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Singular integral equations with applications to travelling waves for doubly nonlinear diffusion

Alejandro Gárriz

March 28, 2022

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Abstract

The goal of this article is to offer a series of results related to the existence and properties of wavefront solutions for doubly nonlinear diffusion-reaction equations involving the p-Laplacian operator in terms of the constitutive functions of the problem. These results are derived from the analysis of singular Volterra integral equations that appear in the study of monotone travelling-wave solutions for such equations. Our results extend the ones due to B. Gilding and R. Kersner for the case $p = 2$ to $p > 1$. The fact that $p \neq 2$ modifies the nature of the singularity in the integral equation, and introduces the need to develop some new tools and ideas for the analysis.

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1 Introduction

This work intends to develop and present a collection of results related to the existence and the properties of travelling-wave solutions of nonlinear reaction-convection-diffusion equations of the form

$$u_t = \Delta_p(a(u)) + (b(u))_x + c(u), \quad x \in \mathbb{R},\, t \geq 0, \tag{1.1}$$

involving the p-Laplacian operator,

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad p > 1.$$

Travelling waves or fronts are special solutions of the form $u_\sigma(x,t) = V_\sigma(x - \sigma t)$ for some speed σ and profile, that may depend on the speed, V_σ. They are important at least in relation with two aspects: finite speed of propagation and large time behaviour.

Equation (1.1) is said to display *finite speed of propagation* if a non-negative solution which has bounded support with respect to the spatial variable at some initial time also presents this property at later times. As proved in [18] (see also [17]), equation (1.1) displays finite speed of propagation if and only if it possesses a travelling-wave solution whose profile's support is bounded from above. For this purpose it is enough to have a *local* travelling wave, with a profile only defined in a half-interval (ω, ∞).

A bounded *global* (defined in the whole real line) travelling-wave profile V_σ that is monotonic, but not constant, and such that

$$V_\sigma(\xi) \to \ell^- \text{ as } \xi \to -\infty \quad \text{and } V_\sigma(\xi) \to \ell^+ \text{ as } \xi \to \infty$$

for some $\ell^-, \ell^+ \in \mathbb{R}$ such that $c(\ell^-) = c(\ell^+) = 0$ is said to be a *wavefront profile* from ℓ^- to ℓ^+ (the associated travelling wave is a *wavefront solution*), see *Figures 1 ∼ 2*.

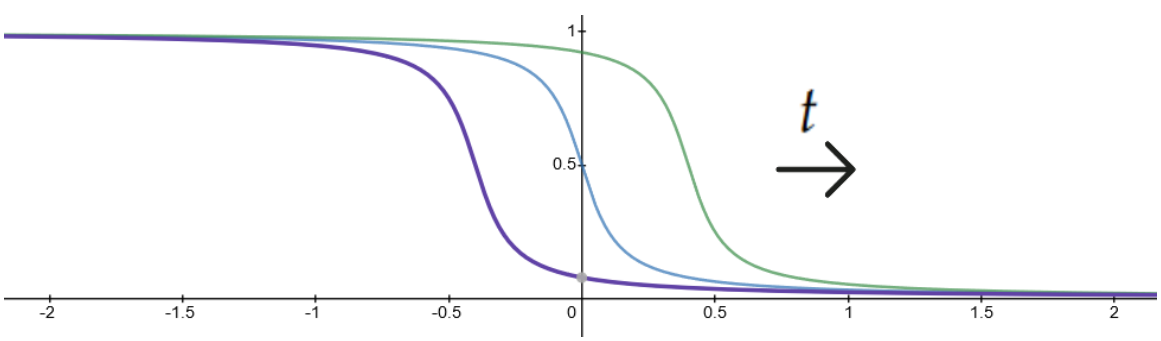

Figure 1: Wavefronts with $c(u) = u(1-u)$, $\ell^- = 1$, $\ell^+ = 0$.

Wavefronts are already known to be important in the description of the large-time behaviour of more general solutions for wide classes of initial data when $p = 2$, $a(u) = u^m$, $m \geq 1$, and $b \equiv 0$ for several reaction nonlinearities c; see for instance [1,

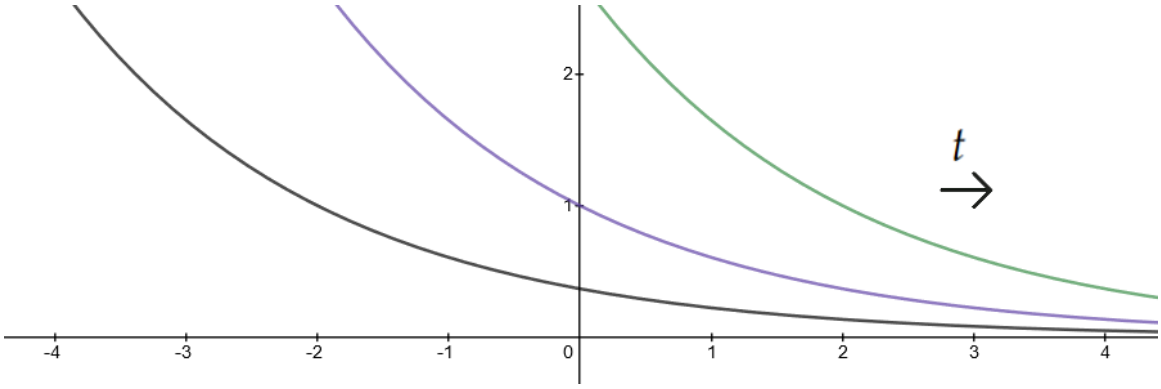

Figure 2: Wavefronts with $c(u) = u^q,\ q > 1,\ \ell^- = \infty,\ \ell^+ = 0$.

4, 5, 7, 10, 11, 12, 21, 22]. They are also expected to give the large time behaviour when $p \neq 2$. As a first step in this direction we have the papers [2, 3, 6, 12], being the interest in these problems one of the main motivations for this work.

Given the relevance of this type of solutions it was for our great surprise to find that not much was known in the particular case of the p-Laplacian operator. In the making of the article [6] the authors found the need to develop the theory for the reaction-diffusion case, and that motivated our study here. This survey intends to be a collection of results and references that will, hopefully, alleviate similar needs in the future when working with nonlinear diffusion with the p-Laplacian operator.

The main tool in our study and the technical focus of this paper is the Volterra integral equation

$$x(t) = f(t) + \int_0^t \frac{g(s)}{x^\alpha(s)}\,\mathrm{d}s, \quad x(t) \geq 0, \quad t \geq 0, \tag{1.2}$$

where $\alpha \in \mathbb{R}_+ := (0, \infty)$, and

$$f \in C(\overline{\mathbb{R}_+}), \quad f(0) \geq 0, \quad g \in L^1((0,\tau)) \text{ for all } \tau \in \mathbb{R}_+.$$

We intend to extend the results obtained by Gilding in [15] for the case $\alpha = 1$ to the case $\alpha > 1$. We are mainly interested in the singular case $f(0) = 0$.

As shown in [18], this equation arises in the study of travelling waves in nonlinear reaction-convection-diffusion processes, so let us explain the connection between (1.2) and travelling waves for (1.1) in detail, a connection that was first explored for the case $p = 2$ in [16]. Let $u_\sigma(x,t) = V_\sigma(\xi)$, $\xi = x - \sigma t$, be a travelling-wave solution to (1.1) taking values in $\bar{I}$, where $I = (0, \ell)$, $0 < \ell \leq \infty$. The profile V (we drop the subscript for simplicity) satisfies the ordinary differential equation

$$\Delta_p(a(V)) + (b(V))_\xi + c(V) + \sigma V_\xi = 0 \tag{1.3}$$

in a weak sense; see Section 3.1 for a precise definition. Taking $g := -(a(V))_\xi = -a'(V)V_\xi$, we arrive to the system of equations

$$V_\xi = \frac{-g}{a'(V)}, \qquad g_\xi = \frac{1}{(p-1)|g|^{p-2}}\left(c - \frac{g(b' + \sigma)}{a'}\right).$$

The function a is assumed to be nondecreasing, so that equation (1.1) is parabolic (may be degenerate or singular). Hence, if we restrict to non-increasing profiles, we may assume that $|g| = g$. Thus, the trajectories in the phase plane of this system satisfy

$$\frac{\mathrm{d}g}{\mathrm{d}V} = \frac{1}{p-1}\left(\frac{b'+\sigma}{g^{p-2}} - \frac{a'c}{g^{p-1}}\right).$$

If (1.3) admits a solution V such that $V \to 0$, and $(a(V))_\xi \to 0$ as $\xi \to \infty$, this V is represented by a trajectory that approaches the point $(V, g) = (0, 0)$. Thus, setting

$$\theta(s) = g^{p-1}(\xi), \qquad s = V(\xi),$$

and integrating the equation of the trajectories through (0,0) we arrive to

$$\theta(s) = b(s) + \sigma s - \int_0^s \frac{a'(r)c(r)}{\theta^\alpha(r)}\,\mathrm{d}r, \qquad \alpha = 1/(p-1) > 0, \tag{1.4}$$

which lies within the class of integral equations of the form (1.2). Note that in terms of the original problem θ represents the flux as a function of the height of the solution.

Conversely, if the flux θ satisfies the *integrability condition*

$$\int_{s_0}^{s_1} \frac{a'(r)}{\theta^\alpha(r)}\,\mathrm{d}r < \infty \quad \text{for all } 0 < s_0 < s_1 < \delta, \tag{1.5}$$

for $\delta = \ell$, a solution of (1.3) can be constructed from a solution of (1.4) by means of

$$\int_\nu^{V(\xi)} \frac{a'(r)}{\theta^\alpha(r)}\,\mathrm{d}r = \xi_0 - \xi \tag{1.6}$$

for some ν in the domain of definition of θ and some $\xi_0 \in \mathbb{R}$. Let us remark that the profile obtained in this way is strictly monotone in $\{\xi \in \mathbb{R} : 0 < V(\xi) < \ell\}$.

