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► **To cite this version:**

Jernej Činč, Serge Troubetzkoy. AN UPPER BOUND ON TOPOLOGICAL ENTROPY OF THE BUNIMOVICH STADIUM BILLIARD MAP. 2022. hal-03621869v1

HAL Id: hal-03621869

<https://hal.science/hal-03621869v1>

Preprint submitted on 28 Mar 2022 (v1), last revised 20 Jul 2023 (v3)

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AN UPPER BOUND ON TOPOLOGICAL ENTROPY OF THE BUNIMOVICH STADIUM BILLIARD MAP

JERNEJ ČINČ AND SERGE TROUBETZKOY

ABSTRACT. We show that the topological entropy of the billiard map in a Bunimovich stadium is at most $\log(3.4908)$.

1. INTRODUCTION

The Bunimovich stadium is a planar domain whose boundary consists of two semicircles joined by parallel segments as in Figure 1. In this article we study the billiard in a Bunimovich stadium, this is the free motion of a point particle in the interior of the stadium with elastic collisions when the particle reaches the boundary. Billiards in stadia were first studied by Bunimovich in [5, 6] where he showed that the billiard has hyperbolic behavior and showed the ergodicity, K-mixing and Bernoulli property of the billiard map and flow with respect to the natural invariant measure (see also [13],[14]).

In this article we will study the topological entropy of the billiard map in a Bunimovich stadium. The topological entropy of a topological dynamical system is a nonnegative number that is a measure of the complexity of the system. Roughly, it measures the exponential growth rate of the number of distinguishable orbits as time advances. We will discuss its exact definition in our setting in the next section.

The study of topological entropy of billiards was initiated in [12]. In this article it was claimed with a one sentence proof that the topological entropy of the billiard map of stadia is at most $\log(4)$. A detailed proof using this strategy was given later by Bäker and Chernov, but they were able to show only a weaker estimate, that the topological entropy is at most $\log(6)$ [2]. Our main result will be a better upper bound on the topological entropy.

Recently, Misiurewicz and Zhang [17] have shown that as the side length tends to infinity the topological entropy of stadia is at least $\log(1 + \sqrt{2})$ by studying the map restricted to a subspace of the phase space which is compact and invariant under the billiard map. Another lower bound of the topological entropy can be derived from the variational principle¹ and the results of Chernov on the asymptotics of the metric entropy when the stadium degenerates to a circle, an infinite stadium, a segment, a point, or the plane in certain controlled ways [11].

Topological entropy of hyperbolic billiards has also been studied in several other articles [3],[7],[10],[20].

2. DEFINITIONS AND STATEMENT OF THE RESULTS

We consider the Bunimovich stadium billiard table B_l , with the radius of the semicircles 1, and the lengths of straight segments $l > 0$. The phase space of this

Date: March 28, 2022.

1991 Mathematics Subject Classification. 37D50, 37B40.

Key words and phrases. Bunimovich stadium billiard, topological entropy.

¹The variational principle holds for the Pesin-Pitskel' definition of entropy [19], see Section 5 for applicability to our situation.

billiard map will be denoted by M_l , it consists of points s in the boundary of B_l and unit vectors pointing into the interior of B_l . We represent the unit vector by measuring its angle θ with respect to the inner pointing normal vector, thus

$$M_l := \{(s, \theta) : s \in \partial B_l, \theta \in (-\pi/2, \pi/2)\}.$$

The billiard map F_l is the first return map of the billiard flow to the set M_l . Note that F_l is continuous, but M_l is not compact since we do not include vectors tangent to the boundary of B_l .

We remark that the map F_l does not extend to a continuous map of the closure of M_l . Thus the usual definitions of the topological entropy due to Adler, Konheim and McAndrew [1], Bowen [4] and Dinaburg [16] can not be applied. There are several definitions of topological entropy which are possible. The definition we take, is a very natural one, we take a natural coding of the billiard, and then consider the entropy of the shift map on the closure of the set of all possible codes. This definition gives an upper bound of another natural definitions of topological entropy on non-compact spaces, the Pesin-Pitskel' [19] topological entropy. In particular, similar results for Sinai billiards (also known as Lorenz gas) were recently obtained by Baladi and Demers [3]. For a more detailed discussion of possible definitions of topological entropy in our setting and their relationship to our definition see Section 5.

