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# Realizing finitely presented groups as projective fundamental groups of SFTs 

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#### Abstract

Subshifts are sets of colourings - or tilings - of the plane, defined by local constraints. Historically introduced as discretizations of continuous dynamical systems, they are also heavily related to computability theory. In this article, we study a conjugacy invariant for subshifts, known as the projective fundamental group and we show that any finitely presented group can be realized as a projective fundamental group of some SFT.

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## 1 Introduction

A d-dimensional subshift is a set of colourings of $\mathbb{Z}^{d}$ by a finite number of colours which avoids some family of forbidden patterns. If the family is finite, it is called a subshift of finite type (SFT). Subshifts can also be defined topologically: they are the closed shift-invariant subsets of $\Sigma^{\mathbb{Z}^{d}}$, where $\Sigma$ is the finite set of colours. Most problems concerning subshifts in dimension $d \geq 2$ are undecidable $[3,12,11]$, which is due to the fact that sets of Wang tilings are SFTs.

Together with the shift action $\sigma$, a subshift forms a dynamical system. Interesting dynamical aspects are usually invariant by conjugacy, which is the isomorphism notion for subshifts: two subshifts are conjugate if there exists a continuous shift-commuting bijection between them. As a matter of fact, most dynamical aspects of subshifts are linked to computability theory or complexity theory in dimensions $d \geq 2$. These links were first explored when the possible values of the topological entropy of multi-dimensional SFTs was characterized as the upper semi-computable numbers [16], afterwards many other conjugacy invariants followed: growth-type invariants [21], periodicity data [18, 17], subactions [14, 2, 8] and so on.

Recently, the links between subshifts and groups have seen a surge in study [1, 19, 22], however, the most well known group associated with a subshift, its automorphism group, is still not very well understood: while we know that SFTs with positive entropy have very complex automorphism groups [15] or that SFTs whose automorphism group has undecidable word problem can be constructed [10], we still do not know whether the automorphism groups of the full shifts on 2 and 3 symbols are the same. Apart from the low complexity setting [7,6] not much is understood about it.

Here, we contribute to the study of another group-related conjugacy invariant called the
projective fundamental group introduced by Geller and Propp [9]. In the classical setting (see for example [13]), the fundamental group $\pi_{1}(S)$ of a surface $S$ is a topological invariant which describes the number of holes and the general shape of $S$. It is constructed using loops on the surface : it is the group of the equivalence classes of loops through continuous deformation together with the composition operation. In the case of subshifts, as they are not surfaces in the classical sense, the definition is not as straightforward but is still based on loops, their composition and their deformations as will be seen in more detail in Subsection 3.1.

The projective fundamental group of a subshift is well defined only in the case of projectively connected subshifts, a property that resembles mixing properties: properties which essentially state that if two patterns appear in different configurations they may appear in the same configuration (see for instance [5] or [24] for more precise definitions of mixing properties in dimensions $d \geq 2$ ). As a matter of fact, projective connectedness is implied by all mixing properties defined in [5].

As a conjugacy invariant, the fundamental group allows to distinguish between some subshifts which share the same entropy and periodicity data. It is also better understood than the automorphism group in the sense that the authors in [9] explicitly compute it for several well-know subshifts: the full shifts on any alphabet always have trivial fundamental group, the square-ice has $\mathbb{Z}$ and factors of full shifts always have a fundamental group with finite order. They also prove that any group of finite order is realizable as a fundamental group of some factor of a full shift.

The main result of this article is that any finitely presented group is the fundamental group of an SFT:

- Theorem 1. Let $G=\langle S \mid R\rangle$ a finitely presented group. Then, there is a subshift of finite type $X$ verifying:
- $X$ is projectively connected,
- the projective fundamental group of $X$ is isomorphic to $G$.

We do not think this is a characterization. As a matter of fact, we do not know how hard the word problem is for the projective fundamental group of SFTs apart from the fact that it is at least $\Sigma_{1}^{0}$-hard since finitely presented groups have $\Sigma_{1}^{0}$-complete word problem [23, 4]. The same is true for projective connectedness, we know it is undecidable but do not know how hard. It is open whether any of these properties is in the arithmetical hierarchy.

The paper is organized as follows. After recalling the symbolic dynamics background in Section 2, we introduce the projective fundamental group in Subsection 3.1, some examples in Subsection 3.2 and finally in Section 4 we prove Theorem 1.

## 2 Definitions

A d-dimensional full shift on some finite alphabet $\Sigma$ is the set $\Sigma^{\mathbb{Z}^{d}}$, together with the shift-action $\sigma_{u}: \Sigma^{\mathbb{Z}^{d}} \rightarrow \Sigma^{\mathbb{Z}^{d}}$ defined by $\sigma_{\mathbf{u}}(x)(\mathbf{v})=x(\mathbf{u}+\mathbf{v})=x_{\mathbf{u}+\mathbf{v}}$, an element of a full shift is called a configuration. A subshift is a closed, shift-invariant subset of some full shift. We call points of a subshift $X$ the configurations belonging to $X$. The underlying topology is the one induced by the Cantor distance, defined on $\Sigma^{\mathbb{Z}^{d}}$ by

$$
d(x, y)=2^{-\min \left\{\|\mathbf{u}\|_{\infty} \mid x_{\mathbf{u}} \neq y_{\mathbf{u}}\right\}}
$$

two configurations are close in this topology if they agree on a large central square.
Alternatively, subshifts can be defined using forbidden patterns. We call pattern any element $P \in \Sigma^{U}$ where $U \subset \mathbb{Z}^{d}$ is finite and is the support of $P$. For a configuration $x$, we
say that $P$ appears in $x$ if there exists $\mathbf{u} \in \mathbb{Z}^{d}$ such that $\sigma_{\mathbf{u}}(x)_{\mid U}=P$. Let $\mathcal{F}$ be a collection (finite or not) of patterns. Then the set

$$
X_{\mathcal{F}}=\left\{x \in \Sigma^{\mathbb{Z}^{d}} \mid \forall P \in \mathcal{F}, P \text { does not appear in } x\right\}
$$

is a subshift. In fact, for any subshift (defined as a closed, shift-invariant set) $X$, there exists a family of patterns $\mathcal{F}$ such that $X=X_{\mathcal{F}}$. A subshift $X$ is a subshift of finite type (SFT) if there exists a finite $\mathcal{F}$ such that $X=X_{\mathcal{F}}$.

For a given subshift $X$, a pattern $P \in \Sigma^{U}$ is locally admissible if it contains no forbidden patterns. It is globally admissible or extensible if it appears in some configuration $x \in X$.

SFT can also be defined using Wang tiles. Let $C$ be a finite set of colours. A set of Wang tiles $\mathcal{T}$ is a finite set of tiles, which are mapping $\tau:\{\mathrm{t}, \mathrm{b}, \mathrm{l}, \mathrm{r}\} \rightarrow C$. A Wang tiling by $\mathcal{T}$ is a mapping $w: \mathbb{Z}^{2} \rightarrow \mathcal{T}$ which verifies for all $(i, j) \in \mathbb{Z}^{2}$,

- $w(i, j)(\mathrm{r})=w(i+1, j)(\mathrm{l})$
- $w(i, j)(\mathrm{t})=w(i, j+1)(\mathrm{b})$

The set $W(\mathcal{T})=\{w, w$ is a wang tiling by $\mathcal{T}\}$ is a SFT.

## 3 Projective Fundamental Group

### 3.1 Intuitions and definitions

The Projective Fundamental Group, introduced by Geller and Propp [9] resembles the usual fundamental group construction of the topological setting: it is defined through paths, loops, and, more generally, a homotopy notion. However, instead of directly considering paths between points of the subshift, they are defined between finite patterns with the same support. By doing so, one actually constructs a family of fundamental groups, potentially different for each finite support $B \subset \mathbb{Z}^{2}$. In order to obtain a single group, the projective fundamental group, one takes their inverse (also known as projective) limit. We will construct a subshift by defining a set $\mathcal{T}$ of Wang tiles. A configuration will then be a mapping $x: \mathbb{Z}^{2} \rightarrow \mathcal{T}$ associating a tile to each point of the plane and which verifies some adjacency rules depending on $\mathcal{T}$. Contrary to the usual convention, we will consider than when embedding such a configuration in the euclidean plane $\mathbb{R}^{2}$, the tile in position $(i, j)$ is a unit square whose bottom-left corner is placed on $(i, j)$, as opposed to its center. This is merely a discussion about conventions, but it will make some definitions substantially simpler.

Fix a support $B \subset \mathbb{Z}^{2}$, in what follows $B$ will be called an aperture window. Most of the time, we will restrict ourselves to the windows $B_{n}=\llbracket-n, n-1 \rrbracket^{2}$. We choose this asymmetrical window to simplify some definitions, but also for consistency with the aforementioned convention. Indeed, when embedded into $\mathbb{R}^{2}, B_{n}$ is a square of side $2 n$, whose center is the point $(0,0) \in \mathbb{R}^{2}$. In any configuration $x$, the tile $x_{(0,0)}$ in position $(0,0)$ will therefore be seen as the square whose bottom-left (resp. top-right) corner is $(0,0)$ (resp. $(1,1))$.