It turns out that, for $\ell < \infty$, the differential equation (1.3) has a wavefront profile from ℓ to 0 with speed σ if and only if the integral equation (1.4) with $\alpha = 1/(p-1)$ has a solution θ such that $\theta(\ell) = 0$ satisfying the integrability condition (1.5) on $\bar{I}$. We omit more explanations regarding this, since the proof of this statement is similar to the one for the case $p = 2$ (that is, $\alpha = 1$) given in [19].

Let us remark that there may be several wavefront profiles leading to the same solution θ of the integral equation. For instance, equation

$$u_t = \Delta_p u + \begin{cases} (p-1)(2u-4u^2)^{\frac{p-2}{2}} 2^p u^p (8u-3) & \text{for } 0 \le u \le 1/2, \\ (p-1)(2(1-u)-4(1-u)^2)^{\frac{p-2}{2}} 2^p (1-u)^p (8u-5) & \text{for } 1/2 < u \le 1, \end{cases}$$

with $p > 2$ admits a family of wavefront profiles from 1 to 0 with speed $\sigma = 0$, see *Figure 3*.

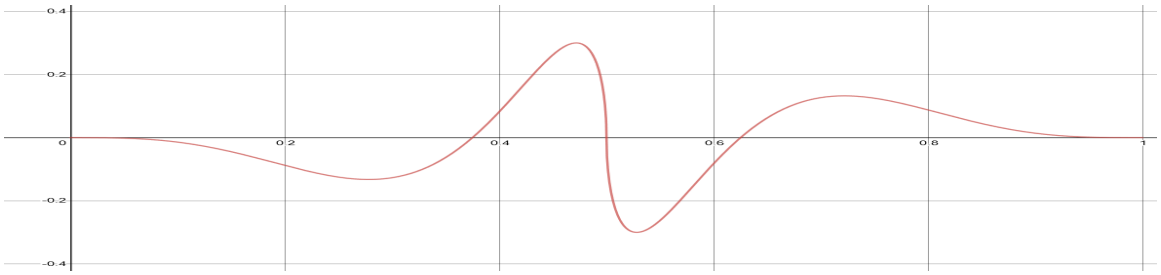

Figure 3: Reaction $c(u)$

.

This family of wavefronts is given by

$$V(\xi) = \begin{cases} \frac{1+2(\xi_0-\xi)^2}{2(1+(\xi_0-\xi)^2)} & \text{for } \xi < \xi_0, \\ 1/2 & \text{for } \xi_0 \le \xi \le \xi_1, \\ \frac{1}{2(1+(\xi-\xi_1)^2)} & \text{for } \xi_1 < \xi, \end{cases}$$

for any pair or real numbers $\xi_0 \le \xi_1$, see *Figure 4*.

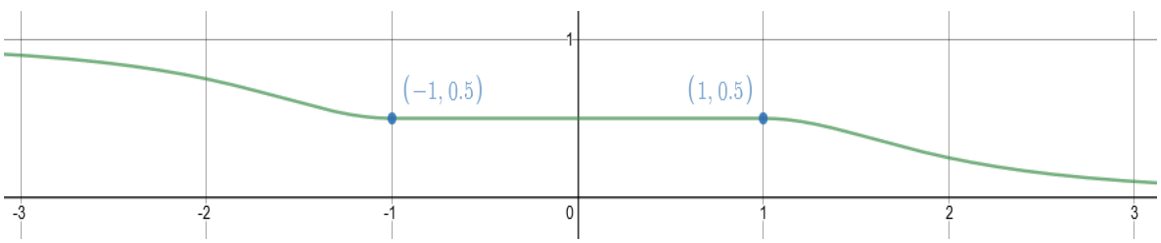

Figure 4: Wavefront $V(\xi)$ for $\xi_0 = -1$ and $\xi_1 = 1$.

The corresponding flux

$$\theta(s) = \begin{cases} \left(2s\sqrt{2s(1-2s)}\right)^{p-1} & \text{for } 0 \le s \le 1/2, \\ \left(2(1-s)\sqrt{2(1-s)(2s-1)}\right)^{p-1} & \text{for } 1/2 < s \le 1, \end{cases}$$

satisfies the integral equation (1.4) regardless the values of ξ_0 and ξ_1, see *Figure 5*.

On the other hand, starting from this solution θ of the integral equation and using (1.6) with $\nu \in (0,1)$ we obtain the wavefront profiles

$$(1.7) \quad V(\xi) = \begin{cases} \frac{1+2(\eta-\xi)^2}{2(1+(\eta-\xi)^2)} & \text{for } \xi \le \eta, \\ \frac{1}{2(1+(\xi-\eta)^2)} & \text{for } \eta < \xi, \end{cases} \qquad \text{where } \eta = \begin{cases} -\left(\frac{1-2\nu}{2\nu}\right)^{1/2} & \text{for } \nu \le 1/2, \\ \left(\frac{2\nu-1}{2(1-\nu)}\right)^{1/2} & \text{for } \nu > 1/2, \end{cases}$$

see *Figure 6*.

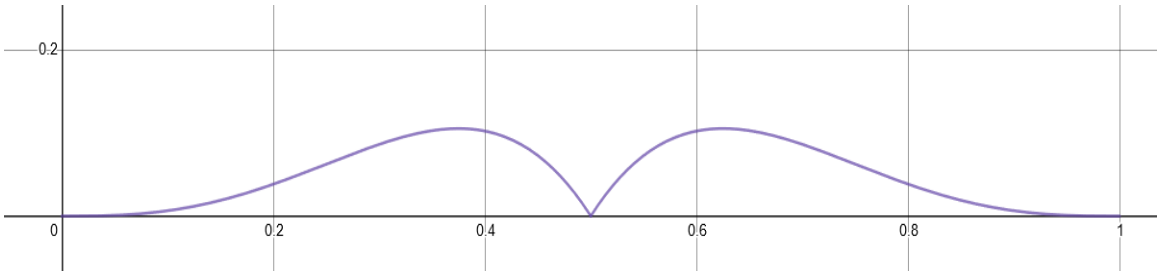

Figure 5: Flux $\theta(s)$, which is unique for all the family of wavefronts.

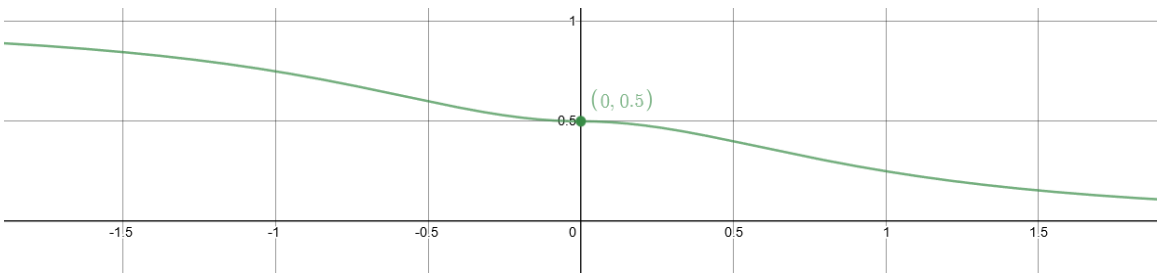

Figure 6: Wavefront $V(\xi)$ for $\xi_0 = \xi_1 = 0$ and $\nu = 1/2$.

We observe that in this example the flux vanishes only at heights where the reaction vanishes, $s = 0, 1/2, 1$. This is true not only in this particular case, but in general, as can be shown by adapting the proof given in [19, Lemma 2.24] when $\alpha = 1$. Note, however, that the opposite implication does not hold: the reaction term may vanish at points where the flux does not.

When $p = 2$ the example above coincides with the one given in [19, Application 2.22]. Nevertheless, in sharp contrast with that case, if $p > 2$ the reaction term is continuous in $(0, \ell)$. Moreover, if $p \geq 4$, c and hence ca' are differentiable in the same open interval. However, the flux θ vanishes at a point in $(0, \ell)$, namely $s = 1/2$. This is not possible when $\alpha = 1$, since for that value of the parameter the differentiability of ca' implies the positivity of the flux; see [19, Lemma 2.40]. The possibility of having a vanishing flux at positive heights for smooth reactions is introduced by the degeneracy of the p-Laplacian for $p > 2$. In the singular case $1 < p < 2$ this possibility does not exist, as can be easily proved in the same way as for $\alpha = 1$.

We also observe that all the wavefront profiles (1.7) arising from the integral equation are translates one of each other, with ν giving the point where V takes the value $1/2$. These wavefront profiles correspond to the ones with $\xi_0 = \xi_1 = \eta$, which are strictly monotonic.

Two wavefront profiles are said to be *indistinct* if one is a translation of the other. Otherwise they are *distinct*. It turns out that to every solution θ of (1.4) satisfying (1.5) in $[0, \ell]$ with $\ell < \infty$ and $\theta(\ell) = 0$ there corresponds precisely one distinct wavefront profile V of equation (1.1) with speed σ from ℓ to 0 which is strictly monotonic where it is positive. Moreover, there corresponds no other (non-strictly monotonic) distinct wavefront profile with speed σ from ℓ to 0 if and only if $\theta > 0$ in

$(0, \ell)$. We omit the proof, since it is similar to the one of [19, Theorem 2.26].

Back to the integral equation (1.2), Gilding and Kersner have already considered it for $\alpha \neq 1$ in [18] for the particular case

$$f(t) = \sigma t + \mu t^{\gamma}, \quad \mu \in \mathbb{R},\ \gamma > 0, \qquad g(t) = c_0 t^{\beta}, \quad c_0 \in \mathbb{R},\ \beta > -1.$$

They characterize there for which values of the parameters $\alpha, \mu, \gamma, c_0, \beta$ there is a local solution of the integral equation for some large enough value σ. This can be translated into the existence of a local travelling wave with a profile whose support is bounded from above, which is enough for their purposes.