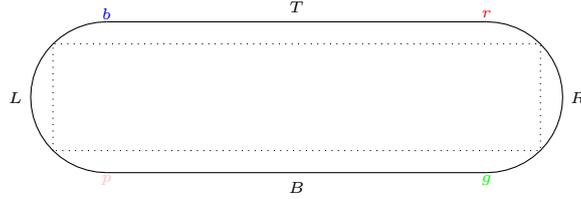


FIGURE 1. Labeling the sides of the stadium and a period 4 orbit.

We now give a precise definition of the topological entropy we consider. We label the four smooth components of the boundary by the alphabet $\{L, T, R, B\}$ (see Figure 1). It is easy to see that the corresponding partition is not a generator, for example the period 4 orbit with code $LLRR$ shown in Figure 1 has the same code traced forwards and backwards.

Let $\bar{c} : M_l \rightarrow A := \{\bar{L}, \underline{L}, T, B, \bar{R}, \underline{R}\}$ be the (multi-valued) *coding map* defined by $\bar{c}(s, \theta) = s$ if $s \in \{T, B\}$, $\bar{c}(s, \theta) = \bar{s}$ if $\theta \geq 0$ and $\bar{c}(s, \theta) = \underline{s}$ if $\theta \leq 0$ for $s \in \{L, R\}$. We consider the cover of the phase space into 6 elements given by this coding. The interiors of each element of the cover are disjoint, thus with the traditional misuse of terminology we will call this cover a *partition*.

We *code* the orbit of a point by the sequence of partition elements it hits, i.e.,

$$c(s, \theta) := (\omega_k) \text{ where } \omega_k = \bar{c}(F_l^k(s, \theta)).$$

Codings are multi-valued, but for any point in the set

$$\tilde{M}_l := \{(s, \theta) \in M_l : F_l^n(s, \theta) \text{ is in the interior of a partition element } \forall n \in \mathbb{Z}\}$$

the coding is unique.

Let Σ be the *set of codes of points from \tilde{M}_l* , and let $\bar{\Sigma}$ be the closure of Σ in the product topology. We let $p(n)$ denote the *complexity of Σ* ; i.e.,

$$p(n) := \#\{(\omega_0, \dots, \omega_{n-1}) \in \Sigma\} = \#\{(\omega_0, \dots, \omega_{n-1}) \in \bar{\Sigma}\}.$$

The growth rate

$$\lim_{n \rightarrow \infty} \frac{\log p(n)}{n}$$

is the *topological entropy of the shift map restricted to the set* $\bar{\Sigma}$.

The 6 element partition is a *generating partition* in the sense that for each $\omega \in \bar{\Sigma} \setminus \{(TB)^\infty\}$ there is a unique $(s, \theta) \in M_l$ whose orbit has code ω [2]², thus it is natural to call this quantity the *topological entropy of the billiard map* F_l , i.e.,

$$h_{top}(F_l) := \lim_{n \rightarrow \infty} \frac{\log p(n)}{n}.$$

In this definition of the topological entropy we first miss a set by restricting to the interiors of partition elements, and then we add some points by taking the closure of Σ . The sequences in $\bar{\Sigma} \setminus \Sigma$ are all the codes of points which hit boundaries of partition elements obtained by using one sided continuity extension in the spatial coordinate. We do not know anything about the entropy of the invariant set $\bar{\Sigma} \setminus \Sigma$, in particular if it is smaller than the difference of estimates from Theorem 1 and from Misiurewicz and Zhang [17] or not.

Our work was originally inspired by [17] where it was shown that

$$\lim_{l \rightarrow \infty} h_{top}(F_l) \geq \log(1 + \sqrt{2}) > \log(2.4142).$$

In fact in [17] the authors identify a certain compact subset of the phase space, such that if we restrict F_l to this set then we get equality in the above limit.

Another inspiration is [2]; the above mentioned fact about the six element partition being a generating partitions immediately implies

$$h_{top}(F_l) \leq \log(6).$$

In this paper we improve the upper bound on $h_{top}(F_l)$. Let $a := \frac{2W(\frac{1}{e})}{1+W(\frac{1}{e})}$ where $W(\frac{1}{e})$ is the unique solution to the equation $1 = we^{w+1}$, see [15] and the beginning of the proof of Lemma 7 for more information on the Lambert W function.

