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Léo Paviet Salomon, Pascal Vanier

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1 Realizing finitely presented groups as projective 2 fundamental groups of SFTs

3 Léo Paviet Salomon ✉

4 Normandie Univ, UNICAEN, ENSICAEN, CNRS, GREYC, 14000 Caen, France

5 Pascal Vanier ✉ 🏠 

6 Normandie Univ, UNICAEN, ENSICAEN, CNRS, GREYC, 14000 Caen, France

7 — Abstract —

8 Subshifts are sets of colourings - or tilings - of the plane, defined by local constraints. Historically
9 introduced as discretizations of continuous dynamical systems, they are also heavily related to
10 computability theory. In this article, we study a conjugacy invariant for subshifts, known as the
11 projective fundamental group and we show that any finitely presented group can be realized as a
12 projective fundamental group of some SFT.

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20 **1** Introduction

21 A d -dimensional subshift is a set of colourings of \mathbb{Z}^d by a finite number of colours which
22 avoids some family of forbidden patterns. If the family is finite, it is called a subshift of finite
23 type (SFT). Subshifts can also be defined topologically: they are the closed shift-invariant
24 subsets of $\Sigma^{\mathbb{Z}^d}$, where Σ is the finite set of colours. Most problems concerning subshifts in
25 dimension $d \geq 2$ are undecidable [3, 12, 11], which is due to the fact that sets of Wang tilings
26 are SFTs.

27 Together with the shift action σ , a subshift forms a dynamical system. Interesting
28 dynamical aspects are usually invariant by conjugacy, which is the isomorphism notion for
29 subshifts: two subshifts are conjugate if there exists a continuous shift-commuting bijection
30 between them. As a matter of fact, most dynamical aspects of subshifts are linked to
31 computability theory or complexity theory in dimensions $d \geq 2$. These links were first
32 explored when the possible values of the topological entropy of multi-dimensional SFTs was
33 characterized as the upper semi-computable numbers [16], afterwards many other conjugacy
34 invariants followed: growth-type invariants [21], periodicity data [18, 17], subactions [14, 2, 8]
35 and so on.

36 Recently, the links between subshifts and groups have seen a surge in study [1, 19, 22],
37 however, the most well known group associated with a subshift, its automorphism group, is
38 still not very well understood: while we know that SFTs with positive entropy have very
39 complex automorphism groups [15] or that SFTs whose automorphism group has undecidable
40 word problem can be constructed [10], we still do not know whether the automorphism
41 groups of the full shifts on 2 and 3 symbols are the same. Apart from the low complexity
42 setting [7, 6] not much is understood about it.

43 Here, we contribute to the study of another group-related conjugacy invariant called the

projective fundamental group introduced by Geller and Propp [9]. In the classical setting (see for example [13]), the fundamental group $\pi_1(S)$ of a surface S is a topological invariant which describes the number of holes and the general shape of S . It is constructed using loops on the surface : it is the group of the equivalence classes of loops through continuous deformation together with the composition operation. In the case of subshifts, as they are not surfaces in the classical sense, the definition is not as straightforward but is still based on loops, their composition and their deformations as will be seen in more detail in Subsection 3.1.

The projective fundamental group of a subshift is well defined only in the case of projectively connected subshifts, a property that resembles mixing properties: properties which essentially state that if two patterns appear in different configurations they may appear in the same configuration (see for instance [5] or [24] for more precise definitions of mixing properties in dimensions $d \geq 2$). As a matter of fact, projective connectedness is implied by all mixing properties defined in [5].

As a conjugacy invariant, the fundamental group allows to distinguish between some subshifts which share the same entropy and periodicity data. It is also better understood than the automorphism group in the sense that the authors in [9] explicitly compute it for several well-know subshifts: the full shifts on any alphabet always have trivial fundamental group, the square-ice has \mathbb{Z} and factors of full shifts always have a fundamental group with finite order. They also prove that any group of finite order is realizable as a fundamental group of some factor of a full shift.

The main result of this article is that any finitely presented group is the fundamental group of an SFT:

► **Theorem 1.** *Let $G = \langle S | R \rangle$ a finitely presented group. Then, there is a subshift of finite type X verifying:*

- X is projectively connected,
- the projective fundamental group of X is isomorphic to G .

We do not think this is a characterization. As a matter of fact, we do not know how hard the word problem is for the projective fundamental group of SFTs apart from the fact that it is at least Σ_1^0 -hard since finitely presented groups have Σ_1^0 -complete word problem [23, 4]. The same is true for projective connectedness, we know it is undecidable but do not know how hard. It is open whether any of these properties is in the arithmetical hierarchy.

The paper is organized as follows. After recalling the symbolic dynamics background in Section 2, we introduce the projective fundamental group in Subsection 3.1, some examples in Subsection 3.2 and finally in Section 4 we prove Theorem 1.

2 Definitions

A **d-dimensional full shift** on some finite alphabet Σ is the set $\Sigma^{\mathbb{Z}^d}$, together with the **shift-action** $\sigma_{\mathbf{u}}: \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$ defined by $\sigma_{\mathbf{u}}(x)(\mathbf{v}) = x(\mathbf{u} + \mathbf{v}) = x_{\mathbf{u} + \mathbf{v}}$, an element of a full shift is called a **configuration**. A **subshift** is a closed, shift-invariant subset of some full shift. We call **points** of a subshift X the configurations belonging to X . The underlying topology is the one induced by the **Cantor distance**, defined on $\Sigma^{\mathbb{Z}^d}$ by

$$d(x, y) = 2^{-\min\{\|\mathbf{u}\|_{\infty} \mid x_{\mathbf{u}} \neq y_{\mathbf{u}}\}},$$

two configurations are close in this topology if they agree on a large central square.

Alternatively, subshifts can be defined using forbidden patterns. We call **pattern** any element $P \in \Sigma^U$ where $U \subset \mathbb{Z}^d$ is finite and is the **support** of P . For a configuration x , we

89 say that P appears in x if there exists $\mathbf{u} \in \mathbb{Z}^d$ such that $\sigma_{\mathbf{u}}(x)|_U = P$. Let \mathcal{F} be a collection
 90 (finite or not) of patterns. Then the set

$$91 \quad X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} \mid \forall P \in \mathcal{F}, P \text{ does not appear in } x \right\}$$

92 is a subshift. In fact, for any subshift (defined as a closed, shift-invariant set) X , there exists
 93 a family of patterns \mathcal{F} such that $X = X_{\mathcal{F}}$. A subshift X is a **subshift of finite type (SFT)**
 94 if there exists a finite \mathcal{F} such that $X = X_{\mathcal{F}}$.

95 For a given subshift X , a pattern $P \in \Sigma^U$ is **locally admissible** if it contains no forbidden
 96 patterns. It is **globally admissible** or **extensible** if it appears in some configuration $x \in X$.

97 SFT can also be defined using **Wang tiles**. Let C be a finite set of colours. A set of
 98 Wang tiles \mathcal{T} is a finite set of tiles, which are mapping $\tau: \{t, b, l, r\} \rightarrow C$. A Wang tiling by
 99 \mathcal{T} is a mapping $w: \mathbb{Z}^2 \rightarrow \mathcal{T}$ which verifies for all $(i, j) \in \mathbb{Z}^2$,

$$100 \quad \blacksquare \quad w(i, j)(r) = w(i + 1, j)(l)$$

$$101 \quad \blacksquare \quad w(i, j)(t) = w(i, j + 1)(b)$$

102 The set $W(\mathcal{T}) = \{w, w \text{ is a wang tiling by } \mathcal{T}\}$ is a SFT.

103 **3 Projective Fundamental Group**

104 **3.1 Intuitions and definitions**

105 The Projective Fundamental Group, introduced by Geller and Propp [9] resembles the usual
 106 fundamental group construction of the topological setting: it is defined through paths, loops,
 107 and, more generally, a homotopy notion. However, instead of directly considering paths
 108 between points of the subshift, they are defined between finite patterns with the same support.
 109 By doing so, one actually constructs a family of fundamental groups, potentially different for
 110 each finite support $B \subset \mathbb{Z}^2$. In order to obtain a single group, the projective fundamental
 111 group, one takes their inverse (also known as projective) limit. We will construct a subshift
 112 by defining a set \mathcal{T} of Wang tiles. A configuration will then be a mapping $x: \mathbb{Z}^2 \rightarrow \mathcal{T}$
 113 associating a tile to each point of the plane and which verifies some adjacency rules depending
 114 on \mathcal{T} . Contrary to the usual convention, we will consider than when embedding such a
 115 configuration in the euclidean plane \mathbb{R}^2 , the tile in position (i, j) is a unit square whose
 116 bottom-left corner is placed on (i, j) , as opposed to its center. This is merely a discussion
 117 about conventions, but it will make some definitions substantially simpler.