In the present paper we obtain both necessary conditions and sufficient conditions in terms of f, g and α for the existence of local solutions to (1.2), in the spirit of [15], which are enough to analyze the question of existence of travelling waves and their behavior for diffusion-reaction equations for reaction nonlinearities with none or one sign change. Though we follow ideas from [15], the new degeneracy/singularity represented by the power α introduces the need to develop nontrivial generalizations of several tools that in addition offer more insight on the analysis, as well as new tools like continued fractions or "polynomials" of fractional order.

Once the study of the integral equation is finished, we apply it to the analysis of travelling-wave solutions of (1.1), following the ideas developed by Gilding and Kersner in the remarkable book [19] for the case $p = 2$. Such examination when $p \neq 2$ has already been considered in [9, 14] for $a(u) = u$, and in [2, 3] for $a(u) = u^m$ using a different technique. However, these papers have several restrictions on the reaction nonlinearity c. Since the ideas behind the proofs of the results in the last section are similar to the ones in [19] (though, in ocasions, the calculations are quite more convoluted) we do not give them here, we leave them as an exercise for the interested reader and refer him to that book for tips.

Organization of the paper. Section 2 represents the core of the article, where we study the integral equation (1.2) in detail. In Section 3.1 we exploit the connection between this integral equation and the diffusion-convection-reaction equation (1.1) to obtain some general results for the latter. Finally, in Section 3.2 we make full profit of the results of Section 2 by applying them to study travelling waves of diffusion-reaction equations (no convection).

2 The integral equation

Throughout this section we will study equation (1.2) in detail in the same fashion as in [15] when $\alpha = 1$. The main differences with that work will appear in subsections 2.2, 2.3 and 2.4

Let us define

$$I(s,x) = \begin{cases} g(s)/x^\alpha & \text{if } x > 0, \\ -\infty & \text{if } g(s) < 0 \text{ and } x = 0, \\ 0 & \text{if } g(s) = 0 \text{ and } x = 0, \\ \infty & \text{if } g(s) > 0 \text{ and } x = 0. \end{cases}$$

Considering the integral in (1.2) as an improper Lebesgue integral, we will say that a function x is a solution of equation (1.2) on $[0,\tau)$ if it is defined, real, nonnegative and continuous in $[0,\tau)$, $G(s,x(s)) \in L^1_{\text{loc}}((0,\tau))$, and

$$\int_0^t I(s,x(s))\,\mathrm{d}s := \lim_{\varepsilon\downarrow 0} \int_\varepsilon^t I(s,x(s))\,\mathrm{d}s$$

exists and satisfies

$$x(t) = f(t) + \int_0^t I(s,x(s))\,\mathrm{d}s \quad \text{for all } t \in (0,\tau).$$

One can check that a solution in this sense satisfies $x(0) = f(0)$.

We devote a first subsection to give some continuation, uniqueness, and existence results for (1.2) which are valid in general. We next pay attention in three different subsections to some particular instances of the integral equation that play an important role in the analysis of wavefront profiles, taking advantage of the sign of the kernel g/x^α, as in [15, Sections 7, 8 and 9]. It is here where the analysis becomes significantly different to the one in [15], requiring the development of new ideas.

2.1 General results

Most of the results in this section will be presented without proofs for the sake of simplicity, since they are somewhat similar to the classic Volterra theory for integral equations, we refer again the reader to [15] for details. This similarity is supported in the fact that the function $x \to x^\alpha$ is monotonic and differentiable if $x > 0$.

We start with a prolongation result, see [15, Lemma 2].

Theorem 2.1. *Let x be a solution of (1.2) on $[0,T)$, $T < \infty$. Then the limits*

$$x(T) := \lim_{t\to T} x(t), \qquad \int_0^T \frac{g(s)}{x^\alpha(s)}\,\mathrm{d}s := \lim_{t\to T} \int_0^t \frac{g(s)}{x^\alpha(s)}\,\mathrm{d}s$$

exist and are finite and

$$x(T) = f(T) + \int_0^T \frac{g(s)}{x^\alpha(s)}\,\mathrm{d}s.$$

Moreover, either $x(T) = 0$ or x is continuously extendible as a solution of equation (1.2) onto a bigger finite interval $[0,T')$.

There is also a comparison principle, see [15, Lemma 4].

Theorem 2.2. *Let x_1 denote a solution of the equation*

$$x_1(t) = f_1(t) + \int_0^t \frac{g_1(s)}{x_1^\alpha(s)}\,\mathrm{d}s \tag{2.1}$$

on some finite interval $[0,\delta)$. Suppose that $f(0) > f_1(0)$, $(f - f_1)$ is nondecreasing on $[0,\delta)$ and $g(t) \geq g_1(t)$ a.e. on $(0,\delta)$. If (1.2) has a solution x on $[0,\delta)$ such that $x(t) > 0$ for all $t \in [0,\delta)$, then $x(t) > x_1(t)$ for all $t \in [0,\delta)$.

Proof. Suppose that the lemma is false. Then there exists a value $t^* \in (0,\delta)$ such that $x(t^*) = x_1(t^*) > 0$ and $x(t) > x_1(t)$ for all $t \in [0,t^*)$.

Let $t_1 \in [0,t^*)$ be such that $x^\alpha(s) - x_1^\alpha(s) < K(x(s) - x_1(s))$ for all $s \in [t_1,t^*)$. This constant K must exist, since the function x^α is differentiable and hence Lipschitz in this interval. After this, we take $t_0 \in [t_1,t^*)$ such that $x_1(t) > 0$ for all $t \in [t_0,t^*]$ and

$$\int_{t_0}^{t^*} \frac{|g(s)|}{x^\alpha(s)x_1^\alpha(s)}\,\mathrm{d}s \leq 1/2K.$$

We finish as in [15]: using the equations satisfied by x and x_1 we see that

$$x(t) - x_1(t) \leq \int_t^{t^*} g(s)\left(\frac{1}{x_1^\alpha(s)} - \frac{1}{x^\alpha(s)}\right)\mathrm{d}s \leq \frac{1}{2}\|x - x_1\|_{L^\infty(t_0,t^*)}, \quad t \in [t_0,t^*],$$

so that $\|x - x_1\|_{L^\infty(t_0,t^*)} = 0$, contradicting that $x(t) > x_1(t)$ for all $t \in [0,t^*)$. □

We now pay attention to the number of solutions, see [15, Theorem 2 and 3]..

Theorem 2.3. *Equation (1.2) has none, one or an uncountable number of solutions. Moreover, if $\operatorname*{ess\,inf}_{0<t<\tau} g(t) \geq 0$ for some $\tau > 0$, then it has at most one solution in $[0,\tau)$.*

In the non-singular case $f(0) > 0$ existence and uniqueness follow from standard theory for nonlinear Volterra integral equations; see for instance [20]. Hence, we only have a difficulty when $f(0) = 0$. The idea to deal with it is to lift the datum f by a constant $\mu > 0$ and then pass to the limit. Thus, we will construct a solution as $\lim_{\mu\downarrow 0} x(t;\mu)$, where $x(t;\mu)$ denotes the unique positive solution to

$$x(t) = \mu + f(t) + \int_0^t \frac{g(s)}{x^\alpha(s)}\,\mathrm{d}s, \tag{2.2}$$

which has an interval of existence $[0,T(\mu))$.

By Theorem 2.2 we have for all $0 < \mu_1 < \mu_2 < \infty$ that $T(\mu_1) \leq T(\mu_2)$ and

$$x(t;\mu_1) < x(t;\mu_2) \quad \text{for all } t \in [0,T(\mu_1)).$$

Moreover, $T(\mu) \to \infty$ as $\mu \to \infty$. These properties allow us to define

$$\tilde{x}(t;0) = \inf\{x(t;\mu) : \mu \in \mathbb{R}_+ \text{ such that } T(\mu) > t\}.$$

Following [15, Section 4, Theorem 4] we can prove that whenever (1.2) has a solution, $\tilde{x}$ is a *maximal* solution of the same equation.

Theorem 2.4. *If equation* (1.2) *has a solution* x *on an interval* $[0,\delta)$ *then* $\tilde{x}(t;0)$ *is a solution of* (1.2) *on an interval* $[0,\tau) \supseteq [0,\delta)$ *and* $\tilde{x}(t;0) \geq x(t)$ *for all* $t \in [0,\delta)$. *Furthermore, either* $\tau = \infty$ *or* $\tilde{x}(t;0) \to 0$ *as* $t \uparrow \tau$.

We can now give sufficient conditions for $\tilde{x}$ to be a solution, necessarily the maximal one, proceeding as in [15, Section 6, Theorems 9 and 10].

Theorem 2.5. *Let* x_1 *be a solution of* (2.1) *on some finite interval* $[0,\tau)$.

(i) *Suppose that* $f(0) \geq f_1(0)$, $(f - f_1)$ *is nondecreasing on* $[0,\tau)$, $g(t) \geq g_1(t)$ *a.e. on* $(0,\tau)$ *and* $g/x_1^\alpha \in L^1_{\mathrm{loc}}((0,\tau))$. *Then* $\tilde{x}(t;0)$ *solves* (1.2) *on* $[0,\tau)$ *and*

$$\tilde{x}(t;0) \geq x_1(t) \text{ for all } t \in [0,\tau).$$

Moreover, if $(f - f_1)(s) < (f - f_1)(t)$ *for all* $s \in [0,t)$ *for a fixed* $t \in [0,\tau)$, *then* $\tilde{x}(t;0) = x_1(t)$ *if and only if* $\tilde{x}(t;0) = 0$.

