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One-sided precedence monitoring schemes for unknown shift sizes using generalized *2-of-*(*h*+1) and *w-of-w* improved runs-rules

J.-C. Malela-Majika[*1], S.C. Shongwe[1] and P. Castagliola[2]

Abstract

Parametric monitoring schemes are expected to perform better than their nonparametric counterparts when the assumption of a specific form of a distribution such as the normal is met. In practice, such assumption is often violated. Consequently, the performance of parametric schemes deteriorates considerably. To solve this problem, more efficient and flexible monitoring schemes based on nonparametric tests are recommended. In this paper, efficient and robust one-sided monitoring schemes are developed using generalized {*1-of-1* or 2-*of-*(*h*+1)} and {*1-of-1* or *w-of-w*} improved runs-rules (IRR) without any distributional assumption in the zero- and steady-states modes. Moreover, the zero- and steady-states run-length properties of the resulting one-sided IRR schemes are formulated and empirically evaluated using the Markov chain technique. Comparisons with other well-known one-sided Shewhart-type nonparametric schemes (e.g. basic precedence scheme and precedence scheme with standard runs-rules) indicate that the proposed schemes have a better performance. Real data are used to demonstrate the design and implementation of the one-sided improved runs-rules precedence schemes. Finally, a discussion on the two-sided IRR precedence schemes is also provided.

Keywords: Distribution-free; Order statistic; Precedence statistic; Runs-rules; Case U; Markov chain; Phase I; Phase II

1. Introduction

Statistical process monitoring (SPM) is a collection of statistical techniques and tools used in industrial and non-industrial processes to distinguish between a process that is operating under chance causes of variation (i.e. the process is said to be in-control (IC)) or under assignable causes of variation (i.e. the process is said to be out-of-control (OOC)). A monitoring scheme is the main tool in SPM used to generally distinguish among the different types of variation, see Montgomery (2005). Monitoring schemes that are generally based on the assumption of normality or any other specific form of a distribution (e.g. exponential, Poisson, Student's *t*-distribution, etc.) are called parametric monitoring schemes. However, in most of the cases, the underlying distribution of the quality process is unknown, or there is not enough information to justify the assumption of normality; then, in such cases, monitoring schemes that do not depend on a particular distributional assumption are preferred. Nonparametric or distribution-free monitoring schemes can serve this broader purpose – see a more thorough discussion of this important class of monitoring schemes in Qiu (2014) and Chakraborti and Graham (2019). A key advantage of nonparametric schemes is that their IC run-length distribution remains the same for all continuous process distributions. This means that, for example, the IC average run-length (ARL_0) or the false alarm rate (FAR) of a nonparametric scheme is the same across all continuous distributions. Therefore, it is said that nonparametric schemes are IC robust. This is not

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true for parametric monitoring schemes in general and consequently, their IC robustness can be a matter of legitimate concern. Moreover, nonparametric charts are often more robust and efficient under some heavy-tailed and skewed underlying distributions. The drawback of nonparametric charts is that they are relatively less sensitive than their parametric counterparts when the underlying distribution of the quality process is known.

When the design parameter(s) of interest are known or specified, this is referred to parameters known (hereafter, Case K); however, when the design parameter(s) are unspecified or unknown, this is referred to parameters unknown (hereafter, Case U). Case K and Case U nonparametric monitoring schemes have received a lot of attention in recent SPM literature. For Case K, see for example, Human et al. (2010), Khilare and Shirke (2010), Kritzinger et al. (2014), Khilare and Shirke (2015), Patil and Shirke (2017), Pawar et al. (2018), etc. For Case U, see for example, Chakraborti and van de Wiel (2008), Chakraborti et al. (2009a), Albers and Kallenberg (2008), Graham et al. (2017), Malela-Majika et al. (2019), etc. In Case U scenario, it is well-known and accepted that there are two distinct phases or stages, namely Phase I (or retrospective phase) and Phase II (or prospective or monitoring phase). Phase I involves the preliminary (including statistical) analysis which includes planning, administration, design of the study, data collection, data management, exploratory work (including graphical and numerical analysis, goodness-of-fit analysis, etc.) to ensure that the process is in a state of IC, and a search for appropriate control limits from the historical data. For more details on control charts for Phase I applications, readers are referred to Chakraborti et al. (2009b) and Jones-Farmer et al. (2014). In Phase II, monitoring schemes are implemented prospectively to continuously monitor any departures from an IC state using the parameters estimated in Phase I. That is, in Phase II, at each sampling time, smaller samples (i.e. test sample) are observed, and a decision is made on each of them to know whether the process is IC or not – see Jensen et al. (2006) and Psarakis et al. (2013) more details on Phase II monitoring schemes.

Shewhart-type charts are the most popular charts because of their simplicity, ease of construction and use, and the fact that they are quite efficient in detecting moderate to large shifts. Hence, Chakraborti, van der Laan and van de Wiel (2004) proposed a class of nonparametric Shewhart-type control charts called the precedence charts, using some order statistic of a Phase II sample as the charting statistic and the control limits are constructed from a Phase I (or reference) sample. The latter paper and the manner in which control charts are formulated have led to a renewed interest in nonparametric methods based on order statistics in industrial statistics and medical researches, particularly in the area of process monitoring and the choice of best treatment, see for example, Chakraborti et al. (2009a), Balakrishnan et al. (2010, 2015), Malela-Majika et al. (2016b), Triantafyllou (2017, 2018), Koutras and Triantafyllou (2018), to count a few. While the majority of attention has been paid to two-sided precedence monitoring schemes, Balakrishnan et al. (2015) argued that there are many real-life situations where it makes more sense to monitor either an increase only, or a decrease only, in a process characteristic(s) of interest. Consequently, following a similar line of argument as in

Balakrishnan et al. (2015) for the construction and use of one-sided schemes; the aim of this paper is to add a contribution to the literature on nonparametric monitoring schemes with supplementary improved runs-rules (IRR) by proposing the one-sided Shewhart-type *1-of-1* or *2-of-*(h+1) (denoted by $\text{IRR}_{2\text{-}of\text{-}(h+1)}$) and the *1-of-1* or *w-of-w* (denoted by $\text{IRR}_{w\text{-}of\text{-}w}$) precedence monitoring schemes, where h and w are positive integers, with $h > 0$ and $w > 1$. When one is interested to test the difference in the location parameter between two populations when no information about the actual distributions is available, the median precedence test is mostly preferred over the minimum and maximum precedence statistics. Moreover, the median is known to be more robust as compared to other statistics and the minimum and maximum statistics are known to be affected by the extreme values; thus the focus is on the median statistic in this paper.

The parametric one- and two-sided schemes supplemented with runs-rules have been recently investigated in many publications, see for instance, Mehmood et al. (2019), Chew et al. (2019), Shongwe et al. (2019a, b), and the references therein, however as far we know, there are very few nonparametric runs-rules schemes: Chakraborti and Eryilmaz (2007), Chakraborti et al. (2009a), Human et al. (2010), Kritzinger et al. (2014), Khilare and Shirke (2015), Malela-Majika et al. (2016a, b), Malela-Majika and Rapoo (2017), Pawar et al. (2018), Malela-Majika et al. (2019), Triantafyllou (2020) as well as the ones discussed in the book by Chakraborti and Graham (2019).

Note that the zero- and steady-state modes are used to characterize the short- and the long-term *RL* properties of a scheme, respectively. That is, the zero-state run-length is defined as the number of sampling points at which the chart first signals given that it begins in some specific initial state. However, the steady-state run-length is the number of sampling points at which the chart first signals given that the process begins and stays IC for a long time, then at some random time, an OOC signal is observed. Now, for runs-rules precedence schemes:

- Chakraborti et al. (2009a) investigated the IC and OOC performances of the two-sided standard runs-rules ($\text{SRR}_{2\text{-}of\text{-}(h+1)}$, with h=1 only) precedence scheme using the Markov chain approach. Note though, only the zero-state mode was considered.
- In an effort to generalize the latter, Malela-Majika et al. (2019) investigated the IC and OOC performances of the two-sided $\text{SRR}_{2\text{-}of\text{-}(h+1)}$ precedence scheme for any integer $h > 0$ using both zero- and steady-state modes. They also showed that when $h > 1$, the $\text{SRR}_{2\text{-}of\text{-}(h+1)}$ precedence schemes tend to have some OOC detection improvements. Moreover, they derived the closed-form expressions for any integer $h > 0$ used to calculate the zero- and steady-state run-length properties via the Markov chain approach.
- While Chakraborti et al. (2009a) and Malela-Majika et al. (2019) used the Markov chain approach to evaluate the run-length properties, Malela-Majika et al. (2016b) investigated the IC and OOC performances of the two-sided $\text{SRR}_{2\text{-}of\text{-}2}$ and $\text{IRR}_{2\text{-}of\text{-}2}$ precedence schemes using Monte Carlo simulations.

Thus, this paper contributes to the SPM literature by:

- Investigating the IC and OOC performances of the one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes for any integers $h > 0$ and $w > 1$.
- Studying both zero- and steady-state run-length performances of the proposed schemes.
- Investigating both specific shifts as well as overall performances of the proposed schemes for a range of shifts in the location parameter.

The remainder of this paper is organized as follows: in Section 2, the one-sided 1-*of*-1 (i.e. basic) median precedence monitoring scheme is introduced and the operation of the proposed schemes are outlined. The general form of the transition probability matrices (TPMs), zero-state and steady-state run-length characteristics of the proposed one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence monitoring schemes are developed in Section 3. Section 4 discusses the IC and OOC zero- and steady-state performances and compares their performances to other one-sided Shewhart-type counterparts. A real-life application of the proposed schemes is given in Section 5. Section 6 gives an extension of the $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence monitoring schemes to the two-sided version. Finally, Section 7 provides concluding remarks and some recommendations.

2. Precedence monitoring schemes with improved runs-rules

2.1 Basic and SRR precedence monitoring schemes

Let $X = \{X_1, X_2, \ldots, X_m\}$ be a Phase I (or reference) sample of size m available from an IC process with an unknown continuous c.d.f. (cumulative distribution function), $F(x)$. Let Y_j^k (with $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots$) denote the k^{th} test sample of size n_k, $n_k = n\ \forall k$, since it is assumed that the Phase II (or test) samples are all of the same size. For instance, $Y_j^1 = \{Y_1, Y_2, \ldots, Y_n\}$ is the first test sample of size n. Let $G^k(y)$ denote the c.d.f. of the distribution of the k^{th} Phase II sample and let $G^k(y) = G(y)\ \forall k$, since the Phase II samples are all assumed to be i.i.d. (independent and identically distributed). Assume that the location model is given by $G(t) = F(t - \delta)$, for all t, where δ is the location difference (or shift in the location parameter). The process is IC in Phase II when $G \equiv F$ (i.e. $\delta = 0$).

The precedence monitoring scheme is a general class of nonparametric monitoring schemes that uses the j^{th} order statistic in the Phase II sample of size n, i.e. $Y_{(j:n)}$. The j^{th} order statistic can be any quartile, decile, percentile, etc.; note though, the most used order statistics are the minimum, lower quartile, median, upper quartile and maximum. In the case of the one-sided Shewhart-type precedence scheme, the charting statistic $Y_{(j:n)}$ is compared separately to either the lower control limit (*LCL*) or the upper control limit (*UCL*). The control limits of the lower and upper one-sided precedence scheme, i.e. *LCL* and *UCL*, are given by the a^{th} and b^{th} order statistics of the Phase I sample, respectively, where $1 \leq a < m$ and $1 < b \leq m$. When n is odd, say, $n = 2r + 1$, then $j = r + 1$ corresponds to the unique test sample median and the corresponding precedence scheme is called median precedence scheme (or, simply called precedence scheme in this paper).

Let p denotes the probability that the precedence scheme does not signal, i.e. $p = P(Y_{(j:n)} \geq X_{(a:m)})$ and $p = P(Y_{(j:n)} \leq X_{(b:m)})$ for the lower and upper one-sided charts, respectively. To check whether the parameter of interest has shifted, we use $LCL = X_{(a:m)}$ and $UCL = X_{(b:m)}$, where a and b are found from $P\left(Y_{(j:n)} \geq X_{(a:m)}\right) \leq 1 - p$ and $P\left(Y_{(j:n)} \leq X_{(b:m)}\right) \leq 1 - p$ for the lower and upper one-sided precedence schemes, respectively. The control limit is found by setting p (or $1 - p$) to some desired high (or low) significant values, say 0.9973 (or 0.0027). Equivalently, a and b are found such that the IC *ARL* (ARL_0) is as close as possible to some high desired values such as 370, 500 and 1000. In Phase II, the j^{th}order statistic from a uniform (0,1) distribution follows a beta distribution with parameters j and $n - j + 1$ (see e.g. Gibbons and Chakraborti, 2003).