Consider $P, P^{\prime}$ two extensible patterns of support $B$ and two points of the grid $\mathbf{v}, \mathbf{v}^{\prime} \in \mathbb{Z}^{2}$. A path between $(P, \mathbf{v})$ and $\left(P^{\prime}, \mathbf{v}^{\prime}\right)$ is a sequence of pairs of vectors and patterns (or equivalently, two sequences of the same length). The first one represents an actual, "geometric" path, called its trajectory, that is to say a sequence of 4 -adjacent vertices of $\mathbb{Z}^{2}$ starting at $\mathbf{v}$ and ending at $\mathbf{v}^{\prime}$. The second associates with each one of those vertices $\mathbf{v}_{\mathbf{t}}$ a pattern $P_{t}$, that needs to be coherent with the path: when moving to the next vertex $\mathbf{v}_{\mathbf{t}+\boldsymbol{1}}$ on the trajectory, the next pattern $P_{t+1}$ needs to be coherent with $P_{t}$. For example, in the full shift,
and for $B=B_{1}$, take the following pairs:

$$
P_{1}=\left(\begin{array}{|cc|}
\hline 0 & 0 \\
0 & 0 \\
\hline
\end{array},(0,0)\right), \quad P_{2}=\left(\begin{array}{|cc|}
\hline 0 & 1 \\
0 & 1
\end{array},(1,0)\right), \quad P_{3}=\left(\begin{array}{|cc|}
\hline 1 & 1 \\
1 & 1
\end{array},(1,0)\right)
$$

The tile in position $(0,0)$ is represented in red. The sequence $\left(P_{1},(0,0)\right),\left(P_{2},(1,0)\right)$ is a valid path, as the overlapping parts of the support are equal in both patterns, but $\left(P_{1},(0,0)\right),\left(P_{3},(0,0)\right)$ is not because e.g., the point $(1,0)$ is tiled by 0 in the first pattern but by 1 in the second one. Moreover, the pattern obtained by "merging" two consecutive patterns also needs to be an extensible pattern.

- Definition 2 (Path). Let $B \subset \mathbb{Z}^{2}$ a finite set, a path of aperture window $B$ is a finite sequence $\left(P_{t}, \mathbf{v}_{\mathbf{t}}\right)_{0 \leq t \leq N}$ of pairs from $\Sigma^{B} \times \mathbb{Z}^{2}$, such that for any $t$ with $0 \leq t \leq N$ :
- $P_{t}$ is an extensible pattern of $X$,
- $\mathbf{v}_{\mathbf{t}}$ is adjacent to $\mathbf{v}_{\mathbf{t}+\mathbf{1}}$, i.e., they are at distance exactly 1 ,
- $P_{t}(u)=P_{t+1}(u)$ for any $u \in\left(B+\mathbf{v}_{\mathbf{t}}\right) \cap\left(B+\mathbf{v}_{\mathbf{t}+\mathbf{1}}\right)$, i.e., consecutive patterns overlap,
- the pattern $P_{t} \cup P_{t+1}$ obtained by merging $P_{t}$ and $P_{t+1}$ is extensible in $X$.

The first and last element of the sequence are respectively called the starting point and the ending point of the path. If they are equal, the path is called a loop. The path $\left(P_{N-t}, \mathbf{v}_{\mathbf{N}-\mathbf{t}}\right)_{0 \leq t \leq N}$ is called its inverse path. If $p$ is a path, its inverse will be denoted by $p^{-1}$

The sequence $\left(\mathbf{v}_{\mathbf{t}}\right)_{0 \leq t \leq N}$ is called the trajectory of the path.
When one path ends where another path starts, they may be composed:

- Definition 3 (Path composition). Given $p=\left(P_{t}, \mathbf{v}_{\mathbf{t}}\right)_{0 \leq t \leq N}$ and $p^{\prime}=\left(P_{t}^{\prime}, \mathbf{v}_{\mathbf{t}}^{\prime}\right)_{0 \leq t \leq N^{\prime}}$ two paths such that $\left(P_{N}, \mathbf{v}_{\mathbf{N}}\right)=\left(P_{0}^{\prime}, \mathbf{v}_{\mathbf{0}}^{\prime}\right)$ we note $p * p^{\prime}$ the path

$$
p * p^{\prime}=\left(P_{0}, \mathbf{v}_{\mathbf{0}}\right) \ldots\left(P_{N}, \mathbf{v}_{\mathbf{N}}\right)\left(P_{1}^{\prime}, \mathbf{v}_{\mathbf{1}}^{\prime}\right) \ldots\left(P_{N^{\prime}}^{\prime}, \mathbf{v}_{\mathbf{N}^{\prime}}^{\prime}\right)
$$

If there exists a point $x \in X$ such that for all $t, x_{\mid B+\mathbf{v}_{\mathbf{t}}}=P_{t}$, we say that $p$ can be traced in the configuration $x$. If a path $p=\left(P_{t}, \mathbf{v}_{\mathbf{t}}\right)_{0 \leq t \leq n}$ is defined with an aperture window $B$, then for all $B^{\prime} \subset B$, we refer to the path $p=\left(P_{t \mid B^{\prime}}, \mathbf{v}_{\mathbf{t}}\right)_{0 \leq t \leq n}$ as the restriction of $p$ to the window $B^{\prime}$.

One can now define a corresponding homotopy notion, using what we call an elementary deformation: let $p=p_{1} * p_{2} * p_{3}$ be a path and suppose that $p_{2}$ can be traced in a single configuration $x \in X$. Then, for any $p_{2}^{\prime}$ traced in $x$ with the same starting and ending point as $p_{2}$, one can deform $p$ into $p_{1} * p_{2}^{\prime} * p_{3}$. As paths might consist of a single point, they can be deformed by inserting or removing loops traced in a single configuration at any step.

- Definition 4 (Homotopy). Two paths $p, p^{\prime}$ are said to be homotopic if there exists a finite sequence of elementary deformations from $p$ to $p^{\prime}$. This defines an equivalence relation between paths, and we denote by $[p]$ the equivalence class of $p$. If $p$ and $p^{\prime}$ are paths with an aperture window $B \subset \mathbb{Z}^{2}$, we denote by $p \sim_{B} p^{\prime}$ the fact that they are homotopic.
- Remark 5. When two paths are homotopic, they necessarily have the same starting and ending points. When $B$ is clear from the context, we will simply note $p \sim p^{\prime}$

With this definition of a path and of homotopy, we can define a fundamental group for each possible aperture window $B \subset \mathbb{Z}^{2}$.

- Definition 6 (Fundamental Group). Let $B \subset \mathbb{Z}^{2}$ be an aperture window, $x_{0} \in X$ and $\mathbf{v} \in \mathbb{Z}^{2}$. The fundamental group of $X$ based at $\left(x_{0}, \mathbf{v}\right)$ for the aperture window $B$, denoted by $\pi_{1}^{B}\left(X,\left(x_{0}, \mathbf{v}\right)\right)$, is the group of all the equivalence classes of loops starting and ending at $\left(x_{0 \mid B}, \mathbf{v}\right)$ for the homotopy equivalence relation, along with the $*$ operation.

Although our paths follow the $\mathbb{Z}^{2}$ grid and seem to be discrete and combinatorial objects, it is legitimate to refer to those objects as homotopy and deformations, which usually suppose some kind of continuity. In fact, this simplification does not entail any loss of generality, compared to paths drawn in $\mathbb{R}^{2}$, and subshifts seen as $\mathbb{Z}^{2}$-invariants subsets of $\Sigma^{\mathbb{R}^{2}}$. For more details, one can refer to [9, Straightening Lemma]. In order to obtain a single object associated with the subshift, we need to get rid of this reference to an aperture window. Geller and Propp consider the projective limit of those groups to define what they refer to as the Projective Fundamental Group of the subshift.

- Definition 7. We define an equivalence relation on $X \times \mathbb{Z}^{2}$, denoted by $\equiv_{B}$, as follows:

$$
(x, \mathbf{v}) \equiv_{B}\left(x^{\prime}, \mathbf{v}^{\prime}\right) \Longleftrightarrow \mathbf{v}=\mathbf{v}^{\prime} \text { and } x_{\mid B+\mathbf{v}}=x_{\mid B+\mathbf{v}}^{\prime} \quad \text { with }(x, \mathbf{v}),\left(x^{\prime}, \mathbf{v}^{\prime}\right) \in X \times \mathbb{Z}^{2}
$$

We denote $X_{B}$ the quotient space $\left(X \times \mathbb{Z}^{2}\right) / \equiv_{B}$. In order to simplify the notation, we will sometimes use $\left(x_{\mid B_{n}+\mathbf{v}}, \mathbf{v}\right)$ to denote the equivalence class of $(x, \mathbf{v})$ for the relation $\equiv_{B_{n}}$.
Notice that with this definition, a path is now an element of $X_{B}^{<\mathbb{N}}$ for some $B \subset \mathbb{Z}^{2}$, which also verifies some additional constraints. One can then consider the fundamental groups of all those spaces, and take their inverse limit to obtain this subshift's projective fundamental group.
$\rightarrow$ Definition 8 (Projective Limit). Let $(\mathcal{I}, \leq)$ a partially ordered set, and let $\left(G_{i}\right)_{i \in \mathcal{I}}$ a family of groups. Suppose that we have a family of homomorphisms $f_{i j}: G_{j} \rightarrow G_{i}$ for all $i \leq j \in \mathcal{I}$, which satisfies the two following properties:

- For all $i \in \mathcal{I}, f_{i, i}=\operatorname{id}_{G_{i}}$
- For all $i \leq j \leq k \in \mathcal{I}, f_{i, k}=f_{i, j} \circ f_{j, k}$

Then, the projective limit (sometimes also called inverse limit) of the projective system $\left(\left(G_{i}\right)_{i \in \mathcal{I}},\left(f_{i, j}\right)_{i \leq j \in \mathcal{I}}\right)$ is a subgroup of the (potentially infinite) direct product of all the $G_{i}$ 's, denoted here by $G_{\infty}$

$$
G_{\infty}=\lim _{i \in \mathcal{I}} G_{i}=\left\{\vec{g} \in \prod_{i \in \mathcal{I}} G_{i} \mid g_{i}=f_{i, j}\left(g_{j}\right) \text { for all } i \leq j \text { in } \mathcal{I}\right\}
$$

- Definition 9 (Projectively connected subshift). A subshift $X$ is projectively connected if for any two points $x, x^{\prime} \in X$, there exists an inverse system of paths $\left(p_{n}\right)_{n \in \mathbb{N}}$, such that $p_{n}$ is a path between $\left(x_{B_{n}},(0,0)\right) \in X_{B_{n}}$ and $\left(x_{B_{n}}^{\prime},(0,0)\right)$, and for each $n>0$, the image of $p_{n}$ under the canonical restriction map to $X_{B_{n-1}}$ is homotopic to $p_{n-1}$.