(ii) *Suppose that* $f(t) \geq f_1(t)$ *for all* $t \in [0,\tau)$ *and* $\min\{0, g(t)\} \geq g_1(t)$ *a.e. on* $(0,\tau)$. *Then* $\tilde{x}(t;0)$ *solves* (1.2) *on* $[0,\tau)$ *and*

$$\tilde{x}(t;0) - f(t) \geq x_1(t) - f_1(t) \text{ for all } t \in [0,\tau).$$

2.2 Existence: negative kernels

We now concentrate on the case of negative kernels,

$$\operatorname*{ess\,sup}_{0<t<\tau} g(t) \leq 0 \quad \text{for some } \tau \in (0,\infty]. \tag{2.3}$$

As before, we only need to consider the case $f(0) = 0$.

Some special functions solving (1.2) for particular data f will play an important role in the analysis. The first one,

$$G(t) := \left|(\alpha+1)\int_0^t g(s)\,\mathrm{d}s\right|^{\frac{1}{\alpha+1}}, \tag{2.4}$$

solves (1.2) if $f \equiv 0$. We may assume that

$$G(t) > 0 \quad \text{for all } t \in (0,\tau), \tag{2.5}$$

since otherwise our equation reduces to $x(t) = f(t)$ on a small neighbourhood of 0, a trivial case. The second kind of special solutions have the form $x(t) = zG(t)$, $z > 0$, and solve the integral equation with $f(t) = kG(t)$ if and only if z is a root of the "fractional polynomial"

$$P_k^+(z) := z^{\alpha+1} - kz^\alpha + 1 = 0, \qquad z > 0.$$

This polynomial attains its minimum in $\mathbb{R}_+$ for $z = z_{min} := \dfrac{k\alpha}{\alpha+1}$, and the number of its (positive) roots depends on the sign of $k - k_0$, where

$$k_0 := (\alpha+1)\alpha^{-\frac{\alpha}{\alpha+1}}.$$

Indeed, a simple analysis shows that if $k > k_0$, there are two positive roots of P_k^+, say z_1 and z_2, while if $k = k_0$ there is only one, z_0, and $z_0 = z_{min}$. It is worth noting that in the case of positive kernels g, as considered in the next subsection, instead of the polynomial P_k^+ we have to deal with

$$P_k^-(z) := z^{\alpha+1} - kz^\alpha - 1 = 0.$$

The change of sign of the zero-order term makes the analysis simpler in this case.

Let us finally define the important auxiliary quantities

$$L(t) := \frac{1}{|\ln G(t)|}, \quad J(t) := \frac{1}{|\ln L(t)|}, \qquad t \in [0,\tau).$$

Note that $|g| = G'G^\alpha$, $L' = G'L^2/G$ and $J' = L'J^2/L$ almost everywhere in $(0,\tau)$. The dependence on t will be omitted in what follows when it is clear.

Theorem 2.6. *Assume that* (2.3) *and* (2.5) *hold. If*

$$f(t) \geq \big(k_0 G - K_0 G L^2(1+J^2)\big)(t) \quad \textit{for all } t \in (0,\tau), \quad \textit{with } K_0 := \frac{\alpha^{\frac{1}{\alpha+1}}}{2(\alpha+1)},$$

then equation (1.2) *has a maximal solution* $\tilde{x}(t;0)$ *on an interval* $[0,\delta) \subset [0,\tau)$, *and*

$$0 \geq \tilde{x}(t;0) - f(t) \geq -\alpha^{\frac{-\alpha}{\alpha+1}}\big(G + \frac{\alpha}{\alpha+1} GL(1+J+J^2)\big)(t) \quad \textit{for all } t \in (0,\delta).$$

Proof. Let $\delta > 0$ be such that $G(t) < \exp(-1)$ for $t \in [0,\delta) \subset [0,\tau)$, and let us define $a := \alpha/(\alpha+1)$ and

$$S := G + aGL(1+J+J^2).$$

A direct computation shows that

$$D := \frac{\partial S}{\partial G} = 1 + aL(1+J+J^2) + aL^2(1+J)(1+2J^2),$$

and through the Taylor series of the function $(1+x)^n$ we also see that

$$\begin{aligned} D^{1/\alpha} =& 1 + \frac{a}{\alpha}(1+J+J^2)L + \frac{a}{\alpha}\left((1+J)(1+2J^2) + \frac{1-\alpha}{2\alpha}a(1+J+J^2)^2\right)L^2 \\ &+ \mathcal{O}(L^3). \end{aligned}$$

It is easy then to check that the function

$$x_1 := \frac{AG}{D^{1/\alpha}}, \quad \text{where } A := \alpha^{\frac{1}{\alpha+1}},$$

is a solution of our equation provided that $f_1 := x_1 + A^{-\alpha}S$. This is equivalent to

$$\begin{aligned} f_1 =& k_0 G - K_0 G L^2(1+J^2) \\ &- D^{-1/\alpha}\left(k_0 G D^{1/\alpha} - K_0 G L^2(1+J^2)D^{1/\alpha} - AG - \frac{1}{A^\alpha} S D^{1/\alpha}\right). \end{aligned}$$

The theorem follows easily once we prove that this last quantity between parentheses is positive. In order to do so, we group terms by its order.

The terms of order G are multiplied by the factor $k_0 - A - A^{-\alpha}$, but this amounts to 0 by definition. The terms of order $GL(1+J+J^2)$ are multiplied by the factor $k_0 a\alpha^{-1} - aA^{-\alpha} - aA^{-\alpha}\alpha^{-1}$, but again this amounts to 0. Note that in order for these two quantities to be 0 we have not used the value of a.

Recall now that $(1+J+J^2)^2 = 1 + 2J + 3J^2 + 2J^3 + J^4$, and let us see the term of order GL^2. It is multiplied by

$$a^2\left(\alpha^{\frac{-1-2\alpha}{\alpha+1}} - \frac{\alpha+1}{2}\right) + a\alpha^{\frac{-\alpha}{\alpha+1}} - K_0,$$

and again this is 0, this time thanks to the definition of a. The terms of order GL^2J and GL^2J^2 suffer the same fate. Finally, the term of order GL^2J^3 is multiplied by a factor of $2K_0 > 0$. The rest of the result now follows from Theorem 2.5. □

Theorem 2.7. *Assume that* (2.3) *and* (2.5) *hold. Let K_0 be as in Theorem 2.6. If*

$$f(t) \le (k_0 G - \beta G L^2)(t) \quad \textit{for all } t \in (0,\tau)$$

for some $\beta > K_0$, equation (1.2) *has no solution.*

Proof. We define

$$H(t) := G^{-1}(t)L^{-1}(t), \quad Y(t) := -\int_0^t \frac{g(s)}{x^\alpha(s)}\,\mathrm{d}s \quad \text{for } t \in (0,\delta').$$

Note that

$$H'(t) = \frac{H(t)g(t)}{G^{\alpha+1}(t)}(L(t)+1) \quad \text{for almost every } t \in (0,\delta').$$

Our goal is to obtain an absurd estimate for Y. For a start, since $x \le f \le k_0 G$, we obtain, substituting in the equation, that $x \le (k_0 - k_0^{-\alpha})G$. We can repeat this process again and again to obtain that $x \le z_0 G$, with

$$z_0 := k_0 - \frac{1}{\left(k_0 - \frac{1}{\left(k_0 - \frac{1}{(\cdots)^\alpha}\right)^\alpha}\right)^\alpha}.$$

Note that z_0 satisfies $z_0 = k_0 - \frac{1}{z_0^\alpha}$, which is precisely the root of P_k^+, hence the notation. Therefore

$$Y(t) \ge -z_0^{-\alpha}\int_0^t g(s)/G^\alpha(s)\,\mathrm{d}s = z_0^{-\alpha}G(t) \quad \text{for all } t \in (0,\delta'). \tag{2.6}$$

On the other hand we have that

$$Y(t)H(t) - Y(\varepsilon)H(\varepsilon) = \int_\varepsilon^t \big(Y'(s)H(s) + Y(s)H'(s)\big)\,\mathrm{d}s$$

for any $0 < \varepsilon < t < \delta$. But substituting H, H', Y, Y' and using (2.6) we get

$$Y(t)H(t) - z_0^{-\alpha}L^{-1}(\varepsilon) \geq \int_\varepsilon^t \left\{ \frac{|g|}{LG}\left(\frac{1}{x^\alpha} + \frac{x(L+1)}{G^{\alpha+1}}\right) - \frac{f(L+1)|g|}{G^{\alpha+2}L} \right\} \mathrm{d}s.$$

Now the term between parentheses can be estimated by studying the function $x^{-\alpha}+Ax$ for $x \geq 0$ and a constant $A \geq 0$. This function attains at the point $x = (\alpha/A)^{1/(\alpha+1)}$ a minimum value of $k_0 A^{\alpha/(\alpha+1)}$. Therefore, and using our hypothesis,

$$Y(t)H(t) - z_0^{-\alpha}L^{-1}(\varepsilon) \geq \int_\varepsilon^t \left\{ \frac{|g|k_0(L+1)^{\frac{\alpha}{\alpha+1}}}{LG^{\alpha+1}} - \frac{(k_0 - \beta L^2)(L+1)|g|}{G^{\alpha+1}L} \right\} \mathrm{d}s,$$

and, after throwing away the appearing term βL^3, we arrive to

$$Y(t)H(t) - z_0^{-\alpha}L^{-1}(\varepsilon) \geq \int_\varepsilon^t \frac{|g|}{LG^{\alpha+1}} \left(k_0((L+1)^{\frac{\alpha}{\alpha+1}} - (L+1)) + \beta L^2\right) ds.$$