Therefore, for an upper and lower one-sided precedence charts, the conditional probabilities that the plotting statistic plots in region A, B, C and D (see Figure 1(a)) are given by

$$\begin{aligned} p_A &= P\left(Y_{(j:n)} \geq X_{(b:m)} \middle| X_{(b:m)} = x_{(b:m)}\right) = I\left(GF^{-1}\left(U_{(b:m)}\right), j, n - j + 1\right), \\ p_B &= 1 - P\left(Y_{(j:n)} \geq X_{(b:m)} \middle| X_{(b:m)} = x_{(b:m)}\right) = 1 - I\left(GF^{-1}\left(U_{(b:m)}\right), j, n - j + 1\right), \\ p_C &= P\left(Y_{(j:n)} \geq X_{(a:m)} \middle| X_{(a:m)} = x_{(a:m)}\right) = I\left(GF^{-1}\left(U_{(a:m)}\right), j, n - j + 1\right), \\ p_D &= 1 - P\left(Y_{(j:n)} \geq X_{(a:m)} \middle| X_{(a:m)} = x_{(a:m)}\right) = 1 - I\left(GF^{-1}\left(U_{(a:m)}\right), j, n - j + 1\right), \end{aligned} \quad (1)$$

respectively, where $I(.,.,.)$ denotes the incomplete beta function and $U_{(e:l)}$ represents the e^{th} order statistic of a sample of size l from the Uniform(0,1) distribution. All the above expressions depend on the c.d.f. F and G only through the transformation function $\Psi = GF^{-1}$. It is important to know that the process is IC if $G = F$. In this case, $\Psi(u) = GF^{-1}(u) = u$ for any $u \in (0,1)$.

To improve the detection ability of the basic Shewhart-type monitoring schemes towards small and moderate shifts, the SPM literature recommends the use of supplementary runs-rules. The $SRR_{2\text{-}of\text{-}(h+1)}$ schemes need at least two plotting statistics to decide if the process is IC or OOC and it uses the charting regions in Figure 1(a). The design of the $SRR_{2\text{-}of\text{-}(h+1)}$ schemes is summarized as follows: For a specific integer value of *h*, take a sample of size *n* and compute the charting statistic. If at some random time t the charting statistic plots on region A (region D) for the first time, then keep track of the charting regions that the scheme plots on from time $t + 1$ until time $t + h$, or alternatively, until the second charting statistic plots on region A (region D), respectively. The $SRR_{w\text{-}of\text{-}w}$ monitoring scheme needs exactly *w* consecutive plotting statistics to decide if the process is IC or OOC and it also uses the charting regions in Figure 1(a). The design of the $SRR_{w\text{-}of\text{-}w}$ scheme is summarized as follows: For a specific integer value *w*, take a sample of size *n* and compute the charting statistic. If at some random time t the charting statistic plots on region A (region D) for the first time, then the $SRR_{w\text{-}of\text{-}w}$ scheme will give an OOC signal, if and only if, the consecutive charting statistics from time $(t + 1)$ to $(t + w\text{-}1)$ plot on region A (region D), respectively.

The charting regions corresponding to the one-sided basic and SRR schemes are as follows:

- Upper one-sided basic and SRR schemes: Regions A and B (see Figure 1(a)),

- Lower one-sided basic and SRR schemes: Regions C and D (see Figure 1(a));

where regions B and C are IC regions for schemes (i.e. basic and SRR), whereas regions A and D are OOC and nonconforming regions for the basic and SRR schemes, respectively.

<Insert Figure 1>

2.2 Operation of the one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes

The $IRR_{2\text{-}of\text{-}(h+1)}$ scheme is the combination of the $SRR_{2\text{-}of\text{-}(h+1)}$ scheme and the basic (i.e. 1-*of*-1) scheme (discussed in Section 2.1). Therefore, the upper (lower) one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ scheme gives a signal when either a single point plots on or above (below) the *UCL* (*LCL*) or 2 out of $h+1$ successive points plot on or above (below) the upper (lower) warning limit. Similarly, the $IRR_{w\text{-}of\text{-}w}$ is the combination of the $SRR_{w\text{-}of\text{-}w}$ scheme and the 1-*of*-1 scheme. The upper (lower) one-sided $IRR_{w\text{-}of\text{-}w}$ scheme gives a signal when either a single point plots on or above (below) the *UCL* (*LCL*) or w successive points plot on or above (below) the upper (lower) warning limit.

The charting regions which are separated by *UCL* or *LCL* and the upper or lower warning limit (*UWL* or *LWL*) corresponding to each one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes are as follows:

- Upper one-sided scheme: Regions 1, 2 and 3 (see Figure 1(b)),
- Lower one-sided scheme: Regions 4, 5 and 6 (see Figure 1(b));

where regions 1 and 6 are OOC regions, whereas regions 2 and 5 are nonconforming regions and regions 3 and 4 are conforming regions. Note that a sample plots on a conforming region when it is under the influence of common causes of variation only; however, when it plots on a nonconforming region, it implies it has some assignable causes of variation present.

The conditional probabilities that the plotting statistic plots in region 1, 2, 3, 4, 5 and 6 of the one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes (see Figure 1(b)) are given by

$$
\begin{aligned}
p_1 &= P\left(Y_{(j:n)} \geq X_{(b_2:m)} \middle| X_{(b_2:m)} = x_{(b_2:m)}\right) = I\left(GF^{-1}\left(U_{(b_2:m)}\right), j, n-j+1\right),\\
p_2 &= I\left(GF^{-1}\left(U_{(b_2:m)}\right), j, n-j+1\right) - I\left(GF^{-1}\left(U_{(b_1:m)}\right), j, n-j+1\right),\\
p_3 &= P\left(Y_{(j:n)} \leq X_{(b_1:m)} \middle| X_{(b_1:m)} = x_{(b_1:m)}\right) = 1 - I\left(GF^{-1}\left(U_{(b_1:m)}\right), j, n-j+1\right),\\
p_4 &= P\left(Y_{(j:n)} \geq X_{(a_1:m)} \middle| X_{(a_1:m)} = x_{(a_1:m)}\right) = I\left(GF^{-1}\left(U_{(a_1:m)}\right), j, n-j+1\right),\\
p_5 &= I\left(GF^{-1}\left(U_{(a_1:m)}\right), j, n-j+1\right) - I\left(GF^{-1}\left(U_{(a_2:m)}\right), j, n-j+1\right),\\
p_6 &= P\left(Y_{(j:n)} \leq X_{(a_2:m)} \middle| X_{(a_2:m)} = x_{(a_2:m)}\right) = 1 - I\left(GF^{-1}\left(U_{(a_2:m)}\right), j, n-j+1\right).
\end{aligned}
\tag{2}
$$

Note that b_1 and b_2 (a_1 and a_2) are computed such that the attained ARL_0 is as close as possible to the nominal ARL_0 (see Section 4.1 for more details).

The operational procedure of the one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes is as summarized in Table 1, where *CRL* (i.e. conforming run-length) is the number of conforming samples that fall in between any two consecutive nonconforming samples.

<Insert Table 1>

3. Run-length properties of the one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence scheme

In this section, mathematical foundations and necessary notations that are needed to derive the run-length (*RL*) properties of the one-sided Shewhart-type $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes using Markov chain technique are given.

3.1 Transition probability matrix (TPM)

To illustrate how to construct the TPMs, one can follow a similar procedure implemented in Fu and Lou (2003, Chapter 4) to formulate general TPMs for one-sided schemes that are valid for all integer values of $h > 0$ and $w > 1$.

Firstly, since in general, the TPM is of the form,

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & | & \mathbf{r} \\ - & - & - \\ \mathbf{0}' & | & 1 \end{pmatrix} \tag{3}$$

where $\mathbf{P}$ is the $(\tau + 1) \times (\tau + 1)$ matrix, $\mathbf{Q}$ is the $\tau \times \tau$ essential TPM, $\mathbf{r} = \mathbf{1} - \mathbf{Q1}$ is a $\tau \times 1$ vector so that each row sum to unity, with $\mathbf{1} = (1\ 1\ ...\ 1)'$ and $\mathbf{0} = (0\ 0\ ...\ 0)'$. Then using the Markov chain procedure discussed in Champ (1992), Shongwe et al. (2019a, b), it can be shown that $\tau = (h+1)$ for the $IRR_{2\text{-}of\text{-}(h+1)}$ schemes whereas $\tau = w$ for the $IRR_{w\text{-}of\text{-}w}$ schemes. Consequently, it follows that for any integer value of $h > 0$, the TPM of the lower or upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ schemes is, in general, given by Equation (4).

	ϕ	η_2	η_3	η_4	⋯	η_h	η_{h+1}	OOC
ϕ	$\pi_\#$	π_*	0	0	⋯	0	0	π_λ
η_2	0	0	$\pi_\#$	0	⋯	0	0	$\pi_* + \pi_\lambda$
η_3	0	0	0	$\pi_\#$	⋯	0	0	$\pi_* + \pi_\lambda$
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
η_{h-1}	0	0	0	0	⋯	$\pi_\#$	0	$\pi_* + \pi_\lambda$
η_h	0	0	0	0	⋯	0	$\pi_\#$	$\pi_* + \pi_\lambda$
η_{h+1}	$\pi_\#$	0	0	0	⋯	0	0	$\pi_* + \pi_\lambda$
OOC	0	0	0	0	⋯	0	0	1

(4)

As explained in Shongwe et al. (2019b), ϕ denotes the IC initial state, η_r (r=2,3,…, h+1) denotes the transient states deduced from the OOC absorbing states.

Note that the elements of the TPM in Equation (4) denote the following

$$\begin{aligned} \pi_\# &= \begin{cases} p_3 & \text{if upper one-sided scheme} \\ p_4 & \text{if lower one-sided scheme,} \end{cases} \\ \pi_* &= \begin{cases} p_2 & \text{if upper one-sided scheme} \\ p_5 & \text{if lower one-sided scheme,} \end{cases} \\ \pi_\lambda &= \begin{cases} p_1 & \text{if upper one-sided scheme} \\ p_6 & \text{if lower one-sided scheme,} \end{cases} \end{aligned} \tag{5}$$

where p_1, p_2, p_3, p_4, p_5 and p_6 are defined in Equation (2).

Similarly, it follows that for any integer value of $w > 1$, the TPM of the lower or upper one-sided $IRR_{w\text{-}of\text{-}w}$ schemes is, in general, given by Equation (6), with the probability elements $\pi_\#$, π_* and π_λ as defined in Equation (5), with η_r (r=2,3,…, w).

	ϕ	η_2	η_3	η_4	$\cdots$	η_{w-1}	η_w	OOC
ϕ	π_*	$\pi_\#$	0	0	$\cdots$	0	0	π_λ
η_2	π_*	0	$\pi_\#$	0	$\cdots$	0	0	π_λ
η_3	π_*	0	0	$\pi_\#$	$\cdots$	0	0	π_λ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
η_{w-2}	π_*	0	0	0	$\cdots$	$\pi_\#$	0	π_λ
η_{w-1}	π_*	0	0	0	$\cdots$	0	$\pi_\#$	π_λ
η_w	π_*	0	0	0	$\cdots$	0	0	$\pi_\#+\pi_\lambda$
OOC	0	0	0	0	$\cdots$	0	0	1

(6)

3.2 General run-length characteristics

The characteristics of the run-length (*RL*) distribution reveal important information about the short-term and long-term performance of a monitoring scheme. Note that once Equation (3) is determined (see the respective TPMs in Equations (4) and (6)), then important properties of the *RL* can be determined via an appropriate Markov chain technique discussed in Fu and Lou (2003). That is, the conditional average run-length (*CARL* = E(*RL*)) defined as

$$CARL = \boldsymbol{\xi}^{\mathrm{T}}\mathbf{R} \tag{7}$$

where $\boldsymbol{\xi}$ denotes either the zero- or steady-state initial ($\tau \times 1$) probability vector discussed in Sections 3.3 and 3.4; and

$$\mathbf{R} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1} \tag{8}$$

i.e. the $\tau \times \tau$ vector, with *ARL* values of being in each separate τ transient or non-absorbing states and **I** is the $\tau \times \tau$ identity matrix.

From Equation (8), with **Q** extracted from Equation (4), using basic algebraic matrix manipulation, it follows that for any integer value $h > 0$, the *ARL* vector of the lower or upper one-sided $\mathrm{IRR}_{2\text{-}of\text{-}(h+1)}$ scheme is given by

$$\mathbf{R} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \vdots \\ \zeta_{h-1} \\ \zeta_h \\ \zeta_{h+1} \end{pmatrix} = \frac{1}{(1-\pi_\#)(1-\pi_\#-\pi_*\pi_\#^h)} \begin{pmatrix} 1-\pi_\#-\pi_*\pi_\#^h+\pi_*\pi_\#^0 \\ 1-\pi_\#-\pi_*\pi_\#^h+\pi_*\pi_\#^h \\ 1-\pi_\#-\pi_*\pi_\#^h+\pi_*\pi_\#^{h-1} \\ 1-\pi_\#-\pi_*\pi_\#^h+\pi_*\pi_\#^{h-2} \\ \vdots \\ 1-\pi_\#-\pi_*\pi_\#^h+\pi_*\pi_\#^3 \\ 1-\pi_\#-\pi_*\pi_\#^h+\pi_*\pi_\#^2 \\ 1-\pi_\#-\pi_*\pi_\#^h+\pi_*\pi_\# \end{pmatrix}. \tag{9}$$

Similarly, using **Q** extracted from Equation (5), the *ARL* vector of the lower or upper one-sided $\mathrm{IRR}_{w\text{-}of\text{-}w}$ scheme is given by

$$\mathbf{R} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \vdots \\ \zeta_{w-2} \\ \zeta_{w-1} \\ \zeta_w \end{pmatrix} = \frac{1}{1-\pi_* - \pi_\#(1-\pi_*^w)} \begin{pmatrix} 1-\pi_*^w \\ 1-\pi_*^{w-1} \\ 1-\pi_*^{w-2} \\ 1-\pi_*^{w-3} \\ \vdots \\ 1-\pi_*^3 \\ 1-\pi_*^2 \\ 1-\pi_* \end{pmatrix}. \tag{10}$$

The unconditional *ARL* (*UARL*) is given by

$$UARL = \int_0^1 \int_0^t CARL\, f_{b_1 b_2}(u,t)du\, dt \tag{11}$$

where $f_{b_1 b_2}(u,t) = \frac{m!}{(b_1-1)!\,(b_2-b_1-1)!(m-b_2)!} t^{b_1-1}(t-u)^{b_2-b_1-1}(1-t)^{m-b_2}$ which is the joint pdf of the b_1^{th} and b_2^{th} order statistics in a sample of size *m* from the Uniform (0,1) distribution. For more details readers are also referred to Chakraborti et al. (2009a).