The main result of our article is the following theorem

Theorem 1. *For any $l > 0$ we have $h_{top}(F_l) < \log \left(2 \left(\frac{2}{a} - 1 \right)^a \right) < \log(3.4908)$.*

We prove Theorem 1 by studying possible word complexity of the 6 elements language that can be associated to the Bunimovich billiard. In Section 3 we use Cassaigne's formula from [8] and prove that $h_{top}(F_l)$ is bounded from above by the limit of logarithmic growth rate of the number of distinct saddle connections of increasing lengths. Cassaigne's formula is very useful in studying low complexity systems, for example polygonal billiards [9]. To the best of our knowledge this is the first application of this formula to positive entropy systems. In Section 4 we give upper bounds for the number of different possible saddle connections using analytical tools, this yields our for $h_{top}(F_l)$.

3. SADDLE CONNECTIONS

We consider the 6 element partition \mathcal{A} defined in the previous section. We will use the word *corner* to refer to the four points where the semi-circles meet the line segments as well as the two centers of the semi-circles. The corners separate partition elements, this is clear for the four points, while for the centers of the semicircles we remark that the forward and backward orbit of any point $\bar{L} \cap \underline{L} = \{(s, \theta) : s \in L, \theta = 0\}$ passes through the center of the left semi-circle (a similar statement holds for points in $\bar{R} \cap \underline{R}$).

²The formal statement in [2] only claims this holds for $\omega \in \Sigma \setminus \{(TB)^\infty\}$. The proof of this stronger statement is identical, in Equation (10) of their proof the authors show that the intersection of closed cells is a single point.

In analogy to polygonal billiards a *saddle connection* is an orbit segment which starts and ends at corners of B_l (not necessarily the same) and the *length of the saddle connection* is the number of links in this trajectory. Let $N(n)$ denote the *number of distinct saddle connections of length at most n* . Our main result is based on the following result.

Theorem 2.

$$h_{top}(F_l) \leq \lim_{n \rightarrow \infty} \frac{\log \left(\sum_{j=0}^{n-1} N(j) \right)}{n}.$$

To prove this we need some techniques that were developed by Cassaigne in [8] and applied to polygonal billiards in [9]. For any $n \geq 1$ let $\mathcal{L}(n)$ be the *set of blocks of length n* in the subshift $\bar{\Sigma}$ (so $p(n) = \#\mathcal{L}(n)$) and $s(n) := p(n+1) - p(n)$. For $u \in \mathcal{L}(n)$ let

$$\begin{aligned} m_\ell(u) &:= \#\{a \in \mathcal{A} : au \in \mathcal{L}(n+1)\}, \\ m_r(u) &:= \#\{b \in \mathcal{A} : ub \in \mathcal{L}(n+1)\}, \\ m_b(u) &:= \#\{(a, b) \in \mathcal{A}^2 : aub \in \mathcal{L}(n+2)\}. \end{aligned}$$

We remark that all three of these quantities are larger than or equal to one. A word $u \in \mathcal{L}(n)$ is called *left special* if $m_\ell(u) > 1$, *right special* if $m_r(u) > 1$ and *bispecial* if it is left and right special. Let

$$\mathcal{BL}(n) := \{u \in \mathcal{L}(n) : u \text{ is bispecial}\}.$$

In a more general setting in [8] (see [9] for an English version) it was shown that for all $k \geq 1$ we have

$$s(k+1) - s(k) = \sum_{v \in \mathcal{BL}(k)} (m_b(v) - m_\ell(v) - m_r(v) + 1).$$

Proof of Theorem 2. Note that $(m_b(v) - m_\ell(v) - m_r(v) + 1) \leq (\#\mathcal{A})^2 = 36$, thus summing over $1 \leq k \leq j-1$ yields

$$s(j) \leq s(1) + 36 \sum_{k=1}^{j-1} \#\mathcal{BL}(k).$$

In our case $\mathcal{BL}(1) = 18$ and $s(1) = p(2) - p(1) = 30 - 6 = 24 < 2B_l(1)$, and thus we can estimate

$$s(j) \leq 38 \sum_{k=1}^{j-1} \#\mathcal{BL}(k).$$

Remember that $s(j) = p(j+1) - p(j)$, thus summing over $1 \leq j \leq n-1$ yields

$$p(n) \leq p(1) + 38 \sum_{j=1}^{n-1} \sum_{k=1}^{j-1} \#\mathcal{BL}(k).$$

Again we can adjust the constant to absorb the term $p(1)$ yielding the estimate

$$p(n) \leq 39 \sum_{j=1}^{n-1} \sum_{k=1}^{j-1} \#\mathcal{BL}(k).$$

To finish the proof of Theorem 2 we need the following result

Proposition 3. *For any $v \in \mathcal{BL}(k)$ and any pair of corners (s, s') there is at most one saddle connection with code v starting at the corner s and ending at the corner s' , and thus there are at most 36 saddle connections of length k with code v .*