It is easy now to check, via Taylor series for example, that

$$(L+1)^{\frac{\alpha}{\alpha+1}} \geq 1 + \frac{\alpha}{\alpha+1}L - \frac{\alpha}{2(\alpha+1)^2}L^2.$$

Therefore, recalling that $|g| = G'G^\alpha$ and that $z_0 = k_0\alpha/(\alpha+1)$, we obtain

$$Y(t)H(t) - z_0^{-\alpha}L^{-1}(\varepsilon) \geq \int_\varepsilon^t \frac{G'}{G}\left((\beta - K_0)\,L + (k_0 - z_0)\right)\,\mathrm{d}s.$$

Therefore, using that $L' = L^2G'/G$ and integrating,

$$Y(t)H(t) \geq (\beta - K_0)\ln\left(\frac{L(t)}{L(\varepsilon)}\right) + (k_0 - z_0)\ln\left(\frac{G(t)}{G(\varepsilon)}\right) + z_0^{-\alpha}L^{-1}(\varepsilon).$$

Since $k_0 - z_0 = z_0^{-\alpha}$, taking ε small enough so that $L^{-1}(\varepsilon) + |\ln(G(\varepsilon))| = 0$, we get

$$Y(t)H(t) \geq (\beta - K_0)\ln\left(\frac{L(t)}{L(\varepsilon)}\right) + z_0^{-\alpha}\ln L^{-1}(t),$$

and we get our contradiction by making ε go to 0 whenever $\beta > K_0$. □

We can give an additional necessary condition for the existence of solutions of our integral equation. For it, we need the following lemma concerning non-negative kernels, which will be also used in the next subsection.

Lemma 2.1. *Suppose that* $\operatorname*{ess\,inf}_{t\in(0,\tau)} g(t) \geq 0$ *and* $f(t) \leq \beta G(t)$ *for all* $t \in [0,\tau)$ *for some* $\tau \in (0,\infty)$ *and* $\beta \in (-\infty,\infty]$*. Then, given any* $\mu > 0$*, there exists* $\rho > 0$ *such that*

$$x(t;\mu) \geq \rho + f(t) + \beta_0^{-\alpha} G(t) \quad \textit{for all } t \in [0, \min\{T(\mu),\tau\}),$$

where β_0 *is the unique positive root of the* fractional polynomial P_β^- *when* $\beta < \infty$ *and* $\beta_0^{-\alpha} := 0$ *when* $\beta = \infty$*.*

Proof. The case $\beta = \infty$ follows easily from (2.2) simply by taking $\rho = \mu$.

For finite β we argue by contradiction. Note that $x(0;\mu) = \mu > 0$. Suppose that there exists $t_0 \in [0, \min\{T(\mu),\tau\})$ such that

$$x(t;\mu) > f(t) + \beta_0^{-\alpha} G(t) \quad \text{for all } t \in (0,t_0), \quad x(t_0;\mu) = f(t_0) + \beta_0^{-\alpha} G(t_0). \tag{2.7}$$

We set

$$Y(t) := \mu + \int_0^t \frac{g(s)}{x^\alpha(s;\mu)}\,\mathrm{d}s.$$

If we multiply equation (2.2) by Y' we get

$$0 = YY' + fY' - x^{1-\alpha} G^\alpha G' \tag{2.8}$$

for every $t \in (t_0,t_1)$. Substituting $x = f + Y$ and recalling (2.7) and that $f \leq \beta G$,

$$0 \leq (\beta G + Y)^\alpha Y' - G^\alpha G' \text{ and } Y(t) > \beta_0^{-\alpha} G(t) \text{ for all } t \in (0,t_0),$$

with

$$G(0) = 0, Y(0) = \mu \text{ and } Y(t_0) = \beta_0^{-\alpha} G(t_0).$$

From here it is easy to arrive to

$$\frac{G(t_0)^{\alpha+1}}{\alpha+1} \leq \int_0^{t_0} (\beta G + Y)^\alpha Y'\,\mathrm{dt} < \int_0^{\mathrm{t}_0} (\beta\beta_0^\alpha + 1)^\alpha \mathrm{Y}^\alpha \mathrm{Y}'\,\mathrm{dt},$$

and, recalling that $\beta_0^{\alpha+1} = \beta\beta_0^\alpha + 1$, we obtain

$$\mu^{\alpha+1} < Y(t_0)^{\alpha+1} - \left(\beta_0^{-\alpha} G(t_0)\right)^{\alpha+1} = 0,$$

which is a contradiction with the fact that $\mu > 0$. □

This lemma corresponds to [15, Lemma 11] for the case $\alpha = 1$, and yields the following theorem, arguing as in Theorem 13 in the same paper.

Theorem 2.8. *Let* β *and* β_0 *be as in the previous lemma, and assume* (2.3)*. If*

$$f(t) < \beta_0^{-\alpha} G(t) \textit{ for some } t \in (0,\tau) \textit{ and}$$
$$f(s) - f(t) \leq \beta\{G^{\alpha+1}(t) - G^{\alpha+1}(s)\}^{\frac{1}{\alpha+1}} \quad \textit{for all } 0 \leq s < t,$$

then equation (1.2) *has no solution on* $[0,\tau)$*.*

Note that we are not asking f to be below $\beta_0^{-\alpha}G$ in the whole interval $(0,\tau)$, but only at some specific time. Let us also remark that two of the main differences with respect to the case $\alpha = 1$ have already appeared: the fractional polynomials $P_k^{\pm}$ and the ordinary differential equation (2.8), that will be studied more profoundly for a particular f later, in Section 2.4. The final theorem of this subsection corresponds to [15, Theorem 14].

Theorem 2.9. *Assume that* (2.3) *and* (2.5) *hold, and* $f(0) = 0$. *If given any* $t \in (0,\tau)$ *there exist constants* $\delta > 0$ *and* $k < k_0$ *such that*

$$f(s) - f(t) \le k\{G^{\alpha+1}(s) - G^{\alpha+1}(t)\}^{\frac{1}{\alpha+1}} \quad \textit{for all } s \in (t, t+\delta),$$

then either the equation (1.2) *has no solution or it has a uncountable number of them.*

2.3 Existence: non-negative kernels

This time we assume that the kernel is non-negative,

$$\operatorname*{ess\,inf}_{t\in(0,\tau)} g(t) \ge 0 \quad \text{for some } \tau \in (0,\infty]. \tag{2.9}$$

Our first existence result involves the function G defined in (2.4).

Theorem 2.10. *Assume* (2.9). *If for some* $-\infty < \gamma \le \beta \le \infty$,

$$\gamma G(t) \le f(t) \le \beta G(t) \quad \textit{for all } t \in [0,\tau),$$

then equation (1.2) *has a unique solution* $\tilde{x}(t;0)$ *on* $[0,\tau)$ *such that*

$$\beta_0^{-\alpha}G(t) \le \tilde{x}(t;0) - f(t) \le \gamma_0^{-\alpha}G(t),$$

where γ_0 *and* β_0 *are respectively the unique positive roots of the polynomials* P_γ^- *and* P_β^-. *If* $\beta = \infty$ *then we take* $1/\beta_0^\alpha = 0$.

Proof. If $\beta = \infty$ we are only saying that $\tilde{x}(t;0) \ge f(t)$, which is trivial, since the kernel is non-negative. So, let us assume that $\beta < \infty$. To prove the right-hand estimate we argue by contradiction. Suppose that it is false, then there must exist an interval $[t_0, t_1] \subset [0,\tau)$ such that

$$Y(t_0) = \gamma_0^{-\alpha}G(t_0) \quad \text{and } Y(t) > \gamma_0^{-\alpha}G(t) \text{ for all } t \in (t_0, t_1], \tag{2.10}$$

where $Y(t) := \tilde{x}(t;0) - f(t)$. Multiplying (1.2) by Y' we get (2.8) for every $t \in (t_0, t_1)$, or, substituting $x = f + Y$ and recalling that $f \ge \gamma G$,

$$0 \ge (\gamma G + Y)^\alpha Y' - G^\alpha G'.$$

But, using (2.10), we get

$$0 \ge (\gamma + \gamma_0^{-\alpha})^\alpha Y' - G'.$$

Since $\gamma + \gamma_0^{-\alpha} = \gamma_0$, it is easy to get now, integrating from t_0 to t_1, a contradiction with (2.10) in t_1. The left-hand estimate follows analogously. □

The last result of this part can be proved as [15, Theorem 16].

Theorem 2.11. *Suppose that* $\operatorname*{ess\,inf}_{0<t<\tau} g(t) \geq 0$ *and that given any* $t \in (0,\tau)$ *there exist* $\delta \in (0,t]$ *and* $\beta < k_0$ *such that*

$$f(s) - f(t) \leq \beta\{G^{\alpha+1}(t) - G^{\alpha+1}(s)\}^{\frac{1}{\alpha+1}} \quad \textit{for all } s \in (\delta, t).$$

Then equation (1.2) *has a unique solution on* $[0,\tau)$*. Moreover, if there exists a point* $\tilde{s} \in (0,\tau)$ *for which* $\theta(\tilde{s}) = 0$*, then* $\theta(s) \equiv G(s) \equiv 0$ *in* $[0,\tilde{s}]$*.*

2.4 A special case

We consider now equation (1.2) when

$$\int_0^t g(s)\,\mathrm{d}s < 0 \text{ for all } t > 0, \quad f(t) = kG(t) \text{ for all } t \geq 0, \quad k \geq k_0. \tag{2.11}$$

Again as in [15, Theorem 17] we are able to prove that our equation admits an uncountable number of solutions characterized by their behaviour as $t \downarrow 0$.