3.3 Zero-state run-length characteristics

In the zero-state mode, we have $\boldsymbol{\xi}^{\mathrm{T}} = \mathbf{q}^{\mathrm{T}}$ = (1,0,0,…,0) where the unique "1" is located at the 1st position so that the zero-state *ARL* (*ZSARL*) is given by $ZSARL = \mathbf{q}^{\mathrm{T}}\mathbf{R}$. That is, using Equation (9), it follows that for any integer value $h > 0$, the conditional $ZSARL$ ($CZSARL$) of the lower or upper one-sided $\mathrm{IRR}_{2\text{-}of\text{-}(h+1)}$ scheme is given by

$$CZSARL = \frac{1-\pi_\# - \pi_*\pi_\#^h + \pi_*\pi_\#^0}{(1-\pi_\#)(1-\pi_\# - \pi_*\pi_\#^h)}. \tag{12}$$

with $\pi_\#$ and π_* defined in Equation (5). Thus, the unconditional $ZSARL$ ($UZSARL$) for any integer value $h > 0$, is then defined by

$$UZSARL = \int_0^1 \int_0^t \left[\frac{1-\pi_\# - \pi_*\pi_\#^h + \pi_*\pi_\#^0}{(1-\pi_\#)(1-\pi_\# - \pi_*\pi_\#^h)}\right] f_{b_1 b_2}(u,t)du\, dt. \tag{13}$$

However, using Equation (10), it follows that for any integer value $w > 1$, the $CZSARL$ of the lower or upper one-sided $\mathrm{IRR}_{w\text{-}of\text{-}w}$ scheme is given by

$$CZSARL = \frac{1-\pi_*^w}{1-\pi_* - \pi_\#(1-\pi_*^w)}, \tag{14}$$

with $\pi_\#$ and π_* defined in Equation (5). Thus, the $UZSARL$ for any integer value $w > 1$, is then defined by

$$UZSARL = \int_0^1 \int_0^t \left[\frac{1-\pi_*^w}{1-\pi_* - \pi_\#(1-\pi_*^w)}\right] f_{b_1 b_2}(u,t)du\, dt. \tag{15}$$

3.4 Steady-state run-length characteristics

In the steady-state mode, we have $\boldsymbol{\xi}^{\mathrm{T}} = \mathbf{s}^{\mathrm{T}} = (\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_\tau)$, i.e. a non-zero initial probability vector, which is obtained by dividing each element of $\mathbf{Q}$, when $\delta = 0$, by its corresponding row sum, so that the modified essential TPM is called the conditional essential TPM, denoted by $\mathbf{Q_0}$. That is, $\mathbf{Q_0}$ is the

altered version of **Q** so that the '*new*' essential TPM is ergodic, see some recent discussions by Knoth (2016) and Shongwe et al. (2019a). That is, $\mathbf{s}^{\mathrm{T}}\mathbf{Q_0} = \mathbf{s}^{\mathrm{T}}$ subject to $\sum_{j=1}^{\tau} s_j = 1$.

Using matrix **Q** in Equation (4) and following the procedure described above, it follows that the initial probability vector of the lower or upper one-sided IRR$_{2\text{-}of\text{-}(h+1)}$ precedence scheme for any value of $h > 0$ is given by

$$\mathbf{s}^{\mathrm{T}} = (\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_\tau) = \frac{1}{1+h\theta}\ (1\ \ \theta\ \ \theta\ \ \ldots\ \theta)\ \text{ with }\ \theta = \frac{\pi_*}{\pi_\# + \pi_*}. \tag{16}$$

Similarly, using matrix **Q** in Equation (5), the initial probability vector of the lower or upper one-sided IRR$_{w\text{-}of\text{-}w}$ precedence scheme for any value of $w > 1$ is given by

$$\mathbf{s}^{\mathrm{T}} = (\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_\tau) = \frac{1-\pi_*}{1-\pi_*^w}\ (\pi_*^0\ \pi_*\ \pi_*^2\ \pi_*^3\ \ldots\ \pi_*^{w-2}\ \pi_*^{w-1}). \tag{17}$$

The conditional $SSARL$ ($CSSARL$) of the lower or upper one-sided IRR$_{2\text{-}of\text{-}(h+1)}$ precedence scheme for any value of $h > 0$ is given by

$$CSSARL = \mathbf{s}^{\mathrm{T}}\mathbf{R} = \frac{1}{1+h\theta}\zeta_1 + \frac{\theta}{1+h\theta}\sum_{i=2}^{h+1}\zeta_i, \tag{18}$$

where θ is defined in Equation (16) and ζ_i $(i = 1,2,\ldots,h+1)$ are defined in Equation (9). Thus, the unconditional $SSARL$ ($USSARL$) for any integer value $h > 0$, is then defined by

$$USSARL = \int_0^1 \int_0^t \left[\frac{1}{1+h\theta}\zeta_1 + \frac{\theta}{1+h\theta}\sum_{i=2}^{h+1}\zeta_i\right] f_{b_1 b_2}(u,t)du\ dt. \tag{19}$$

However, using Equation (10), it follows that for any integer value $w > 1$, the $CSSARL$ of the lower or upper one-sided IRR$_{w\text{-}of\text{-}w}$ scheme is given by

$$CSSARL = \mathbf{s}^{\mathrm{T}}\mathbf{R} = \sum_{i=1}^{w} s_i \cdot \zeta_i, \tag{20}$$

with s_i defined in Equation (17) and ζ_i $(i = 1,2,\ldots,h+1)$ are defined in Equation (10). Thus, the $USSARL$ for any integer value $w > 1$, is then defined by

$$USSARL = \int_0^1 \int_0^t \left[\sum_{i=1}^{w} s_i \cdot \zeta_i\right] f_{b_1 b_2}(u,t)du\ dt. \tag{21}$$

3.5 Overall performance measurement

Many studies in SPM use the *ARL* values to assess the performance of schemes (Li et al., 2014). This measure evaluates the performance of a control chart for a specific shift. Therefore, schemes which are designed on the basis of a specified optimal shift (say, δ_{opt}) will perform poorly if the shift is actually different from δ_{opt}. When researchers are interested in measuring the chart's performance for a range of shifts, $\delta_{min} \leq \delta \leq \delta_{max}$, it is recommended to use measures of the overall performance (see Machado and Costa, 2014); where δ_{min} and δ_{max} are lower and upper bound of δ, respectively. In this paper, we make use of one of the characteristics of the quality loss function (*QLF*), the average

extra quadratic loss (*AEQL*) value, in order to investigate the overall performance of the proposed schemes. A *QLF* describes the relationship between the size of the shift and the quality impact. More recently, a number of researchers tend to supplement the results of the *ARL* with that of QLF because users tend not to know beforehand what exact shift value(s) is targeted for a specific process; for more discussions about this, see Wu et al. (2008), Reynolds and Lou (2010), Tran et al. (2017), Rakitzis et al. (2019), etc. Defining $f(\delta)$ as the probability density function (pdf) of a uniform distribution with parameters δ_{min} and δ_{max}, i.e. shifts occur with equal probability, the unconditional *AEQL* (denoted as $UAEQL$) may be given by:

$$UAEQL = \frac{1}{\delta_{max} - \delta_{min}} \int_{\delta_{min}}^{\delta_{max}} \left(\delta^2 \times UARL(\delta)\right) d\delta. \tag{22}$$

where $UARL(\delta)$ is the unconditional *ARL* given in Equations (13), (15), (19) and (21). When comparing several schemes (with the charting constants computed while the process is IC and the ARL_0 is approximately equal to the nominal ARL_0), the scheme with the minimum $UAEQL$ value is considered to be the most efficient.

To investigate the difference between the zero- and steady state $UAEQL$ values of a monitoring scheme, the percentage difference (denoted as *%Diff*) of the *AEQL* values is calculated using the following formula

$$\%Diff = \left(\frac{UZSAEQL - USSAEQL}{USSAEQL}\right) \times 100, \tag{23}$$

where $UZSAEQL$ and $USSAEQL$ represents the unconditional zero- and steady-state *AEQL* values, respectively. Note that the percentage difference of the *ARL* values can also be calculated in a similar way.

4. Zero-state and steady-state performance of the proposed precedence schemes

4.1 IC design of the proposed monitoring schemes

One of the most important steps in the design and implementation of a monitoring scheme is the computation (or search) of the control limits. The first step in the design of the upper (lower) one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes is based on the determination of the UWL and UCL (*LWL* and *LCL*), respectively. The UWL and UCL (*LWL* and *LCL*) are defined by the b_1^{th} and b_2^{th} (a_1^{th} and a_2^{th}) order statistics, also known as charting constants of the Phase I sample, which means $UWL = X_{(b_1:m)}$ and $UCL = X_{(b_2:m)}$ ($LWL = X_{(a_1:m)}$ and $LCL = X_{(a_2:m)}$) where $b_1 < b_2$ ($a_1 > a_2$), respectively. Due to space restriction, only the IC design and OOC performance of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes will be presented in details.

The zero-state charting constants of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes are determined using Equations (12) and (14), respectively; whereas, the steady-state charting constants of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes are found using

Equations (18) and (20), respectively. For instance, when h = 1 for the $IRR_{2\text{-}of\text{-}(h+1)}$ scheme or w = 2 for the $IRR_{w\text{-}of\text{-}w}$ scheme, i.e. $IRR_{2\text{-}of\text{-}2}$ scheme, the couple (b_1, b_2) = (457, 469) yields the attained $CZSARL_0$ and $CSSARL_0$ values of 500.51 and 500.41, respectively, so that ($\widehat{UWL}$, $\widehat{UCL}$) = ($X_{(457:\,500)}$, $X_{(469:\,500)}$). The charting constants (i.e. b_1 and b_2) and attained $CZSARL_0$ and $CSSARL_0$ values of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes are given in Table 2 for nominal ARL_0 values of 370 (first row) and 500 (second) when $h \in \{1, 2, 5, 10\}$ and $w \in \{2, 5, 10\}$ with $n \in \{5, 7\}$ and $m \in \{100, 200, 500\}$. As expected, it can be observed that for both $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes, for large sample sizes, the attained $CZSARL_0$ and $CSSARL_0$ values are much closer to the nominal ARL_0 value. For fixed values of h, w, and m, and a selected nominal ARL_0 value (e.g. nominal $ARL_0 = 370$ or 500), when n increases/decreases, the charting constants b_1 and b_2 decreases/increases. Moreover, when h, w and m are kept constant, as the n increases (decreases), the magnitudes of both charting constants b_1 and b_2 decrease (increase). Note that, for m, n, h and w fixed, if the charting constant b_1 increases (decreases), the attained $CZSARL_0$ (or $CSSARL_0$) value increases (decreases). A key observation from Table 2 is that for a given nominal ARL_0 values, the zero- and steady-state charting constants are the same and the attained $CZSARL_0$ and $CSSARL_0$ values are approximately equal. When $b_1 = b_2$, the upper one-sided IRR precedence scheme is equivalent to the upper one-sided SRR precedence scheme. From Table 2 it can also be seen that for large Phase I sample sizes, the attained $ZSARL_0$ and $SSARL_0$ are much closer to the nominal ARL_0 as compared to small Phase I sample sizes.

<Insert Table 2>

4.2 IC robustness of the proposed precedence schemes

The ARL_0 of the nonparametric monitoring schemes does not depend on the underlying process distribution. This statement can be verified by checking their IC robustness. To check the robustness property of the proposed monitoring schemes, various distributions are considered and these are: (i) The standard normal distribution, i.e. $N(0,1)$, to investigate the effect of symmetric distributions, (ii) The Student's t-distribution, with degree of freedom v, i.e. $t(v)$, to study the effect of heavy tails, and (iii) The gamma distribution, i.e. $GAM(\alpha,\beta)$, to investigate the effect of skewness. Note that when $v \geq 30$, the $t(v)$ distribution approximate the normal distribution. Thus, to properly investigate the effect of the heavy tail, v is set to a small value. Thus, in this paper we used $v = 5$. For the $GAM(\alpha,\beta)$, when the shape parameter α converges towards infinity when the scale parameter β remains fixed, the gamma distribution reduces to the normal distribution. Therefore, to study the effect of skewness, in this paper both shape and scale parameters are fixed to small integer values such that $\alpha = \beta = 1$ which is practically equivalent to an exponential distribution with parameter β = 1 (i.e. $EXP(1)$). Since the performance of the proposed schemes depends on the Phase I and Phase II probabiltity distributions only through the transformation function, $\Psi(u)$, it is very important to show how to find $\Psi(u)$. Table 3 gives the IC and OOC transformation functions for the $N(0,1)$, $t(5)$ and

GAM(1,1) distributions. It can be observed that when the process is IC, $\psi(u) = u$ regardless of the nature of the p.d.f. under consideration. This property confirms the IC robustness of the precedence schemes. For more details on how to derive the transformation functions, readers are referred to the Appendix.