Once proven, this yields

$$p(n) \leq 39 \cdot 36 \sum_{j=1}^{n-1} N(j)$$

and thus

$$h_{\text{top}}(F_l) = \lim_{n \rightarrow \infty} \frac{\log p(n)}{n} \leq \lim_{n \rightarrow \infty} \frac{\log(39 \cdot 36 \sum_{j=0}^{n-1} N(j))}{n} = \lim_{n \rightarrow \infty} \frac{\log(\sum_{j=0}^{n-1} N(j))}{n}$$

which finishes the proof of Theorem 2. \square

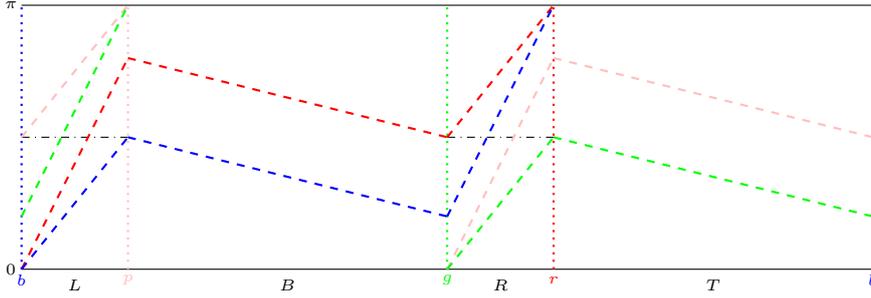


FIGURE 2. The singularity sets of T (dotted) and T^{-1} (dashed). The two segments which are both dotted and dashed are in both singularity sets.

Before proving Proposition 3 we need to introduce some more terminology. Let Γ' be the set of points from M_l perpendicular to the semicircles. Let Γ be the union of Γ' with the set of points where T fails to be C^2 , and analogously Γ^- is the union of Γ' with the set of points where T^{-1} fails to be C^2 . We call Γ and Γ^- the singularity sets for T and T^{-1} respectively.

The singularity sets Γ, Γ^- consist of a finite number of C^1 smooth compact curves in M_l (see Figure 3), which are increasing, decreasing, horizontal, or vertical. Define the singularity set for the map T^n for $n \geq 1$ by $\Gamma^n := \bigcup_{i=1}^n T^{-i+1}(\Gamma)$ and the singularity set of the map T^{-n} for $n \geq 1$ by $\Gamma^{-n} := \bigcup_{i=1}^n T^{i+1}(\Gamma^{-1})$.

Remember that the set \tilde{M}_l defined in Section 2 is the set of points in \tilde{M}_l having unique codes. In such a language, notice that the \tilde{M}_l is the set of points whose orbit does not hit a corner.

For $(s, \theta) \in \tilde{M}_l$ let $c_k(s, \theta) := (\bar{c}(F_l^i(s, \theta)))_{i=0}^{k-1}$ denote the block of length k containing $c(s, \theta)$. For $v \in \mathcal{L}(k)$ we define the set

$$\omega(v) := \overline{\{(s, \theta) \in \tilde{M}_l : v = c_k(s, \theta)\}}$$

and call $\omega(v)$ a k -cell.

The proof of Lemma 2.5 in [2] shows that the set $\omega(v)$ is a simply connected closed set whose boundary consists of a finite collection of piecewise smooth curves whose angles are less than π . These curves belong to the union of the singular sets of T^i for $0 \leq i \leq k-1$. For each $0 \leq i \leq k-1$ the map T^i is continuous on $\omega(v)$.

Proof of Proposition 3. Fix $v \in \mathcal{BL}(k)$. Consider the partition $\bigcup_{avb \in \mathcal{L}(k+2)} \omega(avb)$ of the set $\omega(v)$. This partition is produced by cutting $\omega(v)$ by the singular sets of T^k and T^{-1} .

An intersection in $\omega(v)$ of these two singular sets corresponds to a saddle connection of length k whose code is v . Label the corners of B_l by the alphabet $\mathcal{A}' := \{1, 2, 3, 4, 5, 6\}$. The code of a saddle connection is the sequence from $\mathcal{A} \cup \mathcal{A}'$

of sides a point hits along with the starting and ending corners; thus a saddle connection of length n will have a code of length $n + 1$. To finish the proof we need to show that there is a bijection between saddle connections and their codes.