Theorem 2.12. *Suppose that* (2.11) *holds. We denote the positive roots of* P_k^+ *by* z_1 *and* z_2 *when it has two, and by* z_0 *when it has only one.*

(i) *If* $k > k_0$ *then equation* (1.2) *admits the maximal solution*

$$\tilde{x}(t;0) = z_2 G(t),$$

and for each $\rho \in \mathbb{R}$ *a unique solution* x_ρ *such that*

$$x_\rho(t) = z_1 G(t) + \rho G^{\gamma+1}(t) + \mathcal{O}(G^{2\gamma+1}(t)) \quad \textit{as } t \downarrow 0, \quad \gamma = \frac{\alpha(k - z_1)}{z_1} - 1 > 0,$$

with maximal interval of existence $[0, T_\rho)$*, with* $T_\rho = \infty$ *if* $\rho \geq 0$ *and* T_ρ *finite if* $\rho < 0$*, and no other solutions. Moreover, if* $G(t) \to \infty$ *as* $t \to \infty$ *then for every* $\rho > 0$ *there holds* $x_\rho(t) \sim z_2 G(t)$ *as* $t \to \infty$*.*

(ii) *If* $k = k_0$ *then equation* (1.2) *admits the maximal solution*

$$\tilde{x}(t;0) = z_0 G(t),$$

and for each $\rho \in \mathbb{R}$ *a unique solution* x_ρ *such that*

$$x_\rho(t) = (z_0 G - \eta G L + G L^2(\rho + \gamma J^{-1}))(t) + \mathcal{O}(G L^3 J^{-2}(t)) \quad \textit{as } t \downarrow 0,$$

where

$$\eta = \frac{2z_0}{\alpha+1} = \frac{2k\alpha}{(\alpha+1)^2} \quad \textit{and} \quad \gamma = \frac{2(\alpha+2)\eta}{3(\alpha+1)},$$

with maximal interval of existence $[0, T_\rho)$*, with* T_ρ *finite, and no other solutions.*

In both cases the solutions x_ρ are monotone with respect to the parameter ρ.

Proof. Suppose that x is a solution of our equation on $[0, \delta)$ for some δ positive. We define

$$Y(t) := -\int_0^t \frac{g(s)}{x^\alpha(s)}\,\mathrm{d}s.$$

We can multiply the equation by Y' to obtain

$$G'G^\alpha(kG - Y)^{1-\alpha} - Y'(kG - Y) = 0.$$

This equation is homogeneous, so defining $V = Y/G$ we arrive to

$$\frac{G'}{G} = \frac{(k-V)^\alpha V'}{1-(k-V)^\alpha V}$$

or, in other words,

$$\frac{\mathrm{d}V}{\mathrm{d}G} = \frac{1-(k-V)^\alpha V}{G(k-V)^\alpha}. \tag{2.12}$$

We have to study the trajectories that satisfy this equation.

Let us first study the case where $k > k_0$, which means that there are two different roots of P_k^+. Inspired by the results in [15], we define, for a certain $\gamma \in \mathbb{R}$ to be defined later, a new independent variable $G^* = G^\gamma$, and (2.12) becomes

$$\frac{\mathrm{d}V}{\mathrm{d}G^*} = \frac{1-(k-V)^\alpha V}{\gamma G^*(k-V)^\alpha}.$$

We have to study the trajectories $V(G^*)$ as $G^* \to 0$.

Clearly, we have that $V(0) = k - z_i$, $i = 1$ or 2. In fact, the only trajectory that can start from $k - z_2$ is the constant $V(G^*) = k - z_2$, which, recalling the definition of V, translates to a solution $x(t) = z_2 G(t)$. More diverse is the family of trajectories sprouting from $k - z_1$. We would like to study the limit of $\mathrm{d}V/\mathrm{d}G^*$ as G^* goes to 0.

Suppose that this limit exist, and define

$$\rho = \lim_{G^*\to 0} \frac{1-(k-V)^\alpha V}{\gamma G^*(k-V)^\alpha} = \frac{1}{\gamma z_1^\alpha} \lim_{G^*\to 0} \frac{1-(k-V)^\alpha V}{G^*}.$$

Then, applying L'Hôpital's rule, we can check that

$$\rho = \frac{1}{\gamma z_1^\alpha} \lim_{G^*\to 0} V'(\alpha V(k-V)^{\alpha-1} - (k-V)^\alpha) = \frac{\rho}{\gamma z_1^\alpha}(\alpha z_1^{\alpha-1}(k-z_1) - z_1^\alpha),$$

meaning that ρ is arbitrary whenever

$$\gamma = \frac{\alpha(k-z_1)}{z_1} - 1.$$

To see that this limit ρ in fact exists we study the monotonicity of $(1-(k-V)^\alpha V)/G^*$ with respect to G^*. Indeed,

$$\partial_{G^*}\left(\frac{1-(k-V)^\alpha V}{G^*}\right) = \frac{V'}{G^*}(\alpha V - (\gamma+1)(k-V)) = \frac{V'\alpha k}{z_1 G^*}(V-(k-z_1)).$$

Since the trajectories never cross the trajectory $V(G^*) = k - z_1$ (i.e. $V'(G^*) = 0$), this means that the previous derivative has a constant sign, which gives monotonicity and existence of the limit ρ. The rest of the assertions when $k > k_0$ follow from the study of the phase-plane and the definition of V.

The case $k = k_0$, when we have only one root $z_0 = \alpha k/(\alpha+1)$ of P_k^+, is more difficult, since γ becomes 0. In this case we can say that the zero in the numerator $1-(k-V)^\alpha V$ is one order higher, so to compensate for it we would need a higher order zero in the denominator. We achieve this by considering $\mathrm{d}V/\mathrm{d}L$, recalling the definition of L. Summarizing,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}L} &= \frac{1-(k-V)^\alpha V}{L^2(k-V)^\alpha}, \\ \frac{\mathrm{d}^2V}{\mathrm{d}L^2} &= \frac{V'}{k-V}\left(\frac{(\alpha+1)V-k}{L^2} - \frac{2(k-V)}{L} + \alpha V'\right). \end{aligned} \tag{2.13}$$

Now, from a study similar to the one before, we can see that

$$\lim_{L\to 0} V(L) = k - z_0 \quad \text{and } \eta := \lim_{L\to 0}\frac{\mathrm{d}V}{\mathrm{d}L} = \frac{2z_0}{\alpha+1}$$

for every trajectory. So, the distinction between them has to come from the second order derivative, but a problem appears since through L'Hôpital's rule we get

$$\lim_{L\to 0}\frac{\mathrm{d}^2V}{\mathrm{d}L^2} = \infty.$$

Let us write then a possible expansion of the function $V(L)$ near $L = 0$. We will omit the higher order terms for the sake of simplicity, since they will not play an important role here. For a certain function $Q(L)$ we have, near zero, that

$$V(L) \approx (k-z_0) + \eta L + Q(L)L^2, \quad V'(L) \approx \eta + 2LQ(L) + L^2Q'(L)$$

and

$$V''(L) \approx 2Q(L) + 4LQ'(L) + L^2Q''(L),$$

where we impose that both $LQ(L)$ and $L^2Q'(L)$ go to 0 as L goes to 0 in order for $V'(0)$ to be equal to η. If we substitute in (2.13) we get

$$\begin{aligned} 2Q(L) + 4LQ'(L) + L^2Q''(L) \approx{}& \frac{V'}{k-V}\cdot\frac{(\alpha+1)((k-z_0)+\eta L + Q(L)L^2) - k}{L^2} \\ &- \frac{V'}{k-V}\left(\frac{2(z_0-\eta L - Q(L)L^2)}{L} - \alpha V'\right). \end{aligned}$$

Since $(\alpha+1)(k-z_0)-k=0$ and $(\alpha+1)\eta-2z_0=0$ we can cancel the problematic terms and make L approach 0 to get

$$2Q(L)+4LQ'(L)+L^2Q''(L)\approx\frac{\eta}{z_0}((\alpha+1)Q(L)+(\alpha+2)\eta),$$

which means that we can impose $4LQ'(L)+L^2Q''(L)=\eta^2(\alpha+2)/z_0$. Thus $Q(L)\to\infty$ as $L\to 0$. The function that satisfies all these conditions is

$$Q(L)=\rho-\frac{2(\alpha+2)\eta}{3(\alpha+1)}\ln(L),$$

where ρ is once more an arbitrary real constant. Again an analysis of the trajectories of the phase-plane and the definition of V give the desired result. □

Remark. If we let $\alpha\to 1$ we recover the result from [15].

3 Travelling Waves

In order to show the full force of the theory developed here, and for future reference, we encompass in this annex several results regarding wavefronts for the general equation (1.1) and also for the particular case $b\equiv 0$, the reaction-diffusion case. More results, the details of the proofs and the intermediates lemmata can be found in the doctoral thesis of the author under the title *Large time behaviour in local and non-local diffusion* or in [13].

3.1 Travelling waves. General results

It is time now to translate the results obtained in the article for the integral equation to the frame of travelling waves. Since we are interested mainly in the applications of the integral equation to the study of wavefronts for diffusion equations we will not explain in detail all the steps followed to attain the results presented here.

This said, due to the possible degeneracies/singularities of the equation, in general we will need to deal with weak solutions. A function V defined on an open real interval Ω with values on I is said to be a travelling-wave profile of equation (1.1) corresponding to the speed σ if $V\in C(\Omega), |(a(V))'|^{p-2}(a(V))'$ and $c(V)\in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega |(a(V))'|^{p-2}(a(V))'\varphi'+b(V)\varphi'+\sigma V\varphi'-c(V)\varphi=0 \tag{3.1}$$

for every $\varphi\in C_0^\infty(\Omega)$. If $\Omega=\mathbb{R}$ then it is called a *global* travelling-wave profile.