<Insert Table 3>

From Table 4, it can also be seen that, as expected, for all continuous distributions under consideration, the proposed schemes yield the same IC characteristics (i.e., $CZSARL_0$ or $CSSARL_0$). For instance, the upper control and warning limits, attained $CZSARL_0$ and $CSSARL_0$ values of the $IRR_{2\text{-}of\text{-}3}$ precedence scheme (i.e. when h = 2) are computed under the *N*(0,1), *t*(5) and *GAM*(1,1) distributions for m = 500, n = 5 and a nominal ARL_0 value of 500. It is observed that ($\widehat{UWL}$, $\widehat{UCL}$) = ($X_{(460:500)}$, $X_{(469:500)}$) yields the attained $CZSARL_0$ and $CSSARL_0$ values of 500.61 and 500.60, respectively, for all the distributions under consideration. This shows that the proposed precedence control schemes are IC robust. Therefore, the charting constants and characteristics of the IC *RL* distribution do not depend on the nature of the underlying distribution.

<Insert Table 4>

4.3 OOC Performance

Since the proposed precedence monitoring schemes are IC robust, it is of interest to compare their performance when the process is OOC. For a specific shift, $\delta \neq 0$, the monitoring scheme with a small OOC $UZSARL$ ($UZSARL_\delta$) or small OOC $USSARL$ ($USSARL_\delta$) value is considered to be more sensitive. When comparing the overall performance of several monitoring schemes, the scheme with the smallest $UAEQL$ value is preferred. The $UZSARL_\delta$ values of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes are determined using Equations (13) and (15), respectively; whereas, the $USSARL_\delta$ values of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes are found using Equations (19) and (21), respectively. The $UZSAEQL$ and $USSAEQL$ values are computed using Equation (22). Given that δ_{min}= 0, note that, when δ_{max} = 0.7, the *UAEQL* value gives the measure of the overall performance for small shifts only. When δ_{max} = 1.5, the *UAEQL* value measures the overall performance covering small to moderate shifts. For δ_{max} = 2.5, the *UAEQL* value measures the overall performance covering small to large shifts.

Tables 5 and 6 display the unconditional zero- and steady-state performance results of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ monitoring schemes, respectively. The *UAEQL* and the $\%Diff$ of the *UAEQL* values are computed using Equations (22) and (23), respectively. In these tables, a grey shaded cell shows that the proposed scheme performs better in that particular situation. Table 5 presents the unconditional zero- and steady-state (in brackets) OOC *ARL* performances of the proposed upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ precedence monitoring scheme under the *N*(0,1), *t*(5) and *GAM*(1,1) distributions for h = 1, 2, 5 and 10. However, Table 6 displays the unconditional zero- and steady-state (in brackets) OOC *ARL* performances of the proposed upper one-sided $IRR_{w\text{-}of\text{-}w}$

precedence monitoring scheme under the $N(0,1)$, $t(5)$ and $GAM(1,1)$ distributions for w = 2, 5, 10 and 15. The results in Table 5 shows that the zero-and steady-states performances are almost similar. For both $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes, the percentage difference between the $UZSARL$ and $USSARL$ values doest not exceed 1%. In terms of the overall performance, the percentage difference between the $UZSAEQL$ and $USSAEQL$ values is between 0.02% and 0.09% for the $IRR_{2\text{-}of\text{-}(h+1)}$ scheme and between 0.02% and 0.8% for the $IRR_{w\text{-}of\text{-}w}$ scheme. For small shifts in the location parameter, in terms of the ARL values, the enhanced $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes perform better under the $GAM(1,1)$ distribution followed by $N(0,1)$ distribution when $\delta \leq 0.5$ for both zero- and steady-state modes. For moderate and large shifts, the proposed schemes perform better under the $t(5)$ distribution regardless of the value of h or w. In both zero- and steady-state modes, for small shifts, the proposed $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme is more sensitive under the $N(0,1)$ distribution than the $t(5)$ distribution when $\delta \leq 0.4$.

A thorough examination of the results shows that when $\delta \in (0, 1.1)$ the $IRR_{w\text{-}of\text{-}w}$ scheme is more sensitive for large values of w. However, when $\delta \in (1.1, 1.7)$, the $IRR_{w\text{-}of\text{-}w}$ scheme is more sensitive for small values of w. When $\delta \geq 1.7$, the proposed $IRR_{w\text{-}of\text{-}w}$ precedence scheme is sensitive regardless of the value of w. The proposed IRR schemes (i.e. both $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes) are relatively inefficient under the $GAM(1,1)$ distribution for moderate and large shifts. Unlike the $SRR_{w\text{-}of\text{-}w}$ scheme, the OOC ARL of the $IRR_{w\text{-}of\text{-}w}$ scheme converges toward one for large shifts. Consequently, the $IRR_{w\text{-}of\text{-}w}$ control scheme performs better than the $SRR_{w\text{-}of\text{-}w}$ control scheme for large shifts in the location parameter.

When comparing the zero- and steady state performances of the proposed schemes, it is observed that the steady-state ARL values are slightly smaller than the zero-state ARL values; so that their corresponding $AEQL$ have a difference of no more than 1% (see Tables 5 and 6). Since the zero- and steady performances of the proposed precedence schemes are almost similar and because of the page restriction, in Figure 2, the proposed precedence schemes are compared to the basic precedence scheme in terms of the $ZSAEQL$ values under symmetrical, heavy-tailed and skewed distributions. Thus, in terms of the $ZSAEQL$ values (i.e., overall performance), for δ_{max} = 0.7, the IRR precedence schemes are more sensitive under skewed distributions (see Figure 2). For δ_{max} = 1.5 and δ_{max} = 2.5, the proposed monitoring schemes are more sensitive under heavy-tailed distributions followed by symmetric distributions and relatively insensitive under skewed distributions. It can also be seen that the proposed precedence schemes are superior to the basic precedence scheme regardless of the nature of the underlying distribution.

<Insert Tables 5 and 6>

Table 7 compares the zero- and steady state $AEQL$ values of the proposed $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme with the $SRR_{2\text{-}of\text{-}(h+1)}$ and basic precedence schemes. However, Table 8 compares the zero- and steady state $AEQL$ values of the proposed $IRR_{w\text{-}of\text{-}w}$ precedence scheme with the $SRR_{w\text{-}of\text{-}w}$ and basic

precedence monitoring schemes. In Tables 7 and 8, the best precedence scheme is shaded in grey under different probability distributions.

The results in Tables 7 and 8 as well as Figure 2 yield the following findings:

1. In terms of the *AEQL* values with $\delta_{max} = 0.7$ and 1.5 (i.e. for "small" and "small to moderate" shifts),
    - The IRR precedence schemes outperform the SRR and basic precedence schemes regardless of the value of h or w for both zero- and steady-state modes.
    - The overall performance of the $IRR_{w\text{-}of\text{-}w}$ precedence scheme is an increasing function of w; which means that for "small" as well as "small to moderate" shifts, the larger the value of w, the more sensitive the proposed $IRR_{w\text{-}of\text{-}w}$ precedence scheme (see Figure 2(a) – (b)).
    - The $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence monitoring schemes perform worse under *GAM*(1,1) distribution *except* for "small" shifts in the process location regardless of the value of h (see Figure 2(b)-(c)).
    - For $h > 5$, the performance of the $SRR_{2\text{-}of\text{-}(h+1)}$ schemes deteriorate as h increases. However, the $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme performs uniformly better regardless of the value of h. Therefore, the proposed $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme outperforms the $SRR_{2\text{-}of\text{-}(h+1)}$ precedence monitoring scheme in terms of the overall performance regardless of the value of h (see Table 7).
    - The $SRR_{w\text{-}of\text{-}w}$ precedence scheme is more efficient under the $t(5)$ distribution regardless of the value of w. The sensitivity of the $SRR_{w\text{-}of\text{-}w}$ precedence control schemes increases slowly as h increases. Whereas, the sensitivity of the $IRR_{w\text{-}of\text{-}w}$ control schemes increases rapidly as w increases in the interval [2,5] and for $w > 5$ the overall performance of the $IRR_{w\text{-}of\text{-}w}$ control schemes increases slowly.
    - Under small shifts, the proposed $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme performs better than the basic precedence scheme regardless of the value of h. Under skewed distributions, the $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme is more sensitive followed by the basic precedence scheme regardless of the value of h. However, the $IRR_{w\text{-}of\text{-}w}$ precedence scheme is more sensitive under skewed distributions when $w \in \{2, 3\}$ and for $w > 3$ the $IRR_{w\text{-}of\text{-}w}$ precedence scheme is more sensitive under heavy-tailed distributions regardless of the value of w (see Figure 2(a)).
    - Under "small to moderate" shifts, the proposed IRR schemes perform worse under skewed distributions. However, the IRR precedence schemes outperform the SRR and basic precedence schemes.
2. In terms of the "small to large" shifts, *AEQL*, with $\delta_{max} = 2.5$,
    - Both SRR and IRR schemes perform uniformly better in steady-state mode for small values of h or w. However, as the value of w increases, the performance of the $SRR_{w\text{-}of\text{-}w}$

precedence monitoring scheme deteriorates rapidly. Nevertheless, the $SRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{2\text{-}of\text{-}(h+1)}$ as well as the $IRR_{w\text{-}of\text{-}w}$ precedence schemes preserve their sensitivity regardless of the value of h or w; note though, the $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes outperform the $SRR_{2\text{-}of\text{-}(h+1)}$, $SRR_{w\text{-}of\text{-}w}$ and basic schemes.

- For the proposed $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme, the steady-state overall performance is slightly smaller than the zero-state overall performance by less 1%.
- Both upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes are more sensitive under light and heavy-tailed distributions followed by symmetric distributions. They are relatively insensitive under skewed distributions.
- The $SRR_{w\text{-}of\text{-}w}$ control scheme performs worst for large values of w; however, $IRR_{w\text{-}of\text{-}w}$ and $IRR_{2\text{-}of\text{-}(h+1)}$ monitoring schemes perform uniformly better regardless of the value of w and h, respectively, and outperform the basic precedence scheme from small to large shifts in the location parameter.
- The proposed IRR schemes are more sensitive under heavy-tailed distributions regardless of the value of h and outperform the basic precedence scheme.

<Insert Tables 7 and 8>

<Insert Figure 2>

Note that to confirm the results found in Tables 5-8, Monte Carlo simulations with 50000 replications were used. The discrepancy between the results found using exact formulas and simulations is within 1% which means these results are almost similar. Because of the similarity of the results, the simulations results are not displayed in this paper.

5. Illustrative example

In this section, an illustrative example on the design and implementation of the proposed IRR monitoring schemes is given using a well-known dataset from Montgomery (2005, page 223; Tables 5.2 and 5.3). The data are the inside diameters of piston rings manufactured by a forging process. The data given in Table 5.2 contains fifteen Phase II samples, each of size n = 5. Table 5.3 of Montgomery (2005) contains 125 Phase I observations, that were collected when the process was considered IC (m = 125). These data are considered to be the Phase I (or reference) observations for which a goodness of fit test for normality is not rejected. In this example, the detection ability of the proposed IRR precedence schemes is compared to the detection ability of the SRR and basic schemes. For both zero- and steady-state modes, for a nominal ARL_0 of 500, the zero-state and steady-state *UWL* and *UCL* of the upper one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme for h = 2 (i.e., the *UWL* and *UCL* of the $IRR_{2\text{-}of\text{-}3}$ scheme) are given by the 110th and 117th order statistics, that is $\widehat{UWL} = X_{(110:125)}$ = 74.013 and $\widehat{UCL} = X_{(117:125)}$ = 74.015, respectively. However, for both zero- and steady state modes, the *UCLs* of the upper one-sided $SRR_{2\text{-}of\text{-}(h+1)}$ and basic precedence schemes for h = 2 (i.e., the *UCLs* of $SRR_{2\text{-}of\text{-}3}$ and $RR_{1\text{-}of\text{-}1}$ schemes) are given by the 115th and 122th order statistics, that

is $\widehat{UCL} = X_{(115:125)} = 74.015$ and $\widehat{UCL} = X_{(122:125)} = 74.02$, respectively. A plot of the $IRR_{2\text{-}of\text{-}3}$, $SRR_{2\text{-}of\text{-}3}$ and basic precedence (i.e. median) charting statistics for both cases is shown in Figure 3. It is seen that the $IRR_{2\text{-}of\text{-}3}$ precedence scheme signals for the first time on the 9th sample in the prospective phase (Phase II); whereas, the $SRR_{2\text{-}of\text{-}3}$ and basic precedence schemes signal for the first time on the 13th and 14th samples, respectively.