118 Fix a support $B \subset \mathbb{Z}^2$, in what follows B will be called an **aperture window**. Most
 119 of the time, we will restrict ourselves to the windows $B_n = \llbracket -n, n - 1 \rrbracket^2$. We choose
 120 this asymmetrical window to simplify some definitions, but also for consistency with the
 121 aforementioned convention. Indeed, when embedded into \mathbb{R}^2 , B_n is a square of side $2n$,
 122 whose center is the point $(0, 0) \in \mathbb{R}^2$. In any configuration x , the tile $x_{(0,0)}$ in position $(0, 0)$
 123 will therefore be seen as the square whose bottom-left (resp. top-right) corner is $(0, 0)$ (resp.
 124 $(1, 1)$).

125 Consider P, P' two extensible patterns of support B and two points of the grid $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}^2$.
 126 A **path** between (P, \mathbf{v}) and (P', \mathbf{v}') is a sequence of pairs of vectors and patterns (or
 127 equivalently, two sequences of the same length). The first one represents an actual, "geometric"
 128 path, called its **trajectory**, that is to say a sequence of 4-adjacent vertices of \mathbb{Z}^2 starting
 129 at \mathbf{v} and ending at \mathbf{v}' . The second associates with each one of those vertices \mathbf{v}_t a pattern
 130 P_t , that needs to be coherent with the path: when moving to the next vertex \mathbf{v}_{t+1} on the
 131 trajectory, the next pattern P_{t+1} needs to be coherent with P_t . For example, in the full shift,

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132 and for $B = B_1$, take the following pairs:

$$133 \quad P_1 = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (0,0) \right), \quad P_2 = \left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, (1,0) \right), \quad P_3 = \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, (1,0) \right)$$

134 The tile in position $(0,0)$ is represented in red. The sequence $(P_1, (0,0)), (P_2, (1,0))$ is
 135 a valid path, as the overlapping parts of the support are equal in both patterns, but
 136 $(P_1, (0,0)), (P_3, (0,0))$ is not because e.g., the point $(1,0)$ is tiled by 0 in the first pattern
 137 but by 1 in the second one. Moreover, the pattern obtained by "merging" two consecutive
 138 patterns also needs to be an extensible pattern.

139 ► **Definition 2 (Path).** Let $B \subset \mathbb{Z}^2$ a finite set, a path of **aperture window** B is a finite
 140 sequence $(P_t, \mathbf{v}_t)_{0 \leq t \leq N}$ of pairs from $\Sigma^B \times \mathbb{Z}^2$, such that for any t with $0 \leq t \leq N$:

- 141 ■ P_t is an extensible pattern of X ,
- 142 ■ \mathbf{v}_t is adjacent to \mathbf{v}_{t+1} , i.e., they are at distance exactly 1,
- 143 ■ $P_t(u) = P_{t+1}(u)$ for any $u \in (B + \mathbf{v}_t) \cap (B + \mathbf{v}_{t+1})$, i.e., consecutive patterns overlap,
- 144 ■ the pattern $P_t \cup P_{t+1}$ obtained by merging P_t and P_{t+1} is extensible in X .

145 The first and last element of the sequence are respectively called the **starting point** and
 146 the **ending point** of the path. If they are equal, the path is called a **loop**. The path
 147 $(P_{N-t}, \mathbf{v}_{N-t})_{0 \leq t \leq N}$ is called its **inverse path**. If p is a path, its inverse will be denoted by
 148 p^{-1}

149 The sequence $(\mathbf{v}_t)_{0 \leq t \leq N}$ is called the **trajectory** of the path.

150 When one path ends where another path starts, they may be composed:

151 ► **Definition 3 (Path composition).** Given $p = (P_t, \mathbf{v}_t)_{0 \leq t \leq N}$ and $p' = (P'_t, \mathbf{v}'_t)_{0 \leq t \leq N'}$ two
 152 paths such that $(P_N, \mathbf{v}_N) = (P'_0, \mathbf{v}'_0)$ we note $p * p'$ the path

$$153 \quad p * p' = (P_0, \mathbf{v}_0) \dots (P_N, \mathbf{v}_N) (P'_1, \mathbf{v}'_1) \dots (P'_{N'}, \mathbf{v}'_{N'}).$$

154 If there exists a point $x \in X$ such that for all t , $x|_{B+\mathbf{v}_t} = P_t$, we say that p can be **traced**
 155 in the configuration x . If a path $p = (P_t, \mathbf{v}_t)_{0 \leq t \leq n}$ is defined with an aperture window B ,
 156 then for all $B' \subset B$, we refer to the path $p = (P_t|_{B'}, \mathbf{v}_t)_{0 \leq t \leq n}$ as the **restriction** of p to the
 157 window B' .

158 One can now define a corresponding homotopy notion, using what we call an **elementary**
 159 **deformation**: let $p = p_1 * p_2 * p_3$ be a path and suppose that p_2 can be traced in a single
 160 configuration $x \in X$. Then, for any p'_2 traced in x with the same starting and ending point
 161 as p_2 , one can deform p into $p_1 * p'_2 * p_3$. As paths might consist of a single point, they can
 162 be deformed by inserting or removing loops traced in a single configuration at any step.

163 ► **Definition 4 (Homotopy).** Two paths p, p' are said to be **homotopic** if there exists a
 164 finite sequence of elementary deformations from p to p' . This defines an equivalence relation
 165 between paths, and we denote by $[p]$ the equivalence class of p . If p and p' are paths with an
 166 aperture window $B \subset \mathbb{Z}^2$, we denote by $p \sim_B p'$ the fact that they are homotopic.

167 ► **Remark 5.** When two paths are homotopic, they necessarily have the same starting and
 168 ending points. When B is clear from the context, we will simply note $p \sim p'$

169 With this definition of a path and of homotopy, we can define a fundamental group for
 170 each possible aperture window $B \subset \mathbb{Z}^2$.

171 ► **Definition 6 (Fundamental Group).** Let $B \subset \mathbb{Z}^2$ be an aperture window, $x_0 \in X$ and
 172 $\mathbf{v} \in \mathbb{Z}^2$. The **fundamental group** of X based at (x_0, \mathbf{v}) for the aperture window B , denoted
 173 by $\pi_1^B(X, (x_0, \mathbf{v}))$, is the group of all the equivalence classes of loops starting and ending at
 174 $(x_0|_B, \mathbf{v})$ for the homotopy equivalence relation, along with the $*$ operation.

175 Although our paths follow the \mathbb{Z}^2 grid and seem to be discrete and combinatorial objects,
 176 it is legitimate to refer to those objects as *homotopy* and *deformations*, which usually suppose
 177 some kind of continuity. In fact, this simplification does not entail any loss of generality,
 178 compared to paths drawn in \mathbb{R}^2 , and subshifts seen as \mathbb{Z}^2 -invariants subsets of $\Sigma^{\mathbb{R}^2}$. For
 179 more details, one can refer to [9, Straightening Lemma]. In order to obtain a single object
 180 associated with the subshift, we need to get rid of this reference to an aperture window.
 181 Geller and Propp consider the projective limit of those groups to define what they refer to as
 182 the **Projective Fundamental Group** of the subshift.

183 ► **Definition 7.** We define an equivalence relation on $X \times \mathbb{Z}^2$, denoted by \equiv_B , as follows:

$$184 \quad (x, \mathbf{v}) \equiv_B (x', \mathbf{v}') \iff \mathbf{v} = \mathbf{v}' \text{ and } x|_{B+\mathbf{v}} = x'|_{B+\mathbf{v}} \quad \text{with } (x, \mathbf{v}), (x', \mathbf{v}') \in X \times \mathbb{Z}^2.$$

185 We denote X_B the quotient space $(X \times \mathbb{Z}^2) / \equiv_B$. In order to simplify the notation, we will
 186 sometimes use $(x|_{B_n+\mathbf{v}}, \mathbf{v})$ to denote the equivalence class of (x, \mathbf{v}) for the relation \equiv_{B_n} .

187 Notice that with this definition, a path is now an element of $X_B^{<\mathbb{N}}$ for some $B \subset \mathbb{Z}^2$, which
 188 also verifies some additional constraints. One can then consider the fundamental groups of
 189 all those spaces, and take their inverse limit to obtain this subshift's projective fundamental
 190 group.