The constitutive functions $a,b,c:\bar{I}\to\mathbb{R}$ are only assumed to satisfy:

(i) $a\in C(\bar{I})\cap C^1(I)$, $a'(u)>0$ in I, and $a(0)=0$;

(ii) $b \in C(\bar{I}) \cap C^1(I)$ and $b(0) = 0$;

(iii) c is such that the one sided limits$c(u^-)$ and $c(u^+)$ exist for every $0 < u < \ell$, with $c(u^-) < 0$ if $c(u) < 0$ and $c(u^+) > 0$ if $c(u) > 0$;

$$\int_0^s |c(r)|a'(r)\,dr < \infty \text{ for all } 0 < s \le \ell,\ s < \infty;$$

$c(0) = 0$; and $c(\ell) = 0$ if $\ell < \infty$.

Let us start this section by presenting two equivalence results.

Theorem 3.1. *Suppose $\ell < \infty$. If one of the equations*

$$u_t = \Delta_p(a(u)) + \nabla(b(u)) + c(u), \qquad u_t = \Delta_p u + \nabla(b(u)) + c(u)a'(u)$$

admits a wavefront profile from ℓ to 0 with speed σ, then they both do. Moreover if $a \in C^1(I)$, there is an explicit transformation from a wavefront profile V_1 for the first one onto a wavefront profile V_2 for the second one given by

$$V_2(\xi) = V_1(\Psi(\xi)), \quad \xi \in \mathbb{R}, \qquad \textit{where } \Psi(\xi) = \int_0^\xi a'(V_2(\eta))\,\mathrm{d}\eta.$$

This equivalence was first studied by Engler in [8] when $\alpha = 1$ and $a(u) = u^m$. Gilding and Kersner extended the result in [19] to more general nonlinearities a. Their proof works with only trivial changes for $\alpha \neq 1$. The next theorem will be useful when studying the existence near $u = \ell$ and the equations with one sign change in the reaction. It corresponds to [19, Lemma 3.5]

Theorem 3.2. *Suppose that $\ell < \infty$. Then the following statements are equivalent: equation* (1.4) *has a solution θ on $\bar{I}$ with $\theta(l) = 0$; equation*

$$\Theta(s) = -\sigma s + \tilde{b}(s) - \int_0^s \frac{\tilde{c}(r)\tilde{a}'(r)}{\Theta^\alpha(r)}\,\mathrm{d}r \tag{3.2}$$

has a solution Θ on $\bar{I}$ with $\Theta(\ell) = 0$; and, equations (1.4) *and* (3.2) *both have solutions on $\bar{I}$. In the same way, these statements are also equivalent: equation* (1.4) *has a solution θ satisfying the integrability condition on $\bar{I}$ with $\theta(l) = 0$; equation* (3.2) *has a solution Θ satisfying the integrability condition on $\bar{I}$ with $\Theta(l) = 0$; and, equations* (1.4) *and* (3.2) *both have solutions satisfying the integrability condition on $\bar{I}$.*

The next result is about the admissible wave speeds. It corresponds to [19, Theorem 8.3], and is a consequence of [19, Theorem 8.1].

Theorem 3.3. *Suppose that $\ell < \infty$.*

(i) *If $c < 0$ in $(0, \ell)$, the set S of wave speeds for which* (1.1) *has a wavefront solution from ℓ to 0 is either empty or there exists σ_0 such that $S = (-\infty, \sigma_0]$.*

(ii) *If* $c > 0$ *in* $(0, \ell)$*, either* $S = \emptyset$ *or* $S = [\sigma_0, \infty)$ *for some value* σ_0*.*

(iii) *If* $c \leq 0$ *in* $(0, \ell)$*, the set* S *is either empty, contains a single value or is an interval which is bounded above and contains its right endpoint.*

(iv) *If* $c \geq 0$ *in* $(0, \ell)$*, the set* S *is either empty, contains a single value or is an interval which is bounded below and contains its left endpoint.*

3.2 Reaction-diffusion

In this section we will focus on travelling waves for the equation (1.1) when there is no convection, $b \equiv 0$, which leaves us with the reaction-diffusion equation

$$u_t = \Delta_p(a(u)) + c(u). \tag{3.3}$$

For an equation of this class the integral equation becomes

$$\theta(s) = \sigma s - \int_0^s \frac{a'(r)c(r)}{\theta^\alpha(r)}\,\mathrm{d}r, \qquad \alpha > 0. \tag{3.4}$$

Remark. If $c \equiv 0$ ($b \not\equiv 0$), the integral equation (1.4) becomes $\theta(s) = \sigma s + b(s)$, and the search for nonnegative solutions in an interval $(0, \delta)$ is much easier. Nevertheless, one has to treat the possibility of having $\theta(s^*) = 0$ for some $s^* > 0$ more carefully than in the case $p = 2$, since on the one hand this could happen without breaking the integrability condition (1.5) and on the other hand there is no reaction to "compensate" the degeneracy.

We begin by presenting a connection between the speed σ and the integral

$$\kappa := \int_0^\ell c(s)a'(s)\,\mathrm{d}s, \tag{3.5}$$

a result that corresponds in the semi-linear case $p = 2$, $a(u) = u$ to [19, Theorem 10.1].

Theorem 3.4. *Let* $\ell < \infty$*. Equation* (3.3) *has a wavefront profile from 0 to* ℓ *with speed* σ *only if one of the following holds:*

(a) $\kappa > 0$*,* $\sigma > 0$ *and* $\displaystyle\int_u^\ell c(s)a'(s)\,\mathrm{d}s > 0$ *for all* $u \in (0, \ell)$*.*

(b) $\kappa = 0$*,* $\sigma = 0$*,* $\displaystyle\int_0^u c(s)a'(s)\,\mathrm{d}s \leq 0$ *for all* $u \in (0, \ell)$*, and* $\theta(s) = G(s)$ *satisfies the integrability condition on* I*.*

(c) $\kappa < 0$*,* $\sigma < 0$ *and* $\displaystyle\int_0^u c(s)a'(s)\,\mathrm{d}s < 0$ *for all* $u \in (0, \ell)$*.*

The necessary condition (b) *is also a sufficient condition for existence.*

3.2.1 Fixed sign

Now we focus our attention on the case in which $c > 0$ in $(0, \ell)$. Due to a clever change of variables $u \to \ell - u$ (see [19, Lemma 3.4]) our results will apply, with the needed changes, to the case in which $c < 0$ in $(0, \ell)$.

Theorem 3.5. *Suppose that $\ell < \infty$ and $c > 0$ in $(0, \ell)$. Set*

$$\lambda_1 := \limsup_{s \downarrow 0} \left\{ \frac{1}{s} \int_0^s \frac{c(r)a'(r)}{r^\alpha} \, \mathrm{d}r \right\}.$$

(i) *If $\lambda_1 = \infty$, then equation* (3.3) *has no wavefront profile from ℓ to 0.*

(ii) *If $\lambda_1 < \infty$, there exists a value $\sigma^* > 0$ such that* (3.3) *has exactly one wavefront profile from ℓ to 0 for every wave speed $\sigma \geq \sigma^*$ and no such profile for $\sigma < \sigma^*$.*

This result is analogous to the one in [19, Theorem 10.5]

Our next step is to say something about the supports of the solutions. There are some previous results needed here that are similar to [19, Lemmata 10.9, 10.19 and 10.20]. It is well known that some combinations of filtration and reaction nonlinearities a and c provoke the appearance of free boundaries in the solutions. Thus, for wavefront profiles from ℓ to 0 one or both of the following properties may hold:

$$V(\xi) \equiv 0 \quad \text{for all } \xi \geq \xi^* \quad \text{for some } \xi^* \in \mathbb{R}, \tag{3.6}$$

$$V(\xi) \equiv \ell \quad \text{for all } \xi \leq \xi_* \quad \text{for some } \xi_* \in \mathbb{R}. \tag{3.7}$$

This theorem about the support of the solutions corresponds to [19, Theorem 10.21].

Theorem 3.6. *Suppose that $\ell < \infty$, $c > 0$ in $(0, \ell)$ and $\lambda_1 < \infty$. Let $\sigma^* > 0$ be the critical wave speed for which equation* (3.3) *has exactly one distinct wavefront profile from ℓ to 0 for every speed $\sigma \geq \sigma^*$ and no such solution for $\sigma < \sigma^*$.*

(i) *Suppose in addition that $(ca')^{1/\alpha}$ is differentiable on $[0, \delta]$ for some $\delta \in (0, \ell)$, $(ca')(0) = 0$ and $((ca')^{1/\alpha})'(u) \to ((ca')^{1/\alpha})'(0)$ as $u \downarrow 0$. Then the following alternatives are mutually exclusive.*

   (a) *Every wavefront profile from ℓ to 0 satisfies* (3.6)*. This occurs if and only if*

$$\int_0^\delta \frac{1}{c(s)} \, \mathrm{d}s < \infty.$$

   (b) *Every wavefront profile from ℓ to 0 with wave speed σ^* satisfies* (3.6)*, while every such solution with speed $\sigma > \sigma^*$ satisfies $V(\xi) > 0$ for all $\xi \in \mathbb{R}$. This occurs if and only if*

$$\int_0^\delta \frac{a'(s)}{s^\alpha} \, \mathrm{d}s < \int_0^\delta \frac{1}{c(s)} \, \mathrm{d}s = \infty.$$

(c) *Every wavefront profile from ℓ to 0 satisfies $V(\xi) > 0$ for all $\xi \in \mathbb{R}$. This occurs if and only if*

$$\int_0^\delta \frac{a'(s)}{s^\alpha}\,\mathrm{d}s = \infty.$$

(ii) *Suppose furthermore that $(ca')^{1/\alpha}$ is differentiable on $[\ell-\delta, \ell]$ for some $\delta \in (0, \ell)$, $(ca')(\ell) = 0$ and $((ca')^{1/\alpha})'(u) \to ((ca')^{1/\alpha})'(\ell)$ as $u \uparrow \ell$. Then the following alternatives are mutually exclusive.*

(a) *Every wavefront profile from ℓ to 0 satisfies* (3.7)*. This occurs if and only if*

$$\int_{\ell-\delta}^{\ell} \frac{1}{c(s)}\,\mathrm{d}s < \infty.$$

(b) *Every wavefront profile from ℓ to 0 satisfies $V(\xi) < \ell$ for all $\xi \in \mathbb{R}$. This occurs if and only if*

$$\int_{\ell-\delta}^{\ell} \frac{1}{c(s)}\,\mathrm{d}s = \infty.$$

We close this section studying the behaviour of our wavefront profiles when they approach the critical values 0 and ℓ. This result is analogous to [19, Theorem 10.22] though we find important to sketch the behaviour of the parameter λ_1 in this case. When $\alpha = 1$ the work in [19] shows how λ_1 becomes $(ca')'(0)$ but our case is not so tidy. In fact tracing back the definition of λ_1 we see that in our case, whenever $\lambda_1 < \infty$, we have that this value becomes $[((ca')^{1/\alpha})'(0)]^\alpha$.