However, for both zero- and steady-state modes, for a nominal ARL_0 of 500, the zero-state and steady-state *UWL* and *UCL* of the upper one-sided $IRR_{w\text{-}of\text{-}w}$ precedence scheme for $w = 3$ (i.e., the *UWL* and *UCL* of the $IRR_{3\text{-}of\text{-}3}$ scheme) are given by the 99th and 117th order statistics, that is $\widehat{UWL} = X_{(99:125)} = 74.009$ and $\widehat{UCL} = X_{(117:125)} = 74.015$, respectively. Whereas, for both zero- and steady state modes, the *UCLs* of the upper one-sided $SRR_{w\text{-}of\text{-}w}$ (for $w = 3$) and basic precedence schemes (i.e., the *UCLs* of $SRR_{3\text{-}of\text{-}3}$ and $RR_{1\text{-}of\text{-}1}$ schemes) are given by the 107th and 122th order statistics; that is, $\widehat{UCL} = X_{(107:125)} = 74.012$ and $\widehat{UCL} = X_{(122:125)} = 74.02$, respectively. A plot of the $IRR_{3\text{-}of\text{-}3}$, $SRR_{3\text{-}of\text{-}3}$ and basic precedence (i.e. median) charting statistics for both cases is shown in Figure 4. It is seen that the $IRR_{3\text{-}of\text{-}3}$ precedence scheme signals for the first time on the 9th sample in the prospective phase; whereas, both the $SRR_{3\text{-}of\text{-}3}$ and basic precedence schemes signal for the first time on the 14th sample in the prospective phase.

<Insert Figures 3 and 4>

The above example shows that the addition of runs-rules improves the basic precedence scheme and the IRR precedence schemes outperform both the SRR and basic schemes.

6. Extension of the $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes to two-sided scenario

6.1 TPMs and run-length properties

In this section, a brief summary of the extension to the two-sided version of the $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes, with charting regions shown in Figure 5, is presented. The $IRR_{2\text{-}of\text{-}(h+1)}$ discussed here is an improved version of the two-sided $RR_{2\text{-}of\text{-}(h+1)}$ presented in Malela-Majika et al. (2019).

<Insert Figure 5>

The two-sided $IRR_{2\text{-}of\text{-}(h+1)}$ scheme signals when either a single point falls on Region 1 or 5; or when 2 out of $h + 1$ successive samples fall on Region 2 (Region 5) which are separated by at most $h - 1$ samples falling on Region 3 or 4 (Region 2 or 3), respectively. Following a similar procedure as in Section 3.1, it follows that the TPM of the two-sided $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme for any value of h is given by

	η_1	η_2	$\cdots$	η_{h-3}	η_{h-2}	η_{h-1}	η_h	ϕ	η_{h+2}	η_{h+3}	η_{h+4}	η_{h+5}	$\cdots$	η_{2h-1}	η_{2h}	η_{2h+1}	OOC
η_1								q_3	q_4								$q_2+q_1+q_5$
η_2	q_3								q_4								$q_2+q_1+q_5$
η_3		q_3							q_4								$q_2+q_1+q_5$
$\vdots$			$\ddots$						$\vdots$								$\vdots$
η_{h-2}				q_3					q_4								$q_2+q_1+q_5$
η_{h-1}					q_3				q_4								$q_2+q_1+q_5$
η_h						q_3			q_4								$q_2+q_1+q_5$
ϕ							q_2	q_3	q_4								q_1+q_5
η_{h+2}							q_2			q_3							$q_4+q_1+q_5$
η_{h+3}							q_2				q_3						$q_4+q_1+q_5$
η_{h+4}							q_2					q_3					$q_4+q_1+q_5$
$\vdots$							$\vdots$						$\ddots$				$\vdots$
η_{2h-2}							q_2							q_3			$q_4+q_1+q_5$
η_{2h-1}							q_2								q_3		$q_4+q_1+q_5$
η_{2h}							q_2									q_3	$q_4+q_1+q_5$
η_{2h+1}							q_2	q_3									$q_4+q_1+q_5$
OOC																	1

where

$$q_1 = P\left(Y_{(j:n)} \geq X_{(b_2:m)} \middle| X_{(b_2:m)} = x_{(b_2:m)}\right) = I\left(GF^{-1}\left(U_{(b_2:m)}\right), j, n-j+1\right),$$

$$q_2 = I\left(GF^{-1}\left(U_{(b_2:m)}\right), j, n-j+1\right) - I\left(GF^{-1}\left(U_{(b_1:m)}\right), j, n-j+1\right),$$

$$q_3 = I\left(GF^{-1}\left(U_{(b_1:m)}\right), j, n-j+1\right) - I\left(GF^{-1}\left(U_{(a_1:m)}\right), j, n-j+1\right), \quad (24)$$

$$q_4 = I\left(GF^{-1}\left(U_{(a_1:m)}\right), j, n-j+1\right) - I\left(GF^{-1}\left(U_{(a_2:m)}\right), j, n-j+1\right),$$

$$q_5 = P\left(Y_{(j:n)} \leq X_{(a_2:m)} \middle| X_{(a_2:m)} = x_{(a_2:m)}\right) = 1 - I\left(GF^{-1}\left(U_{(a_2:m)}\right), j, n-j+1\right).$$

Similarly, the TPM of the two-sided IRR$_{w\text{-}of\text{-}w}$ precedence scheme is given by

	η_1	η_2	$\cdots$	η_{w-3}	η_{w-2}	η_{w-1}	φ	η_{w+1}	η_{w+2}	η_{w+3}	$\cdots$	η_{2w-2}	η_{2w-1}	OOC
η_1							q_3	q_4						$q_2+q_1+q_5$
η_2	q_2						q_3	q_4						q_1+q_5
η_3		q_2					q_3	q_4						q_1+q_5
$\vdots$			$\ddots$				$\vdots$	$\vdots$						$\vdots$
η_{w-2}				q_2			q_3	q_4						q_1+q_5
η_{w-1}					q_2		q_3	q_4						q_1+q_5
φ						q_2	q_3	q_4						q_1+q_5
η_{w+1}						q_2	q_3		q_4					q_1+q_5
η_{w+2}						q_2	q_3			q_4				q_1+q_5
$\vdots$						$\vdots$	$\vdots$				$\ddots$			$\vdots$
η_{2w-3}						q_2	q_3					q_4		q_1+q_5
η_{2w-2}						q_2	q_3						q_4	q_1+q_5
η_{2w-1}						q_2	q_3							$q_4+q_1+q_5$
OOC														1

with q_1, …, q_5 defined in Equation (24). For more thorough discussions on these two-sided IRR$_{2\text{-}of\text{-}(h+1)}$ and IRR$_{w\text{-}of\text{-}w}$ TPMs, see for instance Malela-Majika et al. (2018), Shongwe et al. (2019a) and Shongwe (2020).

Next, the $UZSARL$ and $USSARL$ of the two-sided $IRR_{2\text{-}of\text{-}(h+1)}$ scheme for any value of h are defined by

$$UZSARL = \int_0^1 \int_0^t \int_0^u \int_0^s CZSARL\, f_{a_2 a_1 b_1 b_2}(r,s,u,t)\, dr\, ds\, du\, dt \tag{25}$$

and

$$USSARL = \int_0^1 \int_0^t \int_0^u \int_0^s CSSARL\, f_{a_2 a_1 b_1 b_2}(r,s,u,t)\, dr\, ds\, du\, dt, \tag{26}$$

where $a_2 < a_1 < b_1 < b_2$ and

$f_{a_2 a_1 b_1 b_2}(r,s,u,t) = \frac{m!}{(a_2-1)!(a_1-a_2-1)!(b_1-a_1-1)!(b_2-b_1-1)!(m-b_2)!} r^{a_2-1}(s-r)^{a_1-a_2-1}(u-s)^{b_1-a_1-1}(t-u)^{b_2-b_1-1}(1-t)^{m-b_2}$ is the joint pdf of the a_2^{th}, a_1^{th}, b_1^{th} and b_2^{th} order statistics in a sample of size m from the Uniform (0,1) distribution, r, s, u and t are random variables from the Uniform (0,1) distribution, with

$$CZSARL = \frac{\left(1+q_2\left(\frac{1-q_3^h}{1-q_3}\right)\right)\left(1+q_4\left(\frac{1-q_3^h}{1-q_3}\right)\right)}{1-q_3-q_2q_3^h-q_4q_3^h-q_2q_4\left(\frac{1-q_3^{2h}}{1-q_3}\right)}$$

and

$$CSSARL = s_{h+1}\varsigma_{h+1} + \sum_{i=1}^{h} s_i \times \left(\varsigma_i + \varsigma_{(2h+2)-i}\right).$$

Note that

$$\mathbf{s} = \begin{pmatrix} \mathrm{s}_1 \\ \mathrm{s}_2 \\ \mathrm{s}_3 \\ \vdots \\ \mathrm{s}_{h-2} \\ \mathrm{s}_{h-1} \\ \mathrm{s}_h \\ \mathrm{s}_{h+1} \\ \mathrm{s}_{h+2} \\ \mathrm{s}_{h+3} \\ \mathrm{s}_{h+4} \\ \vdots \\ \mathrm{s}_{2h-1} \\ \mathrm{s}_{2h} \\ \mathrm{s}_{2h+1} \end{pmatrix}' = \frac{1}{2\left(\frac{1-\gamma_1^h}{1-\gamma_1}\right)+2\gamma_1^h(1-\gamma_2)^{-1}} \begin{pmatrix} \gamma_1^{h-1} \\ \gamma_1^{h-2} \\ \gamma_1^{h-3} \\ \vdots \\ \gamma_1^2 \\ \gamma_1 \\ 1 \\ 2\gamma_1^h(1-\gamma_2)^{-1} \\ 1 \\ \gamma_1 \\ \gamma_1^2 \\ \vdots \\ \gamma_1^{h-3} \\ \gamma_1^{h-2} \\ \gamma_1^{h-1} \end{pmatrix}'$$

with $\gamma_1 = \frac{q_3}{q+q_3}$, $\gamma_2 = \frac{q_3}{2q+q_3}$ and $q = q_2 = q_4$ (since $\mathbf{s}$ is computed while the process is IC). In addition,

$$\mathbf{R} = \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{h-3} \\ \varsigma_{h-2} \\ \varsigma_{h-1} \\ \varsigma_h \\ \varsigma_{h+1} \\ \varsigma_{h+2} \\ \varsigma_{h+3} \\ \varsigma_{h+4} \\ \varsigma_{h+5} \\ \vdots \\ \varsigma_{2h} \\ \varsigma_{2h+1} \end{pmatrix} = \frac{1}{1 - q_3 - q_2 q_3^h - q_4 q_3^h - q_2 q_4 \left(\frac{1 - q_3^{2h}}{1 - q_3}\right)} \begin{pmatrix} (1 + q_2 \mathrm{N}_1)(1 + q_4 \mathrm{N}_0) \\ (1 + q_2 \mathrm{N}_2)(1 + q_4 \mathrm{N}_0) \\ \vdots \\ (1 + q_2 \mathrm{N}_{h-3})(1 + q_4 \mathrm{N}_0) \\ (1 + q_2 \mathrm{N}_{h-2})(1 + q_4 \mathrm{N}_0) \\ (1 + q_2 \mathrm{N}_{h-1})(1 + q_4 \mathrm{N}_0) \\ (1 + q_4 \mathrm{N}_0) \\ (1 + q_2 \mathrm{N}_0)(1 + q_4 \mathrm{N}_0) \\ (1 + q_2 \mathrm{N}_0) \\ (1 + q_4 \mathrm{N}_{h-1})(1 + q_2 \mathrm{N}_0) \\ (1 + q_4 \mathrm{N}_{h-2})(1 + q_2 \mathrm{N}_0) \\ (1 + q_4 \mathrm{N}_{h-3})(1 + q_2 \mathrm{N}_0) \\ \vdots \\ (1 + q_4 \mathrm{N}_2)(1 + q_2 \mathrm{N}_0) \\ (1 + q_4 \mathrm{N}_1)(1 + q_2 \mathrm{N}_0) \end{pmatrix}.$$

with

$$\mathrm{N}_i = q_3^i \left(\frac{1 - q_3^{h-i}}{1 - q_3}\right).$$

Similarly, the $UZSARL$ and $USSARL$ of the two-sided $\mathrm{IRR}_{w\text{-}of\text{-}w}$ scheme for any value of w are given by Equations (25) and (26), respectively; however, the

$$CZSARL = \frac{(1 - q_2^w)(1 - q_4^w)}{D}$$

and

$$CSSARL = \mathrm{s}_w \varsigma_w + \sum_{j=1}^{w-1} \mathrm{s}_j \times \left(\varsigma_j + \varsigma_{2w-j}\right);$$

where,

$$D = (1 - q_3)\big((1 - q_2)(1 - q_4)\big) - q_3\big((1 - q_2)(q_4 - q_4^w) + (1 - q_4)(q_2 - q_2^w)\big) - (1 + q_3)\big((q_2 - q_2^w)(q_4 - q_4^w)\big).$$

Note that

$$\mathbf{R} = \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \\ \varsigma_3 \\ \vdots \\ \varsigma_{w-2} \\ \varsigma_{w-1} \\ \varsigma_w \\ \varsigma_{w+1} \\ \varsigma_{w+2} \\ \vdots \\ \varsigma_{2w-3} \\ \varsigma_{2w-2} \\ \varsigma_{2w-1} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} (1 - q_2)(1 - q_4^w) \\ (1 - q_2^2)(1 - q_4^w) \\ (1 - q_2^3)(1 - q_4^w) \\ \vdots \\ (1 - q_2^{w-2})(1 - q_4^w) \\ (1 - q_2^{w-1})(1 - q_4^w) \\ (1 - q_2^w)(1 - q_4^w) \\ (1 - q_4^{w-1})(1 - q_2^w) \\ (1 - q_4^{w-2})(1 - q_2^w) \\ \vdots \\ (1 - q_4^3)(1 - q_2^w) \\ (1 - q_4^2)(1 - q_2^w) \\ (1 - q_4)(1 - q_2^w) \end{pmatrix},$$

and

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{w-2} \\ s_{w-1} \\ s_w \\ s_{w+1} \\ s_{w+2} \\ \vdots \\ s_{2w-3} \\ s_{2w-2} \\ s_{2w-1} \end{pmatrix} = \left(\frac{1-q}{1-q^w}\right) \begin{pmatrix} q^{w-1} \\ q^{w-2} \\ q^{w-3} \\ \vdots \\ q^2 \\ q \\ 1-\left(\frac{q-q^w}{1-q}\right) \\ q \\ q^2 \\ \vdots \\ q^{w-3} \\ q^{w-2} \\ q^{w-1} \end{pmatrix}.$$

For more thorough discussions of these expressions, see Shongwe et al. (2019a) and Shongwe (2020).

