



HAL
open science

Primitive elements of the Hopf algebra of free quasi-symmetric functions

Loïc Foissy

► **To cite this version:**

Loïc Foissy. Primitive elements of the Hopf algebra of free quasi-symmetric functions. Combinatorics and physics. Mini-workshop on renormalization, December 15–16, 2006, Conference on combinatorics and physics, March 19–23, 2007, Bonn, Germany., 2006, Bonn, Germany. hal-03623934

HAL Id: hal-03623934

<https://hal.science/hal-03623934>

Submitted on 1 Apr 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Primitive elements of the Hopf algebra of free quasi-symmetric functions

L. Foissy

*Laboratoire de Mathématiques - FRE3111, Université de Reims
Moulin de la Housse - BP 1039 - 51687 REIMS Cedex 2, France
e-mail : loic.foissy@univ-reims.fr*

ABSTRACT: Using the dendriform and the bidendriform Cartier-Quillen-Milnor-Moore theorem, we construct a basis of the space of primitive elements of the Hopf algebra of free quasi-symmetric functions, indexed by a certain set of trees, and inductively computable.

Contents

1	Bidendriform bialgebras and \mathbf{FQSym}	2
1.1	Bidendriform bialgebras	2
1.2	An example: the Hopf algebra \mathbf{FQSym}	3
2	Recovering $Prim_{coAss}(\mathbf{FQSym})$ from $Prim_{coDend}(\mathbf{FQSym})$	4
2.1	Dendriform Cartier-Quillen-Milnor-Moore theorem and variations	4
2.2	Free brace algebras	6
3	Recovering $Prim_{coDend}(\mathbf{FQSym})$ from $Prim_{coAss}(\mathbf{FQSym})$	7
4	An inductive basis of $Prim_{coAss}(\mathbf{FQSym})$	8

Introduction

The Hopf algebra of free quasi-symmetric functions \mathbf{FQSym} , also known as the Malvenuto-Reutenauer Hopf algebra, is introduced in [7]. It is a graded self-dual Hopf algebra, with the set of all permutations as a basis. Certain interesting properties are shown in [2]: in particular, it is shown that it is both free and cofree, and a basis of the space of its primitive elements is given, using the self-duality and a monomial basis. Note that computing the primitive elements of degree n by this method implies to inverse a certain $n! \times n!$ matrix.

The aim of this paper is to describe another basis of $Prim_{coAss}(\mathbf{FQSym})$, which can be inductively computed. We use for this the dendriform structure of \mathbf{FQSym} . Recall that a dendriform algebra is an associative algebra such that its product can be split into two non-associative products \prec and \succ , with good compatibilities [5, 6, 8]. It is known that \mathbf{FQSym} , or more precisely its augmentation ideal, is dendriform. More precisely, it is a dendriform Hopf algebra, in the sense of [8]. This implies, by the dendriform Cartier-Quillen-Milnor-Moore theorem, that $Prim_{coAss}(\mathbf{FQSym})$ is a brace algebra.

We introduced in [4] the notion of bidendriform bialgebra and showed that \mathbf{FQSym} is bidendriform. The bidendriform Cartier-Quillen-Milnor-Moore theorem implies that \mathbf{FQSym} is freely generated, as a dendriform algebra, by the space $Prim_{coDend}(\mathbf{FQSym})$ of primitive elements

in the codendriform sense. Combining this result with the dendriform Cartier-Quillen-Milnor-Moore theorem, we show that $Prim_{coAss}(\mathbf{FQSym})$ is, as a brace algebra, freely generated by $Prim_{coDend}(\mathbf{FQSym})$. We recall in section 2 a description of free brace algebras. If $(v_i)_{i \in I}$ is a basis of the vector space V , then the free brace algebra generated by V has a basis indexed by planar rooted trees decorated by I , and the brace structure is described in this basis by the help of graftings. Hence, for any basis of $Prim_{coDend}(\mathbf{FQSym})$, it is possible to recover a basis of $Prim_{coAss}(\mathbf{FQSym})$, indexed by a certain set of planar rooted trees.

Let, for all $n \in \mathbb{N}$:

$$\begin{cases} p_n &= \dim(Prim_{coAss}(\mathbf{FQSym})_n), \\ q_n &= \dim(Prim_{coDend}(\mathbf{FQSym})_n). \end{cases}$$

We then prove in section 3 that for $n \geq 2$, $q_n = (n-2)p_{n-1}$. We then give $n-2$ applications from $Prim_{coAss}(\mathbf{FQSym})_{n-1}$ to $Prim_{coDend}(\mathbf{FQSym})_n$, which give all elements of $Prim_{coDend}(\mathbf{FQSym})_n$. These applications are given by the insertion of $n+1$ at a given place in elements of the symmetric group S_n , seen as words in letters $1, \dots, n$.