Let $X_{\infty}$ be the inverse limit of the system $\left(X_{B}\right)_{B \subset \mathbb{Z}^{2}}$ along with the canonical restriction maps $\left(x_{\mid B+\mathbf{v}}, v\right) \in X_{B} \mapsto\left(x_{\mid B^{\prime}+\mathbf{v}}, v\right)$ for each $B^{\prime} \subseteq B \subset \mathbb{Z}^{2}$.

- Lemma 10 (Sequence Lemma [9]). $X_{\infty}$ is the inverse limit of the system $\left(X_{B_{n}}\right)_{n \in \mathbb{N}}$.
- Definition 11 (Projective Fundamental Group). The projective fundamental group based on the point $\left(x_{0}, \mathbf{v}\right) \in X \times \mathbb{Z}^{2}$ of a subshift $X$ is the inverse limit of $\left(\pi_{1}^{B}\left(X,\left(x_{0}, \mathbf{v}\right)\right)\right)_{B \subset \mathbb{Z}^{2}}$, and is denoted by $\pi_{1}^{p r o j}\left(X,\left(x_{0}, \mathbf{v}\right)\right)$. If $X$ is projectively connected, then its projective fundamental group does not depend on the chosen basepoint $\left(x_{0}, \mathbf{v}\right)$, and we denote it simply by $\pi_{1}^{p r o j}(X)$.

Definition 12 (Projective path-class). A projective path class is a family $\left(\left[p_{n}\right]\right)_{n \in \mathbb{N}^{*}}$, where each $p_{n}$ is a path of aperture window $B_{n}$, such that for any $n \leq n^{\prime}$, the restriction of the path $p_{n^{\prime}}$ to the window $B_{n}$ is homotopic to $p_{n}$. Elements of $\pi_{1}^{\text {proj }}(X)$ can then naturally be called projective loop-classes.

### 3.2 First example

We slightly modify an example of [9]. Consider the two-dimensional (sofic) subshift on the alphabet $\{0,1\}$ of all the configurations containing at most one 1 . We show how some paths can be deformed to the trivial path. It is then easy to show that all paths are homotopic to the trivial path. Take an aperture window of size 1, i.e., only one cell is visible at a time. Consider the following path $p$, starting at $(0,(0,0))$ (we see a 0 at the origin of the $\mathbb{Z}^{2}$ plane). The path then moves in the $\mathbb{Z}^{2}$ grid while only seeing 0 's, and comes back to the origin where it now sees a 1 . Then it moves away from the origin while only seeing 0 's, and finally comes back to $(0,0)$ with a 0 in the window. For simplicity, we also suppose that the path does not pass through the origin at any other time. To sum up, the path is a loop, starting and ending at $(0,(0,0))$, which only sees 0 along the way except at one time ( $t_{2}$ on the figure) where it sees a 1 at the origin. This is illustrated in Figure 1a.

(a) Example of a path that cannot be traced in a single configuration.

(b) A homotopic deformation to a path that can entirely be traced in the all-0 configuration.

Figure 1 Example of a path and of a deformation of this path. Notice that the central 0 and 1 windows at $t_{0}$ and $t_{2}$ are actually located at the same point of the plane, although the figure depicts them on top of each other for the sake of clarity. Red wires can be traced in $x_{0}$, and blue wires in $x_{1}$. The wire of alternating colours can be traced within both, and so it is both homotopic to the initial path, and to the trivial path.

Let $x_{0}, x_{1}$ respectively be the all-zero configuration, and the configuration containing a 1 at the origin. The path $p$ can be homotopically deformed in the following way: between the times $t_{1}$ and $t_{3}$, it can be considered to be entirely in $x_{1}$. It can thus be deformed in this configuration by completely avoiding the origin, and joining the same points, as in Figure 1b. By definition of $x_{1}$, this new path will now see only 0 's. The resulting loop then also sees 0 's at any point, and so it can be homotopically contracted to the trivial path in the configuration $x_{0}$. This proof can be extended to make any 1 on a path "disappear" and show how any path can be contracted. In this case, it is sufficient to show that the projective fundamental group based at $x_{0}$ of this subshift is trivial.

## 4 Realization of projective fundamental groups

We are now going to prove our main result: any finitely presented group is the fundamental projective group of some SFT.

Theorem 1. Let $G=\langle S \mid R\rangle$ a finitely presented group. Then, there is a subshift of finite type $X$ verifying:

- $X$ is projectively connected,
- the projective fundamental group of $X$ is isomorphic to $G$.


### 4.1 The construction

The subshift $X$ that we construct will informally consist of oriented wires, drawn on an empty background, each wire corresponding to a generator $\mathrm{s} \in S$ of the group $G=\langle S \mid R\rangle$.

We only authorize the wires to go up, perhaps in some kind of "zigzag" manner, but never down or horizontally. More precisely, we define the following tiles: first of all, a tile that we call empty, visually represented by $\square$ and denoted by $\mathcal{T}_{\text {empty }}$. We denote by $x_{\square} \in X$ the configuration which only contains empty tiles, and its patterns are called empty patterns. Then, for each element $\mathrm{s} \in \bar{S}=S \cup\left\{\mathrm{~s}^{-1} \mid \mathrm{s} \in S\right\}$, we also consider the set $\mathcal{T}_{\mathrm{s}}$ of the 5 following tiles:


Notice that if $\mathrm{s} \neq \mathrm{s}^{\prime}$, then $\mathcal{T}_{\mathrm{s}} \cap \mathcal{T}_{\mathrm{s}^{\prime}}=\emptyset$. Distinct $T_{\mathrm{s}}$ will be represented by wires of different colours in the figures. Those tiles will, intuitively, be used to represent generators of the group in valid configurations of $X$. In order to have some kind of representation for the relations, we need to use additional tiles. We start by adding all the trivial relators $\mathrm{ss}^{-1}$ and $\mathrm{s}^{-1} \mathrm{~s}$ to $R$ for all $\mathrm{s} \in S$. Now, for each relator $\mathrm{r}_{1} \mathrm{r}_{2} \ldots \mathrm{r}_{\mathrm{n}} \in R$, we add the tiles of Figure 2 to our set of tiles.


Figure 2 The relation tiles.

Notice that the wire exiting from the right side of the tile Figure 2a does not have the same colour as the one exiting from the top. This colour is denoted by $\overline{r_{1}}$, to differentiate it from the actual $r_{0}$ wires. In the other tiles, $\overline{R_{i}}=\overline{r_{1} r_{2}} \ldots \overline{r_{i}}$. Hence, for each relator $r_{1} \ldots r_{n}$, we have one tile of type Figure 2a and one of type Figure 2c, and $n-2$ tiles of type Figure 2b. Let $\mathcal{T}_{r}$ be this set of $n$ tiles. Note that we do not introduce any tile of the form Figure 2d, and also notice that if $\mathrm{u} \in R$ is such that it is the prefix of two different relators, i.e.,, there exists $\mathrm{v}, \mathrm{v}^{\prime} \in \bar{S}^{*}$ such that $\mathrm{uv} \in R, \mathrm{uv}^{\prime} \in R$ then the colours $\bar{u}$ are shared by the tiles used to represent those relators and so $\mathcal{T}_{\mathrm{uv}} \cap \mathcal{T}$ uv $^{\prime} \neq \emptyset . X$ is the subshift generated by the tileset $\mathcal{T}=\mathcal{T}_{\text {empty }} \cup \bigcup_{\mathrm{s} \in \bar{S}} \mathcal{T}_{\mathrm{s}} \bigcup_{\mathrm{r} \in R} \mathcal{T}_{\mathrm{r}}$ along with the obvious adjacency rules.

We now formalize what we really mean by a wire.

- Definition 13 (Wire). A wire is a sequence $\mathcal{U}=\left(T_{t}, \mathbf{v}_{\mathbf{t}}\right)_{t \in I}, I \subseteq \mathbb{Z}$ an interval, of pair of non-empty tiles and $\mathbb{Z}^{2}$ vectors, such that
- $\left\|\mathbf{v}_{\mathbf{t}+\mathbf{1}}-\mathbf{v}_{\mathbf{t}}\right\|_{1}=1$,
- The tile $T_{t+1}$ in position $\mathbf{v}_{\mathbf{t}+\mathbf{1}}$ extends the tile $T_{t}$ in position $\mathbf{v}_{\mathbf{t}}$.