6.2 IC and OOC performance analysis

In this section, a brief analysis of the performance of the two-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence monitoring schemes is presented. Table 9 displays the zero-state performances of the two-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence monitoring schemes for a nominal ARL_0 of 500 under the $N(0,1)$, $t(5)$ and $GAM(1,1)$ distributions when $(m,n) = (500,5)$, $h \in \{1,5\}$, $w \in \{2,5\}$ and $(\delta_{min},\delta_{max}) = (0,2)$. At each shift value or range of shift values, the distribution with the best performance is boldfaced. From Table 9, it is observed that under heavy-tailed distributions, the two-sided precedence schemes present better *ARL* and *AEQL* results as compared to the normal distribution. Moreover, under skewed distributions, the two-sided precedence scheme has the worst performance for small shifts in the process location as well as in terms of the *AEQL*. Similar results are observed for the corresponding steady-state mode.

<Insert Table 9>

7. Summary and recommendations

Malela-Majika et al. (2019) proposed two-sided $SRR_{2\text{-}of\text{-}(h+1)}$ precedence monitoring schemes to monitor both the increase and decrease in the location process parameter. In this paper, one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence monitoring schemes are proposed to monitor either an increase or decrease in the location process parameter without any distributional assumption. The sensitivity and robustness of the these schemes are investigated using zero- and steady-state properties of the run-length distribution through the Markov chain technique. It is found that the proposed one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes are IC robust and superior in performance to the one-sided $SRR_{2\text{-}of\text{-}(h+1)}$, $SRR_{w\text{-}of\text{-}w}$ and basic precedence schemes. The $IRR_{w\text{-}of\text{-}w}$ precedence scheme is more sensitive than the $IRR_{2\text{-}of\text{-}(h+1)}$ precedence scheme when compared head to head (i.e. when $w = h + 1$) for all $h > 1$ and $w > 2$.

Practitioners and operaters in industrial or non-industrial environments are adviced to use the proposed one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes over the one-sided $SRR_{2\text{-}of\text{-}(h+1)}$, $SRR_{w\text{-}of\text{-}w}$ and basic precedence schemes regardless of the situation. Thus, for "small" shifts in the location process parameter, we recommend $h = 1$ & $w = 3$ or 4 under symmetrical distributions and $h = 1$ & $w = 3$, 4 or 5 under heavy-tailed and skewed distributions. For "small and moderate" shifts as well as "small to large shifts", we recommend $h = 1$ and $w = 2$, 3 or 4) regardless of the nature of the underlying distribution. Since the SPM literature recommends the use of small values of h or w for simplicity in the design of monitoring schemes supplemented with runs-rules, it is recommended to use the $IRR_{2\text{-}of\text{-}(h+1)}$ or $IRR_{w\text{-}of\text{-}w}$ precedence scheme two reasons, which are: (i) simplicity in the design and implementation of monitoring schemes, (ii) higher efficiency in monitoring quality processes regardless of the size of shifts and nature of the underlying process distribution. For the sake of completeness, a discussion on the two-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes is provided. In future, the performance of the one-sided 2-*of*-(h+1) and *w*-*of*-*w* precedence schemes will be investigated using improved modified runs-rules and the improved modified side-sensitive design for synthetic schemes will be investigated.

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Appendix: Transformation functions

Recall that in Sections 2 and 4 it is stated that for the precedence scheme, its performance in terms of the characteristics of the RL such as ARL depends on the Phase I and Phase II probability distributions only through the transformation function $\Psi(u)$. Once the latter is specified, the ARL can be calculated for specified values of n, j, b_1 and b_2 (a_1 and a_2). In this Appendix, we show how to get the transformation function for the underlying process distributions considered in this paper.

Conversion function under the normal distribution

If $F \sim N(0,1)$ and $G \sim N(\delta, 1)$. Then, $\psi(u)$ can be determined as follows

$$F(x) = \Phi(x - \mu_1) = \Phi(x - 0) = \Phi(x) \tag{A.1}$$

and

$$G(x) = \Phi(x - \mu_2) = \Phi(x - \delta), \tag{A.2}$$

where μ_1 and μ_2 (with $\mu_2 = \mu_1 + \delta$ and $\mu_1 = 0$) represent the location parameters (or means) of the Phase I and Phase II samples, respectively.

Since $\psi(u) = GF^{-1}(u)$ it follows that

$$\psi(u) = \Phi\left(-\delta + \Phi^{-1}(u)\right) = \Phi(-\delta + \Phi^{-1}(\mu))$$

Thus,

$$\psi(u) = \Phi(-\delta + \Phi^{-1}(u)), \tag{A.3}$$

where δ represents a shift in the process mean. Therefore, for an IC process $\delta = 0$ and it follows that

$$\psi(u) = \Phi(\Phi^{-1}(u)) = u. \tag{A.4}$$

Conversion function under the gamma distribution

If $F \sim GAM(1,1)$and $G \sim GAM(1,\beta)$ with $\beta \neq 1$. Then, $\psi(u)$ can be determined as follows

$$F(x) = 1 - \exp(-x), \tag{A.5}$$

$$F^{-1}(x) = -\ln(1-x) \tag{A.6}$$

and

$$G(x) = 1 - \exp(\tfrac{-x}{\beta}) \tag{A.7}$$

Then,

$$\psi(u) = GF^{-1}(u) = 1 - \exp\left(\frac{1}{\beta}\ln(1-u)\right). \tag{A.8}$$

A shift in the mean is given by $\delta = \mu_2 - \mu_1$, that is, $\beta - 1 = \delta$, then $\beta = \delta + 1$. Therefore,

$$\psi(u) = 1 - \exp\left(\tfrac{1}{\delta+1}\ln(1-u)\right). \tag{A.9}$$

For the IC process, $\delta = 0$. Then,

$\psi(u) = 1 - \exp(\ln(1-u)) = 1 - (1-u) = u.$

Note that the transformation functions for other probability distributions can be derived in a similar way.

Conversion function under the Student's *t*-distribution distribution

If $F \sim F_\nu(x)$ and $G \sim F_\nu(x - \sqrt{2}\delta)$ where

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\left[tan^{-1}\left(\frac{x}{\sqrt{\nu}}\right) + \frac{x\sqrt{\nu}}{\nu+x^2}\sum_{j=0}^{\frac{\nu-3}{2}} \frac{c_j}{\left(1+\frac{x^2}{\nu}\right)^j}\right] \text{ if } \nu \text{ is odd} \tag{A.10}$$

and

$$F(x) = \frac{1}{2} + \frac{x}{2\sqrt{\nu+x^2}}\sum_{j=0}^{\frac{\nu-2}{2}} \frac{d_j}{\left(1+\frac{x^2}{\nu}\right)^j} \text{ if } \nu \text{ is even,} \tag{A.11}$$

where $c_j = \left(\frac{2j}{2j+1}\right)c_{j-1};\ c_0 = 1$ and $d_j = \left(\frac{2j-1}{2j}\right)d_{j-1};\ d_0 = 1$ and ν is a positive integer.

It can be shown without loss of generality that $\psi(u)$ is given as follows

$$\psi(u) = F_\nu\left(-\sqrt{2}\delta + F_\nu^{-1}(u)\right). \tag{A.12}$$

For the IC process,

$$\psi(u) = F_\nu\left(-\sqrt{2}(0) + F_\nu^{-1}(u)\right) = F_\nu\left(F_\nu^{-1}(u)\right) = u.$$

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Table 1. Operation of the Phase II one-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ precedence schemes

Step	Operation
1	Specify the desired value of h or w.
2	Determine: $UCL = X_{(b_2:m)}$ / $UWL = X_{(b_1:m)}$ or $LCL = X_{(a_2:m)}$ / $LWL = X_{(a_1:m)}$, see Figure 1(b).
3	At each inspection point i, collect a Phase II sample of size n and compute $Y^i_{(j:n)}$.
4	If $Y^i_{(j:n)} \in$ Regions 1 or 6, go to Step 8; otherwise, go to Step 5.
5	If the $Y^i_{(j:n)} \in$ Regions 3 or 4, return to Step 3; otherwise, go to Step 6.
6	**(a)** If $Y^i_{(j:n)} \in$ Region 2, go to Step 7(a). **(b)** If $Y^i_{(j:n)} \in$ Region 5, go to Step 7(b).
7	**(a)** - For the $IRR_{w\text{-}of\text{-}w}$ scheme, if $Y^{i+1}_{(j:n)}, Y^{i+2}_{(j:n)}, \ldots, Y^{i+w-1}_{(j:n)} \in$ Region 2, go to Step 8; otherwise return to Step 3. - For the $IRR_{2\text{-}of\text{-}(h+1)}$ scheme, calculate *CRL*, and if $CRL < h$, go to Step 8; otherwise return to Step 3. **(b)** - For the $IRR_{w\text{-}of\text{-}w}$ scheme, if $Y^{i+1}_{(j:n)}, Y^{i+2}_{(j:n)}, \ldots, Y^{i+w-1}_{(j:n)} \in$ Region 5, go to Step 8; otherwise return to Step 3. - For the $IRR_{2\text{-}of\text{-}(h+1)}$ scheme, calculate *CRL*, and if $CRL < h$, go to Step 8; otherwise return to Step 3.
8	Issue an OOC signal. Take necessary corrective action to find and remove the assignable causes and return to Step 3.

Table 2. Charting constants (b_1, b_2) and the corresponding attained $CZSARL_0$ ($CSSARL_0$ −in brackets) values of the upper one-sided $IRR_{2\text{-of-}(h+1)}$ and $IRR_{w\text{-of-}w}$ precedence schemes when $n \in \{5,7\}, j \in \{3,4\}$ $h \in \{1,2,5,10\}$, $w \in \{2,5,10\}$ for nominal ARL_0 of 370 (first row) and 500 (second row) for each $m \in \{100,200,500\}$

h	m	$IRR_{2\text{-of-}(h+1)}$					
		$(n,j) = (5,3)$			$(n,j) = (7,4)$		
		b_1	b_2	$CZSARL_0(CSSARL_0)$	b_1	b_2	$CZSARL_0(CSSARL_0)$
1	100	85	93	367.41 (367.24)	83	89	375.14 (375.06)
		91	93	494.49 (494.48)	82	90	492.43 (492.27)
	200	165	189	369.47 (369.02)	160	181	374.70 (374.40)
		169	189	498.29 (497.97)	166	181	496.39 (496.26)
	500	423	469	369.19 (368.95)	403	453	368.24 (367.98)
		457	469	500.51 (500.50)	430	453	499.61 (499.59)
2	100	87	93	392.63 (392.41)	81	90	377.61 (377.17)
		86	94	492.52 (492.11)	81	91	488.62 (488.06)
	200	170	188	363.82 (363.30)	163	181	370.74 (384.34)
		172	189	493.51 (493.50)	169	181	505.20 (595.02)
	500	431	469	369.95 (369.60)	411	453	368.60 (368.21)
		460	469	500.61 (500.60)	435	453	500.33 (500.29)
5	100	86	94	365.81 (364.64)	80	93	358.97 (356.98)
		88	94	529.12 (528.36)	81	93	479.56 (477.67)
	200	174	188	372.85 (371.85)	165	182	362.03 (360.80)
		176	189	517.02 (516.10)	169	182	508.90 (508.14)
	500	440	469	373.26 (372.56)	420	453	369.21 (368.45)
		463	469	500.71 (500.69)	440	453	500.38 (500.30)
10	100	89	93	382.65 (381.75)	86	89	375.13 (374.66)
		89	94	534.15 (532.78)	86	90	521.31 (520.52)
	200	175	189	365.54 (363.14)	166	185	388.60 (385.38)
		178	189	510.07 (508.35)	168	185	515.60 (512.69)
	500	445	469	369.56 (368.27)	426	453	372.98 (371.63)
		464	469	499.74 (499.69)	443	453	500.39 (500.25)