Combining the results of the second and third sections, we define inductively in the fourth section a new basis of $Prim_{coAss}(\mathbf{FQSym})_n$, indexed by certain planar decorated rooted trees. The trees which are only a root give a basis of $Prim_{coDend}(\mathbf{FQSym})_n$.

Notations.

1. K is a commutative field of any characteristic.
2. If V is a K -vector field which is \mathbb{N} -graded, we shall denote by V_k the space of homogeneous elements of V of degree k .

1 Bidendriform bialgebras and \mathbf{FQSym}

1.1 Bidendriform bialgebras

We introduced in [4] the following definition:

Definition 1 A bidendriform bialgebra is a family $(A, \prec, \succ, \Delta_\prec, \Delta_\succ)$ such that:

1. A is a K -vector space and:

$$\begin{array}{l} \prec: \left\{ \begin{array}{l} A \otimes A \longrightarrow A \\ a \otimes b \longrightarrow a \prec b, \end{array} \right. \quad \left| \quad \Delta_\prec: \left\{ \begin{array}{l} A \longrightarrow A \otimes A \\ a \longrightarrow \Delta_\prec(a) = a'_\prec \otimes a''_\prec, \end{array} \right. \\ \succ: \left\{ \begin{array}{l} A \otimes A \longrightarrow A \\ a \otimes b \longrightarrow a \succ b, \end{array} \right. \quad \left| \quad \Delta_\succ: \left\{ \begin{array}{l} A \longrightarrow A \otimes A \\ a \longrightarrow \Delta_\succ(a) = a'_\succ \otimes a''_\succ. \end{array} \right. \end{array}$$

2. (Dendriform axioms). (A, \prec, \succ) is a dendriform algebra: for all $a, b, c \in A$,

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \quad (1)$$

$$(a \succ b) \prec c = a \succ (b \prec c), \quad (2)$$

$$5a \prec b + a \succ b \succ c = a \succ (b \succ c). \quad (3)$$

3. (Codendriform axioms). $(A, \Delta_\prec, \Delta_\succ)$ is a codendriform coalgebra: for all $a \in A$,

$$(\Delta_\prec \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec + Id \otimes \Delta_\succ) \circ \Delta_\prec(a), \quad (4)$$

$$(\Delta_\succ \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec) \circ \Delta_\succ(a), \quad (5)$$

$$(\Delta_\prec \otimes Id + \Delta_\succ \otimes Id) \circ \Delta_\succ(a) = (Id \otimes \Delta_\succ) \circ \Delta_\succ(a). \quad (6)$$

4. (Bidendriform axioms). For all $a, b \in A$,

$$\Delta_{\succ}(a \succ b) = a'b'_{\succ} \otimes a'' \succ b''_{\succ} + a' \otimes a'' \succ b + b'_{\succ} \otimes a \succ b''_{\succ} + ab'_{\succ} \otimes b''_{\succ} + a \otimes b, \quad (7)$$

$$\Delta_{\succ}(a \prec b) = a'b'_{\succ} \otimes a'' \prec b''_{\succ} + a' \otimes a'' \prec b + b'_{\succ} \otimes a \prec b''_{\succ}, \quad (8)$$

$$\Delta_{\prec}(a \succ b) = a'b'_{\prec} \otimes a'' \succ b''_{\prec} + ab'_{\prec} \otimes b''_{\prec} + b'_{\prec} \otimes a \succ b''_{\prec}, \quad (9)$$

$$\Delta_{\prec}(a \prec b) = a'b'_{\prec} \otimes a'' \prec b''_{\prec} + a'b \otimes a'' + b'_{\prec} \otimes a \prec b''_{\prec} + b \otimes a. \quad (10)$$

Remarks.

1. If A is a bidendriform bialgebra, then $K \oplus A$ is naturally a Hopf algebra, extending $\prec + \succ$ and $\Delta_{\prec} + \Delta_{\succ}$ on $K \oplus A$.
2. If A is a bidendriform bialgebra, it is also a dendriform hopf algebra in the sense of [8, 9], with coassociative coproduct given by $\tilde{\Delta} = \Delta_{\prec} + \Delta_{\succ}$. The compatibilities of dendriform Hopf algebras are given by (7) + (9) and (8) + (10).