We also require that $\mathcal{U}$ does not contain two consecutive relation tiles.

- Definition 14 (Coherent wire). We say that a wire is coherent if there exists a configuration $x \in X$ such that for any tile $\left(T_{i}, \mathbf{v}_{\mathbf{i}}\right)$ of the wire, $x_{\mathbf{v}_{\mathbf{i}}}=T_{i}$.

Valid configurations of $X$ contain non-intersecting infinite wires, and some relation tiles with wires originating from them. Any relation tile belongs to a horizontal line of $r$ relation tiles corresponding to a valid relator $r_{1} \ldots r_{r}$.

One important concept associated to paths on this subshift is the idea that paths can cross wires. Informally, this is what happens when the window, and its particular, its center, moves from one side to the other of a given wire in a path. As the tiles are considered to have their corners on the $\mathbb{Z}^{2}$ lattice, the central point of any pattern of support some translation of $B_{n}$ is never directly "on" a wire.

- Definition 15 (Crossing a wire tile). Let $n>0$, and let $\mathbf{v}, \mathbf{v}^{\prime} \in \mathbb{Z}^{2}$ be two adjacent points, and $P, P^{\prime}$ two patterns of respective support $\mathbf{v}+B_{n}, \mathbf{v}^{\prime}+B_{n}$ such that $(P, \mathbf{v}),\left(P^{\prime}, \mathbf{v}^{\prime}\right)$ is a valid path. For $(i, j) \in B_{n}$, let $T_{(i, j)}$ be the tile whose bottom-left corner is on $(i, j)$ in $P$. We say that this path crosses a wire tile if
- $\mathbf{v}^{\prime}-\mathbf{v}=e_{0}=(1,0)\left(\right.$ resp. $\left.-e_{0}\right)$ and the tile $T_{\mathbf{v}}$ (resp. $T_{\mathbf{v}-e_{0}}$ ) was of one of the following form:
- $\mathbf{v}^{\prime}-\mathbf{v}=e_{1}=(0,1)\left(\right.$ resp. $\left.-e_{1}\right)$ at the next step $t+1$ and the tile $T_{\mathbf{v}}\left(\right.$ resp. $\left.T_{\mathbf{v}-e_{1}}\right)$ was of one of the following form:

In the following, we let $B_{n}=\{-n, \ldots, n-1\}^{2}$. Unless stated otherwise, all the aperture windows considered will be of this form.

- Definition 16 (Coherent path). A path $p=\left(P_{i}, \mathbf{v}_{\mathbf{i}}\right)_{i \leq N}$ is coherent if all its patterns are equal on the points where their support overlap, and furthermore, the pattern obtained by merging all the $P_{i}$ is globally admissible in $X$.
- Definition 17 (Seeing a wire). A path $p=\left(P_{i}, \mathbf{t}_{\mathbf{i}}\right)_{i \leq N}$ sees a wire $\mathcal{U}$ if there exists a timestep $i \leq N$, and $\left(T_{j}, \mathbf{v}_{\mathbf{j}}\right) \in r$ such that the tile in position $\mathbf{v}_{\mathbf{j}}$ in $P_{i}$ is $T_{j}$.
- Remark 18. In a finite aperture window, some tiles belonging to a same wire need not be adjacent. This is the case exactly when the wires "leaves" the window on its right or left edge, and re-enters it higher on this same edge.
- Remark 19. If a path $p$ sees one relation tile, because such a tile is necessarily part of larger pattern involving other relation tiles and other wires in any point of $X$, we also say that $p$ sees those extra wires.
- Definition 20 (Crossing a wire). A path crosses a wire if crosses one of its tiles.
- Definition 21 (Coherent path decomposition). A coherent decomposition of a path $p$ is a sequence $p_{1}, \ldots, p_{D}$ of coherent paths such that $p=p_{1} * p_{2} \ldots * p_{L}$, and $L$ is called the length of the decomposition.


### 4.2 Only Crossed Wires Matter

Our final goal is to prove that the projective fundamental group of this subshift $X$ is the group $G=\langle S \mid R\rangle$. To do so, the idea will be to associate an element of the group to each path, according to the wires that it crosses. The following lemmas are used, in some sense, to prove that the only thing that determines the homotopy class of a path is indeed the sequence of wires that it crosses, regardless of the underlying geometry of the path. All the lemmas consider paths that both start and end in empty patterns. This is not really a restriction, as we will later prove that the subshift $X$ is projectively connected, and so we will only consider loops based on $x_{\square}$. Unless stated otherwise, all the considered paths are using $B_{n}$ as an aperture window, and patterns have support $B_{n}$.

- Lemma 22 (Wire Order Lemma). Let $x \in X$, and let $\mathcal{U}, \mathcal{V}$ be two wires in $x$. Suppose that $\mathcal{U}, \mathcal{V}$ do not contain relation tiles.
- For all $z \in \mathbb{Z}$, there exists between one and two $z_{\mathcal{U}}^{0} \in \mathbb{Z}$ such that $\mathcal{U}$ passes through the position $\left(z_{\mathcal{U}}^{0}, z\right)$. If there are two such $z_{\mathcal{U}}^{0}$, then they are necessarily adjacent, e.g., $\square$ side-by-side.
- Let $z^{1} \in \mathbb{Z}$, and $z_{\mathcal{U}}^{0}, z_{\mathcal{V}}^{0} \in \mathbb{Z}$ as in the previous point respectively for $\mathcal{U}$ and $\mathcal{V}$. If $z_{\mathcal{U}}^{0}<z_{\mathcal{V}}^{0}$, then for all $z_{\mathcal{U}}, z_{\mathcal{V}}, z \in \mathbb{Z}$ such that $\left(z_{\mathcal{U}}, z\right) \in \mathcal{U},\left(z_{\mathcal{V}}, z\right) \in \mathcal{V}$, we have $z_{\mathcal{U}}<z_{\mathcal{V}}$. Intuitively, this means that wires can globally be ordered from left to right.
If $\mathcal{U}$ or $\mathcal{V}$ contains a relation tile, then the previous claims are true only for $z^{1}$ large enough.
- Lemma 23. Let $P$ be a globally admissible pattern. Let $\mathcal{U}$ be a wire in $P$ without relation tiles. Suppose that $\mathcal{U}$ passes to the right of $(0,0)$ in $P$. Then, $\mathcal{U}$ neither enters nor exits $P$ on its left edge.

Proof. This directly follows from the fact that all the tiles with a wire have it move up when moving left, and $B_{n}$ is a square.

- Corollary 24. If $P$ is a globally admissible pattern that sees a wire $\mathcal{U}$ with no relation tiles, and $x \in X$ is such that $x_{\mid B_{n}}=P$, then $\sigma^{4 n}(x)_{\mid B_{n}}$ and $\sigma^{-4 n}(x)_{\mid B_{n}}$ do not see $\mathcal{U}$.
- Lemma 25 (No Relation Tile Lemma). Let $p$ a coherent path starting and ending on an empty pattern. Then there exists $p^{\prime} \sim p$ that does not see any relation tile.

Proof of the No Relation Tile Lemma. Let $x$ a configuration in which $p$ can be traced, and which does not contain any other wire than the ones seen by $p$. Let $\left(P_{N}, \mathbf{v}_{\mathbf{N}}\right)$ the final point of $p$. Up to a translation of both $p$ and $x$ we can always assume that $p$ starts at $(0,0)$, and without loss of generality, suppose that $\mathbf{v}_{\mathbf{N}}$ is on the right, i.e., it has a non-negative x -coordinate. This is a legitimate assumption, up to considering the path $p^{-1}$ instead of $p$, which also starts and ends with empty patterns. Deform $p$ into a path $p^{\prime}$ in $x$, whose trajectory only consists of moving right, and then up or down, depending on whether $\mathbf{v}_{\mathbf{N}}$ is above or below $(0,0)$. Let $i_{\min }$ (resp. $i_{\max }$ ) be the leftmost (resp. righmost) position of a relation tile seen by $p$, and let $j$ be the topmost one. We can deform $p^{\prime}$ as follows:

- Move left until the position $i_{\min }-2 n$ (or don't move if $i_{\min }-n \geq 0$ ).
- Move up until the position $j+2 n$
- Move right until $i_{\max }+2 n$
- Finally, move to $\mathbf{v}_{\mathbf{N}}$, by moving vertically first and then horizontally.

Let $p^{\prime \prime}$ the resulting path. Then, $p^{\prime \prime}$ does not see any relation tile. Figure 3 shows this process in a simple case, with the first and third steps being trivial.

The next two lemmas are the main tools needed in the proof of the theorem. Informally, they show that the only thing that matters on a path is the set of wires that it crosses, and that we can moreover consider all those wires independently from one another.

- Lemma 26 (Single Wire Lemma). Let $p=\left(P_{i}, \mathbf{v}_{\mathbf{i}}\right)_{0 \leq i \leq N}$ be a path starting and ending with empty patterns. There exists a path $p^{\prime}$, homotopic to $p$, such that the union of any two consecutive patterns in $p^{\prime}$ contains at most a single wire.