191 ► **Definition 8 (Projective Limit).** Let (\mathcal{I}, \leq) a partially ordered set, and let $(G_i)_{i \in \mathcal{I}}$ a family
 192 of groups. Suppose that we have a family of homomorphisms $f_{ij}: G_j \rightarrow G_i$ for all $i \leq j \in \mathcal{I}$,
 193 which satisfies the two following properties:

- 194 ■ For all $i \in \mathcal{I}$, $f_{i,i} = \text{id}_{G_i}$
- 195 ■ For all $i \leq j \leq k \in \mathcal{I}$, $f_{i,k} = f_{i,j} \circ f_{j,k}$

196 Then, the **projective limit** (sometimes also called **inverse limit**) of the projective
 197 system $((G_i)_{i \in \mathcal{I}}, (f_{i,j})_{i \leq j \in \mathcal{I}})$ is a subgroup of the (potentially infinite) direct product of all
 198 the G_i 's, denoted here by G_∞

$$199 \quad G_\infty = \varprojlim_{i \in \mathcal{I}} G_i = \left\{ \vec{g} \in \prod_{i \in \mathcal{I}} G_i \mid g_i = f_{i,j}(g_j) \text{ for all } i \leq j \text{ in } \mathcal{I} \right\}$$

200 ► **Definition 9 (Projectively connected subshift).** A subshift X is **projectively connected** if
 201 for any two points $x, x' \in X$, there exists an inverse system of paths $(p_n)_{n \in \mathbb{N}}$, such that p_n
 202 is a path between $(x_{B_n}, (0, 0)) \in X_{B_n}$ and $(x'_{B_n}, (0, 0))$, and for each $n > 0$, the image of p_n
 203 under the canonical restriction map to $X_{B_{n-1}}$ is homotopic to p_{n-1} .

204 Let X_∞ be the inverse limit of the system $(X_B)_{B \subset \mathbb{Z}^2}$ along with the canonical **restriction**
 205 **maps** $(x|_{B+\mathbf{v}}, v) \in X_B \mapsto (x|_{B'+\mathbf{v}}, v)$ for each $B' \subseteq B \subset \mathbb{Z}^2$.

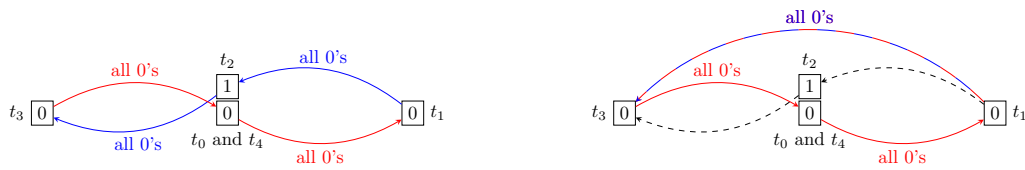
206 ► **Lemma 10 (Sequence Lemma [9]).** X_∞ is the inverse limit of the system $(X_{B_n})_{n \in \mathbb{N}}$.

207 ► **Definition 11 (Projective Fundamental Group).** The **projective fundamental group** based on
 208 the point $(x_0, \mathbf{v}) \in X \times \mathbb{Z}^2$ of a subshift X is the inverse limit of $(\pi_1^B(X, (x_0, \mathbf{v})))_{B \subset \mathbb{Z}^2}$, and
 209 is denoted by $\pi_1^{\text{proj}}(X, (x_0, \mathbf{v}))$. If X is projectively connected, then its projective fundamental
 210 group does not depend on the chosen basepoint (x_0, \mathbf{v}) , and we denote it simply by $\pi_1^{\text{proj}}(X)$.

211 ► **Definition 12 (Projective path-class).** A **projective path class** is a family $([p_n])_{n \in \mathbb{N}^*}$,
 212 where each p_n is a path of aperture window B_n , such that for any $n \leq n'$, the restriction of
 213 the path $p_{n'}$ to the window B_n is homotopic to p_n . Elements of $\pi_1^{\text{proj}}(X)$ can then naturally
 214 be called **projective loop-classes**.

215 **3.2 First example**

216 We slightly modify an example of [9]. Consider the two-dimensional (sofic) subshift on the
 217 alphabet $\{0, 1\}$ of all the configurations containing at most one 1. We show how some paths
 218 can be deformed to the trivial path. It is then easy to show that *all* paths are homotopic to
 219 the trivial path. Take an aperture window of size 1, *i.e.*, only one cell is visible at a time.
 220 Consider the following path p , starting at $(0, (0, 0))$ (we see a 0 at the origin of the \mathbb{Z}^2 plane).
 221 The path then moves in the \mathbb{Z}^2 grid while only seeing 0's, and comes back to the origin where
 222 it now sees a 1. Then it moves away from the origin while only seeing 0's, and finally comes
 223 back to $(0, 0)$ with a 0 in the window. For simplicity, we also suppose that the path does
 224 not pass through the origin at any other time. To sum up, the path is a loop, starting and
 225 ending at $(0, (0, 0))$, which only sees 0 along the way except at one time (t_2 on the figure)
 226 where it sees a 1 at the origin. This is illustrated in Figure 1a.



(a) Example of a path that cannot be traced in a single configuration.

(b) A homotopic deformation to a path that can entirely be traced in the all-0 configuration.

■ **Figure 1** Example of a path and of a deformation of this path. Notice that the central 0 and 1 windows at t_0 and t_2 are actually located at the same point of the plane, although the figure depicts them on top of each other for the sake of clarity. Red wires can be traced in x_0 , and blue wires in x_1 . The wire of alternating colours can be traced within both, and so it is both homotopic to the initial path, and to the trivial path.

227 Let x_0, x_1 respectively be the all-zero configuration, and the configuration containing a
 228 1 at the origin. The path p can be homotopically deformed in the following way: between
 229 the times t_1 and t_3 , it can be considered to be entirely in x_1 . It can thus be deformed
 230 in this configuration by completely avoiding the origin, and joining the same points, as in
 231 Figure 1b. By definition of x_1 , this new path will now see only 0's. The resulting loop then
 232 also sees 0's at any point, and so it can be homotopically contracted to the trivial path in
 233 the configuration x_0 . This proof can be extended to make any 1 on a path "disappear" and
 234 show how any path can be contracted. In this case, it is sufficient to show that the projective
 235 fundamental group based at x_0 of this subshift is trivial.

236 **4 Realization of projective fundamental groups**

237 We are now going to prove our main result: any finitely presented group is the fundamental
 238 projective group of some SFT.

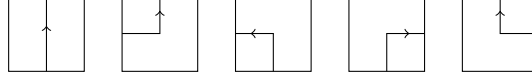
239 ► **Theorem 1.** *Let $G = \langle S|R \rangle$ a finitely presented group. Then, there is a subshift of finite
 240 type X verifying:*

- 241 ■ X is projectively connected,
- 242 ■ the projective fundamental group of X is isomorphic to G .

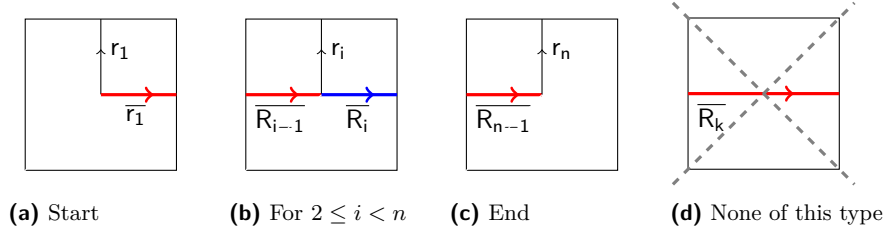
243 **4.1 The construction**

244 The subshift X that we construct will informally consist of oriented wires, drawn on an
 245 empty background, each wire corresponding to a generator $s \in S$ of the group $G = \langle S|R \rangle$.

246 We only authorize the wires to go up, perhaps in some kind of "zigzag" manner, but never
 247 down or horizontally. More precisely, we define the following tiles: first of all, a tile that we
 248 call **empty**, visually represented by \square and denoted by $\mathcal{T}_{\text{empty}}$. We denote by $x_{\square} \in X$
 249 the configuration which only contains empty tiles, and its patterns are called **empty patterns**.
 250 Then, for each element $s \in \bar{S} = S \cup \{s^{-1} | s \in S\}$, we also consider the set \mathcal{T}_s of the 5 following
 251 tiles:



252
 253 Notice that if $s \neq s'$, then $\mathcal{T}_s \cap \mathcal{T}_{s'} = \emptyset$. Distinct \mathcal{T}_s will be represented by wires of different
 254 colours in the figures. Those tiles will, intuitively, be used to represent generators of the
 255 group in valid configurations of X . In order to have some kind of representation for the
 256 relations, we need to use additional tiles. We start by adding all the trivial relators ss^{-1} and
 257 $s^{-1}s$ to R for all $s \in S$. Now, for each relator $r_1 r_2 \dots r_n \in R$, we add the tiles of Figure 2 to
 258 our set of tiles.