If A is a bidendriform algebra, we define:

$$\text{Prim}_{\text{codend}}(A) = \text{Ker}(\Delta_{\prec}) \cap \text{Ker}(\Delta_{\succ}).$$

The following result is proved in [4] (theorem 35 and corollary 17):

Theorem 2 (Bidendriform Cartier-Quillen-Milnor-Moore theorem) *Let A be a \mathbb{N} -graded bidendriform bialgebra, such that $A_0 = (0)$. Then A is freely generated as a dendriform algebra by $\text{Prim}_{\text{coDend}}(A)$. Moreover, consider the following formal series:*

$$R(X) = \sum_{n=1}^{+\infty} \dim(A_n) X^n, \quad Q(X) = \sum_{n=1}^{+\infty} \dim(\text{Prim}_{\text{coDend}}(A_n)) X^n.$$

$$\text{Then } Q(X) = \frac{R(X)}{(R(X) + 1)^2}.$$

1.2 An example: the Hopf algebra \mathbf{FQSym}

See [1, 2, 7]. The algebra \mathbf{FQSym} is the vector space generated by the elements $(\mathbf{F}_u)_{u \in \mathbb{S}}$, where \mathbb{S} is the disjoint union of the symmetric groups S_n ($n \in \mathbb{N}$). Its product and its coproduct are given in the following way: for all $u \in S_n, v \in S_m$, putting $u = (u_1 \dots u_n)$,

$$\begin{aligned} \Delta(\mathbf{F}_u) &= \sum_{i=0}^n \mathbf{F}_{st(u_1 \dots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \dots u_n)}, \\ \mathbf{F}_u \cdot \mathbf{F}_v &= \sum_{\zeta \in sh(n, m)} \mathbf{F}_{(u \times v) \cdot \zeta^{-1}}, \end{aligned}$$

where $sh(n, m)$ is the set of (n, m) -shuffles, and st is the standardisation. Its unit is $1 = \mathbf{F}_{\emptyset}$, where \emptyset is the unique element of S_0 . Moreover, \mathbf{FQSym} is a \mathbb{N} -graded Hopf algebra, by putting $|\mathbf{F}_u| = n$ if $u \in S_n$.

Examples.

$$\begin{aligned} \mathbf{F}_{(12)} \mathbf{F}_{(123)} &= \mathbf{F}_{(12345)} + \mathbf{F}_{(13245)} + \mathbf{F}_{(13425)} + \mathbf{F}_{(13452)} + \mathbf{F}_{(31245)} \\ &\quad + \mathbf{F}_{(31425)} + \mathbf{F}_{(31452)} + \mathbf{F}_{(34125)} + \mathbf{F}_{(34152)} + \mathbf{F}_{(34512)}; \end{aligned}$$

$$\begin{aligned} \Delta(\mathbf{F}_{(12543)}) &= 1 \otimes \mathbf{F}_{(12543)} + \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1432)} + \mathbf{F}_{(12)} \otimes \mathbf{F}_{(321)} \\ &\quad + \mathbf{F}_{(123)} \otimes \mathbf{F}_{(21)} + \mathbf{F}_{(1243)} \otimes \mathbf{F}_{(1)} + \mathbf{F}_{(12543)} \otimes 1. \end{aligned}$$

Let $(\mathbf{FQSym})_+ = Vect(\mathbf{F}_u / u \in S_n, n \geq 1)$ be the augmentation ideal of \mathbf{FQSym} . We define $\prec, \succ, \Delta_\prec$ and Δ_\succ on $(\mathbf{FQSym})_+$ in the following way: for all $u \in S_n, v \in S_m$, by putting $u = (u_1 \dots u_n)$,

$$\begin{aligned} \mathbf{F}_u \prec \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m) \\ \zeta^{-1}(n+m)=n}} \mathbf{F}_{(u \times v) \cdot \zeta^{-1}}, \\ \mathbf{F}_u \succ \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m) \\ \zeta^{-1}(n+m)=n+m}} \mathbf{F}_{(u \times v) \cdot \zeta^{-1}}, \\ \Delta_\prec(\mathbf{F}_u) &= \sum_{i=u^{-1}(n)}^{n-1} \mathbf{F}_{st(u_1 \dots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \dots u_n)}, \\ \Delta_\succ(\mathbf{F}_u) &= \sum_{i=1}^{u^{-1}(n)-1} \mathbf{F}_{st(u_1 \dots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \dots u_n)}. \end{aligned}$$

Examples.

$$\begin{aligned} \mathbf{F}_{(12)} \prec \mathbf{F}_{(123)} &= \mathbf{F}_{(13452)} + \mathbf{F}_{(31452)} + \mathbf{F}_{(34152)} + \mathbf{F}_{(34512)}, \\ \mathbf{F}_{(12)} \succ \mathbf{F}_{(123)} &= \mathbf{F}_{(12345)} + \mathbf{F}_{(13245)} + \mathbf{F}_{(13425)} + \mathbf{F}_{(31245)} + \mathbf{F}_{(31425)} + \mathbf{F}_{(34125)}, \\ \Delta_\prec(\mathbf{F}_{(12543)}) &= \mathbf{F}_{(123)} \otimes \mathbf{F}_{(21)} + \mathbf{F}_{(1243)} \otimes \mathbf{F}_{(1)}, \\ \Delta_\succ(\mathbf{F}_{(12543)}) &= \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1432)} + \mathbf{F}_{(12)} \otimes \mathbf{F}_{(321)}. \end{aligned}$$