Lemma 27 (No Uncrossed Wire Lemma). Let $p$ be a path starting and ending with empty patterns, and $\mathcal{U}$ some wire seen but not crossed by $p$. There exists a path $p^{\prime}$, homotopic to $p$, which does not see $\mathcal{U}$.


Figure 3 A path that can be traced in a single configuration can always be deformed so as not to see relation tiles

The proof of the No Uncrossed Wire Lemma can be found in Appendix A.
Proof of Single Wire Lemma. The result is also proved by induction on the length of a path decomposition of $p$.

Base case: $\mathbf{L}=1 \quad p$ can be traced entirely in a configuration $x \in X$. Using the No Relation Tile Lemma, we may assume that $p$ does not see any relation tile. Without loss of generality, we may assume that $x$ does not contain any wire that is not seen by $p$ and that $p$ starts at $(0,0)$ and ends at $\mathbf{v}_{\mathbf{N}}=\left(v_{N}^{0}, v_{N}^{1}\right)$, with $v_{N}^{0} \geq 0, v_{N}^{1} \geq 0$. We can assume that $p$ does not see any relation tile using the No Relation Tile Lemma. The Wire Order Lemma ensures that each wire is crossed at most once. For simplicity, we assume that the trajectory is a straight line, from $(0,0)$ to some $\mathbf{v}_{\mathbf{N}} \in \mathbb{Z}^{2}$, with $v_{N}^{0}>0, v_{N}^{1}=0$. Let $\mathcal{U}_{0}, \ldots, \mathcal{U}_{k}$ be the wires seen from right to left by $p$ (so $p$ sees $\mathcal{U}_{k}$ first, then $\mathcal{U}_{k-1}$ and so on until $\mathcal{U}_{0}$ ).

Now consider a configuration $x^{\prime}$ verifying (see Figure 4):

- $x^{\prime}$ does not contain any other wire than the $\mathcal{U}_{i}$ 's
- for $0 \leq i \leq k$, let $\left(z_{i},-n\right)$ the position of the only tile of $\mathcal{U}_{i}$ whose second coordinate is $-n$, and whose wire enters it from its bottom edge. Then, for $-n-4 i k \leq z \leq-n$, we define $x^{\prime}\left(z_{i}, z\right)$ to be a tile of the form $\mathbb{W}$, and all the tiles of $\mathcal{U}_{i}$ below that are of the form $\boxed{\square}$ and . This uniquely determines all the $\mathcal{U}_{i}$ 's below $p$.
For $z \in \mathbb{Z}$, no pattern of support $B_{n}$ centered at $(z,-4 n(k+1))$ can see tiles belonging to two different wires at the same time in $x^{\prime}$. Therefore, we can deform $p$ in $x^{\prime}$ into $p^{\prime}$, where $p^{\prime}$ starts by moving down for $4 n(k+1)$ steps, then right until crossing $\mathcal{U}_{0}$, and finally up and either right or left as needed to reach $\mathbf{v}_{\mathbf{N}}$. Any two consecutive patterns on this path see at most one wire.


## Base case: $\mathbf{L}=2 \quad p=p_{1} * p_{2}$

Using the same notation, let $\left(P_{t}, \mathbf{v}_{\mathbf{t}}\right)$ the endpoint of $p_{1}$ and the starting point of $p_{2}$, with $\mathbf{v}_{\mathbf{t}}=\left(v_{t}^{0}, v_{t}^{1}\right)$. Without loss of generality, we assume $v_{t}^{0} \geq 0, v_{t}^{1} \geq 0$. Let $\mathbf{v}_{N}$ be the $\mathbb{Z}^{2}$ point at which $p$ ends - by assumption, the associated pattern $P_{N}$ is only made of empty tiles.


Figure 4 Deformation of $p$ into $p^{\prime}$ in a single configuration to see only one wire per pattern.

Let $x_{1}, x_{2} \in X$ two configurations such that $p_{1}, p_{2}$ can respectively be traced entirely within them. Consider the path obtained by deforming $p_{1}$ in $x_{1}$, so that its trajectory is following the same one as in the previous case $L=1$ : it starts from $(0,0)$, then goes right until $\left(v_{t}^{0}, 0\right)$, and finally up until $\mathbf{v}_{\mathbf{t}}$. To lighten the notation, we still call this path $p_{1}$, and in the rest of the proof $p_{1}$ will refer to this deformed path.

Consider the loop $q=q_{1} * q_{1}^{-1}$ in $x_{2}$ that starts from $P_{t}, \mathbf{v}_{\mathbf{t}}$, and which follows the inverse trajectory to $p_{1}$, reaches $(0,0)$, continues to the left until seeing an empty pattern, and then comes back by the inverse path. Let $p_{1}^{\prime}=p_{1} * q_{1}$ and let $p_{2}^{\prime}=q_{1}^{-1} * p_{2}$, so that $p=p_{1}^{\prime} * p_{2}^{\prime}$. By construction, $p_{2}^{\prime}$ can be traced entirely within $x_{2}$, and so can be appropriately deformed according to the case $L=1$. Like $p, p_{1}^{\prime}$ has a decomposition of length 2, but we can further simplify it. Indeed, we show that there exists a loop $r=r_{1} * r_{1}^{-1}$, based on $\left(P_{t}, \mathbf{v}_{\mathbf{t}}\right)$, such that $r_{1}$ ends in an empty pattern and each of $p_{1} * r_{1}$ and $r_{1}^{-1} * q_{1}^{-1}$ can be traced within a single configuration. This is enough to prove the case $L=2$, using three times the case $L=1$. To prove the existence of such a loop $r$, we construct a part of the upper-half of some configuration $x_{\text {cone }}$ as follows:

- $x_{\text {cone }}{\mid \mathbf{v}_{\mathbf{t}}+B_{n}}=P_{t}$
- all the wires exiting $P_{t}$ on its left half, i.e., on a tile placed on $\left(v^{0}, v^{1}\right)$ with $v^{0}<v_{t}^{0}$, are then only made of tiles shaped as $\boxed{\square}$ and $\boxed{6}$.
- all the wires exiting $P_{t}$ on its right half, i.e., on a tile placed on ( $v^{0}, v^{0}$ ) with $v^{0} \geq v_{t}^{0}$, are then only made of tiles shaped as $\square$ and $\vartheta$
- $x_{\text {cone }}$ does not contain any other wire than the ones seen in $P_{t}$
- Let $\mathcal{U}_{\text {left }}, \mathcal{U}_{\text {right }}$ respectively be the leftmost and rightmost wires of $P_{t}$. The tiles located to the left of $\mathcal{U}_{\text {left }}$, to the right of $\mathcal{U}_{\text {right }}$, and below $v_{t}^{1}-n$, are not defined in $x_{\text {cone }}$.
$x_{\text {cone }}$ is then an infinite pattern, whose shape is somewhat similar to a cone.
Let $x_{1}^{\prime}$ be the configuration obtained by extending $x_{\text {cone }}$ as follows:
- If $(i, j)$ belongs to $x_{\text {cone }}$, then $x_{1}^{\prime}(i, j)=x_{\text {cone }}(i, j)$.
- If $j \leq v_{t}^{1}-n$, then for any $i, x_{1}^{\prime}(i, j)=x_{1}=(i, j)$.
- All the wires on the left of $\mathcal{U}_{\text {left }}$ in $x_{1}$ are extended in $x_{1}^{\prime}$ above the $v_{t}^{1}-n$ line by only using tiles of the shape $\square^{\square}$ and
- All the wires on the right of $\mathcal{U}_{\text {right }}$ in $x_{1}$ are extended in $x_{1}^{\prime}$ above the $v_{t}^{1}-n$ line by only using tiles of the shape $\square$ and $\boxtimes$.
- All the tiles that are not already defined in $x_{1}^{\prime}$ are empty.

Using Wire Order Lemma, $x_{1^{\prime}}$ is well-defined and belongs to $X$. We define $x_{2}^{\prime}$ in the same way. Let $r_{1}$ be the path obtained by following the trajectory starting from $\mathbf{v}_{\mathbf{t}}$, and moving up for $4 n$ timesteps, up to $\left(v_{t}^{0}, v_{t}^{1}+4 n\right)$ in $x_{1}^{\prime}$. The loop $r=r_{1} * r_{1}^{-1}$ then extends both $p_{1}^{\prime}$ and $q_{1}^{-1}$. Indeed, $p_{1}^{\prime} * r_{1}$ is a coherent path, as it can entirely be traced in $x_{1}^{\prime}$, and $r_{1}^{-1} * q_{1}^{-1}$ can be traced in $x_{2}^{\prime}$. Finally, we have that

$$
p \sim_{B_{n}} \underbrace{p_{1} * r_{1}}_{\text {traced in } x_{1}^{\prime}} * \underbrace{r_{1}^{-1} * q_{1}}_{\text {traced in } x_{2}^{\prime}} * \underbrace{q_{1}^{-1} * p_{2}}_{\text {traced in } x_{2}}
$$

General case: L > $2 \quad p=p_{1} * \ldots * p_{L}$.
Consider the timestep $t$ at which $p_{1}$ ends and $p_{2}$ starts. By definition of a coherent decomposition, there exists $x_{2} \in X$ such that $p_{2}$ can be entirely traced within $x_{2}$. We can suppose that $x_{2}$ does not contain any other wire than the ones seen by $p_{2}$. Consider a loop $r=r_{1} * r_{1}^{-1}$ that moves to an empty pattern in $x_{2}$ by moving left (this is always possible according to Lemma 23) and then comes back. We have
$p=p_{1} * p_{2} \ldots * p_{L}=\underbrace{p_{1} * r_{1}}_{p_{1}^{\prime}} * \underbrace{r_{1}^{-1} * p_{2} \ldots * p_{L}}_{p^{\prime}}$
$p_{1}^{\prime}$ and $p^{\prime}$ are then respectively paths of length 2 and $L-1$, and so using the induction hypothesis, they can be deformed so at not to see $\mathcal{U}$. The resulting path then only sees one wire at a time.

