Robust Global Asymptotic Stabilization of Linear Cascade Systems with Hysteresis Interconnection

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***Abstract*— We address the problem of setpoint regulation for cascaded minimum-phase linear systems interconnected through a scalar hysteresis, modeled as a Prandtl-Ishlinskii operator. Employing well-posed constrained differential inclusions to represent the hysteretic dynamics, we formulate the control problem in terms of stabilization of a compact set of equilibria depending on the hysteresis states. For our design, we firstly consider a proportional-integral controller for linear systems with hysteretic input, and provide model-free sufficient conditions based on high-gain arguments for closed-loop stability. Then, the controller is dynamically extended to obtain an inversion-free stabilizer of the overall cascade. For the presented schemes, we prove robust global asymptotic stability of a compact set that ensures setpoint regulation, regardless of the hysteresis states.**

I. Introduction

Smart materials are becoming widespread nowadays due to their unique sensing and actuation capabilities [1], leading to innovative solutions in several application domains such as next-generation mechatronic technologies. For control systems involving smart materials, one of the major challenges is the typical presence of hysteretic behaviors, causing performance deterioration if not correctly addressed. In this context, the control-theoretical literature has dedicated several works to hystereses represented with the Prandtl-Ishlinskii (PI) operator [2], particularly useful given the existence of its analytical inverse [3]. Numerous control strategies have been developed based on the explicit inversion of the PI operator [4], [5], [6]. However, the computational burden or the inversion inaccuracy has led to inversion-free or implicit inversion solutions, involving integral control [7], adaptive control [8], or adaptive conditional servo-compensation [9].

All the works mentioned above consider systems with hysteretic inputs. In this note, instead, we address inversion-free setpoint regulation for linear cascade systems where a finite-dimensional PI operator affects the interconnection. This problem is inspired by operator-based models of thermal shape memory alloys (SMAs), comprising a thermal and a mechanical subsystem where the temperature influences the elastic behavior through a hysteresis [10], [11]. Although highly nonlinear and coupled models typically describe thermal SMAs, we believe that the methodologies developed in this work are instrumental in addressing those systems, paving the way towards robust controllers for a large class of smart actuators. Specifically, we consider a cascade where the system affecting the hysteresis input is an integrator (see Fig. 1). This choice is made to streamline the presentation, as the proposed strategy is straightforwardly extended to any linear system that can be robustly stabilized by output feedback. On the other hand, we require that the system with hysteretic input has minimum phase and relative degree 1, while we aim to relax this assumption in future works.

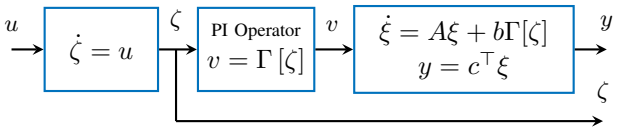

Fig. 1. Block representation of system (1).

Our design is based on the representation of the PI operator as a sum of stop operators, which can then be modeled as a well-posed differential inclusion constrained in a compact set [12, Ch. 2]. In this setting, we develop a setpoint regulation framework based on the analysis tools for hybrid dynamical systems [13]. We firstly address the scenario wherein the hysteresis input can be assigned as a control input. In that context, we show that a proportional controller ensures practical regulation, whereas the inclusion of an integral action enables global exponential stabilization. As compared to the integrator-based approach of [7], we do not rely on an LMI to tune the control gains. Instead, we provide high-gain arguments that ensure quadratic stability without any knowledge of the system matrices and of the hysteresis. The proportional-integral controller is then combined with a high-gain filter of the regulation error to obtain the design for the overall cascade. For the closed-loop dynamics, we prove the existence of a globally robustly asymptotically stable attractor ensuring setpoint regulation, regardless of the hysteresis states.

The paper is organized as follows. First, in Section II, we introduce the considered class of systems, present the state-space hysteresis model, and define the control problem. Section III is dedicated to the control design for linear systems with hysteretic input. Next, Section IV develops the overall control scheme and provides the main stability result. Finally, Section V reports numerical simulations and Section VI concludes the paper.

II. Linear Cascade Systems with Hysteresis Interconnection

A. Model Description

We consider a class of systems comprising two linear subsystems, having states $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}$, interconnected

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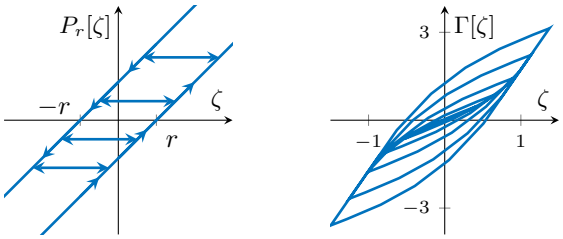

Fig. 2. Left: representation of a play operator. Right: behavior of a Prandtl-Ishlinskii operator with input $\zeta(t) = 0.04t\sin(t)$.

through a scalar hysteresis $\Gamma[\cdot]$, as shown in Fig. 1:

$$\begin{aligned}\dot{\xi} &= A\xi + b\Gamma[\zeta] \\ \dot{\zeta} &= u\end{aligned} \qquad y_a := \begin{bmatrix} y \\ \zeta \end{bmatrix} := \begin{bmatrix} c^\top \xi \\ \zeta \end{bmatrix}, \tag{1}$$

with control input $u \in \mathbb{R}$ and available output $y_a \in \mathbb{R}^2$. In (1), matrix A and vectors b and c have appropriate dimensions, while hysteresis $\Gamma[\cdot]$ has been indicated in square brackets to highlight its intrinsically dynamic behavior.