We will give a brief sketch describing the bijection between the set of codes and possible trajectories of the billiard map. Suppose by way of contradiction that two trajectories start at the same corner s' and end at the corner s'' (possibly $s'' = s'$) and have the same code. For concreteness the starting points are (s', θ_1) and (s', θ_2) . We consider the set $B := \{(s, \theta) : \theta_1 \leq \theta \leq \theta_2 \text{ and } s \in \partial B_l\}$. A smooth curve from the phase space M_l equipped with a continuous family of unit normal vectors is called a *wave front*, in particular the set B is a wave front. A wave front is said to *focus* (or *collapse*) when the trajectories of the front intersect within the billiard. By assumption the wave front B refocuses at a corner; we will show that this is in fact impossible. We refer to Subsection 8.4. in [13] for a more complete description of what follows.

Focusing occurs in Bunimovich billiards when the wave front reflects from one of the two semi-circles. Suppose that an infinitesimal wave front collides with ∂M_l at a point X and denote its pre-collisional and post-collisional curvature by \mathcal{B}^- and \mathcal{B}^+ respectively. Furthermore, say that X experiences two consecutive collisions with ∂M_l at times t and $t + \tau$. Unstable wave fronts in Bunimovich billiards satisfy the condition $\mathcal{B}^+ < -2/\tau$ (equation (8.1) in Subsection 8.2. in [13]). That is $\mathcal{B}^- > 0$ implies that $\mathcal{B}^+ > 0$ for the flat boundaries and it implies that $\mathcal{B}^+ < -2/\tau$ for the semi-circles. A focusing wave front with curvature $\mathcal{B}_t < 0$ passes through a focusing point and defocuses at the time $t^* = t - 1/\mathcal{B}_t$ and thus $t - 1/\mathcal{B}_t < t + \tau/2$ which implies that $\mathcal{B}_t < -2/\tau$. This means that the front must defocus before it reaches the midpoint between the consecutive collisions. By Theorem 8.9. from [13] it holds that the families of unstable cones remain unstable under the iteration of the map. This implies that all wave fronts are in the unstable cones which implies that we indeed have unique coding of trajectories of the billiard map. \square

4. PROOF OF THEOREM 1

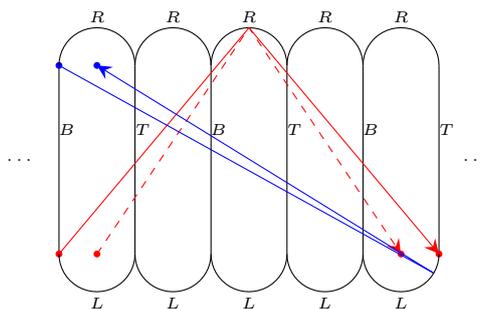


FIGURE 3. Unfolding the stadium. There are eight saddle connections with signed composition $2, 1, 2$, two of them having code $T\bar{B}R\bar{T}B$ are drawn in red, there are two more saddle connections with this code, the other four have code $T\bar{B}\bar{L}T\bar{B}$. If we reverse the arrows we obtain the saddle connection with signed composition $-2, 1, -2$. In blue we show a saddle connection with code $T\bar{B}T\bar{B}\bar{L}B\bar{T}B\bar{T}$ and signed composition $4, 1, -4$. If we reverse the arrows the blue saddle connection has the same code and signed composition.

To count the number of saddle connections we unfold the stadium (see Figure 3). Consider an integer $j \geq 1$. We say $(n_1, m_1, n_2, m_2, \dots, n_k, m_k)$ is a *signed composition* of j if $n_i \in \mathbb{Z}$, $m_i \in \mathbb{N}$ with $m_i \geq 1$ for all i such that $\sum |n_i| + \sum m_i = j$. Let $Q(j)$ denote the number of *signed compositions* of j . Recall that $N(n)$ denotes the number of distinct saddle connections of length at most n .

Lemma 4. *For each n and $l > 0$ we have*

$$N(n) \leq 36 \sum_{j=0}^{n-1} Q(j).$$

Proof. Fix a corner of B_l and consider the saddle connections of length at most n starting at this corner. We consider the associate signed composition in the following way: the non-negative integer $|n_i|$ counts the consecutive hits in the flat sides of B_l and m_i counts the consecutive hits in a semicircle. The sign of n_i tells us which way we are moving in the unfolding, left or right, when changing from one semicircle to the other. In this way each saddle connection yields a signed composition.