- Lemma 28 (Cross Anywhere Lemma). Let pa path starting and ending with empty patterns. If $p$ sees no relation tiles, but sees and crosses a single wire $\mathcal{U}$ exactly once, then for all $\mathbf{v}=\left(v^{0}, v^{1}\right) \in \mathbb{Z}^{2}, p$ is homotopic to a path $p^{\prime}$ which crosses $\mathcal{U}$ exactly on $\mathbf{v}$.

The proof can be found in Appendix A.

### 4.3 Projective connectedness

- Lemma 29. $X$ is projectively connected.

Proof. To prove this, it suffices to show that for any configuration $x \in X$, we can find a sequence $\left(p_{n}\right)_{n \in \mathbb{N}}$ where each $p_{n}=\left(P_{i}^{n}, \mathbf{v}_{\mathbf{i}}{ }^{n}\right)_{0 \leq i \leq N_{n}}$ is a path in $X_{B_{n}}$ between $\left(x_{\mid B_{n}},(0,0)\right)$ and $\left(x_{\square \mid B_{n}},(0,0)\right)$, and such that the canonical restriction of each $p_{n+1}$ is homotopic to $p_{n}$.

Suppose first that $x$ contains no wire in some octant. In order to construct a path from $x$ to $x_{\square}$, we will simply use this empty octant as follows: For $n$ large enough such that $B_{n}$ intersects the empty octant of $x$, we define a path $p_{n}$ by moving straight into it while staying in the configuration $x$, far enough so that the aperture window is entirely contained in this empty octant. At this point, we can simply come back to the point $(0,0)$ within the
configuration $x_{\square}$. It is moreover easy to deform the restriction of a path $p_{n+1}$ defined in this way into a path $p_{n}$ of the same kind.

Assume now that $x$ contains wires arbitrarily far in any octant, and let $N$ be large enough so that $x_{\mid B_{N}}$ contains a wire on the left of $(0,0)$. Let $\mathcal{U}$ be the first such wire, and let $(-k, 0)$ be the point at which $\mathcal{U}$ crosses the x -axis (if there are two such points, pick the one on the right, see the Wire Order Lemma). We will define $p_{n}$ for $n \geq N$. Let $x_{n}$ be the configuration obtained by extending $x_{\mid B_{n}}$, only using tiles of the form $\square, \boxtimes, 母, \boxtimes$ for its wires: $\mathcal{U}$ and the wires on its left are extended so that they are moving to the left, and wires on its right are extended to move the right, as in the proof of the Single Wire Lemma. We can then define a path $p_{n}$ between $\left(x_{\mid B_{n}},(0,0)\right)$ and $\left(x_{\square \mid B_{n}},(0,0)\right)$ as follows:

- Move left until $(-k+1,0)$ in $x_{n}$
- Follow $\mathcal{U}$ until reaching the height $n$, also in $x_{n}$
- Move up for $4 n$ steps, reaching an empty pattern, still in $x_{n}$
- Come back to $(0,0)$, in $x_{\square}$

Let $\rho_{n}$ be the canonical restriction of $p_{n+1}$ to the aperture window $B_{n}$. We need to show that $p_{n} \sim_{B_{n}} \rho_{n}$. To do so, notice that neither $p_{n}$ nor $\rho_{n}$ moves down. Hence, we will deform them both in the lower part of a configuration, so that we can apply one of our lemmas for paths between empty patterns to conclude that they are homotopic. Let then $r_{n} * r_{n}^{-1}$ be the loop in $x_{n}$ that starts at $(0,0)$, moves down for $4 n$ steps and comes back. The paths $r_{n}^{-1} * p_{n}$ and $r_{n}^{-1} * \rho_{n}$ are both paths between empty patterns, and cross exactly the same wires, namely, the ones crossed by $r_{n}^{-1}$. Using the No Relation Tile Lemma, then the No Uncrossed Wire Lemma and finally the Cross Anywhere Lemma, they are therefore homotopically equivalent. Hence, for any $n \geq N, p_{n} \sim_{B_{n}} \rho_{n}$. Thus, there exists a projective path class between $x$ and $x_{\square}$, and so $X$ is projectively connected.

### 4.4 Computing the projective fundamental group

We can now compute $\pi_{1}^{p r o j}(X)$, which is independent of the basepoint since $X$ is projectively connected. Hence, unless stated otherwise, all the loops in this proof are based on $\left(x_{\square},(0,0)\right)$. With any such loop $p$, we associate a word $\llbracket p \rrbracket$ on the alphabet $\bar{S}$ in the following way:

- If $p$ does not cross any wire, we associate the empty word with it, $\llbracket p \rrbracket=\varepsilon$.
- If $p$ crosses a single wire $\mathcal{U}$ on a tile $T$, then:
- If $\mathcal{U}$ is not a horizontal wire found on relation tile, and $s \in \bar{S}$ is the generator corresponding to $\mathcal{U}$ (see Subsection 4.1)
* if $p$ crosses it from left to right, or from top to bottom on a tile shaped as $\forall$, or from bottom to top on a tile $\boxed{G}$, then $\llbracket p \rrbracket=\mathrm{s} \in \bar{S}$.
* if $p$ crosses it in any other direction, we set $\llbracket p \rrbracket=\mathrm{s}^{-1} \in \bar{S}$
- Otherwise, $\mathcal{U}$ is a horizontal wire on a relation tile. Let $\overline{\mathrm{R}_{\mathrm{i}}}=\overline{r_{0} \ldots r_{i}}$ be its colour.
* If it is crossed from top to bottom, then $\llbracket p \rrbracket=\mathrm{r}_{\mathrm{i}}^{-1} \ldots \mathrm{r}_{0}^{-1} \in \bar{S}^{*}$
* Otherwise, $\llbracket p \rrbracket=\mathrm{R}_{\mathrm{i}}=\mathrm{r}_{0} \ldots \mathrm{r}_{\mathrm{i}}$
- If $p=p_{1} * p_{2}$, then $\llbracket p \rrbracket=\llbracket p_{1} \rrbracket \cdot \llbracket p_{2} \rrbracket \in \bar{S}^{*}$ where $\cdot$ represents the concatenation in $\bar{S}^{*}$.

Some examples are given in Figure 5a and Figure 5b.
For any two words $\mathrm{w}, \mathrm{w}^{\prime}$ on $\bar{S}$, we will note $\mathrm{w} \equiv \mathrm{w}^{\prime}$ if they are equal as words on this alphabet, and $\mathrm{w}={ }_{G} \mathrm{w}^{\prime}$ if they represent the same element of the group $G$. Let $\leftrightarrow_{R}$ be be the relation defined as the symmetric closure of $\left\{(u w v, u v) \mid w \in R\right.$ and $\left.u, v \in(\bar{S})^{*}\right\}$, corresponding to the operation of inserting and removing relators to words. We can always

(a) The word associated with this loop is $\mathrm{bb}^{-1} \mathrm{a}^{-1} \mathrm{abcc}^{-1} \mathrm{~b}^{-1}={ }_{G} 1_{G}$

(b) Widget for the relator $a b c=1_{G}$. From top to bottom, the words associated with the paths (1), (2) and (3) are respectively $a b c=$ $1_{G}, a a^{-1}(a b) c=1_{G},(a b) c=1_{G}$. For clarity, the relation tiles are not adjacent on the figure
suppose that it is reflexive by adding the empty word $\varepsilon$ to the relators. We denote $\leftrightarrow_{R}^{*}$ its transitive closure. By definition, $\mathrm{w} \leftrightarrow_{R}^{*} \mathrm{w}^{\prime} \Longleftrightarrow \mathrm{w}={ }_{G} \mathrm{w}^{\prime}$ (see e.g., [20, Theorem 1.1]). For example, if we take $\mathrm{a} \in S$, we have aa ${ }^{-1}={ }_{G} 1_{G}$, but aa ${ }^{-1} \not \equiv \varepsilon$.

In order to prove that the projective fundamental group of this subshift is $G$, we will prove that the operation $\llbracket p \rrbracket$ entirely characterizes a loop up to homotopy, in the sense that loops associated with the same element of $G$ are exactly a projective loop-class:

- Lemma 30. For $n>0$ and any two loops $p_{n}, p_{n}^{\prime}$ starting at $\left(x_{\square \mid B_{n}},(0,0)\right)$,

$$
p_{n} \sim_{B_{n}} p_{n}^{\prime} \Longrightarrow \llbracket p_{n} \rrbracket=_{G} \llbracket p_{n}^{\prime} \rrbracket
$$

The full proof can be found in Appendix A.