For plant (1), we design a controller ensuring asymptotic setpoint regulation of constant reference signals for the output y of the ξ-subsystem, without any knowledge of matrices A, b, and c (except for the sign of $c^\top b$) or of the hysteresis $\Gamma[\cdot]$. This objective is addressed under the following assumptions.

Assumption 1. *Pair (A, b) is reachable and pair $(c^\top, A)$ is observable.*

Assumption 2. *System $\dot{\xi} = A\xi + bv$ with output $y = c^\top \xi$ has relative degree 1 and is minimum phase. Furthermore, it holds that $c^\top b > 0$.*

B. Prandtl-Ishlinskii Operator

We model hysteresis $\Gamma[\cdot]$ in (1) as a finite-dimensional Prandtl-Ishlinskii (PI) operator, given by a weighted sum of basic hysteresis functions known as play operators. Specifically, following [14] and [7], for any continuous input $\zeta(\cdot)$ that is monotone in each interval $t \in [t_{i-1}, t_i]$ of the partition $0 = t_0 < t_1 < \ldots < t_j = T$, the output $\phi(\cdot)$ of a play operator $P_r[\cdot]$, with constant radius $r \geq 0$, is given by:

$$\phi(t) = \max\{\min\{\zeta(t) + r, \phi(t_{i-1})\}, \zeta(t) - r\}. \tag{2}$$

See Fig. 2 for a depiction of the play operator. The finite-dimensional Prandtl-Ishlinskii operator is then defined as [7]:

$$\Gamma[\zeta] := \mu_0 \zeta + \sum_{i=1}^{p} \mu_i P_{r_i}[\zeta], \tag{3}$$

where $P_{r_i}[\cdot]$, $i \in \{1, \ldots, p\}$, are play operators with constant radii r_i, while μ_i, $i \in \{0, \ldots, p\}$, are constant scalar weights. Note that the linear term multiplying μ_0 corresponds to the output of a play operator with radius $r_0 = 0$. For simplicity, we consider the case where the radii satisfy $0 < r_1 < \ldots < r_p$, while for weights μ_i we require the following property.

Assumption 3. *The weights of the Prandtl-Ishlinskii operator in* (3) *satisfy $\mu_i > 0$, for all $i \in \{0, \ldots, p\}$.*

In Fig. 2, we report the behavior of a Prandtl-Ishlinskii operator for an illustrative input signal.

C. State-Space Representation via Constrained Differential Inclusions

We provide a state-space representation of the PI operator (3) based on the reformulation proposed in [12, Ch. 2] of the play operator as a constrained differential inclusion. Specifically, we employ the complement of the play operator $P_r[\cdot]$, called stop operator and denoted by $S_r[\cdot]$. Indeed, for all $i \in \{1, \ldots, p\}$ and for an appropriate initialization of the operators, it holds that $P_{r_i}[\zeta] + S_{r_i}[\zeta] = \zeta$, where $\delta_i := S_{r_i}[\zeta]$ has a dynamic behavior modeled as [12, Ch. 2]:

$$\begin{aligned}\dot{\delta}_i &\in \begin{cases} \overline{\text{co}}\{\dot{\zeta}, \max\{\dot{\zeta}, 0\}\} & \delta_i = -r_i, \\ \dot{\zeta} & \delta_i \in (-r_i, r_i), \\ \overline{\text{co}}\{\dot{\zeta}, \min\{\dot{\zeta}, 0\}\} & \delta_i = r_i, \end{cases} \\ \delta_i &\in \Delta_i := [-r_i, r_i],\end{aligned} \tag{4}$$

where $\overline{\text{co}}\{\cdot\}$ denotes the closed convex hull operation. System (4) can be rewritten in compact form as the constrained differential inclusion $\dot{\delta}_i \in F_{r_i}(\delta_i, \dot{\zeta})$, $\delta_i \in \Delta_i$, where the set-valued map $F_{r_i}(\delta_i, \dot{\zeta})$ is outer semicontinuous, locally bounded relative to $\Delta_i \times \mathbb{R}$, and for each $(\delta_i, \dot{\zeta}) \in \Delta_i \times \mathbb{R}$ it is nonempty and convex. These properties are particularly meaningful because, with the selection of an appropriate control strategy, it is possibile to obtain a closed-loop system whose data satisfy the so-called hybrid basic assumptions [13, As. 6.5]. Under these assumptions, global asymptotic stability of a compact set implies robust global $\mathcal{KL}$ asymptotic stability in the presence of fairly general perturbations of the dynamics [13, Def. 7.18]. From (4), $\Gamma[\zeta]$ in (3) can be defined through the dynamical system

$$\dot{\delta} \in F_\Gamma(\delta, \dot{\zeta}) := \begin{bmatrix} F_{r_1}(\delta_1, \dot{\zeta}) \\ \vdots \\ F_{r_p}(\delta_p, \dot{\zeta}) \end{bmatrix}, \quad \delta \in \Delta := \prod_{i=1}^{p} \Delta_i, \tag{5}$$

having state $\delta := (\delta_1, \ldots, \delta_p)$ and output map

$$\Gamma[\zeta] := \sum_{i=0}^{p} \mu_i \zeta - \sum_{i=1}^{p} \mu_i \delta_i = \gamma\zeta - \mu^\top \delta, \quad \gamma := \sum_{i=0}^{p} \mu_i, \tag{6}$$

where $\mu := [\mu_1 \cdots \mu_p]^\top$ is a constant vector of positive parameters. Note that $\gamma > 0$ in (6) by Assumption 3.