w	$IRR_{w\text{-of-}w}$					
	$(n,j) = (5,3)$			$(n,j) = (5,3)$		
	b_1	b_2	$CZSARL_0$ ($CSSARL_0$)	b_1	b_2	$CZSARL_0$ ($CSSARL_0$)
2	85	93	367.41 (367.24)	78	91	383.19 (382.74)
	91	93	494.49 (494.48)	80	91	527.97 (527.64)
	176	186	372.97 (372.93)	163	180	376.85 (376.68)
	172	188	494.04 (493.86)	166	181	496.39 (496.26)
	444	466	369.70 (369.67)	403	453	368.24 (367.98)
	457	469	500.56 (500.50)	430	453	499.61 (499.59)
5	66	93	365.91 (365.34)	56	94	379.25 (376.49)
	89	93	499.88 (499.88)	58	93	486.77 (484.45)
	122	190	380.82 (378.88)	121	182	375.12 (373.69)
	126	190	503.17 (501.59)	126	182	493.91 (492.99)
	328	468	370.86 (370.24)	311	453	370.05 (369.12)
	323	473	500.82 (499.71)	346	453	500.70 (500.61)
10	52	93	384.09 (383.09)	49	90	384.99 (383.03)
	86	93	499.88 (499.88)	53	90	515.41 (514.52)
	110	186	372.19 (371.84)	110	179	370.50 (370.21)
	135	187	500.09 (500.09)	106	181	506.83 (506.04)
	278	466	370.73 (370.52)	246	453	371.90 (369.95)
	266	470	499.49 (498.78)	279	453	500.35 (500.15)

Table 3. IC and OOC transformation function under the $N(\mu, \sigma)$, $GAM(\alpha,\beta)$ and $t(\nu)$ distributions

Distribution	Parameters	$F(x)$	$F^{-1}(x)$	$G(x)$	$\psi(u)$ (OOC case)	$\psi(u)$ (IC case)
$N(\mu, \sigma)$	$\mu = 0$ and $\sigma = 1$	$\Phi(x)$ $x \in (-\infty, \infty)$	$\Phi^{-1}(x)$	$\Phi(x - \delta)$	$\Phi(-\delta + \Phi^{-1}(u))$	u
$GAM(\alpha, \beta)$	$\alpha = \beta = 1$	$1 - \exp(-x)$ $x \in [0, \infty)$	$-\ln(1 - x)$	$1 - \exp(\frac{-x}{\delta + 1})$	$1 - \exp(\frac{1}{\delta + 1}\ln(1 - u))$	u
$t(\nu)$	$\nu = 5$	$F_\nu(x)$ $x \in (-\infty, \infty)$	$F_\nu^{-1}(x)$	$F_\nu(x - \sqrt{2}\delta)$	$F_\nu(-\sqrt{2}\delta + F_\nu^{-1}(u))$	u

Table 4 Charting constants (b_1, b_2) and attained $CZSARL_0$ ($CSSARL_0$ – in brackets) values of the upper one-sided $IRR_{2\text{-of-}(h+1)}$ and $IRR_{w\text{-of-}w}$ precedence schemes for different values of h and w when $m = 500$ and $n = 5$

$IRR_{2\text{-of-}(h+1)}$			**$IRR_{w\text{-of-}w}$**		
$\boldsymbol{h}$	$\boldsymbol{(b_1, b_2)}$	$CZSARL_0$ ($CSSARL_0$)	$\boldsymbol{w}$	$\boldsymbol{(b_1, b_2)}$	$CZSARL_0$ ($CSSARL_0$)
1	(457,469)	500.51 (500.50)	2	(457, 469)	500.51 (500.50)
2	(460,469)	500.61 (500.60)	3	(428, 469)	500.71 (500.69)
3	(461,469)	500.21 (500.19)	4	(399, 469)	500.36 (500.34)
4	(462,469)	500.30 (500.28)	5	(375, 469)	500.34 (500.32)
5	(463,469)	500.71 (500.69)	6	(354, 469)	500.08 (500.05)
6	(463,469)	500.19 (500.09)	7	(337, 469)	500.16 (500.13)
7	(463,469)	499.54 (499.49)	8	(322, 469)	500.11 (500.07)
8	(464,469)	500.51 (500.48)	9	(309, 469)	500.09 (500.04)
9	(464,469)	500.12 (500.09)	10	(298, 469)	500.23 (500.17)
10	(464,469)	499.69 (499.69)			

Table 5 The OOC unconditional zero-state (and steady-state, in brackets) *ARL* and *AEQL* performance of the upper one-sided $IRR_{2\text{-of-}(h+1)}$ precedence scheme for $h \in \{1,2,5,10\}$ when $m = 500$ and $n = 5$ for a nominal $ZSARL_0$ ($SSARL_0$) of 500

h	**1**			**2**			**5**			**10**		
Shift(δ)	***N*(0,1)**	***t*(5)**	***GAM*(1,1))**	***N*(0,1)**	***t*(5)**	***GAM*(1,1))**	***N*(0,1)**	***t*(5)**	***GAM*(1,1))**	***N*(0,1)**	***t*(5)**	***GAM*(1,1))**
0.1	282.78(282.76)	294.48(294.47)	235.52(235.51)	282.83(282.82)	294.48(294.55)	235.53(235.52)	282.89(282.87)	294.67(294.65)	235.55(235.54)	282.08(282.04)	293.77(293.73)	234.86(234.82)
0.2	164.40(164.39)	173.38(173.38)	126.64(126.64)	164.41(164.40)	173.45(173.44)	126.62(126.61)	164.44(164.43)	173.56(173.54)	126.62(126.60)	163.81(163.77)	172.78(172.74)	126.13(126.10)
0.3	98.35 (98.35)	102.41(102.41)	75.43 (75.43)	98.34 (98.34)	102.47(102.46)	75.40 (75.39)	98.37 (98.36)	102.58(102.57)	75.39 (75.38)	97.88 (97.85)	101.96(101.92)	75.05 (75.02)
0.4	60.56 (60.56)	60.89 (60.89)	48.68 (48.68)	60.54 (60.54)	60.94 (60.93)	48.65 (48.64)	60.57 (60.56)	61.06 (61.04)	48.64 (48.63)	60.22 (60.19)	60.60 (60.56)	48.40 (48.38)
0.5	38.39 (38.39)	36.60 (36.59)	33.49 (33.48)	38.37 (38.37)	36.64 (36.63)	33.46 (33.45)	38.40 (38.39)	36.76 (36.75)	33.46 (33.45)	38.17 (38.15)	36.47 (36.44)	33.30 (33.28)
0.6	25.06 (25.06)	22.36 (22.36)	24.26 (24.26)	25.05 (25.05)	22.40 (22.39)	24.23 (24.23)	25.09 (25.08)	22.53 (22.52)	24.24 (24.24)	24.95 (24.93)	22.37 (22.35)	24.14 (24.12)
0.7	16.86 (16.86)	13.99 (13.99)	18.34 (18.33)	16.85 (16.85)	14.03 (14.02)	18.32 (18.31)	16.90 (16.89)	14.16 (14.15)	18.33 (18.32)	16.82 (16.81)	14.10 (14.08)	18.26 (18.25)
0.8	11.69 (11.69)	9.03 (9.03)	14.36 (14.35)	11.69 (11.68)	9.06 (9.06)	14.34 (14.34)	11.74 (11.73)	9.19 (9.18)	14.36 (14.35)	11.71 (11.70)	9.19(9.18)	14.32 (14.30)
0.9	8.36 (8.35)	6.06 (6.06)	11.57 (11.57)	8.36 (8.35)	6.09 (6.09)	11.56 (11.56)	8.41 (8.40)	6.20 (6.20)	11.58 (11.58)	8.41 (8.40)	6.23 (6.22)	11.56 (11.55)
1.0	6.16 (6.15)	4.25 (4.25)	9.57 (9.57)	6.16 (6.16)	4.28 (4.28)	9.56 (9.56)	6.21 (6.21)	4.38 (4.38)	9.58 (9.58)	6.23 (6.22)	4.41 (4.40)	9.57 (9.56)
1.1	4.67 (4.67)	3.13 (3.12)	8.08 (8.07)	4.68 (4.68)	3.15 (3.15)	8.07 (8.07)	4.73 (4.73)	3.22 (3.22)	8.09 (8.09)	4.75 (4.74)	3.26 (3.25)	8.09 (8.08)
1.2	3.66 (3.65)	2.41 (2.41)	6.94 (6.94)	3.67 (3.66)	2.43 (2.43)	6.94 (6.94)	3.71 (3.70)	2.49 (2.48)	6.96 (6.96)	3.73 (3.72)	2.51 (2.50)	6.96 (6.96)
1.3	2.94 (2.94)	1.95 (1.95)	6.06 (6.06)	2.95 (2.95)	1.96 (1.96)	6.06 (6.06)	2.99 (2.98)	2.00 (2.00)	6.08 (6.08)	3.00 (3.00)	2.01 (2.01)	6.09 (6.08)
1.4	2.43 (2.43)	1.64 (1.64)	5.36 (5.36)	2.44 (2.44)	1.65 (1.65)	5.36 (5.36)	2.47 (2.47)	1.67 (1.67)	5.39 (5.38)	2.48 (2.48)	1.68 (1.68)	5.39 (5.39)
1.5	2.06 (2.06)	1.43 (1.43)	4.80 (4.80)	2.07 (2.07)	1.44 (1.44)	4.80 (4.80)	2.09 (2.09)	1.45 (1.45)	4.83 (4.82)	2.10 (2.10)	1.46 (1.46)	4.83 (4.83)
1.6	1.79 (1.79)	1.29 (1.29)	4.34 (4.34)	1.80 (1.79)	1.30 (1.30)	4.35 (4.34)	1.81 (1.81)	1.30 (1.30)	4.37 (4.36)	1.82 (1.82)	1.31 (1.30)	4.38 (4.37)
1.7	1.59 (1.58)	1.19 (1.19)	3.97 (3.97)	1.59 (1.59)	1.20 (1.20)	3.97 (3.97)	1.60 (1.60)	1.20 (1.20)	3.99 (3.99)	1.61 (1.60)	1.20 (1.20)	4.00 (3.99)
1.8	1.44 (1.43)	1.13 (1.13)	3.65 (3.65)	1.44 (1.44)	1.13 (1.13)	3.65 (3.65)	1.45 (1.44)	1.13 (1.13)	3.67 (3.67)	1.45 (1.45)	1.13 (1.13)	3.68 (3.68)
1.9	1.32 (1.32)	1.08 (1.08)	3.38 (3.38)	1.32 (1.32)	1.08 (1.08)	3.39 (3.38)	1.33 (1.33)	1.08 (1.08)	3.40 (3.40)	1.33 (1.33)	1.08 (1.08)	3.41 (3.41)
2.0	1.23 (1.23)	1.05 (1.05)	3.15 (3.15)	1.23 (1.23)	1.05 (1.05)	3.16 (3.16)	1.24 (1.24)	1.05 (1.05)	3.17 (3.17)	1.24 (1.24)	1.05 (1.05)	3.18 (3.18)
2.1	1.17 (1.17)	1.03 (1.04)	2.96 (2.96)	1.17 (1.17)	1.03 (1.03)	2.96 (2.96)	1.17 (1.17)	1.03 (1.03)	2.98 (2.97)	1.17 (1.17)	1.03 (1.00)	2.98 (2.98)
2.2	1.12 (1.12)	1.02 (1.02)	2.79 (2.79)	1.12 (1.12)	1.02 (1.02)	2.79 (2.79)	1.12 (1.12)	1.02 (1.02)	2.80 (2.80)	1.12 (1.12)	1.02 (1.00)	2.81 (2.81)
2.3	1.08 (1.08)	1.01 (1.01)	2.64 (2.64)	1.08 (1.08)	1.01 (1.01)	2.64 (2.64)	1.08 (1.08)	1.01 (1.01)	2.66 (2.65)	1.08 (1.08)	1.01 (1.01)	2.66 (2.66)
2.4	1.06 (1.06)	1.01 (1.01)	2.51 (2.51)	1.06 (1.06)	1.01 (1.00)	2.51 (2.51)	1.06 (1.06)	1.01 (1.01)	2.52 (2.52)	1.06 (1.06)	1.01 (1.01)	2.53 (2.53)
2.5	1.04 (1.04)	1.00 (1.00)	2.39 (2.39)	1.04 (1.04)	1.00 (1.00)	2.40 (2.40)	1.04 (1.04)	1.00 (1.00)	2.41 (2.41)	1.04 (1.04)	1.00 (1.00)	2.41 (2.41)
AEQL	61.15 (61.13)	**52.10(52.09)**	102.66 (102.64)	61.21 (61.20)	**52.22 (52.20)**	102.69 (102.66)	61.45 (61.43)	**52.53 (52.50)**	103.05 (103.00)	61.41 (61.36)	**52.48 (52.44)**	103.09 (103.00)
%Diff	0.03%	0.02%	0.02%	0.02%	0.04%	0.03%	0.03%	0.06%	0.05%	0.08%	0.08%	0.09%