■ **Figure 2** The relation tiles.

259 Notice that the wire exiting from the right side of the tile Figure 2a does *not* have the
 260 same colour as the one exiting from the top. This colour is denoted by \bar{r}_1 , to differentiate it
 261 from the actual r_0 wires. In the other tiles, $\bar{R}_i = \bar{r}_1 \bar{r}_2 \dots \bar{r}_i$. Hence, for each relator $r_1 \dots r_n$, we
 262 have one tile of type Figure 2a and one of type Figure 2c, and $n - 2$ tiles of type Figure 2b.
 263 Let \mathcal{T}_r be this set of n tiles. Note that we do not introduce any tile of the form Figure 2d,
 264 and also notice that if $u \in R$ is such that it is the prefix of two different relators, *i.e.*, there
 265 exists $v, v' \in \bar{S}^*$ such that $uv \in R, uv' \in R$ then the colours \bar{u} are shared by the tiles used to
 266 represent those relators and so $\mathcal{T}_{uv} \cap \mathcal{T}_{uv'} \neq \emptyset$. X is the subshift generated by the tiling
 267 $\mathcal{T} = \mathcal{T}_{\text{empty}} \cup \bigcup_{s \in \bar{S}} \mathcal{T}_s \cup \mathcal{T}_r$ along with the obvious adjacency rules.

268 We now formalize what we really mean by a wire.

269 ► **Definition 13 (Wire).** A *wire* is a sequence $\mathcal{U} = (T_t, \mathbf{v}_t)_{t \in I}, I \subseteq \mathbb{Z}$ an interval, of pair of
 270 non-empty tiles and \mathbb{Z}^2 vectors, such that

271 ■ $\|\mathbf{v}_{t+1} - \mathbf{v}_t\|_1 = 1,$

272 ■ The tile T_{t+1} in position \mathbf{v}_{t+1} extends the tile T_t in position \mathbf{v}_t .

273 We also require that \mathcal{U} does not contain two consecutive relation tiles.

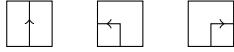
274 ► **Definition 14 (Coherent wire).** We say that a wire is **coherent** if there exists a configuration
 275 $x \in X$ such that for any tile (T_i, \mathbf{v}_i) of the wire, $x_{\mathbf{v}_i} = T_i$.

276 Valid configurations of X contain non-intersecting infinite wires, and some relation tiles
 277 with wires originating from them. Any relation tile belongs to a horizontal line of r relation
 278 tiles corresponding to a valid relator $r_1 \dots r_r$.

279 One important concept associated to paths on this subshift is the idea that paths can
 280 cross wires. Informally, this is what happens when the window, and its particular, its center,
 281 moves from one side to the other of a given wire in a path. As the tiles are considered to have
 282 their *corners* on the \mathbb{Z}^2 lattice, the central point of any pattern of support some translation
 283 of B_n is never directly "on" a wire.

284 ► **Definition 15** (Crossing a wire tile). *Let $n > 0$, and let $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}^2$ be two adjacent points,*
 285 *and P, P' two patterns of respective support $\mathbf{v} + B_n, \mathbf{v}' + B_n$ such that $(P, \mathbf{v}), (P', \mathbf{v}')$ is a*
 286 *valid path. For $(i, j) \in B_n$, let $T_{(i,j)}$ be the tile whose bottom-left corner is on (i, j) in P . We*
 287 *say that this path crosses a wire tile if*

288 ■ $\mathbf{v}' - \mathbf{v} = e_0 = (1, 0)$ (resp. $-e_0$) and the tile $T_{\mathbf{v}}$ (resp. $T_{\mathbf{v}-e_0}$) was of one of the following
 289 form:



291 ■ $\mathbf{v}' - \mathbf{v} = e_1 = (0, 1)$ (resp. $-e_1$) at the next step $t + 1$ and the tile $T_{\mathbf{v}}$ (resp. $T_{\mathbf{v}-e_1}$) was
 292 of one of the following form:



294 In the following, we let $B_n = \{-n, \dots, n - 1\}^2$. Unless stated otherwise, all the aperture
 295 windows considered will be of this form.

296 ► **Definition 16** (Coherent path). *A path $p = (P_i, \mathbf{v}_i)_{i \leq N}$ is coherent if all its patterns are*
 297 *equal on the points where their support overlap, and furthermore, the pattern obtained by*
 298 *merging all the P_i is globally admissible in X .*

299 ► **Definition 17** (Seeing a wire). *A path $p = (P_i, \mathbf{t}_i)_{i \leq N}$ sees a wire \mathcal{U} if there exists a*
 300 *timestep $i \leq N$, and $(T_j, \mathbf{v}_j) \in r$ such that the tile in position \mathbf{v}_j in P_i is T_j .*

301 ► **Remark 18.** In a finite aperture window, some tiles belonging to a same wire need not be
 302 adjacent. This is the case exactly when the wires "leaves" the window on its right or left
 303 edge, and re-enters it higher on this same edge.

304 ► **Remark 19.** If a path p sees *one* relation tile, because such a tile is necessarily part of
 305 larger pattern involving other relation tiles and other wires in any point of X , we also say
 306 that p sees those extra wires.

307 ► **Definition 20** (Crossing a wire). *A path crosses a wire if it crosses one of its tiles.*

308 ► **Definition 21** (Coherent path decomposition). *A coherent decomposition of a path p is a*
 309 *sequence p_1, \dots, p_D of coherent paths such that $p = p_1 * p_2 \dots * p_L$, and L is called the*
 310 *length of the decomposition.*

311 4.2 Only Crossed Wires Matter

312 Our final goal is to prove that the projective fundamental group of this subshift X is the
 313 group $G = \langle S|R \rangle$. To do so, the idea will be to associate an element of the group to each
 314 path, according to the wires that it crosses. The following lemmas are used, in some sense,
 315 to prove that the only thing that determines the homotopy class of a path is indeed the
 316 sequence of wires that it crosses, regardless of the underlying geometry of the path. All
 317 the lemmas consider paths that both start and end in empty patterns. This is not really a
 318 restriction, as we will later prove that the subshift X is projectively connected, and so we
 319 will only consider loops based on x_{\square} . Unless stated otherwise, all the considered paths are
 320 using B_n as an aperture window, and patterns have support B_n .

321 ► **Lemma 22** (Wire Order Lemma). *Let $x \in X$, and let \mathcal{U}, \mathcal{V} be two wires in x . Suppose that*
 322 *\mathcal{U}, \mathcal{V} do not contain relation tiles.*

323 ■ *For all $z \in \mathbb{Z}$, there exists between one and two $z_{\mathcal{U}}^0 \in \mathbb{Z}$ such that \mathcal{U} passes through the*
 324 *position $(z_{\mathcal{U}}^0, z)$. If there are two such $z_{\mathcal{U}}^0$, then they are necessarily adjacent, e.g., $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$*
 325 *side-by-side.*

326 ■ *Let $z^1 \in \mathbb{Z}$, and $z_{\mathcal{U}}^0, z_{\mathcal{V}}^0 \in \mathbb{Z}$ as in the previous point respectively for \mathcal{U} and \mathcal{V} . If $z_{\mathcal{U}}^0 < z_{\mathcal{V}}^0$,*
 327 *then for all $z_{\mathcal{U}}, z_{\mathcal{V}}, z \in \mathbb{Z}$ such that $(z_{\mathcal{U}}, z) \in \mathcal{U}, (z_{\mathcal{V}}, z) \in \mathcal{V}$, we have $z_{\mathcal{U}} < z_{\mathcal{V}}$. Intuitively,*
 328 *this means that wires can globally be ordered from left to right.*

329 *If \mathcal{U} or \mathcal{V} contains a relation tile, then the previous claims are true only for z^1 large enough.*

330 ► **Lemma 23.** *Let P be a globally admissible pattern. Let \mathcal{U} be a wire in P without relation*
 331 *tiles. Suppose that \mathcal{U} passes to the right of $(0, 0)$ in P . Then, \mathcal{U} neither enters nor exits P*
 332 *on its left edge.*

333 **Proof.** This directly follows from the fact that all the tiles with a wire have it move up when
 334 moving left, and B_n is a square. ◀

335 ► **Corollary 24.** *If P is a globally admissible pattern that sees a wire \mathcal{U} with no relation tiles,*
 336 *and $x \in X$ is such that $x|_{B_n} = P$, then $\sigma^{4n}(x)|_{B_n}$ and $\sigma^{-4n}(x)|_{B_n}$ do not see \mathcal{U} .*

337 ► **Lemma 25** (No Relation Tile Lemma). *Let p a coherent path starting and ending on an*
 338 *empty pattern. Then there exists $p' \sim p$ that does not see any relation tile.*

339 **Proof of the No Relation Tile Lemma.** Let x a configuration in which p can be traced, and
 340 which does not contain any other wire than the ones seen by p . Let (P_N, \mathbf{v}_N) the final point
 341 of p . Up to a translation of both p and x we can always assume that p starts at $(0, 0)$,
 342 and without loss of generality, suppose that \mathbf{v}_N is on the right, *i.e.*, it has a non-negative
 343 x-coordinate. This is a legitimate assumption, up to considering the path p^{-1} instead of
 344 p , which also starts and ends with empty patterns. Deform p into a path p' in x , whose
 345 trajectory only consists of moving right, and then up or down, depending on whether \mathbf{v}_N is
 346 above or below $(0, 0)$. Let i_{\min} (resp. i_{\max}) be the leftmost (resp. rightmost) position of a
 347 relation tile seen by p , and let j be the topmost one. We can deform p' as follows:

348 ■ Move left until the position $i_{\min} - 2n$ (or don't move if $i_{\min} - n \geq 0$).