The state-space representation of system (1) is given by the following well-posed constrained differential inclusion:

$$\begin{aligned}\dot{\xi} &= A\xi + b(\gamma\zeta - \mu^\top \delta) \\ \dot{\delta} &\in F_\Gamma(\delta, u) \\ \dot{\zeta} &= u\end{aligned} \qquad \begin{bmatrix} \xi \\ \delta \\ \zeta \end{bmatrix} \in \mathbb{R}^n \times \Delta \times \mathbb{R}. \tag{7}$$

D. Problem Statement

For systems expressed in the form (7), we consider the problem of output-feedback setpoint regulation of constant references for the output $y := c^\top \xi$. More specifically, our objective is to design an output feedback control law for u, based on the available signals $y_a := (c^\top \xi, \zeta)$ and under Assumptions 1, 2, and 3, such that:

$$\lim_{t \to \infty} y(t) = y^\star, \tag{8}$$

for any constant reference $y^\star \in \mathbb{R}$. We intend to achieve this objective through robust asymptotic stabilization of a compact set depending on $y^\star$, so that suitable stability properties will be ensured, in addition to the convergence in (8). Among other things, stability will imply uniformity of the convergence (8). To this aim, we inspect the effect of δ on the equilibria of (7) ensuring $y = y^\star$. By Assumptions 1 and 2, the system of equations

$$A\xi + bv = 0, \quad c^\top \xi - y^\star = 0, \tag{9}$$

admits a unique equilibrium pair $(\xi^\star, v^\star)$. To evaluate the steady-state value for ζ, we consider then equation

$$\gamma\zeta - \mu^\top \delta = v^\star, \tag{10}$$

which can be solved because $\gamma > 0$ by Assumption 3, but whose solution is not unique due to the presence of the hysteresis state δ. More precisely, all equilibria of (7) satisfying (10) correspond to elements of the compact set

$$\mathcal{E}_\delta \coloneqq \{(\delta, \zeta) \in \Delta \times \mathbb{R} : \gamma\zeta = \mu^\top \delta + v^\star\}. \tag{11}$$

Note that $\mathcal{E}_\delta$ is a set of equilibria because, by Assumption 3 and $\dot\delta_i \in \overline{\text{co}}\{0, \dot\zeta\}$, the definitions of γ and μ in (6) (with $\mu_0 > 0$) imply that $(\delta, \zeta) \in \mathcal{E}_\delta$ (satisfying $\gamma\dot\zeta = \mu^\top \dot\delta$) holds only if $\dot\delta = 0$ and $\dot\zeta = 0$.

Therefore, we can reformulate goal (8) by ensuring appropriate attractivity properties of the following compact *regulation set*:

$$\begin{aligned}\mathcal{E} \coloneqq &\{(\xi, \delta, \zeta) \in \mathbb{R}^n \times \Delta \times \mathbb{R} : \\ &A\xi + b(\zeta - \mu^\top \delta) = 0, c^\top \xi = y^\star\} = \{\xi^\star\} \times \mathcal{E}_\delta,\end{aligned} \tag{12}$$

for which we will also achieve a suitable stability property.

We are now ready to provide a precise statement for the considered control problem.

Problem 1. *Under Assumptions 1, 2, and 3, design a controller for the input u such that the closed-loop system admits a compact attractor that is robustly globally $\mathcal{KL}$ asymptotically stable in the sense of [13, Def. 7.18] and whose projection in the plant state directions coincides with the regulation set* (12)*.*

III. Setpoint Regulation for Systems with Hysteretic Input

This section considers a simplified scenario wherein ζ of (7) can be assigned algebraically as a differentiable control input (so that $\dot\zeta$ is well defined). In other words, we address the setpoint regulation objective $y \to y^\star$ in (8) for system

$$\begin{aligned}\dot\xi &= A\xi + b(\gamma\zeta - \mu^\top \delta) \\ \dot\delta &\in F_\Gamma(\delta, \dot\zeta) \\ y &= c^\top \xi\end{aligned} \qquad \begin{bmatrix}\xi \\ \delta\end{bmatrix} \in \mathbb{R}^n \times \Delta, \tag{13}$$

with control input $\zeta \in \mathbb{R}$. Given the equilibrium pair $(\xi^\star, v^\star)$ of (9), define $\tilde\xi \coloneqq \xi - \xi^\star$, $e \coloneqq y - y^\star$. From $A\xi^\star + bv^\star = 0$ due to (9), we compute the error dynamics for the ξ-subsystem as

$$\dot{\tilde\xi} = A\tilde\xi + b(\gamma\zeta - \mu^\top \delta - v^\star), \qquad e = c^\top \tilde\xi. \tag{14}$$