Fix $j \geq 1$ and a signed composition of j . As we showed in Proposition 3, for each pair of corners there is at most one saddle connection with this signed composition. Thus, since there are 6 corners, there are at most 36 codes of saddle connections which correspond to a given signed composition. \square

Let $(n_1, m_1, n_2, m_2, \dots, n_k, m_k)$ be a signed composition of j with $2k$ terms. Denote by $Q(j, k)$ the *number of such possible compositions of $j \in \mathbb{N}$ with $2k$ terms*. Let

$$r_i := |n_i| + m_i, \text{ then } \vec{r}_k(j) := (r_1, \dots, r_k)$$

is a *composition of j with k terms*. In what follows we will first estimate $Q(j, k)$ and then $Q(j)$.

Fix a composition $\vec{r}_k(j)$. Each $n_i \in \{-r_i + 1, \dots, -1, 0, 1, \dots, r_i - 1\}$ yields a different signed composition, there are

$$f(\vec{r}_k(j)) := \prod_{i=1}^k (2r_i - 1)$$

preimages of $\vec{r}_k(j)$ in total, i.e.,

$$Q(j, k) = \sum_{\ell \geq 1} \ell \times \#\{\vec{r}_k(j) : f(\vec{r}_k(j)) = \ell\}.$$

We start by estimating the number of terms in this sum, i.e., the largest possible value of ℓ . If $s = q_1 + \dots + q_k$, then the arithmetic-geometric mean inequality

$$\sqrt[k]{q_1 \cdots q_k} \leq \frac{q_1 + \dots + q_k}{k}$$

yields

$$q_1 \cdots q_k \leq \left(\frac{s}{k}\right)^k.$$

Notice that equality is obtained if and only if all the $q_1 = q_2 = \dots = q_k$, and thus $q_i = s/k$. Setting $q_i = 2r_i - 1$ and $s = 2j - k$ yields

$$f(\vec{r}_k(j)) \leq \left(\frac{2j}{k} - 1\right)^k$$

with equality if and only if $r_i = \frac{2j}{k} - 1$.

Thus

$$\begin{aligned} Q(j, k) &\leq \left(\frac{2j}{k} - 1\right)^k \sum_{\ell \geq 1} \#\{\vec{r}_k(j) : f(\vec{r}_k(j)) = \ell\} \\ &= \left(\frac{2j}{k} - 1\right)^k \times \#\{\vec{r}_k(j)\} \\ &= \left(\frac{2j}{k} - 1\right)^k \binom{j}{k}. \end{aligned}$$

Fix $j \geq 1$ and let g_j be the function defined by

$$k \in \{1, \dots, j\} \mapsto \left(\frac{2j}{k} - 1\right)^k \binom{j}{k}.$$

Lemma 5. *The function $k \in \{1, \dots, j\} \mapsto \binom{j}{k}$ is increasing for $1 \leq k \leq \frac{j+1}{2}$ and decreasing for $\frac{j+1}{2} \leq k \leq j$.*

Proof. The inequalities $1 \leq k-1 < k \leq \frac{j+1}{2}$ imply that $\frac{j-k+1}{k} \geq 1$ and thus

$$\binom{j}{k} = \binom{j}{k-1} \frac{j-k+1}{k} \geq \binom{j}{k-1}$$

and thus the function is increasing for $1 \leq k \leq \frac{j+1}{2}$.

The decreasing statement holds since $\binom{j}{k} = \binom{j}{j-k}$. \square

For each $1 \leq j$ let

$$h_j(x) = \left(\frac{2j}{x} - 1\right)^x = e^{x \ln\left(\frac{2j}{x} - 1\right)}.$$

Lemma 6. *For each $j \geq 2$ there exists a unique $x_j > 1$ such that h_j is increasing for $x \in [1, x_j]$ and decreasing for $x \in [x_j, j]$.*

Proof. Throughout the proof the functions under consideration are restricted to the domain $[1, j]$. We begin by calculating the derivative of h_j ,

$$h'_j(x) = h_j(x) \left(\frac{-2j}{2j-x} + \ln\left(\frac{2j}{x} - 1\right) \right).$$

Let $k_j(x) := \frac{-2j}{2j-x} + \ln\left(\frac{2j}{x} - 1\right)$, then the signs of h'_j and k_j are the same since h_j is positive. We study the sign of k_j by taking its derivative:

$$k'_j(x) = \frac{-4j}{x(2j-x)^2} < 0$$

for $x \in [1, j]$.

