1, \dots, p_{n-2})) = \Phi_i(p_i) = 0$. As Φ_i is obviously monic on \mathbf{FQSym}_{n-1} (because $i \leq n-2$), $p_i = 0$. So Φ is monic.

Last step. As $dim\left((Prim_{coAss}(\mathbf{FQSym})_{n-1})^{n-2}\right) = dim(Prim_{coDend}(\mathbf{FQSym})_n)$, from proposition 7, Φ is bijective. \square

4 An inductive basis of $Prim_{coAss}(\mathbf{FQSym})$

We now combine results of the second and third sections to obtain an basis of the space $Prim_{coAss}(\mathbf{FQSym})$. We first define inductively some set of partially planar decorated trees $\mathbb{T}(n)$ in the following way:

1. $\mathbb{T}(0)$ is the set of non decorated planar trees. The weight of an element of $\mathbb{T}(0)$ is the number of its vertices.
2. Suppose that $\mathbb{T}(n)$ is defined. Then $\mathbb{T}(n+1)$ is the set of planar trees defined by :
 - (a) The elements of $\mathbb{T}(n+1)$ are partially decorated planar trees.
 - (b) The vertices of the elements of $\mathbb{T}(n+1)$ can eventually be decorated by a pair (t, k) , with $t \in \mathbb{T}(n)$ and k an integer in $\{1, \dots, weight(t) - 1\}$.
 - (c) The weight of an element of $\mathbb{T}(n)$ is the sum of the number of its vertices and of the weights of the trees of $\mathbb{T}(n)$ that appear in its decorations.

Inductively, for all $n \in \mathbb{N}$, $\mathbb{T}(n) \subseteq \mathbb{T}(n+1)$. We put $\mathbb{T} = \bigcup_{n \in \mathbb{N}} \mathbb{T}(n)$.

Examples.

1. Elements of \mathbb{T} of weight 1: \bullet .
2. Elements of \mathbb{T} of weight 2: \downarrow .
3. Elements of \mathbb{T} of weight 3: $\mathbb{V}, \downarrow\downarrow, \bullet(\downarrow,1)$.
4. Elements of \mathbb{T} of weight 4:

$$(a) \quad \mathbb{V}, \downarrow\mathbb{V}, \mathbb{V}\downarrow, \mathbb{V}\downarrow\downarrow,$$

$$(b) \quad \bullet(\mathbb{V},1), \bullet(\mathbb{V},2), \bullet(\downarrow,1), \bullet(\downarrow,2), \bullet(\bullet(\downarrow,1),1), \bullet(\bullet(\downarrow,1),2).$$

$$(c) \quad \downarrow(\downarrow,1), \downarrow(\downarrow,1).$$

We can then define a basis $(p_t)_{t \in \mathbb{T}}$ of $Prim_{coAss}(\mathbf{FQSym})$ inductively in the following way:

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. If $t = \bullet(t',i)$, then $p_t = \Phi_i(p_{t'})$.
3. If t is not a single root, let t_1, \dots, t_{n-1} be the children of its roots, from left to right, and t_n its root. Then $p_t = \langle p_{t_{n-1}}, \dots, p_{t_1}, p_{t_n} \rangle$.

By the preceding results:

Theorem 10 $(p_t)_{t \in \mathbb{T}}$ is a basis of $Prim_{coAss}(\mathbf{FQSym})$. A basis of $Prim_{coDend}(\mathbf{FQSym})$ is given by the p_t 's, where t is a single root.

Examples.

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. $p_{\downarrow} = -\mathbf{F}_{(21)} + \mathbf{F}_{(12)}$.
3. (a) $p_{\bullet(\downarrow,1)} = -\mathbf{F}_{(231)} + \mathbf{F}_{(132)}$.
 (b) $p_{\mathbb{V}} = \mathbf{F}_{(231)} - \mathbf{F}_{(132)} - \mathbf{F}_{(312)} + \mathbf{F}_{(213)}$.
 (c) $p_{\downarrow\downarrow} = \mathbf{F}_{(321)} - \mathbf{F}_{(231)} - \mathbf{F}_{(213)} + \mathbf{F}_{(123)}$.
4. (a) $p_{\bullet(\bullet(\downarrow,1),1)} = -\mathbf{F}_{(2431)} + \mathbf{F}_{(1432)}$.
 (b) $p_{\bullet(\bullet(\downarrow,1),2)} = -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)}$.
 (c) $p_{\bullet(\mathbb{V},1)} = \mathbf{F}_{(2431)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(3412)} + \mathbf{F}_{(2413)}$.
 (d) $p_{\bullet(\mathbb{V},2)} = \mathbf{F}_{(2341)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} + \mathbf{F}_{(2143)}$.
 (e) $p_{\bullet(\downarrow,1)} = \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2413)} + \mathbf{F}_{(1423)}$.
 (f) $p_{\bullet(\downarrow,2)} = \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} - \mathbf{F}_{(2143)} + \mathbf{F}_{(1243)}$.
 (g) $p_{\mathbb{V}\downarrow} = -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)} + \mathbf{F}_{(3142)} + \mathbf{F}_{(3412)} - \mathbf{F}_{(2143)} - \mathbf{F}_{(2413)} - \mathbf{F}_{(4213)} + \mathbf{F}_{(3214)}$.
 (h) $p_{\downarrow\mathbb{V}} = -\mathbf{F}_{(2431)} - \mathbf{F}_{(4231)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(3241)} + \mathbf{F}_{(1432)} + \mathbf{F}_{(4132)} + \mathbf{F}_{(4312)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(2314)}$.

$$\begin{aligned}
\text{(i)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= -\mathbf{F}_{(3241)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)} + \mathbf{F}_{(4213)} - \mathbf{F}_{(1243)} - \mathbf{F}_{(1423)} - \mathbf{F}_{(4123)} - \\
&\quad \mathbf{F}_{(2314)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(1324)} + \mathbf{F}_{(3124)}. \\
\text{(j)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= -\mathbf{F}_{(3421)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - \mathbf{F}_{(3241)} + \mathbf{F}_{(2314)} - \mathbf{F}_{(1324)} - \mathbf{F}_{(3124)} + \mathbf{F}_{(2134)}. \\
\text{(k)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= -\mathbf{F}_{(4321)} + \mathbf{F}_{(3421)} + \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} + \mathbf{F}_{(3214)} - \mathbf{F}_{(2314)} - \mathbf{F}_{(2134)} + \mathbf{F}_{(1234)}. \\
\text{(l)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= \mathbf{F}_{(2341)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - 2\mathbf{F}_{(1342)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(4132)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} + \\
&\quad \mathbf{F}_{(1243)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)}. \\
\text{(m)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2314)} + \mathbf{F}_{(1324)}.
\end{aligned}$$

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