Due to the relative degree 1 assumption, under a suitable change of coordinates [15, Rem. 4.3.1], [16, §1.5], system (14) can be rewritten as

$$\begin{aligned}\dot z &= A_z z + b_z e \\ \dot e &= c_z^\top z + \alpha e + \beta\left(\zeta - (\mu^\top \delta + v^\star)/\gamma\right),\end{aligned} \tag{15}$$

where $z \in \mathbb{R}^{n-1}$ is the state of the internal dynamics, A_z, b_z, c_z, and α are matrices of suitable dimensions, while $\beta \coloneqq \gamma c^\top b > 0$ is the high-frequency gain. Due to Assumption 2, A_z is Hurwitz, thus there exists $P_z = P_z^\top > 0$ such that:

$$P_z A_z + A_z^\top P_z = -I_{n-1}. \tag{16}$$

A. Proportional Control Law

Consider the proportional controller

$$\zeta = -ke, \tag{17}$$

where k is a positive gain. The resulting closed-loop system, obtained from the interconnection between (13) and (17), is the following constrained differential inclusion

$$\begin{aligned}\dot z &= A_z z + b_z e \\ \dot e &= c_z^\top z - (\beta k - \alpha)e - \beta(\mu^\top \delta + v^\star)/\gamma \\ \dot\delta &\in F_\Gamma(\delta, -k(c_z^\top z - (\beta k - \alpha)e - \beta(\mu^\top \delta + v^\star)/\gamma)),\end{aligned} \tag{18}$$

with $(z, e, \delta) \in \mathbb{R}^n \times \Delta$. Since δ belongs to the compact set Δ, the (z, e)-subsystem is a linear system with bounded input disturbance $\mu^\top \delta + v^\star$. Due to its structure, it is possible to choose k sufficiently large to ensure that its solutions are bounded. To show this property, consider the Lyapunov function $V_k(z, e) \coloneqq z^\top P_z z + e^2$, with $P_z = P_z^\top > 0$ selected as in (16). Choose the proportional gain

$$k > k^\star \coloneqq \tfrac{1}{\beta}(\alpha + |P_z b_z + c_z|^2). \tag{19}$$

Then, along the solutions of (18), we obtain

$$\begin{aligned}\dot V_k = &-|z|^2 + 2z^\top(P_z b_z + c_z)e - 2(\beta k - \alpha)e^2 \\ &- 2\beta e(\mu^\top \delta + v^\star)/\gamma \\ \le &-\tfrac{1}{2}|z|^2 - 2\beta|e|((k - k^\star)|e| - |\mu^\top \delta + v^\star|/\gamma),\end{aligned}$$

which implies, from the boundedness of $\delta \in \Delta$ and $v^\star$:

$$\begin{aligned}|e| > e_M \coloneqq \frac{\max_{\delta\in\Delta}\{|\mu^\top \delta + v^\star|\}}{\gamma(k - k^\star)} &\implies \dot V_k < 0, \\ |z| > \frac{2\max_{\delta\in\Delta}\{|\mu^\top \delta + v^\star|\}}{\gamma\sqrt{(k - k^\star)/\beta}} &\implies \dot V_k < 0,\end{aligned} \tag{20}$$

with the second inequality obtained by splitting the analysis in $|e| \ge e_M$ and $|e| \le e_M$. From (20), (z, e) eventually converges to compact sublevel sets of V_k, thus the solutions of (18) are bounded. Moreover, (z, e) approaches an arbitrarily small neighborhood of $(0, 0)$ (i.e., ξ approaches an arbitrarily small neighborhood of $\xi^\star$) for $k > 0$ sufficiently large.

B. Proportional-Integral Control Law

We have shown that controller (17) ensures global boundedness of solutions but cannot achieve $(z,e)\to 0$. Thus, in place of (17), consider a controller with an integral action:

$$\dot\sigma = -he, \qquad \zeta = \sigma - ke, \tag{21}$$

where $\sigma\in\mathbb{R}$ is the integrator state and k, h are the proportional and integral gains. Define $\tilde\sigma := \sigma - (\mu^\top\delta + v^\star)/\gamma$. Using (15), the interconnection among (13) and (21) yields:

$$\begin{aligned}
&\dot z = A_z z + b_z e\\
&\dot e = c_z^\top z - (\beta k-\alpha)e + \beta\tilde\sigma\\
&\dot{\tilde\sigma} \in -he - \frac{\mu^\top}{\gamma} F_\Gamma(\delta, -he - k(c_z^\top z - (\beta k-\alpha)e + \beta\tilde\sigma))\\
&\dot\delta \in F_\Gamma(\delta, -he - k(c_z^\top z - (\beta k-\alpha)e + \beta\tilde\sigma)),
\end{aligned} \tag{22}$$

with $(z,e,\tilde\sigma,\delta)\in\mathbb{R}^{n+1}\times\Delta$. We now expand the dynamics of $\tilde\sigma$ in terms of the components δ_i that lie in the linear region. Namely, inspired by (4), introduce

$$q_i(\delta_i,\dot\zeta) := \begin{cases} 0, & \text{if } |\delta_i| = r_i \text{ and } \delta_i\dot\zeta \geq 0,\\ 1, & \text{otherwise,}\end{cases} \tag{23}$$

so that we can write $\dot\delta_i = q_i(\delta_i,\dot\zeta)\dot\zeta$, then define

$$q(\delta,\dot\zeta) := \frac{1}{\gamma}\sum_{i=1}^{p}\mu_i q_i(\delta_i,\dot\zeta). \tag{24}$$