For $j \geq 2$ we have $k_j(1) = \frac{-2j}{2j-1} + \ln(2j-1) > 0$. Furthermore, $k_j(j) = -2 < 0$; thus there is a unique $x_j \in (1, j)$ which is the solution of the equation $k_j(x) = 0$ such that $\text{sgn}(h'_j(x)) = \text{sgn}(k_j(x)) > 0$ for $x \in (1, x_j)$ and $\text{sgn}(h'_j(x)) = \text{sgn}(k_j(x)) < 0$ for $x \in (x_j, j)$ and thus $h_j(x)$ is maximized when $x = x_j$. \square

To prove the next lemma we will use the Lambert W function; see [15] for an introduction. The Lambert W function is a multivalued function which for a given complex number z gives all the complex numbers w which satisfy the equation $we^w = z$. If z is a positive real number then there is a single real solution w of this equation which we denote $W(z)$.

Lemma 7. For each $j \geq 2$ there exists a constant $a \in (0.435, 0.436)$ such that $x_j = a \cdot j$.

Proof. The equation $k_j(x_j) = 0$ is equivalent to $\frac{2j-x}{x} = e^1 e^{\left(\frac{x}{2j-x}\right)}$. Substituting $w = \frac{x}{2j-x}$ yields $1/w = e^1 e^w$ or equivalently $\frac{1}{e} = w e^w$. Since $\frac{1}{e}$ is positive there is a single solution to this equation $w = W\left(\frac{1}{e}\right)$, and thus $x_j = \frac{2W\left(\frac{1}{e}\right)}{1+W\left(\frac{1}{e}\right)} \cdot j =: a \cdot j$. \square

Lemma 8. The maximum value of h_j is at most $\left(\frac{2}{a} - 1\right)^a < 1.7454$.

Proof. The maximum value of h_j is $h_j(x_j) = h_j(a \cdot j) = \left(\left(\frac{2}{a} - 1\right)^a\right)^j$. If $a_1 < a < 1$ then

$$\left(\frac{2}{a} - 1\right)^a < \left(\frac{2}{a_1} - 1\right)^{a_1}.$$

Notice that $k_j(0.435j) \approx 0.0023 > 0$ and $k_j(0.436j) \approx -0.0014 < 0$. Remembering from Lemma 6 that k_j is decreasing yields $0.435 < a < 0.436$. The above estimate yields $\left(\frac{2}{a} - 1\right)^a < 1.7454$. \square

Combining Lemma 5 and Lemma 8 yields:

Corollary 9. For $j \geq 2$ the maximum value of function g_j is bounded from above by $\left(\frac{2}{a} - 1\right)^a \binom{j}{\lfloor \frac{j}{2} \rfloor} < 1.7454^j \binom{j}{\lfloor \frac{j}{2} \rfloor}$.

To prove the next result we consider the gamma function $\Gamma(z)$, we will use the Legendre duplication formula

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z)$$

as well as Gautschi's inequality

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}$$

which holds for any positive real x and $s \in (0, 1)$.

Lemma 10. For any even $j \geq 2$ we have

$$\binom{j}{\lfloor \frac{j}{2} \rfloor} \leq \sqrt{\frac{2}{j}} \cdot \frac{2^j}{\sqrt{\pi}}$$

while for odd $j > 2$ we have

$$\binom{j}{\lfloor \frac{j}{2} \rfloor} \leq \sqrt{\frac{2}{j+1}} \cdot \frac{2^j}{\sqrt{\pi}}.$$

Proof. If $j = 2n$ is even then $\binom{j}{\lfloor \frac{j}{2} \rfloor} = \binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma(n+1)^2}$. Using the duplication formula with $z = n + \frac{1}{2}$ yields

$$\frac{\Gamma(2n+1)}{\Gamma(n+1)^2} = \frac{\Gamma(2z)}{\Gamma\left(z + \frac{1}{2}\right)^2} = \frac{\Gamma(z)}{\Gamma\left(z + \frac{1}{2}\right)} \cdot \frac{2^{2z-1}}{\sqrt{\pi}} = \frac{\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j}{2} + 1\right)} \cdot \frac{2^j}{\sqrt{\pi}}.$$

Next we apply Gautschi's inequality with $s = \frac{1}{2}$ and $x = \frac{j}{2}$; it yields

$$\frac{\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j}{2} + 1\right)} \cdot \frac{2^j}{\sqrt{\pi}} < \sqrt{\frac{2}{j}} \cdot \frac{2^j}{\sqrt{\pi}}.$$