Theorem 3.7. *Let V denote a wavefront profile of equation* (3.3) *from ℓ to 0 with wave speed σ.*

(i) *Suppose that the conditions of Theorem 3.6* (i) *hold and define*

$$\xi^* := \sup\{\xi \in \mathbb{R} : V(\xi) > 0\}. \tag{3.8}$$

If $\sigma = \sigma^*$, then $\frac{(a(V))'(\xi)}{V^\alpha(\xi)} \to -z_2^\alpha$ as $\xi \uparrow \xi^*$, where z_2 is the biggest root of Q_γ, with

$$Q_\gamma(z) := z^{\alpha+1} - \sigma z^\alpha + \gamma$$

and $\gamma = [((ca')^{1/\alpha})'(0)]^\alpha$, whereas if $\sigma > \sigma^*$, then $\frac{V'(\xi)}{c(V(\xi))} \to -\frac{z_1^\alpha}{((ca')^{1/\alpha})'(0)}$ as $\xi \uparrow \xi^*$, where z_1 is the smallest root of the fractional polynomial Q_γ.

(ii) *Suppose that the conditions of Theorem 3.6* (ii) *hold and define*

$$\xi_* := \inf\{\xi \in \mathbb{R} : V(\xi) < \ell\}. \tag{3.9}$$

Then, $\frac{V'(\xi)}{c(V(\xi))} \to -\frac{z_0^\alpha}{((ca')^{1/\alpha})'(\ell)}$ as $\xi \downarrow \xi_*$, where z_0 is the smallest root of Q_γ with $\gamma = [((ca')^{1/\alpha})'(\ell)]^\alpha$.

3.2.2 One sign change

It is time to close up this work and we do so by studying the properties of wavefront profiles appearing when the reaction term has one sign change, in essence, $c \leq 0$ in $[0, a]$ and $c \geq 0$ in $[a, \ell]$, for some $a \in (0, \ell)$. In the opposite case, when the reaction starts being non-negative and ends being non-positive, Theorem 3.4 prevents the existence of wavefront profiles connecting ℓ with 0. Therefore, we stick to the first case, and to work with it we define the value

$$H(s) := \left| (\alpha + 1) \int_s^\ell c(r) a'(r) \, \mathrm{d}r \right|^{1/(\alpha+1)},$$

which plays a similar role as the function G but coming from the value ℓ. We also recall the equation

$$\Theta(s) = -\sigma s + \tilde{b}(s) - \int_0^s \frac{\tilde{c}(r)\tilde{a}'(r)}{\Theta^\alpha(r)} \, \mathrm{d}r \tag{3.10}$$

from Theorem 3.2, and the reader may have already guessed that we are going to use it to split the analysis of our equation in two halves separated by the value $u = a$. This creates a new difficulty, because during the study of the existence of wavefront profiles we would like to paste the two halves together at some point and with the same speed of propagation. Continuity of wavefront profiles with respect to the parameter σ is stated in [19, Lemma 8.5], and the possibility of sticking both halves back together in [19, Lemma 8.9].

Theorem 3.8. *Suppose that $c \leq 0$ in $[0, a]$ and $c \geq 0$ in $[a, \ell]$, for some $a \in (0, \ell)$, $\ell < \infty$. Let κ be defined as in (3.5). Suppose also that one of the following holds:*

(i) *$\kappa > 0$ and $c > 0$ in (a, ℓ);*

(ii) *$\kappa = 0$, $G(u) > 0$ in $(0, a)$ and $H(u) > 0$ in (a, ℓ);*

(iii) *$\kappa < 0$ and $c < 0$ in $(0, a)$.*

Then there exists a real number σ^* for which equation (3.3) has exactly one distinct wavefront profile from ℓ to 0 with speed σ^* and no such solution for any other wave speed.

We finish by studying the behaviour close to the critical values $S = 0, \ell$ and the support of the solutions, as we did in the previous section.

Theorem 3.9. *Suppose that $\ell < \infty$ and let V denote the wavefront profile from ℓ to 0 of equation (3.3) with wave speed σ.*

(i) *Suppose also that $c \leq 0$ on $(0, \delta]$ for some $\delta \in (0, \ell/2)$, $(ca')^{1/\alpha}$ is differentiable on $[0, \delta]$, $(ca')(0) = 0$ and $((ca')^{1/\alpha})'(u) \to ((ca')^{1/\alpha})'(0)$ as $u \downarrow 0$.*

(a) *If* $\kappa > 0$ *then* V *satisfies* (3.6) *if and only if*

$$\int_0^\delta \frac{a'(s)}{s^\alpha}\,\mathrm{d}s < \infty.$$

(b) *If* $\kappa = 0$ *and* $G > 0$ *on* $(0,\delta]$ *then* V *satisfies* (3.6) *if and only if*

$$\int_0^\delta \frac{a'(s)}{G^\alpha(s)}\,\mathrm{d}s < \infty.$$

(c) *If* $\kappa < 0$ *and* $c < 0$ *on* $(0,\delta]$ *then* V *satisfies* (3.6) *if and only if*

$$\int_0^\delta \frac{1}{|c(s)|}\,\mathrm{d}s < \infty.$$

(ii) *Suppose that* $c \geq 0$ *on* $[\ell-\delta,\ell)$ *for some* $\delta \in (0,\ell/2)$, $(ca')^{1/\alpha}$ *is differentiable on* $[\ell-\delta,\ell]$, $(ca')(\ell) = 0$ *and* $((ca')^{1/\alpha})'(u) \to ((ca')^{1/\alpha})'(\ell)$ *as* $u \uparrow \ell$.

(a) *If* $\kappa > 0$ *and* $c > 0$ *on* $[\ell-\delta,\ell)$ *then* V *satisfies* (3.7) *if and only if*

$$\int_{\ell-\delta}^\ell \frac{1}{c(s)}\,\mathrm{d}s < \infty.$$

(b) *If* $\kappa = 0$ *and* $H > 0$ *on* $[\ell-\delta,\ell)$ *then* V *satisfies* (3.7) *if and only if*

$$\int_{\ell-\delta}^\ell \frac{a'(s)}{H^\alpha(s)}\,\mathrm{d}s < \infty.$$

(c) *If* $\kappa < 0$ *then* V *satisfies* (3.6) *if and only if*

$$\int_{\ell-\delta}^\ell \frac{a'(s)}{(\ell-s)^\alpha}\,\mathrm{d}s < \infty.$$

This theorem is a corollary of the following, which is at the same time consequence of Theorems 3.4, 3.8 and 3.2.

Theorem 3.10. *Let* V *denote a wavefront profile of equation* (3.3) *from* ℓ *to 0 with wave speed* σ. *Define* ξ^* *and* ξ_* *as in* (3.8)–(3.9).

(i) *Suppose that the conditions of Theorem 3.9* (i) *hold.*

(a) *If* $\kappa > 0$ *then* $\frac{(a(V))'(\xi)}{V^\alpha(\xi)} \to -z_2^\alpha$ *as* $\xi \uparrow \xi^*$, *where* z_2 *is the biggest root of* Q_γ *with* $\gamma = [((ca')^{1/\alpha})'(0)]^\alpha$.

(b) *If* $\kappa = 0$ *and* $G > 0$ *on* $(0,\delta]$ *then* $\frac{(a(V))'(\xi)}{G(V^\alpha(\xi))} \to -1$ *as* $\xi \uparrow \xi^*$.

(c) *If* $\kappa < 0$ *and* $c < 0$ *on* $(0,\delta]$ *then* $\frac{V'(\xi)}{|c(V(\xi))|} \to \frac{z_2^\alpha}{((ca')^{1/\alpha})'(0)}$ *as* $\xi \uparrow \xi^*$.

(ii) *Suppose that the conditions of Theorem 3.6* (ii) *hold.*

(a) *If* $\kappa > 0$ *and* $c > 0$ *on* $[\ell - \delta, \ell)$ *then* $\frac{V'(\xi)}{|c(V(\xi))|} \to -\frac{z_1^\alpha}{((ca')^{1/\alpha})'(0)}$ *as* $\xi \downarrow \xi_*$*, where* z_1 *is the smallest root of* Q_γ *with* $\gamma = [((ca')^{1/\alpha})'(\ell)]^\alpha$.

(b) *If* $\kappa = 0$ *and* $H > 0$ *on* $[\ell - \delta, \ell)$ *then* $\frac{(a(V))'(\xi)}{H(V^\alpha(\xi))} \to -1$ *as* $\xi \downarrow \xi_*$.

(c) *If* $\kappa < 0$ *then* $\frac{(a(V))'(\xi)}{(\ell - V(\xi))^\alpha} \to z_1^\alpha$ *as* $\xi \downarrow \xi_*$.

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