Note that, by Assumption 3 and from (6),

$$0\leq q\leq q_{\max} := \frac{1}{\gamma}\sum_{i=1}^{p}\mu_i < 1. \tag{25}$$

Using (22) and (24), we obtain

$$\begin{aligned}
\dot{\tilde\sigma} &= -he - \mu^\top\dot\delta/\gamma = -he - q\dot\zeta\\
&= -he - q(-he - k(c_z^\top z - (\beta k-\alpha)e + \beta\tilde\sigma))\\
&= -(h(1-q) + kq(\beta k-\alpha))e + k\beta q\tilde\sigma + qkc_z^\top z.
\end{aligned} \tag{26}$$

Thus, the $(z,e,\tilde\sigma)$-subsystem in (22) can be written as

$$\begin{bmatrix}\dot z\\ \dot e\\ \dot{\tilde\sigma}\end{bmatrix} = \underbrace{\begin{bmatrix} A_z & b_z & 0\\ c_z^\top & -(\beta k-\alpha) & \beta\\ qkc_z^\top & \begin{pmatrix}-h(1-q)+\\ -kq(\beta k-\alpha)\end{pmatrix} & k\beta q\end{bmatrix}}_{:=A_{\rm s}(q)}\begin{bmatrix} z\\ e\\ \tilde\sigma\end{bmatrix}, \tag{27}$$

with the time-varying input $q\in[0,q_{\max}]\subset[0,1)$ due to (25). For system (22), we provide the following relevant result.

Theorem 1. *Let $k=\nu\bar k$ and $h=\nu^2\bar h$, with gains $\nu>0$, $\bar h>0$, and $\bar k$ such that*

$$\bar k \geq \frac{\alpha + \sqrt{\beta\bar h(2-q_{\max})}}{\beta(1-q_{\max})}. \tag{28}$$

Then, there exists $\nu^\star>0$ such that, for all $\nu>\nu^\star$, the compact attractor $\mathcal{A}_\sigma := \{(z,e,\tilde\sigma,\delta)\in\mathbb{R}^{n+1}\times\Delta : (z,e,\tilde\sigma)=0\}$ is globally exponentially stable for (22)*.*

Proof: Define $\bar\sigma := \tilde\sigma/\nu$ and $\chi := (e,\bar\sigma)\in\mathbb{R}^2$. Then, using (27), system (22) can be rewritten as

$$\begin{aligned}
&\dot z = A_z z + [b_z\ \ 0]\chi\\
&\dot\chi = \nu A(q,\nu^{-1})\chi + B(q)z, \quad q\in[0,q_{\max}]\\
&\dot\delta \in F_\Gamma(\delta, -he - k(c_z^\top z - (\beta k-\alpha)e + \beta\nu\bar\sigma)),
\end{aligned} \tag{29}$$

with $(z,\chi,\delta)\in\mathbb{R}^{n+1}\times\Delta$, $B(q) := [c_z\ \ \bar kqc_z]^\top$ and

$$A(q,\nu^{-1}) := \begin{bmatrix} -(\beta\bar k - \alpha\nu^{-1}) & \beta\\ -\bar h(1-q) - \bar kq(\beta\bar k - \alpha\nu^{-1}) & \bar kq\beta\end{bmatrix}. \tag{30}$$

Consider the auxiliary switching system:

$$\dot\psi = A(q,\nu^{-1})\psi, \qquad q(t)\in\mathcal{Q}, \tag{31}$$

where $\mathcal{Q} := \{q\in\mathbb{R} : q=\sum_{i\in\mathcal{I}}\mu_i, \mathcal{I}\subset\{1,\dots,p\}\}$. Below, we show that the origin of system (31) is GES for all $\nu\geq 1$ and for any switching sequence $t\mapsto q(t)$. Notice that the characteristic polynomial of $A(q,\nu^{-1})$ is $\lambda^2 + \lambda(\beta\bar k(1-q) - \alpha\nu^{-1}) + \beta\bar h(1-q)$, therefore, by (28), $A(q,\nu^{-1})$ is Hurwitz for all $(q,\nu)\in[0,q_{\max}]\times[1,\infty)$ because $q_{\max}<1$ and

$$\bar k > \frac{\alpha}{\beta(1-q_{\max})} \geq \frac{\alpha\nu^{-1}}{\beta(1-q)}, \qquad \bar h>0. \tag{32}$$

By [7] and [17], (31) admits a Common Quadratic Lyapunov Function (CQLF) if there exists $P_\chi = P_\chi^\top > 0$ such that:

$$\begin{aligned}
P_\chi A(0,\nu^{-1}) + A(0,\nu^{-1})^\top P_\chi \leq -I_2\\
P_\chi A(q_{\max},\nu^{-1}) + A(q_{\max},\nu^{-1})^\top P_\chi \leq -I_2.
\end{aligned} \tag{33}$$

Since $\operatorname{rank}(A(0,\nu^{-1}) - A(q_{\max},\nu^{-1})) = 1$, from [18], we have that matrix P_χ exists if and only if the matrix pencil $\Pi(g) := \beta\bar h A(0,\nu^{-1})^{-1} + gA(q_{\max},\nu^{-1})$ is invertible, for all $g\geq 0$. Straightforward computations show that

$$\begin{aligned}
&\det(\Pi(g)) = \beta\bar h(1-q_{\max})g^2 + \beta\bar h\\
&+((\beta\bar k - \alpha\nu^{-1})(\beta\bar k(1-q_{\max}) - \alpha\nu^{-1}) - \beta\bar h(2-q_{\max}))g.
\end{aligned}$$