The following result is proved in [4], theorem 38:

Theorem 3 $((\mathbf{FQSym})_+, \prec, \succ, \Delta_\prec, \Delta_\succ)$ is a connected bidendriform bialgebra.

Moreover, $(\mathbf{FQSym})_+$ is \mathbb{N} -graded, by putting the elements of S_n homogeneous of degree n . By theorem 2, with $q_n = \dim(Prim_{coDend}(\mathbf{FQSym})_n)$, we obtain:

n	1	2	3	4	5	6	7	8	9	10	11	12
q_n	1	0	1	6	39	284	2 305	20 682	203 651	2 186 744	25 463 925	319 989 030

2 Recovering $Prim_{coAss}(\mathbf{FQSym})$ from $Prim_{coDend}(\mathbf{FQSym})$

2.1 Dendriform Cartier-Quillen-Milnor-Moore theorem and variations

Recall that a brace algebra is a K -vector space A together with a n -multilinear operation for all $n \geq 2$:

$$\langle \dots \rangle : \begin{cases} A^{\otimes n} & \longrightarrow A \\ a_1 \otimes \dots \otimes a_n & \longrightarrow \langle a_1, \dots, a_n \rangle, \end{cases}$$

satisfying certain relations; see [8, 9] for more details. For example:

$$\langle a_1, \langle a_2, a_3 \rangle \rangle = \langle a_1, a_2, a_3 \rangle + \langle \langle a_1, a_2 \rangle, a_3 \rangle + \langle a_2, a_1, a_3 \rangle.$$

The following theorem is proved in [8, 9]; more precisely, the first point of this theorem is proposition 2-8 and theorem 3-4 of [8] and the second point is theorem 4-6 of [9]:

Theorem 4 (Dendriform Cartier-Quillen-Milnor-Moore theorem) *Let A be a dendriform Hopf algebra. We denote $Prim_{coAss}(A) = Ker(\tilde{\Delta})$.*

1. $Prim_{coAss}(A)$ is a brace algebra, with brackets given by:

$$\langle p_1, \dots, p_n \rangle = \sum_{i=0}^{n-1} (-1)^{n-1-i} (p_1 \prec (p_2 \prec (\dots \prec p_i) \dots) \succ p_n \prec (\dots (p_{i+1} \succ p_{i+2}) \succ \dots) \succ p_{n-1}).$$

2. If A is freely generated as a dendriform algebra by a subvector space $V \subseteq Prim_{coAss}(A)$, then $Prim_{coAss}(A)$ is freely generated as a brace algebra by V .

Let us precise the relation between $Prim_{coAss}(A)$ and $Prim_{coDend}(A)$ if A is a bidendriform bialgebra. Combining the dendriform and the bidendriform Cartier-Quillen-Milnor-Moore theorems:

Theorem 5 *Let A be a \mathbb{N} -graded bidendriform bialgebra, with $A_0 = (0)$. Then $Prim_{coAss}(A)$ is, as a brace algebra, freely generated by $Prim_{coDend}(A)$.*

Proof. By the bidendriform Cartier-Quillen-Milnor-Moore theorem, A is freely generated as a dendriform algebra by $Prim_{coDend}(A)$. By the second point of the dendriform Cartier-Quillen-Milnor-Moore theorem, $Prim_{coAss}(A)$ is freely generated as a brace algebra by the space $Prim_{coDend}(A)$. \square

Proposition 6 *If A is \mathbb{N} -graded dendriform Hopf algebra, such that $A_0 = (0)$, then A is generated as a dendriform algebra by $Prim_{coAss}(A)$. Moreover, consider the following formal series:*

$$R(X) = \sum_{n=1}^{\infty} \dim(A_n) X^n, \quad P(X) = \sum_{n=1}^{+\infty} \dim(Prim_{coDend}(A)_n) X^n.$$

$$\text{Then } P(X) = \frac{R(X)}{1 + R(X)}.$$

Proof. *First step.* Let $p_1, \dots, p_n \in Prim_{coAss}(A)$. We define by induction on n :

$$\omega(p_1, \dots, p_n) = \begin{cases} p_1 & \text{if } n = 1, \\ p_n \prec \omega(p_1, \dots, p_{n-1}) & \text{if } n \geq 2. \end{cases}$$