349 ■ Move up until the position $j + 2n$

350 ■ Move right until $i_{\max} + 2n$

351 ■ Finally, move to \mathbf{v}_N , by moving vertically first and then horizontally.

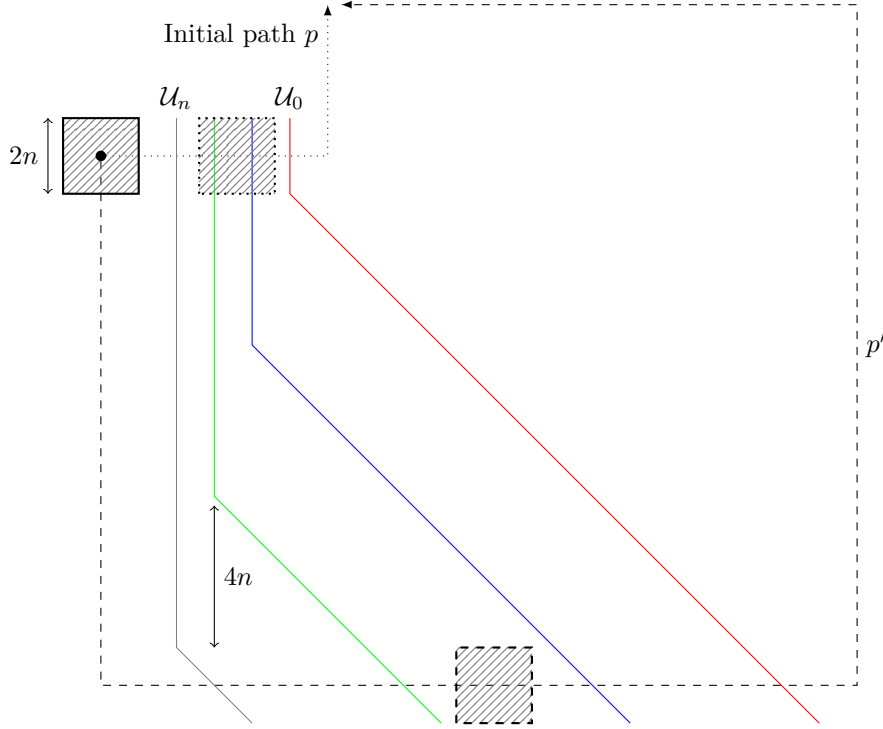
352 Let p'' the resulting path. Then, p'' does not see any relation tile. Figure 3 shows this
 353 process in a simple case, with the first and third steps being trivial.

354 ◀

355 The next two lemmas are the main tools needed in the proof of the theorem. Informally,
 356 they show that the only thing that matters on a path is the set of wires that it crosses, and
 357 that we can moreover consider all those wires independently from one another.

358 ► **Lemma 26** (Single Wire Lemma). *Let $p = (P_i, \mathbf{v}_i)_{0 \leq i \leq N}$ be a path starting and ending*
 359 *with empty patterns. There exists a path p' , homotopic to p , such that the union of any two*
 360 *consecutive patterns in p' contains at most a single wire.*

361 ► **Lemma 27** (No Uncrossed Wire Lemma). *Let p be a path starting and ending with empty*
 362 *patterns, and \mathcal{U} some wire seen but not crossed by p . There exists a path p' , homotopic to p ,*
 363 *which does not see \mathcal{U} .*



■ **Figure 4** Deformation of p into p' in a single configuration to see only one wire per pattern.

390 Let $x_1, x_2 \in X$ two configurations such that p_1, p_2 can respectively be traced entirely within
 391 them. Consider the path obtained by deforming p_1 in x_1 , so that its trajectory is following
 392 the same one as in the previous case $L = 1$: it starts from $(0, 0)$, then goes right until $(v_t^0, 0)$,
 393 and finally up until \mathbf{v}_t . To lighten the notation, we still call this path p_1 , and in the rest of
 394 the proof p_1 will refer to this deformed path.

395 Consider the loop $q = q_1 * q_1^{-1}$ in x_2 that starts from P_t, \mathbf{v}_t , and which follows the inverse
 396 trajectory to p_1 , reaches $(0, 0)$, continues to the left until seeing an empty pattern, and then
 397 comes back by the inverse path. Let $p'_1 = p_1 * q_1$ and let $p'_2 = q_1^{-1} * p_2$, so that $p = p'_1 * p'_2$.
 398 By construction, p'_2 can be traced entirely within x_2 , and so can be appropriately deformed
 399 according to the case $L = 1$. Like p , p'_1 has a decomposition of length 2, but we can further
 400 simplify it. Indeed, we show that there exists a loop $r = r_1 * r_1^{-1}$, based on (P_t, \mathbf{v}_t) , such
 401 that r_1 ends in an empty pattern and each of $p_1 * r_1$ and $r_1^{-1} * q_1^{-1}$ can be traced within
 402 a single configuration. This is enough to prove the case $L = 2$, using three times the case
 403 $L = 1$. To prove the existence of such a loop r , we construct a part of the upper-half of some
 404 configuration x_{cone} as follows:

- 405 ■ $x_{\text{cone}}|_{\mathbf{v}_t + B_n} = P_t$
- 406 ■ all the wires exiting P_t on its left half, *i.e.*, on a tile placed on (v^0, v^1) with $v^0 < v_t^0$, are
 407 then only made of tiles shaped as $\begin{array}{|c|} \hline \square \\ \hline \end{array}$ and $\begin{array}{|c|} \hline \square \\ \hline \end{array}$.
- 408 ■ all the wires exiting P_t on its right half, *i.e.*, on a tile placed on (v^0, v^0) with $v^0 \geq v_t^0$,
 409 are then only made of tiles shaped as $\begin{array}{|c|} \hline \square \\ \hline \end{array}$ and $\begin{array}{|c|} \hline \square \\ \hline \end{array}$
- 410 ■ x_{cone} does not contain any other wire than the ones seen in P_t
- 411 ■ Let $\mathcal{U}_{\text{left}}, \mathcal{U}_{\text{right}}$ respectively be the leftmost and rightmost wires of P_t . The tiles located
 412 to the left of $\mathcal{U}_{\text{left}}$, to the right of $\mathcal{U}_{\text{right}}$, and below $v_t^1 - n$, are not defined in x_{cone} .

- 413 x_{cone} is then an infinite pattern, whose shape is somewhat similar to a cone.
 414 Let x'_1 be the configuration obtained by extending x_{cone} as follows:
 415 ■ If (i, j) belongs to x_{cone} , then $x'_1(i, j) = x_{\text{cone}}(i, j)$.
 416 ■ If $j \leq v_t^1 - n$, then for any i , $x'_1(i, j) = x_1 = (i, j)$.
 417 ■ All the wires on the left of $\mathcal{U}_{\text{left}}$ in x_1 are extended in x'_1 above the $v_t^1 - n$ line by only
 418 using tiles of the shape \square and \square .
 419 ■ All the wires on the right of $\mathcal{U}_{\text{right}}$ in x_1 are extended in x'_1 above the $v_t^1 - n$ line by only
 420 using tiles of the shape \square and \square .
 421 ■ All the tiles that are not already defined in x'_1 are empty.