As $\det(\Pi(0)) > 0$, it is sufficient to verify that $(\beta\bar k - \alpha\nu^{-1})(\beta\bar k(1-q_{\max}) - \alpha\nu^{-1}) - \beta\bar h(2-q_{\max}) > 0$, which is ensured for any $\nu\geq 1$ by choosing $\bar h>0$ and $\bar k$ according to (28). Returning to system (29), define

$$V_\sigma(z,\chi) := z^\top P_z z + \chi^\top P_\chi\chi, \tag{34}$$

which is positive definite and radially unbounded with respect to $\mathcal{A}_\sigma$. Along the solutions of (29), it holds that

$$\begin{aligned}
\dot V_\sigma &\leq -|z|^2 - \nu|\chi|^2 + 2z^\top(P_z b_z[1\ \ 0] + B(q)^\top P_\chi)\chi\\
&\leq -\tfrac{1}{2}|z|^2 - (\nu - 4|P_z b_z|^2 - 4|P_\chi B(q)|^2)|\chi|^2.
\end{aligned}$$

Therefore, selecting ν according to

$$\nu > \nu^\star := 4|P_z b_z|^2 + 4|P_\chi|^2|c_z|^2(1+\bar k^2 q_{\max}^2) \tag{35}$$

ensures that V_σ is a CQLF for $\mathcal{A}_\sigma$, which implies GES. □

From (28), (35), there always exist $k>0$, $h>0$ sufficiently large such that attractor $\mathcal{A}_\sigma$ of Theorem 1 is GES. The next corollary is a direct consequence of the proof of Theorem 1.

Corollary 1. *Choose gains k and h so that Theorem 1 establishes GES of $\mathcal{A}_\sigma$. Then, there exists a CQLF for* (22)*. Namely, there exists $P_{\rm s} = P_{\rm s}^\top > 0$ such that $P_{\rm s}A_{\rm s}(q) + A_{\rm s}(q)^\top P_{\rm s} \leq -I_{n+1}$, for all $q\in[0,q_{\max}]\subset[0,1)$, with $A_{\rm s}(q)$ defined in* (27)*.*

IV. Setpoint Regulation for Cascade Systems with Hysteresis Interconnection

We return to the problem of setpoint regulation for system (1), i.e., we address the problem of designing an input u such that $y \to y^\star$. The key idea proposed here is to first generate a reference $\zeta^\star$ that solves the regulation problem for the reduced plant (13), then impose $\zeta \to \zeta^\star$ by a suitable selection of u for the complete plant (7). To achieve this objective, we require $\dot{\zeta}^\star$ to be well defined and available for control, which implies that (21) cannot be employed directly as $\dot{e}$ is unknown. Thus, we assign $\zeta^\star$ through a modified version of (21) comprising a filter of the tracking error:

$$\dot{\sigma} = -h\eta, \qquad \dot{\eta} = -\ell(\eta - e), \qquad \zeta^\star = \sigma - k\eta, \quad (36)$$

where $\sigma \in \mathbb{R}$ is the integrator state, $\eta \in \mathbb{R}$ is the filter state, while ℓ, h, and k are positive scalar gains. For convenience in the analysis, we define the error variables

$$\tilde{\sigma} \coloneqq \sigma - (\mu^\top \delta + v^\star)/\gamma, \qquad \tilde{\eta} \coloneqq \eta - e. \quad (37)$$

Then, using (15), (24), and $\zeta = \sigma - k\eta + \zeta - \zeta^\star$, we can write the interconnection between plant (7) and controller (36) as

$$\begin{aligned}
\dot{z} &= A_z z + b_z e \\
\dot{e} &= -(\beta k - \alpha)e + \beta\tilde{\sigma} + c_z^\top z - \beta k\tilde{\eta} + \beta(\zeta - \zeta^\star) \\
\dot{\tilde{\sigma}} &= -h\eta - q\dot{\zeta} \\
\dot{\tilde{\eta}} &= -(\ell\tilde{\eta} - (\beta k - \alpha)e + \beta\tilde{\sigma} + c_z^\top z - \beta k\tilde{\eta} + \beta(\zeta - \zeta^\star)) \\
\dot{\zeta} &= u \\
\dot{\delta} &\in F_\Gamma(\delta, \dot{\zeta}),
\end{aligned} \quad (38)$$

with $(z, e, \tilde{\sigma}, \tilde{\eta}, \zeta, \delta) \in \mathbb{R}^{n+3} \times \Delta$.