An easy induction on n allows to show the following result, using (8)+(10):

$$\tilde{\Delta}(\omega(p_1, \dots, p_n)) = \sum_{i=1}^{n-1} \omega(p_1, \dots, p_i) \otimes \omega(p_{i+1}, \dots, p_n).$$

We denote by $\tilde{\Delta}^n : A \rightarrow A^{n+1}$ the iterated coproducts of A . It comes by induction:

$$\tilde{\Delta}^m(\omega(p_1, \dots, p_n)) = \begin{cases} 0 & \text{if } m \geq n, \\ p_1 \otimes \dots \otimes p_n & \text{if } m = n - 1. \end{cases}$$

Second step. We consider the tensor (non counitary) coalgebra $C = T(Prim_{coAss}(A))$:

$$C = \bigoplus_{n=1}^{\infty} Prim_{coAss}(A)^{\otimes n}.$$

It is a coalgebra for the deconcatenation coproduct. As $Prim_{coAss}(A)$ is \mathbb{N} -graded, C is a graded coalgebra with formal series:

$$S(X) = \frac{1}{1 - P(X)} - 1 = \frac{P(X)}{1 - P(X)}.$$

By the first step, the following application is a morphism of graded coalgebras:

$$\Psi : \begin{cases} C & \longrightarrow A \\ p_1 \otimes \dots \otimes p_n & \longrightarrow \omega(p_1, \dots, p_n). \end{cases}$$

Third step. Suppose that $\text{Ker}(\Psi)$ is non zero. As it is a coideal of C , it contains primitive elements of C , that is to say elements of $\text{Prim}_{\text{coAss}}(A)$. As Ψ is obviously monic on $\text{Prim}_{\text{coAss}}(A)$, this is impossible. So $\text{Ker}(\Psi) = (0)$ and Ψ is monic.

Let $a \in A$. As $A_0 = (0)$, for a certain $N(a) \in \mathbb{N}^*$, $\tilde{\Delta}^{N(a)}(a) = 0$. We prove that $a \in \text{Im}(\Psi)$ by induction on $N(a)$. If $N(a) = 1$, then $a \in \text{Prim}_{\text{coAss}}(A)$ and the result is obvious. Suppose that the result is true for all $b \in A$ such that $N(b) < N(a)$. As $\tilde{\Delta}^{N(a)}(a) = 0$, necessarily $\tilde{\Delta}^{N(a)-1}(a) \in \text{Prim}_{\text{coAss}}(A)^{\otimes N(a)}$. We put:

$$\tilde{\Delta}^{N(a)-1}(a) = a_1 \otimes \dots \otimes a_n, \quad b = a - \omega(a_1, \dots, a_n).$$

By the first step, $\tilde{\Delta}^{N(a)-1}(b) = 0$, so $N(b) < N(a)$. By the induction hypothesis, $b \in \text{Im}(\Psi)$. As $\omega(a_1, \dots, a_n) \in \text{Im}(\Psi)$, $a \in \text{Im}(\Psi)$.

Last step. As Ψ is an isomorphism of graded coalgebras, $S(X) = R(X)$. Hence:

$$R(X) = \frac{P(X)}{1 - P(X)},$$

$$\text{so } R(X) - R(X)P(X) = P(X) \text{ and } P(X) = \frac{R(X)}{1 + R(X)}. \quad \square$$

2.2 Free brace algebras

Using a description of the free dendriform algebra generated by a set \mathcal{D} with planar decorated forests, we gave a description of the free brace algebra $\text{Brace}(\mathcal{D})$ in [3]. A basis of this brace algebra is given by the set $T^{\mathcal{D}}$ of planar rooted trees decorated by \mathcal{D} . For example:

$$\begin{aligned} \text{Brace}(\mathcal{D})_1 &= \text{Vect}(\bullet_a, a \in \mathcal{D}), \\ \text{Brace}(\mathcal{D})_2 &= \text{Vect}(\uparrow_a^b, a, b \in \mathcal{D}), \\ \text{Brace}(\mathcal{D})_3 &= \text{Vect}({}^c\mathbb{V}_a^b, \downarrow_a^c, a, b, c \in \mathcal{D}), \\ \text{Brace}(\mathcal{D})_4 &= \text{Vect}({}^d\mathbb{V}_a^c, {}^d\mathbb{V}_a^b, {}^d\mathbb{V}_a^c, {}^d\mathbb{V}_a^c, \downarrow_a^d, a, b, c, d \in \mathcal{D}), \dots \end{aligned}$$