- Lemma 31. For any window $B_{n}$, and for any pair of loops $p_{n}$, $p_{n}^{\prime}$ starting at $\left(x_{\square \mid B_{n}},(0,0)\right)$,

$$
\llbracket p_{n} \rrbracket=_{G} \llbracket p_{n}^{\prime} \rrbracket \Longrightarrow p_{n} \sim_{B_{n}} p_{n}^{\prime} .
$$

Proof. Using the No Relation Tile Lemma, we can always start by deforming $p_{n}$ and $p_{n}^{\prime}$ so that they do not see any relation tile. As each elementary deformation is by definition occuring in some given configuration, Lemma 30 ensures that we still have $\llbracket p_{n} \rrbracket=_{G} \llbracket p_{n}^{\prime} \rrbracket$. We will first prove that $\llbracket p_{n} \rrbracket \equiv \llbracket p_{n}^{\prime} \rrbracket \Longrightarrow p_{n} \sim_{B_{n}} p_{n}^{\prime}$, which is a stronger assumption. Next, we prove that given $p_{n}$ and $p_{n}^{\prime}$ with $\llbracket p_{n} \rrbracket={ }_{G} \llbracket p_{n}^{\prime} \rrbracket$, there exists a loop $p_{n}^{\prime \prime}$ such that $p_{n} \sim_{B_{n}} p_{n}^{\prime \prime}$ and $\llbracket p_{n}^{\prime \prime} \rrbracket \equiv \llbracket p_{n}^{\prime} \rrbracket$. We then have that $p_{n}^{\prime \prime} \sim_{B_{n}} p_{n}^{\prime}$ according to the first part of the proof, and so $p_{n} \sim_{B_{n}} p_{n}^{\prime}$.

- We show that $\llbracket p_{n} \rrbracket \equiv \llbracket p_{n}^{\prime} \rrbracket \Longrightarrow p_{n} \sim_{B_{n}} p_{n}^{\prime}$. The paths $p_{n}$ and $p_{n}^{\prime}$ can be deformed using No Uncrossed Wire Lemma so that they cross all the wires that they see. The Single Wire Lemma can then be used to deform them so that there is at most one of those wires per pattern. Let $\hat{p_{n}}$ and $\hat{p_{n}^{\prime}}$ be the resulting paths, which by assumption cross the same wires. Using Cross Anywhere Lemma for each of those crossed wires, we can finally deform $\hat{p_{n}}$ into $\hat{p_{n}^{\prime}}$, and so $p_{n} \sim_{B_{n}} p_{n}^{\prime}$.
- Now, we show the existence of a loop $p_{n}^{\prime \prime}$ verifying $p_{n} \sim_{B_{n}} p_{n}^{\prime \prime}$ and $\llbracket p_{n}^{\prime \prime} \rrbracket \equiv \llbracket p_{n}^{\prime} \rrbracket$. By definition of $={ }_{G}$, there exists a finite sequence $\left(u_{\mathrm{i}}\right)_{0 \leq i \leq N}$ of words on the alphabet $\bar{S}$ such that $\llbracket p_{n} \rrbracket \equiv \mathrm{u}_{0}, \llbracket p_{n}^{\prime} \rrbracket \equiv \mathrm{u}_{\mathrm{N}}$, and for all $i<N, \mathrm{u}_{\mathrm{i}} \leftrightarrow_{R} \mathrm{u}_{\mathrm{i}+1}$. To prove the result, it is therefore enough to show that for any word v such that $\llbracket p_{n} \rrbracket \leftrightarrow_{R} \mathrm{v}$, we can deform $p_{n}$ in another loop $p_{n}^{\vee}$ such that $\llbracket p_{n}^{\vee} \rrbracket \equiv \vee$.
Suppose that $v$ is obtained from $\llbracket p_{n} \rrbracket$ by deleting a relator. More formally, there exists words $\mathrm{u}_{1}, \mathrm{u}_{2}$ and a relator $\mathrm{r} \in R$ such that $\mathrm{v} \equiv \mathrm{u}_{1} \mathrm{u}_{2}$ and $p_{n} \equiv \mathrm{u}_{1} \mathrm{r} \mathrm{u}_{2}$. Using the Single Wire Lemma, the No Uncrossed Wire Lemma and the Cross Anywhere Lemma as above, we can deform $p_{n}$ into a path that crosses wires corresponding to the letters of $\mathrm{u}_{1} \mathrm{ru}_{2}$, in order, on a horizontal line. Let $p_{\mathrm{u}_{1}}\left(\right.$ resp. $p_{\mathrm{r}}, p_{\mathrm{u}_{2}}$ ) the part of this path which crosses the wires corresponding to $\mathrm{u}_{1}$ (resp. $\mathrm{r}, \mathrm{u}_{2}$ ), starting and ending with empty patterns. Let $x_{\mathrm{r}} \in X$ be such that $p_{\mathrm{r}}$ can be traced in $x_{\mathrm{r}}$, and in which all those wires originate from the same set of relation tiles (see Figure 5 b ). We can then deform $p_{\mathrm{r}}$ in $x_{\mathrm{r}}$ into a path $p_{\mathrm{r}}^{\prime}$ that passes pass below the relation tiles. The resulting path $p_{n}^{\vee}=p_{\mathrm{u}_{1}} * p_{\mathrm{r}}^{\prime} * p_{\mathrm{u}_{2}}$ is then a solution.
- Theorem 32. $\pi_{1}^{\text {proj }}(X)=G$

Proof. Let $n>0$ and let $\Phi_{n}: \pi_{1}^{B_{n}}\left(X,\left(x_{\square},(0,0)\right)\right) \rightarrow G$ be the function which associates with a loop-class with aperture window $B_{n}$ the corresponding element of $G$. Lemma 30 and Lemma 31 show that it is well-defined and injective. Let $[p],\left[p^{\prime}\right]$ be two loop-classes based on $\left(x_{\square \mid B_{n}},(0,0)\right)$. We have shown that $[p] \sim_{B_{n}}\left[p^{\prime}\right] \Longleftrightarrow \Phi_{n}([p])={ }_{G} \Phi_{n}\left(\left[p^{\prime}\right]\right)$. Now notice that $\Phi_{n}\left(\left[p * p^{\prime}\right]\right)={ }_{G} \Phi_{n}(p) \cdot{ }_{G} \Phi_{n}\left(p^{\prime}\right)$, i.e., $\Phi_{n}$ is a group morphism. Being obviously surjective, it is in fact an isomorphism.

Furthermore, notice that for any loop-class $\left[p_{n+1}\right]$ based on $\left(x_{\square \mid B_{n+1}},(0,0)\right)$, if $p_{n+1}$ projects down to $p$ then $\Phi_{n+1}\left(\left[p_{n+1}\right]\right)={ }_{G} \Phi_{n}([p])$. This shows that the inverse limit of the system $\left(\pi_{1}^{B_{n}}\left(X,\left(x_{\square},(0,0)\right)\right)\right)_{n>0}$ is isomorphic to $G$.

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## A Proofs

Proof of No Uncrossed Wire Lemma. We proceed by induction on the length of a coherent decomposition of the path, and we assume that $\mathcal{U}$ is on the right side of the patterns. Using Single Wire Lemma, we can assume that all the patterns of $p$ contain at most a single wire.

Base case: $\mathbf{L}=1 \quad p$ can be traced entirely in a configuration $x \in X$.
In that case, we can simply deform $p$ in $x$ by changing its trajectory so that it always stays more than $n$ units left from $\mathcal{U}$. This path can then be traced in the configuration $x^{\prime}$, equal to $x$ except for the tiles of $\mathcal{U}$ in $x$ that are empty tiles in $x^{\prime}$.

Base case: $\mathbf{L}=2 \quad p=p_{1} * p_{2}$
Let $\left(P_{t}, \mathbf{v}_{\mathbf{t}}\right)$ be the final point of $p_{1}$ and the first one of $p_{2}$. We also assume that the second coordinate of $\mathbf{v}_{\mathbf{t}}=\left(v_{t}^{0}, v_{t}^{1}\right)$ is non-negative. Let $\mathbf{v}_{\mathbf{N}}=\left(v_{N}^{0}, v_{N}^{1}\right)$ be the final point of the path.

Let $x_{1} \in X$ (resp. $x_{2}$ ) be a configuration, containing only the wires seen by $p_{1}$ (resp. $p_{2}$ ), such that $p_{1}$ (resp. $p_{2}$ ) can entirely be traced within it. Let $\mathcal{U}$ be the uncrossed wire. We can always assume that $\mathcal{U}$ appears in $P_{t}$, otherwise we could consider it to be made of two distinct paths, each appearing entirely either only in $p_{1}$ or in $p_{2}$, and so we could directly apply the case $L=1$.

We deform $p_{1}$ into $p_{1}^{\prime}$ inside $x_{1}$ :

- Starting from $(0,0)$, it first moves to the right, until $\mathcal{U}$ appears on the central tile of the pattern seen by $p_{1}$.
- It then moves up, left or right, following $\mathcal{U}$ : up if the central tile is $\mathbb{\checkmark}$, left then up if it is $\boxed{\Delta}$, and so on.
- Finally, once it attains the height $v_{t}^{1}$, it moves left until $\mathbf{v}_{\mathbf{t}}$ if needed, which takes at most $n$ steps.

We can also deform $p_{2}$ into another path $p_{2}^{\prime}$ as follows:

- Starting from $\mathbf{v}_{\mathbf{t}}$, move left for $\max \left(2 n,\left(v_{t}^{0}-v_{N}^{0}\right)\right)$ steps. This ensures that we are far enough so as to not see $\mathcal{U}$ anymore.
- Then, move vertically to height $v_{N}^{1}$.
- Finally, move right until $\mathbf{v}_{\mathbf{N}}$.