A. Stability Analysis of the Reduced System

We study system (38) under the assumption that

$$\zeta = \zeta^\star = \sigma - k\eta \quad (39)$$

is satisfied at all times. This reduction argument is instrumental to introducing the controller of Section IV-B. Define:

$$x_\mathrm{f} \coloneqq \ell\tilde{\eta} - (\beta k - \alpha)e + \beta\tilde{\sigma} + c_z^\top z. \quad (40)$$

With (36), (37) we have $\dot{\zeta} = \dot{\zeta}^\star = -h\eta + k\ell\tilde{\eta}$, and then using (39), (40) we can follow similar computations to (26) exploiting $\mu^\top\dot{\delta}/\gamma = q\dot{\zeta}$ to show that

$$\begin{aligned}
\dot{\tilde{\sigma}} = &-(h(1-q) + kq(\beta k - \alpha))e + k\beta q\tilde{\sigma} + qkc_z^\top z \\
&-qkx_\mathrm{f} - \ell^{-1}h(1-q)(x_\mathrm{f} + (\beta k - \alpha)e - \beta\tilde{\sigma} - c_z^\top z).
\end{aligned} \quad (41)$$

Define $x_\mathrm{s} \coloneqq (z, e, \tilde{\sigma}) \in \mathbb{R}^{n+1}$. Then, in the coordinates $x \coloneqq (x_\mathrm{s}, x_\mathrm{f})$, system (38) with condition (39) reads as

$$\begin{aligned}
\dot{x}_\mathrm{s} &= A_\mathrm{s}(q)x_\mathrm{s} + B_\mathrm{s}(q)x_\mathrm{f} + \ell^{-1}D_\mathrm{s}(q)x \\
\dot{x}_\mathrm{f} &= -\ell x_\mathrm{f} + D_\mathrm{f}(q, \ell^{-1})x \\
\dot{\delta} &\in F_\Gamma(\delta, \dot{\zeta})
\end{aligned} \quad (42)$$

where $A_\mathrm{s}(q)$ is found in (27), $B_\mathrm{s}(q) \coloneqq [0^\top\ 0\ -qk]^\top$, while $D_\mathrm{s}(q)$ and $D_\mathrm{f}(q, \ell^{-1})$ are matrices of appropriate dimensions. In particular, $D_\mathrm{f}(q, \ell^{-1})$ is an affine function of ℓ^{-1}, so that its entries are bounded as $\ell \to \infty$.

Using ℓ, the analysis of (42) can be performed via timescale separation, where the x_f-subsystem (the fast subsystem) is made arbitrarily fast with respect to the x_s-subsystem (the slow subsystem) by selecting $\ell > 0$ sufficiently large. These arguments lead to the following result.

Proposition 1. *Choose gains k and h so that Theorem 1 establishes GES of $\mathcal{A}_\sigma$. Then, there exists $\ell^\star > 0$ such that, for all $\ell > \ell^\star$, attractor $\mathcal{A}_\eta \coloneqq \{(x_\mathrm{s}, x_\mathrm{f}, \delta) \in \mathbb{R}^{n+2} \times \Delta : (x_\mathrm{s}, x_\mathrm{f}) = 0\}$ is GES for system* (42)*.*

Proof: Pick P_s from Corollary 1, then define $d \coloneqq 2|P_\mathrm{s}|^2k^2q_{\max}^2 \geq 2|P_\mathrm{s}B_\mathrm{s}(q)|^2$ for all $q \in [0, q_{\max}]$ and

$$V_\eta(x) \coloneqq x_\mathrm{s}^\top P_\mathrm{s} x_\mathrm{s} + d\ell^{-1}x_\mathrm{f}^2, \quad (43)$$

which is positive definite and radially unbounded with respect to $\mathcal{A}_\eta$. Then, we obtain:

$$\begin{aligned}
\dot{V}_\eta \leq &-|x_\mathrm{s}|^2 + 2x_\mathrm{s}^\top P_\mathrm{s}B_\mathrm{s}x_\mathrm{f} - 2dx_\mathrm{f}^2 \\
&+ 2\ell^{-1}(x_\mathrm{s}^\top P_\mathrm{s}D_\mathrm{s} + dx_\mathrm{f}D_\mathrm{f})x \\
\leq &-\min\{\tfrac{1}{2}, d\}|x|^2 + 2\ell^{-1}(|P_\mathrm{s}D_\mathrm{s}| + d|D_\mathrm{f}|)|x|^2.
\end{aligned} \quad (44)$$

Recalling that $\lim_{\ell\to\infty} \ell^{-1}(|P_\mathrm{s}D_\mathrm{s}(q)| + d|D_\mathrm{f}(q, \ell^{-1})|) = 0$, for $\ell > 0$ sufficiently large, $\dot{V}_\eta$ can be made quadratically negative definite, ensuring that V_η is a CQLF for $\mathcal{A}_\eta$. □

B. Main Result

Differently from (39), we now address the general case of the augmented plant (38) by removing the assumption that $\zeta = \zeta^\star$. Thus, we define the error $\tilde{\zeta} \coloneqq \zeta - \zeta^\star$, whose dynamics are given, by (38) with (36) and (37), as $\dot{\tilde{\zeta}} = u - \dot{\zeta}^\star = u + h\eta - k\ell(\eta - e)$. Therefore, we can ensure $\tilde{\zeta} \to 0$ by selecting $u = \dot{\zeta}^\star - k_\zeta\tilde{\zeta}$, with gain k_ζ, leading to:

$$\dot{\tilde{\zeta}} = -k_\zeta\tilde{\zeta}. \quad (45)$$

Using (36), this yields the following controller for plant (7):

$$\begin{aligned}
\dot{\eta} &= -\ell(\eta - e) \\
\dot{\sigma} &= -h\eta \\
u &= -h\eta + k\ell(\eta - e) - k_\zeta(\zeta - \sigma + k\eta),
\end{aligned} \quad (46)$$

with positive gains ℓ, h, k, and k_ζ. The following statement, which confirms that controller (46) provides a solution for Problem 1, is the main result of this work.