The brace bracket satisfies, for all $t_1, \dots, t_{n-1} \in T^{\mathcal{D}}$, $d \in \mathcal{D}$:

$$\langle t_1, \dots, t_{n-1}, \bullet_d \rangle = B_d(t_{n-1} \dots t_1),$$

where $B_d(t_{n-1} \dots t_1)$ is the tree obtained by grafting the trees t_{n-1}, \dots, t_1 (in this order) on a common root decorated by d . For example, if $a, b, c, d \in \mathcal{D}$,

$$\langle \bullet_a, \downarrow_b^c, \bullet_d \rangle = {}^c\downarrow_b^a.$$

As a consequence, if A is a connected bidendriform bialgebra and if $(q_d)_{d \in \mathcal{D}}$ is a basis of $\text{Prim}_{\text{coDend}}(A)$, then a basis of $\text{Prim}_{\text{coAss}}(A)$ is given by $(p_t)_{t \in T^{\mathcal{D}}}$ defined inductively by:

$$\begin{cases} p_{\bullet_d} &= q_d, \\ p_{B_d^+(t_1 \dots t_n)} &= \langle p_{t_n}, \dots, p_{t_1}, q_d \rangle. \end{cases}$$

3 Recovering $Prim_{coDend}(\mathbf{FQSym})$ from $Prim_{coAss}(\mathbf{FQSym})$

For all $n \in \mathbb{N}^*$, we put:

$$\begin{cases} p_n &= \dim(Prim_{coAss}(\mathbf{FQSym})_n), \\ q_n &= \dim(Prim_{coDend}(\mathbf{FQSym})_n). \end{cases}$$

Proposition 7 For all $n \geq 2$, $q_n = (n-2)p_{n-1}$.

Proof. We put:

$$R(X) = \sum_{n=1}^{\infty} n!X^n, \quad P(X) = \sum_{n=1}^{\infty} p_n X^n, \quad Q(X) = \sum_{n=1}^{\infty} q_n X^n.$$

By theorem 2 and proposition 6:

$$P(X) = \frac{R(X)}{1+R(X)}, \quad Q(X) = \frac{R(X)}{(1+R(X))^2}.$$

Hence:

$$P'(X) = \frac{R'(X)}{(1+R(X))^2}.$$

Moreover:

$$\begin{aligned} R'(X) &= \sum_{n=1}^{\infty} nn!X^{n-1} \\ &= \sum_{n=1}^{\infty} (n+1)!X^{n-1} - \sum_{n=1}^{\infty} n!X^{n-1} \\ &= \frac{R(X) - X}{X^2} - \frac{R(X)}{X} \\ &= \frac{R(X) - X(1+R(X))}{X^2}. \end{aligned}$$

We deduce:

$$X^2 P'(X) = \frac{R(X) - X(1+R(X))}{(1+R(X))^2} = Q(X) - \frac{X}{1+R(X)} = Q(X) - X + XP(X).$$

So:

$$X^2 P'(X) + XP(X) = \sum_{n=1}^{\infty} (n-1)p_n X^{n+1} = Q(X) - X = \sum_{n=2}^{\infty} q_n X^n.$$

As a conclusion, for all $n \geq 2$, $q_n = (n-2)p_{n-2}$. □

Definition 8 Let $i \in \mathbb{N}^*$. We define $\Phi_i : \mathbf{FQSym} \rightarrow \mathbf{FQSym}$ in the following way: for all $n \in \mathbb{N}$, for all $\sigma = (\sigma_1, \dots, \sigma_n) \in S_n$,

$$\Phi_i(\mathbf{F}_\sigma) = \begin{cases} 0 & \text{if } i \geq n, \\ \mathbf{F}_{(\sigma_1, \dots, \sigma_i, n+1, \sigma_{i+1}, \dots, \sigma_n)} & \text{if } i < n. \end{cases}$$

Theorem 9 Let $n \geq 2$. The following application is bijective:

$$\Phi : \begin{cases} (Prim_{coAss}(\mathbf{FQSym})_{n-1})^{n-2} & \longrightarrow Prim_{coDend}(\mathbf{FQSym})_n \\ (p_1, \dots, p_{n-2}) & \longrightarrow \Phi_1(p_1) + \dots + \Phi_{n-2}(p_{n-2}). \end{cases}$$

Proof. *First step.* Let us first prove that Φ takes its values in $Prim_{coDend}(\mathbf{FQSym})$. Let $p \in Prim_{coAss}(\mathbf{FQSym})$ and $1 \leq i \leq n-2$. For all $k \in \mathbb{N}$, let π_k be the projection on \mathbf{FQSym}_k . By definition of Δ_{\prec} and Δ_{\succ} , for all $\sigma \in S_{n-1}$:

$$\begin{aligned}\Delta_{\prec}(\Phi_i(\mathbf{F}_\sigma)) &= \left(\sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(\mathbf{F}_\sigma), \\ \Delta_{\succ}(\Phi_i(\mathbf{F}_\sigma)) &= \left(\sum_{j=1}^i \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(\mathbf{F}_\sigma).\end{aligned}$$

By linearity, we obtain:

$$\begin{aligned}\Delta_{\prec}(p) &= \left(\sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(p) = 0, \\ \Delta_{\succ}(p) &= \left(\sum_{j=1}^i \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(p) = 0.\end{aligned}$$

This proves the first step.