Now suppose that $j = 2n + 1$ is odd, then

$$\binom{j}{\lfloor \frac{j}{2} \rfloor} = \binom{2n+1}{n} = \frac{\Gamma(2n+2)}{\Gamma(n+1)\Gamma(n+2)}.$$

Using the duplication formula with $z = n + 1$ yields

$$\frac{\Gamma(2n+2)}{\Gamma(n+1)\Gamma(n+2)} = \frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+1)} = \frac{\Gamma(z + \frac{1}{2})}{\Gamma(z+1)} \cdot \frac{2^{2z-1}}{\sqrt{\pi}} = \frac{\Gamma(\frac{j}{2} + 1)}{\Gamma(\frac{j+1}{2} + 1)} \cdot \frac{2^j}{\sqrt{\pi}}.$$

Again we apply Gautschi's inequality, here with $x = \frac{j+1}{2}$ and $s = \frac{1}{2}$ which yields

$$\frac{\Gamma(\frac{j}{2} + 1)}{\Gamma(\frac{j+1}{2} + 1)} \cdot \frac{2^j}{\sqrt{\pi}} < \sqrt{\frac{2}{j+1}} \cdot \frac{2^j}{\sqrt{\pi}}$$

□

Now we are ready to prove the main theorem of the paper.

Proof of Theorem 1. From Corollary 9 and Lemma 10 it follows that

$$Q(j, k) \leq \left(\left(\frac{2}{a} - 1 \right)^a \right)^j \binom{j}{\lfloor \frac{j}{2} \rfloor} \leq \left(\left(\frac{2}{a} - 1 \right)^a \right)^j \frac{2^j}{\sqrt{j\pi/2}} = \frac{(2(\frac{2}{a} - 1)^a)^j}{\sqrt{j\pi/2}}.$$

Therefore,

$$Q(j) \leq j \max_k(Q(j, k)) \leq \frac{(2(\frac{2}{a} - 1)^a)^j \sqrt{j}}{\sqrt{\pi/2}}.$$

Thus we obtain

$$N(n) \leq 36 \sum_{j=1}^{n-1} Q(j) \leq 36 \sum_{j=1}^{n-1} \frac{(2(\frac{2}{a} - 1)^a)^j \sqrt{j}}{\sqrt{\pi/2}} \leq \left(2 \left(\frac{2}{a} - 1 \right)^a \right)^n C \sqrt{n-1},$$

where C is a positive constant. Recall that $p(n)$ denotes the complexity of Σ . Therefore using Theorem 2,

$$p(n) \leq \sum_{j=0}^{n-1} \left(2 \left(\frac{2}{a} - 1 \right)^a \right)^j C \sqrt{j-1} \leq \left(2 \left(\frac{2}{a} - 1 \right)^a \right)^n C' \sqrt{n-1}$$

where C' is another positive constant. From the definition of topological entropy we obtain

$$h_{top}(F_l) \leq \log \left(2 \left(\frac{2}{a} - 1 \right)^a \right) \leq \log(3.4908).$$

□

5. OTHER POSSIBLE DEFINITIONS OF TOPOLOGICAL ENTROPY

Another very natural definition of topological entropy was given by Pesin and Pitskel' in [19] and the closely related capacity topological entropy was defined by Pesin in [18][Page 75]. Applying these definitions to the map F_l restricted to \tilde{M}_l yields two quantities, the Pesin-Pitskel' topological entropy $h_{\tilde{M}_l}(F_l)$ and the capacity topological entropy $Ch_{\tilde{M}_l}(F_l)$. Formally, in [18] the capacity topological entropy is defined in a slightly more restrictive setting than in [19], but it can be defined in the setting of [19] and the relationship $h_{\tilde{M}_l}(F_l) \leq Ch_{\tilde{M}_l}(F_l)$ from [18] still holds. But our definition of $h_{top}(F_l)$ coincides with Pesin's definition of $Ch_{\tilde{M}_l}(F_l)$. Thus we have the following corollary

Corollary 11. *For any $l > 0$ the Pesin-Pitskel' topological entropy $h_{\tilde{M}_l}(F_l)$ is bounded from above by $\log(2(\frac{2}{a} - 1)^a) < \log(3.4908)$.*

ACKNOWLEDGEMENTS

We thank Peter Bálint, Lyonia Bunimovich, Michał Misiurewicz and Yasha Pesin for useful discussions and helpful remarks. J. Činč was partially supported by the FWF Schrödinger Fellowship stand-alone project J 4276-N35 and by the IDUB program no. 1484 “Excellence initiative – research university” for the AGH University of Science and Technology.

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