Theorem 2. *Choose any gain $k_\zeta > 0$ and positive gains k, h, ℓ such that $\mathcal{A}_\eta$ of Proposition 1 is GES. Then, attractor $\mathcal{A} \coloneqq \{(\xi, \delta, \zeta, \eta, \sigma) \in \mathbb{R}^n \times \Delta \times \mathbb{R}^3 : (\xi, \delta, \zeta) \in \mathcal{E}, \eta = 0, \sigma = \zeta\}$, with $\mathcal{E}$ being the regulation set of* (12)*, is robustly globally $\mathcal{KL}$ asymptotically stable for the interconnection between system* (7) *and controller* (46)*.*

Proof: The closed-loop system can be seen as the cascade interconnection of (45) and (42) perturbed by $\tilde{\zeta}$. Namely, the x-subsystem can be rewritten as

$$\dot{x} = A_x(q)x + B_x(q)\tilde{\zeta}. \quad (47)$$

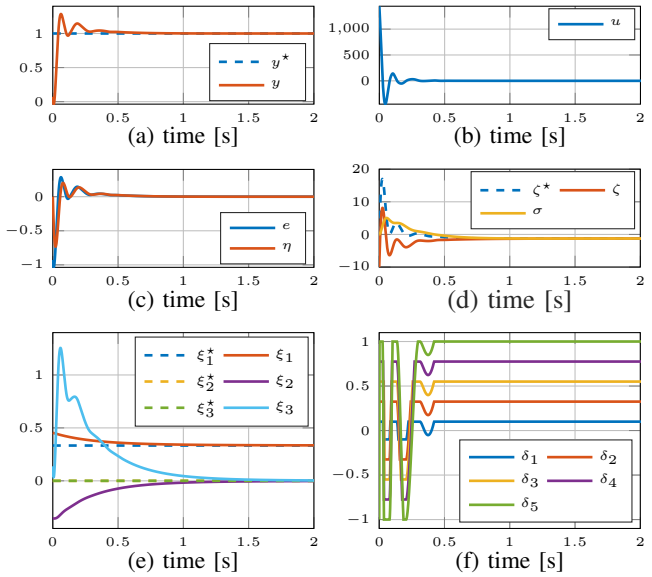

Fig. 3. Closed-loop simulation results. (a): reference $y^\star$ and output y. (b): input u. (c): error e and filter state η. (d): reference $\zeta^\star$, state ζ, and integrator state σ. (e): equilibrium $\xi^\star$ and states ξ. (f): hysteresis states δ.

Consider the Lyapunov function $V := V_\eta + \rho\tilde{\zeta}^2$, where V_η is defined in (43) and $\rho > 0$. By Proposition 1, there exists a constant matrix $Q_x = Q_x^\top > 0$ such that

$$\dot{V} \leq -x^\top Q_x x + \tilde{\zeta} B_x(q) x - 2\rho k_\zeta \tilde{\zeta}^2. \tag{48}$$

Choosing $\rho > 0$ sufficiently large, we conclude that $\mathcal{A}$ is GES. Finally, we note that system (7), (46) can be regarded as a hybrid system, with empty jump set and jump map, satisfying the hybrid basic conditions. Thus, from [13, Thm. 7.21], $\mathcal{A}$ is robustly globally $\mathcal{KL}$ asymptotically stable. □

V. Numerical Results

We perform a numerical analysis to illustrate our theoretical results. Following the structure depicted in (1) and Assumptions 1 and 2, we consider a ξ-subsystem having transfer function

$$G(s) = \frac{(s+3)(s+1)}{(s-5)(s^2+2s+4)}. \tag{49}$$

Matrices A, b, c are then obtained as the minimal realization of (49) in controllability canonical form. The Prandtl-Ishlinskii operator $\Gamma[\zeta]$, whose behavior is shown in Fig. 2, is implemented according to (5) using $p = 5$ stop operators. The weights are given by $\mu = [0.1 \quad 0.325 \quad 0.55 \quad 0.775 \quad 1]^\top$, $\mu_0 = 1$, while the radii values are $r_i = \mu_i$, $i \in \{1, \dots, 5\}$. The feedback law is the one in (46) with tuning parameters selected as $\nu = 20$, $k = 20$, $h = 200$, $\ell = 75$, and $k_\zeta = 5$. Simulations have all the system states initialized randomly, and setpoint $y^\star = 1$. The results obtained for a single run are shown in Fig. 3.

VI. Conclusion

We provided a robust control strategy for linear cascade systems with a hysteretic interconnection. Employing a well-posed constrained differential inclusion to represent the Prandtl-Ishlinskii operator, we formulated the regulation problem by defining a compact set of equilibria depending on the hysteresis states. Then, relying on high-gain arguments and the properties of well-posed hybrid dynamical systems, we proved that the closed-loop system admits a robustly globally asymptotically stable compact attractor for a simple selection of the controller gains. This result was achieved without any parametric knowledge of the system or the hysteresis. Future works will focus on generalizing the cascade dynamics and the hysteresis model, and applying the results to thermal shape memory alloy technologies.

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