Second step. We now prove that Φ is monic. Let $(p_1, \dots, p_{n-2}) \in Ker(\Phi)$. Let be $1 \leq i \leq n-2$. We define:

$$\varpi_i : \begin{cases} \mathbf{FQSym}_n & \longrightarrow \mathbf{FQSym}_n \\ \mathbf{F}_\sigma & \longrightarrow \begin{cases} 0 & \text{if } \sigma^{-1}(n) \neq i+1, \\ \mathbf{F}_\sigma & \text{if } \sigma^{-1}(n) = i+1. \end{cases} \end{cases}$$

Then, in an obvious way, $\varpi_i(\Phi(p_1, \dots, p_{n-2})) = \Phi_i(p_i) = 0$. As Φ_i is obviously monic on \mathbf{FQSym}_{n-1} (because $i \leq n-2$), $p_i = 0$. So Φ is monic.

Last step. As $dim\left((Prim_{coAss}(\mathbf{FQSym})_{n-1})^{n-2}\right) = dim(Prim_{coDend}(\mathbf{FQSym})_n)$, from proposition 7, Φ is bijective. \square

4 An inductive basis of $Prim_{coAss}(\mathbf{FQSym})$

We now combine results of the second and third sections to obtain an basis of the space $Prim_{coAss}(\mathbf{FQSym})$. We first define inductively some set of partially planar decorated trees $\mathbb{T}(n)$ in the following way:

1. $\mathbb{T}(0)$ is the set of non decorated planar trees. The weight of an element of $\mathbb{T}(0)$ is the number of its vertices.
2. Suppose that $\mathbb{T}(n)$ is defined. Then $\mathbb{T}(n+1)$ is the set of planar trees defined by :
 - (a) The elements of $\mathbb{T}(n+1)$ are partially decorated planar trees.
 - (b) The vertices of the elements of $\mathbb{T}(n+1)$ can eventually be decorated by a pair (t, k) , with $t \in \mathbb{T}(n)$ and k an integer in $\{1, \dots, weight(t) - 1\}$.
 - (c) The weight of an element of $\mathbb{T}(n)$ is the sum of the number of its vertices and of the weights of the trees of $\mathbb{T}(n)$ that appear in its decorations.

Inductively, for all $n \in \mathbb{N}$, $\mathbb{T}(n) \subseteq \mathbb{T}(n+1)$. We put $\mathbb{T} = \bigcup_{n \in \mathbb{N}} \mathbb{T}(n)$.

Examples.

1. Elements of \mathbb{T} of weight 1: \bullet .
2. Elements of \mathbb{T} of weight 2: \downarrow .
3. Elements of \mathbb{T} of weight 3: $\mathbb{V}, \downarrow\downarrow, \bullet(\downarrow,1)$.
4. Elements of \mathbb{T} of weight 4:

$$(a) \quad \mathbb{V}, \downarrow\mathbb{V}, \mathbb{V}\downarrow, \mathbb{V}\downarrow\downarrow,$$

$$(b) \quad \bullet(\mathbb{V},1), \bullet(\mathbb{V},2), \bullet(\downarrow,1), \bullet(\downarrow,2), \bullet(\bullet(\downarrow,1),1), \bullet(\bullet(\downarrow,1),2).$$

$$(c) \quad \downarrow(\downarrow,1), \downarrow(\downarrow,1).$$

We can then define a basis $(p_t)_{t \in \mathbb{T}}$ of $Prim_{coAss}(\mathbf{FQSym})$ inductively in the following way:

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. If $t = \bullet(t',i)$, then $p_t = \Phi_i(p_{t'})$.
3. If t is not a single root, let t_1, \dots, t_{n-1} be the children of its roots, from left to right, and t_n its root. Then $p_t = \langle p_{t_{n-1}}, \dots, p_{t_1}, p_{t_n} \rangle$.

By the preceding results:

Theorem 10 $(p_t)_{t \in \mathbb{T}}$ is a basis of $Prim_{coAss}(\mathbf{FQSym})$. A basis of $Prim_{coDend}(\mathbf{FQSym})$ is given by the p_t 's, where t is a single root.