Let $\mathbf{w}_{\mathbf{1}}$ be the last point of $p_{1}^{\prime}$ before seeing $\mathcal{U}$, and $\mathbf{w}_{\mathbf{2}}$ the first point of $p_{2}^{\prime}$ after having seen $\mathcal{U}$ for the last time. The Single Wire Lemma ensures that the patterns seen at both $\mathbf{w}_{1}$ and $\mathbf{w}_{2}$ are empty. This gives a decomposition

$$
p \sim p_{1}^{\prime} * p_{2}^{\prime} \sim p_{\mathrm{start}} * p_{\mathcal{U}} * p_{\mathrm{end}}
$$

where $p_{\text {start }}$ ends at $\mathbf{w}_{\mathbf{1}}, p_{\mathcal{U}}$ is the part of the path between $\mathbf{w}_{\mathbf{1}}$ and $\mathbf{w}_{\mathbf{2}}$, and $p_{\text {end }}$ starts at $\mathrm{w}_{2}$.
$p_{\mathcal{U}}$ can be traced entirely in a configuration $x_{3}$ whose only wire is $\mathcal{U}$. In this configuration, it can be homotopically deformed to $p_{\mathcal{U}}^{\prime}$ which never sees $\mathcal{U}$ according to the case $n=1$.

The final path $p^{\prime}=p_{\text {start }} * p_{\mathcal{U}}^{\prime} * p_{\text {end }}$ does not see $\mathcal{U}$.

General case: L >2 $\quad$ = $p_{1} * \ldots * p_{L}$
In that case, the proof is exactly the same as in Single Wire Lemma: we insert a loop before $p_{2}$ starts that extends it, and from a decomposition of length $L$ we obtain two decompositions of length respectively 2 and $L-1$, which are solved inductively.

Proof of Cross Anywhere Lemma. Let $p=\left(P_{i}, \mathbf{v}_{\mathbf{i}}\right)_{0 \leq i \leq N}$ be such a path, and let $t$ be the timestep at which $p$ crosses $\mathcal{U}$. Without loss of generality, we can then assume that the wire is crossed from left to right, i.e. $\mathcal{U}$ is on the right side of $P_{t-1}$ and on the left side of $P_{t}$. Let $x$ be any configuration containing $P_{t-1} \cup P_{t}$. We can suppose that $\mathbf{v}_{\mathbf{t}}=e_{0}+\mathbf{v}_{\mathbf{t}-\mathbf{1}}$, by deforming $p$ in $x$ if needed, and that $\mathbf{v}_{\mathbf{t}-\mathbf{1}}=(0,0)$. Let $r_{1}$ be the path starting from $\left(P_{t-1},(0,0)\right)$ which moves left for $4 n+2\left|v^{0}\right|$ steps in $x$, and let $r=r_{1} * r_{1}^{-1}$. Let $q_{1}$ be the path starting from $\left(P_{t},(1,0)\right)$ which moves right for $4 n+2\left|v^{0}\right|$ steps in $x$, and let $q=q_{1} * q_{1}^{-1}$. We can deform $p$ in $x$ by inserting the loops $r$ and $q$ respectively at the timesteps $t-1$ and $t$. Using the No Uncrossed Wire Lemma twice, this path can itself be deformed into $p_{\text {start }} * p^{\prime} * p_{\text {end }}$ with $p^{\prime}=r_{1}^{-1} *\left(P_{t},(0,0)\right) * q_{1}$, and $p_{\text {start }}, p_{\text {end }}$ paths that only see empty patterns. The trajectory of $p^{\prime}$ is a straight horizontal line on the $x$-axis of length $8 n+2\left|v^{0}\right|+1$. Let $x^{\prime}$ the configuration obtained by extending $\mathcal{U}$ as seen by $p^{\prime}$ using only tiles of the form $\mathbb{W}$ Without loss of generality, suppose that $v^{1} \leq 0$. We can deform $p^{\prime}$ in $x^{\prime}$ so that it moves up for $8 n+2\left|v^{0}\right|$ steps, then right for $8 n+2\left|v^{0}\right|+1$ as before and finally down to the endpoint of $p^{\prime}$. Call $p^{\prime \prime}$ the horizontal part of this path. There exists a configuration $x^{\prime \prime}$ in which $\mathcal{U}$ passes by $\mathbf{v}$ and in which $p^{\prime}$ can be traced. Then, $p^{\prime \prime}$ can be deformed in $x^{\prime \prime}$ to cross $\mathcal{U}$ on $\mathbf{v}$. This finally gives the result.

Proof of Lemma 30. As any two homotopic loops can be obtained from one another by a sequence of elementary deformations, we can restrict ourselves to the special case of a single deformation that is a loop based at $\left(P_{t}, \mathbf{v}_{\mathbf{t}}\right)$. By definition, this deformation is made in a single configuration $x \in X$. We consider two disjoint cases, according to the presence of relation tiles in $x$.

- Suppose that $x$ does not contain any relation tile. Any bi-infinite wire splits the space in two disjoint regions (a "left" one and a "right" one). Each time a loop crosses such a wire, it has to cross it in the other direction to come back to its initial region. Because wires do not intersect, the associated word will be some kind of Dyck word, where each $\mathrm{s} \in \bar{S}$ can act as an opening or a closing bracket (in which case, the associated closing (resp. opening) bracket is $\mathrm{s}^{-1}$ ), so it is clearly equal to $1_{G}$ in $G$. This is the simple case depicted in Figure 5a.
- Now, suppose that $x$ does contain some relation tiles. In this case, notice that any two relation tiles are either part of the same relator and are therefore linked by a finite sequence of horizontal relation tiles, or they are independent (not linked by any wire). Hence, we can consider each one of those patterns separately. Consider such a pattern, with relation tiles that implement a relator $\mathrm{r}=\mathrm{r}_{0} \ldots \mathrm{r}_{\mathrm{k}} \in R$, and a configuration $x^{\prime}$ that only contains this pattern. Figure 5 b represents this in a configuration corresponding to relation $\mathrm{abc}=1$. We show that, due to how $\llbracket \rrbracket$ has been defined, all the homotopyequivalent paths in $x^{\prime}$ are associated with the same element of $G$. Let $\mathcal{U}_{0}, \ldots, \mathcal{U}_{k}$ be the wires corresponding respectively to $\mathrm{r}_{0}, \ldots, \mathrm{r}_{\mathrm{k}}$, and suppose that the relation tiles in $x^{\prime}$ are placed on $(0,0), \ldots,(k, 0)$. We will show that for any $p$ joining $(0,0)$ to $(k+1,0)$ in $x^{\prime}$, $\llbracket p \rrbracket={ }_{G} 1_{G}$. Let $\mathcal{R} \subset \mathbb{Z}^{2}$ be the set of points above the $(\mathbb{Z}, 1)$ line and between $\mathcal{U}_{0}$ and $\mathcal{U}_{k}$. We can always suppose that no wire is crossed consecutively in opposite directions, as the word associated to a path that crosses a wire in a direction and immediately crosses it in the other direction is $\mathrm{ss}^{-1}={ }_{G} 1_{G}$ for some $\mathbf{s} \in \bar{S}^{*}$. We can also suppose that $p$ only enters and then leaves $\mathcal{R}$ once. Otherwise, we can simply split it into several such paths and prove the claim for each of them independently.
= If $p$ crosses $\mathcal{U}_{0}, \ldots, \mathcal{U}_{k}$, then $\llbracket p \rrbracket \equiv \mathrm{r}_{0} \ldots \mathrm{r}_{\mathrm{k}}={ }_{G} 1_{G}$ by definition.
- If $p$ crosses $\mathcal{U}_{0}, \ldots, \mathcal{U}_{i}, \mathcal{U}_{\overline{r_{0}} \ldots \mathrm{r}_{\mathrm{i}}}$, where $\mathcal{U}_{\overline{\mathrm{r}_{0} \ldots r_{\mathrm{i}}}}$ is a wire of a relation tile which is necessarily crossed from top to bottom, by definition, $\llbracket p \rrbracket \equiv \mathrm{r}_{0} \ldots \mathrm{r}_{\mathrm{i}}\left(\mathrm{r}_{\mathrm{i}}^{-1} \ldots \mathrm{r}_{0}^{-1}\right)={ }_{G} 1_{G}$
= Otherwise, $p$ crosses $\mathcal{U}_{\overline{r_{0} \ldots r_{\mathrm{i}}}}, \mathcal{U}_{i+1}, \ldots, \mathcal{U}_{j}, \mathcal{U}_{\overline{r_{0} \ldots r_{\mathrm{j}}}}$, the first relation tile being crossed from bottom to top to enter $\mathcal{R}$ and the last one being crossed from top to bottom to exit it. By definition, $\llbracket p \rrbracket \equiv\left(\mathrm{r}_{0} \ldots \mathrm{r}_{\mathrm{i}}\right) \mathrm{r}_{\mathrm{i}+1} \ldots \mathrm{r}_{\mathrm{j}}\left(\mathrm{r}_{\mathrm{j}}^{-1} \ldots \mathrm{r}_{0}^{-1}\right)={ }_{G} 1_{G}$
This shows that all the paths traced in a single configuration are associated with the same element of the group $G$. As all homotopies are deformations in a given configuration, this implies that for any homotopically equivalent paths $p, p^{\prime}$, we have $\llbracket p \rrbracket={ }_{G} \llbracket p^{\prime} \rrbracket$.