422 Using Wire Order Lemma, $x_{1'}$ is well-defined and belongs to X . We define x'_2 in the
 423 same way. Let r_1 be the path obtained by following the trajectory starting from \mathbf{v}_t , and
 424 moving up for $4n$ timesteps, up to $(v_t^0, v_t^1 + 4n)$ in x'_1 . The loop $r = r_1 * r_1^{-1}$ then extends
 425 both p'_1 and q_1^{-1} . Indeed, $p'_1 * r_1$ is a coherent path, as it can entirely be traced in x'_1 , and
 426 $r_1^{-1} * q_1^{-1}$ can be traced in x'_2 . Finally, we have that

$$427 \quad p \sim_{B_n} \underbrace{p_1 * r_1}_{\text{traced in } x'_1} * \underbrace{r_1^{-1} * q_1}_{\text{traced in } x'_2} * \underbrace{q_1^{-1} * p_2}_{\text{traced in } x_2}$$

428 **General case: $L > 2$** $p = p_1 * \dots * p_L$.

429 Consider the timestep t at which p_1 ends and p_2 starts. By definition of a coherent
 430 decomposition, there exists $x_2 \in X$ such that p_2 can be entirely traced within x_2 . We can
 431 suppose that x_2 does not contain any other wire than the ones seen by p_2 . Consider a loop
 432 $r = r_1 * r_1^{-1}$ that moves to an empty pattern in x_2 by moving left (this is always possible
 433 according to Lemma 23) and then comes back. We have

$$434 \quad p = p_1 * p_2 \dots * p_L = \underbrace{p_1 * r_1}_{p'_1} * \underbrace{r_1^{-1} * p_2 \dots * p_L}_{p'}$$

435 p'_1 and p' are then respectively paths of length 2 and $L - 1$, and so using the induction
 436 hypothesis, they can be deformed so as not to see \mathcal{U} . The resulting path then only sees one
 437 wire at a time. \blacktriangleleft

438 **► Lemma 28 (Cross Anywhere Lemma).** *Let p a path starting and ending with empty patterns.*
 439 *If p sees no relation tiles, but sees and crosses a single wire \mathcal{U} exactly once, then for all*
 440 *$\mathbf{v} = (v^0, v^1) \in \mathbb{Z}^2$, p is homotopic to a path p' which crosses \mathcal{U} exactly on \mathbf{v} .*

441 The proof can be found in Appendix A.

442 4.3 Projective connectedness

443 **► Lemma 29.** *X is projectively connected.*

444 **Proof.** To prove this, it suffices to show that for any configuration $x \in X$, we can find a
 445 sequence $(p_n)_{n \in \mathbb{N}}$ where each $p_n = (P_i^n, \mathbf{v}_i^n)_{0 \leq i \leq N_n}$ is a path in X_{B_n} between $(x|_{B_n}, (0, 0))$
 446 and $(x|_{\square_{B_n}}, (0, 0))$, and such that the canonical restriction of each p_{n+1} is homotopic to p_n .

447 Suppose first that x contains no wire in some octant. In order to construct a path from
 448 x to x_\square , we will simply use this empty octant as follows: For n large enough such that
 449 B_n intersects the empty octant of x , we define a path p_n by moving straight into it while
 450 staying in the configuration x , far enough so that the aperture window is entirely contained
 451 in this empty octant. At this point, we can simply come back to the point $(0, 0)$ within the

452 configuration x_{\square} . It is moreover easy to deform the restriction of a path p_{n+1} defined in this
 453 way into a path p_n of the same kind.

454 Assume now that x contains wires arbitrarily far in any octant, and let N be large enough
 455 so that $x|_{B_N}$ contains a wire on the left of $(0, 0)$. Let \mathcal{U} be the first such wire, and let $(-k, 0)$
 456 be the point at which \mathcal{U} crosses the x-axis (if there are two such points, pick the one on the
 457 right, see the Wire Order Lemma). We will define p_n for $n \geq N$. Let x_n be the configuration
 458 obtained by extending $x|_{B_n}$, only using tiles of the form $\square, \uparrow, \downarrow, \leftarrow, \rightarrow$ for its wires: \mathcal{U} and
 459 the wires on its left are extended so that they are moving to the left, and wires on its right
 460 are extended to move the right, as in the proof of the Single Wire Lemma. We can then
 461 define a path p_n between $(x|_{B_n}, (0, 0))$ and $(x_{\square|_{B_n}}, (0, 0))$ as follows:

- 462 ■ Move left until $(-k + 1, 0)$ in x_n
- 463 ■ Follow \mathcal{U} until reaching the height n , also in x_n
- 464 ■ Move up for $4n$ steps, reaching an empty pattern, still in x_n
- 465 ■ Come back to $(0, 0)$, in x_{\square}

466 Let ρ_n be the canonical restriction of p_{n+1} to the aperture window B_n . We need to show that
 467 $p_n \sim_{B_n} \rho_n$. To do so, notice that neither p_n nor ρ_n moves down. Hence, we will deform them
 468 both in the lower part of a configuration, so that we can apply one of our lemmas for paths
 469 between empty patterns to conclude that they are homotopic. Let then $r_n * r_n^{-1}$ be the loop
 470 in x_n that starts at $(0, 0)$, moves down for $4n$ steps and comes back. The paths $r_n^{-1} * p_n$ and
 471 $r_n^{-1} * \rho_n$ are both paths between empty patterns, and cross exactly the same wires, namely,
 472 the ones crossed by r_n^{-1} . Using the No Relation Tile Lemma, then the No Uncrossed Wire
 473 Lemma and finally the Cross Anywhere Lemma, they are therefore homotopically equivalent.
 474 Hence, for any $n \geq N$, $p_n \sim_{B_n} \rho_n$. Thus, there exists a projective path class between x and
 475 x_{\square} , and so X is projectively connected.

476

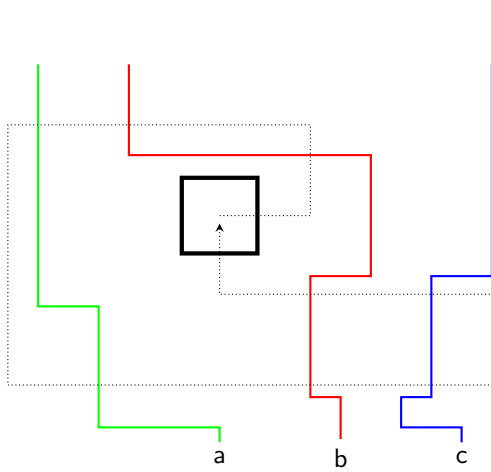
477 4.4 Computing the projective fundamental group

478 We can now compute $\pi_1^{proj}(X)$, which is independent of the basepoint since X is projectively
 479 connected. Hence, unless stated otherwise, all the loops in this proof are based on $(x_{\square}, (0, 0))$.
 480 With any such loop p , we associate a word $\llbracket p \rrbracket$ on the alphabet \bar{S} in the following way:

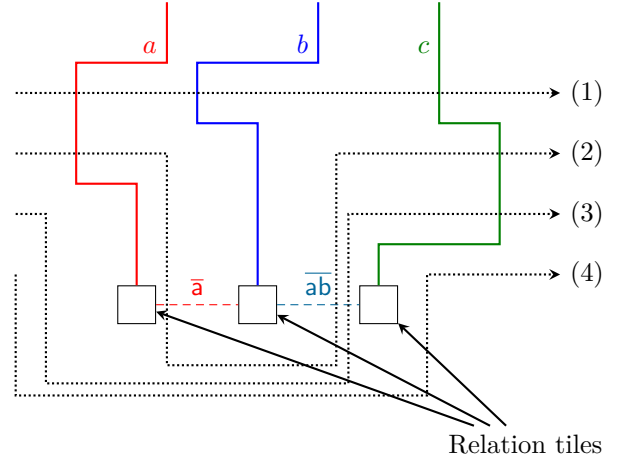
- 481 ■ If p does not cross any wire, we associate the empty word with it, $\llbracket p \rrbracket = \varepsilon$.
- 482 ■ If p crosses a single wire \mathcal{U} on a tile T , then:
 - 483 ■ If \mathcal{U} is not a horizontal wire found on relation tile, and $s \in \bar{S}$ is the generator
 484 corresponding to \mathcal{U} (see Subsection 4.1)
 - 485 * if p crosses it from left to right, or from top to bottom on a tile shaped as \uparrow , or
 486 from bottom to top on a tile \downarrow , then $\llbracket p \rrbracket = s \in \bar{S}$.
 - 487 * if p crosses it in any other direction, we set $\llbracket p \rrbracket = s^{-1} \in \bar{S}$
 - 488 ■ Otherwise, \mathcal{U} is a horizontal wire on a relation tile. Let $\bar{R}_i = \overline{r_0 \dots r_i}$ be its colour.
 - 489 * If it is crossed from top to bottom, then $\llbracket p \rrbracket = r_i^{-1} \dots r_0^{-1} \in \bar{S}^*$
 - 490 * Otherwise, $\llbracket p \rrbracket = R_i = r_0 \dots r_i$
- 491 ■ If $p = p_1 * p_2$, then $\llbracket p \rrbracket = \llbracket p_1 \rrbracket \cdot \llbracket p_2 \rrbracket \in \bar{S}^*$ where \cdot represents the concatenation in \bar{S}^* .