Examples.

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. $p_{\downarrow} = -\mathbf{F}_{(21)} + \mathbf{F}_{(12)}$.
3. (a) $p_{\bullet(\downarrow,1)} = -\mathbf{F}_{(231)} + \mathbf{F}_{(132)}$.
 (b) $p_{\mathbb{V}} = \mathbf{F}_{(231)} - \mathbf{F}_{(132)} - \mathbf{F}_{(312)} + \mathbf{F}_{(213)}$.
 (c) $p_{\downarrow\downarrow} = \mathbf{F}_{(321)} - \mathbf{F}_{(231)} - \mathbf{F}_{(213)} + \mathbf{F}_{(123)}$.
4. (a) $p_{\bullet(\bullet(\downarrow,1),1)} = -\mathbf{F}_{(2431)} + \mathbf{F}_{(1432)}$.
 (b) $p_{\bullet(\bullet(\downarrow,1),2)} = -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)}$.
 (c) $p_{\bullet(\mathbb{V},1)} = \mathbf{F}_{(2431)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(3412)} + \mathbf{F}_{(2413)}$.
 (d) $p_{\bullet(\mathbb{V},2)} = \mathbf{F}_{(2341)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} + \mathbf{F}_{(2143)}$.
 (e) $p_{\bullet(\downarrow,1)} = \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2413)} + \mathbf{F}_{(1423)}$.
 (f) $p_{\bullet(\downarrow,2)} = \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} - \mathbf{F}_{(2143)} + \mathbf{F}_{(1243)}$.
 (g) $p_{\mathbb{V}\downarrow} = -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)} + \mathbf{F}_{(3142)} + \mathbf{F}_{(3412)} - \mathbf{F}_{(2143)} - \mathbf{F}_{(2413)} - \mathbf{F}_{(4213)} + \mathbf{F}_{(3214)}$.
 (h) $p_{\downarrow\mathbb{V}} = -\mathbf{F}_{(2431)} - \mathbf{F}_{(4231)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(3241)} + \mathbf{F}_{(1432)} + \mathbf{F}_{(4132)} + \mathbf{F}_{(4312)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(2314)}$.

$$\begin{aligned}
\text{(i)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= -\mathbf{F}_{(3241)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)} + \mathbf{F}_{(4213)} - \mathbf{F}_{(1243)} - \mathbf{F}_{(1423)} - \mathbf{F}_{(4123)} - \\
&\quad \mathbf{F}_{(2314)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(1324)} + \mathbf{F}_{(3124)}. \\
\text{(j)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= -\mathbf{F}_{(3421)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - \mathbf{F}_{(3241)} + \mathbf{F}_{(2314)} - \mathbf{F}_{(1324)} - \mathbf{F}_{(3124)} + \mathbf{F}_{(2134)}. \\
\text{(k)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= -\mathbf{F}_{(4321)} + \mathbf{F}_{(3421)} + \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} + \mathbf{F}_{(3214)} - \mathbf{F}_{(2314)} - \mathbf{F}_{(2134)} + \mathbf{F}_{(1234)}. \\
\text{(l)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= \mathbf{F}_{(2341)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - 2\mathbf{F}_{(1342)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(4132)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} + \\
&\quad \mathbf{F}_{(1243)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)}. \\
\text{(m)} \quad p \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} &= \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2314)} + \mathbf{F}_{(1324)}.
\end{aligned}$$

References

- [1] Marcelo Aguiar and Frank Sottile, *Structure of the Malvenuto-Reutenauer Hopf algebra of permutations*, Adv. Math. **191** (2005), no. 2, 225–275, math.CO/0203282.
- [2] Gérard Duchamp, Florent Hivert, and Jean-Yves Thibon, *Some generalizations of quasi-symmetric functions and noncommutative symmetric functions*, Springer, Berlin, 2000, math.CO/0105065.
- [3] Loïc Foissy, *Les algèbres de Hopf des arbres enracinés, II*, Bull. Sci. Math. **126** (2002), 249–288.
- [4] ———, *Bidendriform bialgebras, trees, and quasi-symmetric functions*, J. Pure Appl. Algebra **209** (2007), 429–459.
- [5] Jean-Louis Loday, *Dialgebras*, Lecture Notes in Math., no. 1763, Springer, Berlin, 2001, math.QA/0102053.
- [6] Jean-Louis Loday and Maria O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139** (1998), no. 2, 293–309.
- [7] Claudia Malvenuto and Christophe Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982.
- [8] Maria O. Ronco, *Primitive elements of a free dendriform algebra*, Contemp. Math. **267** (2000), 245–263.
- [9] ———, *Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras*, J. Algebra **254** (2002), no. 1, 152–172.