492 Some examples are given in Figure 5a and Figure 5b.

493 For any two words w, w' on \bar{S} , we will note $w \equiv w'$ if they are equal as words on this
 494 alphabet, and $w =_G w'$ if they represent the same element of the group G . Let \leftrightarrow_R be
 495 be the relation defined as the symmetric closure of $\left\{ (uwv, uv) \mid w \in R \text{ and } u, v \in (\bar{S})^* \right\}$,
 496 corresponding to the operation of inserting and removing relators to words. We can always



(a) The word associated with this loop is $bb^{-1}a^{-1}abcc^{-1}b^{-1} =_G 1_G$



(b) Widget for the relator $abc = 1_G$. From top to bottom, the words associated with the paths (1), (2) and (3) are respectively $abc = 1_G$, $aa^{-1}(ab)c = 1_G$, $(ab)c = 1_G$. For clarity, the relation tiles are not adjacent on the figure

497 suppose that it is reflexive by adding the empty word ε to the relators. We denote \leftrightarrow_R^* its
 498 transitive closure. By definition, $w \leftrightarrow_R^* w' \iff w =_G w'$ (see e.g., [20, Theorem 1.1]). For
 499 example, if we take $a \in S$, we have $aa^{-1} =_G 1_G$, but $aa^{-1} \neq \varepsilon$.

500 In order to prove that the projective fundamental group of this subshift is G , we will
 501 prove that the operation $\llbracket p \rrbracket$ entirely characterizes a loop up to homotopy, in the sense that
 502 loops associated with the same element of G are exactly a projective loop-class:

503 ► **Lemma 30.** For $n > 0$ and any two loops p_n, p'_n starting at $(x_{\square|B_n}, (0, 0))$,

$$504 \quad p_n \sim_{B_n} p'_n \implies \llbracket p_n \rrbracket =_G \llbracket p'_n \rrbracket$$

505 The full proof can be found in Appendix A.

506 ► **Lemma 31.** For any window B_n , and for any pair of loops p_n, p'_n starting at $(x_{\square|B_n}, (0, 0))$,

$$507 \quad \llbracket p_n \rrbracket =_G \llbracket p'_n \rrbracket \implies p_n \sim_{B_n} p'_n.$$

508 **Proof.** Using the No Relation Tile Lemma, we can always start by deforming p_n and p'_n
 509 so that they do not see any relation tile. As each elementary deformation is by definition
 510 occurring in some given configuration, Lemma 30 ensures that we still have $\llbracket p_n \rrbracket =_G \llbracket p'_n \rrbracket$. We
 511 will first prove that $\llbracket p_n \rrbracket \equiv \llbracket p'_n \rrbracket \implies p_n \sim_{B_n} p'_n$, which is a stronger assumption. Next, we
 512 prove that given p_n and p'_n with $\llbracket p_n \rrbracket =_G \llbracket p'_n \rrbracket$, there exists a loop p''_n such that $p_n \sim_{B_n} p''_n$
 513 and $\llbracket p''_n \rrbracket \equiv \llbracket p'_n \rrbracket$. We then have that $p''_n \sim_{B_n} p'_n$ according to the first part of the proof, and
 514 so $p_n \sim_{B_n} p'_n$.

515 ■ We show that $\llbracket p_n \rrbracket \equiv \llbracket p'_n \rrbracket \implies p_n \sim_{B_n} p'_n$. The paths p_n and p'_n can be deformed using
 516 No Uncrossed Wire Lemma so that they cross all the wires that they see. The Single
 517 Wire Lemma can then be used to deform them so that there is at most one of those
 518 wires per pattern. Let \hat{p}_n and \hat{p}'_n be the resulting paths, which by assumption cross the
 519 same wires. Using Cross Anywhere Lemma for each of those crossed wires, we can finally
 520 deform \hat{p}_n into \hat{p}'_n , and so $p_n \sim_{B_n} p'_n$.

521 ■ Now, we show the existence of a loop p_n'' verifying $p_n \sim_{B_n} p_n''$ and $\llbracket p_n'' \rrbracket \equiv \llbracket p_n' \rrbracket$. By
 522 definition of $=_G$, there exists a finite sequence $(u_i)_{0 \leq i \leq N}$ of words on the alphabet \bar{S}
 523 such that $\llbracket p_n \rrbracket \equiv u_0$, $\llbracket p_n' \rrbracket \equiv u_N$, and for all $i < N$, $u_i \leftrightarrow_R u_{i+1}$. To prove the result, it is
 524 therefore enough to show that for any word v such that $\llbracket p_n \rrbracket \leftrightarrow_R v$, we can deform p_n in
 525 another loop p_n^v such that $\llbracket p_n^v \rrbracket \equiv v$.

526 Suppose that v is obtained from $\llbracket p_n \rrbracket$ by deleting a relator. More formally, there exists
 527 words u_1, u_2 and a relator $r \in R$ such that $v \equiv u_1 u_2$ and $p_n \equiv u_1 r u_2$. Using the Single
 528 Wire Lemma, the No Uncrossed Wire Lemma and the Cross Anywhere Lemma as above,
 529 we can deform p_n into a path that crosses wires corresponding to the letters of $u_1 r u_2$, in
 530 order, on a horizontal line. Let p_{u_1} (resp. p_r, p_{u_2}) the part of this path which crosses the
 531 wires corresponding to u_1 (resp. r, u_2), starting and ending with empty patterns. Let
 532 $x_r \in X$ be such that p_r can be traced in x_r , and in which all those wires originate from
 533 the same set of relation tiles (see Figure 5b). We can then deform p_r in x_r into a path p_r'
 534 that passes pass below the relation tiles. The resulting path $p_n^v = p_{u_1} * p_r' * p_{u_2}$ is then a
 535 solution.

536

537 ► **Theorem 32.** $\pi_1^{proj}(X) = G$

538 **Proof.** Let $n > 0$ and let $\Phi_n : \pi_1^{B_n}(X, (x_\square, (0, 0))) \rightarrow G$ be the function which associates
 539 with a loop-class with aperture window B_n the corresponding element of G . Lemma 30 and
 540 Lemma 31 show that it is well-defined and injective. Let $[p], [p']$ be two loop-classes based on
 541 $(x_{\square|B_n}, (0, 0))$. We have shown that $[p] \sim_{B_n} [p'] \iff \Phi_n([p]) =_G \Phi_n([p'])$. Now notice that
 542 $\Phi_n([p * p']) =_G \Phi_n(p) \cdot_G \Phi_n(p')$, *i.e.*, Φ_n is a group morphism. Being obviously surjective, it
 543 is in fact an isomorphism.

544 Furthermore, notice that for any loop-class $[p_{n+1}]$ based on $(x_{\square|B_{n+1}}, (0, 0))$, if p_{n+1}
 545 projects down to p then $\Phi_{n+1}([p_{n+1}]) =_G \Phi_n([p])$. This shows that the inverse limit of the
 546 system $(\pi_1^{B_n}(X, (x_\square, (0, 0))))_{n>0}$ is isomorphic to G . ◀

547 ——— References ———

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A Proofs

612

613 **Proof of No Uncrossed Wire Lemma.** We proceed by induction on the length of a coherent
 614 decomposition of the path, and we assume that \mathcal{U} is on the right side of the patterns. Using
 615 Single Wire Lemma, we can assume that all the patterns of p contain at most a single wire.

616 **Base case: $L = 1$** p can be traced entirely in a configuration $x \in X$.

617 In that case, we can simply deform p in x by changing its trajectory so that it always
 618 stays more than n units left from \mathcal{U} . This path can then be traced in the configuration x' ,
 619 equal to x except for the tiles of \mathcal{U} in x that are empty tiles in x' .

620 **Base case: $L = 2$** $p = p_1 * p_2$

621 Let (P_t, \mathbf{v}_t) be the final point of p_1 and the first one of p_2 . We also assume that the
 622 second coordinate of $\mathbf{v}_t = (v_t^0, v_t^1)$ is non-negative. Let $\mathbf{v}_N = (v_N^0, v_N^1)$ be the final point of
 623 the path.

624 Let $x_1 \in X$ (resp. x_2) be a configuration, containing only the wires seen by p_1 (resp. p_2),
 625 such that p_1 (resp. p_2) can entirely be traced within it. Let \mathcal{U} be the uncrossed wire. We
 626 can always assume that \mathcal{U} appears in P_t , otherwise we could consider it to be made of two
 627 distinct paths, each appearing entirely either only in p_1 or in p_2 , and so we could directly
 628 apply the case $L = 1$.

629 We deform p_1 into p'_1 inside x_1 :

- 630 ■ Starting from $(0, 0)$, it first moves to the right, until \mathcal{U} appears on the central tile of the
 631 pattern seen by p_1 .
- 632 ■ It then moves up, left or right, following \mathcal{U} : up if the central tile is \square , left then up if it
 633 is \square , and so on.
- 634 ■ Finally, once it attains the height v_t^1 , it moves left until \mathbf{v}_t if needed, which takes at most
 635 n steps.

636 We can also deform p_2 into another path p'_2 as follows:

- 637 ■ Starting from \mathbf{v}_t , move left for $\max(2n, (v_t^0 - v_N^0))$ steps. This ensures that we are far
 638 enough so as to not see \mathcal{U} anymore.
- 639 ■ Then, move vertically to height v_N^1 .
- 640 ■ Finally, move right until \mathbf{v}_N .

641 Let \mathbf{w}_1 be the last point of p'_1 before seeing \mathcal{U} , and \mathbf{w}_2 the first point of p'_2 after having seen
 642 \mathcal{U} for the last time. The Single Wire Lemma ensures that the patterns seen at both \mathbf{w}_1 and
 643 \mathbf{w}_2 are empty. This gives a decomposition

$$644 \quad p \sim p'_1 * p'_2 \sim p_{\text{start}} * p_{\mathcal{U}} * p_{\text{end}}$$

645 where p_{start} ends at \mathbf{w}_1 , $p_{\mathcal{U}}$ is the part of the path between \mathbf{w}_1 and \mathbf{w}_2 , and p_{end} starts at
 646 \mathbf{w}_2 .

647 $p_{\mathcal{U}}$ can be traced entirely in a configuration x_3 whose only wire is \mathcal{U} . In this configuration,
 648 it can be homotopically deformed to $p'_{\mathcal{U}}$ which never sees \mathcal{U} according to the case $n = 1$.

649 The final path $p' = p_{\text{start}} * p'_{\mathcal{U}} * p_{\text{end}}$ does not see \mathcal{U} .

650 **General case: $L > 2$** $p = p_1 * \dots * p_L$

651 In that case, the proof is exactly the same as in Single Wire Lemma: we insert a loop
 652 before p_2 starts that extends it, and from a decomposition of length L we obtain two
 653 decompositions of length respectively 2 and $L - 1$, which are solved inductively. ◀

654 **Proof of Cross Anywhere Lemma.** Let $p = (P_i, \mathbf{v}_i)_{0 \leq i \leq N}$ be such a path, and let t be the
655 timestep at which p crosses \mathcal{U} . Without loss of generality, we can then assume that the wire
656 is crossed from left to right, *i.e.* \mathcal{U} is on the right side of P_{t-1} and on the left side of P_t . Let x
657 be any configuration containing $P_{t-1} \cup P_t$. We can suppose that $\mathbf{v}_t = e_0 + \mathbf{v}_{t-1}$, by deforming
658 p in x if needed, and that $\mathbf{v}_{t-1} = (0, 0)$. Let r_1 be the path starting from $(P_{t-1}, (0, 0))$ which
659 moves left for $4n + 2|v^0|$ steps in x , and let $r = r_1 * r_1^{-1}$. Let q_1 be the path starting from
660 $(P_t, (1, 0))$ which moves right for $4n + 2|v^0|$ steps in x , and let $q = q_1 * q_1^{-1}$. We can deform
661 p in x by inserting the loops r and q respectively at the timesteps $t - 1$ and t . Using the
662 No Uncrossed Wire Lemma twice, this path can itself be deformed into $p_{start} * p' * p_{end}$
663 with $p' = r_1^{-1} * (P_t, (0, 0)) * q_1$, and p_{start}, p_{end} paths that only see empty patterns. The
664 trajectory of p' is a straight horizontal line on the x -axis of length $8n + 2|v^0| + 1$. Let x'
665 the configuration obtained by extending \mathcal{U} as seen by p' using only tiles of the form \square .
666 Without loss of generality, suppose that $v^1 \leq 0$. We can deform p' in x' so that it moves up
667 for $8n + 2|v^0|$ steps, then right for $8n + 2|v^0| + 1$ as before and finally down to the endpoint
668 of p' . Call p'' the horizontal part of this path. There exists a configuration x'' in which \mathcal{U}
669 passes by \mathbf{v} and in which p' can be traced. Then, p'' can be deformed in x'' to cross \mathcal{U} on \mathbf{v} .
670 This finally gives the result. \blacktriangleleft

671 **Proof of Lemma 30.** As any two homotopic loops can be obtained from one another by a
672 sequence of elementary deformations, we can restrict ourselves to the special case of a single
673 deformation that is a loop based at (P_t, \mathbf{v}_t) . By definition, this deformation is made in a
674 single configuration $x \in X$. We consider two disjoint cases, according to the presence of
675 relation tiles in x .

- 676 ■ Suppose that x does not contain any relation tile. Any bi-infinite wire splits the space
677 in two disjoint regions (a "left" one and a "right" one). Each time a loop crosses such a
678 wire, it has to cross it in the other direction to come back to its initial region. Because
679 wires do not intersect, the associated word will be some kind of Dyck word, where each
680 $s \in \bar{S}$ can act as an opening or a closing bracket (in which case, the associated closing
681 (resp. opening) bracket is s^{-1}), so it is clearly equal to 1_G in G . This is the simple case
682 depicted in Figure 5a.
- 683 ■ Now, suppose that x does contain some relation tiles. In this case, notice that any two
684 relation tiles are either part of the same relator and are therefore linked by a finite
685 sequence of horizontal relation tiles, or they are independent (not linked by any wire).
686 Hence, we can consider each one of those patterns separately. Consider such a pattern,
687 with relation tiles that implement a relator $r = r_0 \dots r_k \in R$, and a configuration x' that
688 only contains this pattern. Figure 5b represents this in a configuration corresponding
689 to relation $abc = 1$. We show that, due to how $\llbracket \cdot \rrbracket$ has been defined, all the homotopy-
690 equivalent paths in x' are associated with the same element of G . Let $\mathcal{U}_0, \dots, \mathcal{U}_k$ be the
691 wires corresponding respectively to r_0, \dots, r_k , and suppose that the relation tiles in x' are
692 placed on $(0, 0), \dots, (k, 0)$. We will show that for any p joining $(0, 0)$ to $(k + 1, 0)$ in x' ,
693 $\llbracket p \rrbracket =_G 1_G$. Let $\mathcal{R} \subset \mathbb{Z}^2$ be the set of points above the $(\mathbb{Z}, 1)$ line and between \mathcal{U}_0 and \mathcal{U}_k .
694 We can always suppose that no wire is crossed consecutively in opposite directions, as
695 the word associated to a path that crosses a wire in a direction and immediately crosses
696 it in the other direction is $ss^{-1} =_G 1_G$ for some $s \in \bar{S}^*$. We can also suppose that p only
697 enters and then leaves \mathcal{R} once. Otherwise, we can simply split it into several such paths
698 and prove the claim for each of them independently.
 - 699 ■ If p crosses $\mathcal{U}_0, \dots, \mathcal{U}_k$, then $\llbracket p \rrbracket \equiv r_0 \dots r_k =_G 1_G$ by definition.
 - 700 ■ If p crosses $\mathcal{U}_0, \dots, \mathcal{U}_i, \mathcal{U}_{\overline{r_0 \dots r_i}}$, where $\mathcal{U}_{\overline{r_0 \dots r_i}}$ is a wire of a relation tile which is necessarily
701 crossed from top to bottom, by definition, $\llbracket p \rrbracket \equiv r_0 \dots r_i (r_i^{-1} \dots r_0^{-1}) =_G 1_G$

702 – Otherwise, p crosses $\mathcal{U}_{r_0 \dots r_i}, \mathcal{U}_{i+1}, \dots, \mathcal{U}_j, \mathcal{U}_{r_0 \dots r_j}$, the first relation tile being crossed from
703 bottom to top to enter \mathcal{R} and the last one being crossed from top to bottom to exit it.
704 By definition, $\llbracket p \rrbracket \equiv (r_0 \dots r_i) r_{i+1} \dots r_j (r_j^{-1} \dots r_0^{-1}) =_G 1_G$
705 This shows that all the paths traced in a single configuration are associated with the same
706 element of the group G . As all homotopies are deformations in a given configuration, this
707 implies that for any homotopically equivalent paths p, p' , we have $\llbracket p \rrbracket =_G \llbracket p' \rrbracket$. ◀