Table 6 The OOC unconditional zero-state (and steady-state, in brackets) *ARL* and *AEQL* values of the upper one-sided $IRR_{w\text{-of-}w}$ precedence scheme for $w \in \{2,5,10,15\}$ when $m = 500$ and $n = 5$ for a nominal $ZSARL_0$ ($SSARL_0$) of 500

w	**2**			**5**			**10**			**15**		
Shift(δ)	***N*(0,1)**	***t*(5)**	***GAM*(1,1))**	***N*(0,1)**	***t*(5)**	***GAM*(1,1))**	***N*(0,1)**	***t*(5)**	***GAM*(1,1))**	***N*(0 ,1)**	***t*(5)**	***GAM*(1,1))**
0.1	282.78(282.76)	294.48(294.47)	235.52(235.51)	280.96(280.93)	290.45(290.41)	234.66(334.64)	279.41(279.34)	286.46 (286.37)	234.38(234.33)	278.21(278.19)	284.26(284.09)	234.26(234.18)
0.2	164.40(164.39)	173.38(173.38)	126.64(126.64)	161.24(161.21)	164.53(164.47)	125.53(125.51)	158.19(158.09)	154.93 (154.75)	125.38(125.17)	156.34(156.16)	150.53(150.20)	125.19(125.09)
0.3	98.35 (98.35)	102.41(102.41)	75.43 (75.43)	94.39 (94.34)	90.06 (89.99)	74.30 (74.27)	90.56 (90.42)	78.98 (78.73)	74.06 (74.00)	88.57 (88.32)	75.62 (75.19)	74.10 (74.00)
0.4	60.56 (60.56)	60.89 (60.89)	48.68 (48.68)	56.40 (56.35)	48.20 (48.11)	47.63 (47.61)	52.83 (52.67)	40.53 (40.25)	47.49 (47.43)	51.44 (51.15)	39.65 (39.17)	47.60 (47.51)
0.5	38.39 (38.39)	36.60 (36.59)	33.49 (33.48)	34.56 (34.51)	26.20 (26.11)	32.57 (32.55)	31.94 (31.77)	22.66 (22.38)	32.53 (32.46)	31.36 (31.07)	23.31 (22.87)	32.69 (32.59)
0.6	25.06 (25.06)	22.36 (22.36)	24.26 (24.26)	21.88 (21.83)	15.11 (15.03)	23.49 (23.46)	20.33 (20.17)	14.21 (13.97)	23.52 (23.46)	20.40 (20.12)	15.37 (15.01)	23.71 (23.62)
0.7	16.86 (16.86)	13.99 (13.99)	18.34 (18.33)	14.42 (14.37)	9.49 (9.42)	17.70 (17.67)	13.75 (13.60)	9.82 (9.63)	17.79 (17.74)	14.14 (13.90)	10.95 (10.68)	17.99 (17.91)
0.8	11.69 (11.69)	9.03 (9.03)	14.36 (14.35)	9.94 (9.90)	6.49 (6.43)	13.84 (13.81)	9.84 (9.71)	7.24 (7.09)	13.97 (13.91)	10.33 (10.15)	8.08 (7.91)	14.16 (14.09)
0.9	8.36 (8.35)	6.06 (6.06)	11.57 (11.57)	7.18 (7.14)	4.76 (4.71)	11.16 (11.13)	7.38 (7.28)	5.50 (5.41)	11.31 (11.26)	7.82 (7.70)	6.02 (5.93)	11.48 (11.42)
1.0	6.16 (6.15)	4.25 (4.25)	9.57 (9.57)	5.41 (5.38)	3.68 (3.64)	9.23 (9.21)	5.73 (5.65)	4.23 (4.18)	9.39 (9.34)	6.05 (5.98)	4.49 (4.43)	9.55 (9.50)
1.1	4.67 (4.67)	3.13 (3.12)	8.08 (8.07)	4.24 (4.21)	2.93 (2.91)	7.81 (7.79)	4.55 (4.50)	3.27 (3.25)	7.97 (7.93)	4.75 (4.71)	3.38 (3.36)	8.11 (8.07)
1.2	3.66 (3.65)	2.41 (2.41)	6.94 (6.94)	3.42 (3.40)	2.39 (2.38)	6.73 (6.71)	3.68 (3.65)	2.57 (2.56)	6.89 (6.86)	3.78 (3.77)	2.60 (2.59)	7.00 (6.97)
1.3	2.94 (2.94)	1.95 (1.95)	6.06 (6.06)	2.84 (2.82)	1.99 (1.98)	5.90 (5.88)	3.01 (3.00)	2.06 (2.06)	6.04 (6.02)	3.06 (3.05)	2.07 (2.07)	6.14 (6.11)
1.4	2.43 (2.43)	1.64 (1.64)	5.36 (5.36)	2.40 (2.39)	1.69 (1.68)	5.24 (5.22)	2.50 (2.50)	1.71 (1.71)	5.37 (5.35)	2.52 (2.52)	1.71 (1.71)	5.45 (5.43)
1.5	2.06 (2.06)	1.43 (1.43)	4.80 (4.80)	2.07 (2.06)	1.47 (1.47)	4.70 (4.69)	2.12 (2.12)	1.47 (1.47)	4.83 (4.81)	2.13 (2.13)	1.48 (1.48)	4.89 (4.87)
1.6	1.79 (1.79)	1.29 (1.29)	4.34 (4.34)	1.81 (1.80)	1.31 (1.31)	4.27 (4.26)	1.83 (1.83)	1.31 (1.31)	4.38 (4.36)	1.84 (1.84)	1.31 (1.31)	4.43 (4.42)
1.7	1.59 (1.58)	1.19 (1.19)	3.97 (3.97)	1.61 (1.60)	1.20 (1.21)	3.91 (3.90)	1.62 (1.62)	1.20 (1.20)	4.01 (3.99)	1.62 (1.62)	1.21 (1.21)	4.04 (4.04)
1.8	1.44 (1.43)	1.13 (1.13)	3.65 (3.65)	1.45 (1.45)	1.13 (1.13)	3.61 (3.60)	1.46 (1.46)	1.13 (1.13)	3.69 (3.68)	1.46 (1.46)	1.13 (1.13)	3.72 (3.72)
1.9	1.32 (1.32)	1.08 (1.08)	3.38 (3.38)	1.33 (1.33)	1.08 (1.08)	3.35 (3.34)	1.33 (1.33)	1.08 (1.09)	3.43 (3.42)	1.33 (1.33)	1.09 (1.09)	3.45 (3.44)
2.0	1.23 (1.23)	1.05 (1.05)	3.15 (3.15)	1.24 (1.24)	1.05 (1.05)	3.13 (3.13)	1.24 (1.24)	1.05 (1.06)	3.20 (3.19)	1.24 (1.24)	1.05 (1.05)	3.22 (3.21)
2.1	1.17 (1.17)	1.03 (1.04)	2.96 (2.96)	1.17 (1.17)	1.03 (1.03)	2.95 (2.94)	1.17 (1.17)	1.03 (1.03)	3.00 (3.00)	1.17 (1.17)	1.03 (1.02)	3.01 (3.01)
2.2	1.12 (1.12)	1.02 (1.02)	2.79 (2.79)	1.12 (1.12)	1.02 (1.02)	2.78 (2.77)	1.12 (1.12)	1.02 (1.02)	2.83 (2.83)	1.12 (1.12)	1.02 (1.02)	2.84 (2.84)
2.3	1.08 (1.08)	1.01 (1.01)	2.64 (2.64)	1.09 (1.09)	1.01 (1.01)	2.64 (2.63)	1.08 (1.08)	1.01 (1.01)	2.70 (2.68)	1.09 (1.09)	1.01 (1.01)	2.69 (2.68)
2.4	1.06 (1.06)	1.01 (1.01)	2.51 (2.51)	1.06 (1.06)	1.01 (1.01)	2.51 (2.51)	1.06 (1.06)	1.01 (1.01)	2.55 (2.54)	1.06 (1.06)	1.01 (1.01)	2.55 (2.55)
2.5	1.04 (1.04)	1.00 (1.00)	2.39 (2.39)	1.04 (1.04)	1.00 (1.00)	2.40 (2.39)	1.04 (1.04)	1.00 (1.00)	2.43 (2.43)	1.03 (1.03)	1.00 (1.00)	2.43 (2.43)
AEQL	61.15 (61.13)	**52.10(52.09)**	102.66 (102.64)	57.92 (57.79)	46.45 (46.32)	100.96 (100.74)	57.66 (57.41)	45.98 (45.73)	102.59 (102.30)	58.11 (57.81)	46.74 (46.40)	103.09 (103.00)
%Diff	0.03%	0.02%	0.02%	0.22%	0.28%	0.22%	0.44%	0.55%	0.28%	0.52%	0.73%	0.09%

Table 7 The unconditional zero-state (and steady-state, in brackets) *AEQL* values of the upper one-sided $IRR_{2\text{-of-}(h+1)}$, $SRR_{2\text{-of-}(h+1)}$ and basic precedence schemes for the $N(0,1)$, $t(5)$ and $GAM(1,1)$ distributions under different types shifts when $h \in \{1,2,5,10,15\}$

		$N(0,1)$			$t(5)$			$GAM(1,1)$		
[#]**Shifts**	***h***	**$IRR_{2\text{-of-}(h+1)}$**	**$SRR_{2\text{-of-}(h+1)}$**	***Basic***	**$IRR_{2\text{-of-}(h+1)}$**	**$SRR_{2\text{-of-}(h+1)}$**	***Basic***	**$IRR_{2\text{-of-}(h+1)}$**	**$SRR_{2\text{-of-}(h+1)}$**	***Basic***
Small only	1	**78.33 (78.31)**	79.85 (79.83)	80.44	**75.57 (75.56)**	76.23 (76.20)	79.02	**68.71 (68.68)**	79.56 (79.49)	70.08
Small to moderate		**67.01 (67.00)**	68.09 (68.06)	69.44	**56.55 (56.54)**	57.09 (57.05)	59.93	**85.03 (85.01)**	85.72 (85.64)	87.11
Small to large		**61.15 (61.14)**	64.53 (64.50)	62.78	**52.10 (52.09)**	55.21 (55.18)	54.19	**102.67 (102.64)**	105.53 (105.42)	104.97
Small only	2	**78.30 (78.29)**	78.89 (78.73)	80.44	**75.64 (75.62)**	75.71 (75.69)	79.02	**68.65 (68.63)**	79.22 (79.19)	70.08
Small to moderate		**67.06 (67.04)**	68.54 (68.49)	69.44	**56.72 (56.71)**	56.74 (56.68)	59.93	**84.98 (84.94)**	85.17 (85.14)	87.11
Small to large		**61.21 (61.20)**	64.48 (64.45)	62.78	**52.22 (52.20)**	54.05 (54.01)	54.19	**102.69 (102.66)**	104.99 (104.88)	104.97
Small only	5	**78.37 (78.35)**	78.66 (78.62)	80.44	**75.90 (75.88)**	76.76 (76.68)	79.02	**68.66 (68.64)**	79.05 (79.00)	70.08
Small to moderate		**67.36 (67.34)**	68.48 (68.45)	69.44	**57.21 (57.20)**	57.88 (57.73)	59.93	**85.15 (85.12)**	86.01 (85.59)	87.11
Small to large		**61.45 (61.42)**	68.32 (68.32)	62.78	**52.53 (52.51)**	60.89 (60.76)	54.19	**103.05 (103.00)**	105.18 (105.09)	104.97
Small only	10	**77.98 (77.98)**	79.69 (79.58)	80.44	**75.43 (75.41)**	76.12 (76.09)	79.02	**68.36 (68.34)**	78.58 (78.51)	70.08
Small to moderate		**67.25 (67.24)**	69.28 (69.23)	69.44	**57.12 (57.10)**	57.53 (57.50)	59.93	**85.00 (85.00)**	85.47 (85.43)	87.11
Small to large		**61.40 (61.36)**	70.23 (70.20)	62.78	**52.48 (52.44)**	62.03 (62.00)	54.19	**103.09 (102.99)**	106.97 (106.87)	104.97
Small only	15	**78.27 (78.21)**	80.43 (80.36)	80.44	**75.53 (75.50)**	76.11 (76.06)	79.02	**68.56 (68.51)**	78.53 (78.49)	70.08
Small to moderate		**67.63 (67.56)**	69.14 (69.11)	69.44	**57.21 (57.20)**	57.24 (57.21)	59.93	**85.37 (85.29)**	85.65 (85.57)	87.11
Small to large		**61.65 (61.60)**	72.07 (72.03)	62.78	**52.78 (52.73)**	63.81 (63.70)	54.19	**103.49 (103.38)**	107.13 (107.06)	104.97

[#]Note that $\delta \in [\delta_{min}, \delta_{max}]$ with δ_{min}=0; δ_{max}=0.7 for 'small only', δ_{max}=1.5 for 'small to moderate', and δ_{max}=2.5 for 'small to large'.

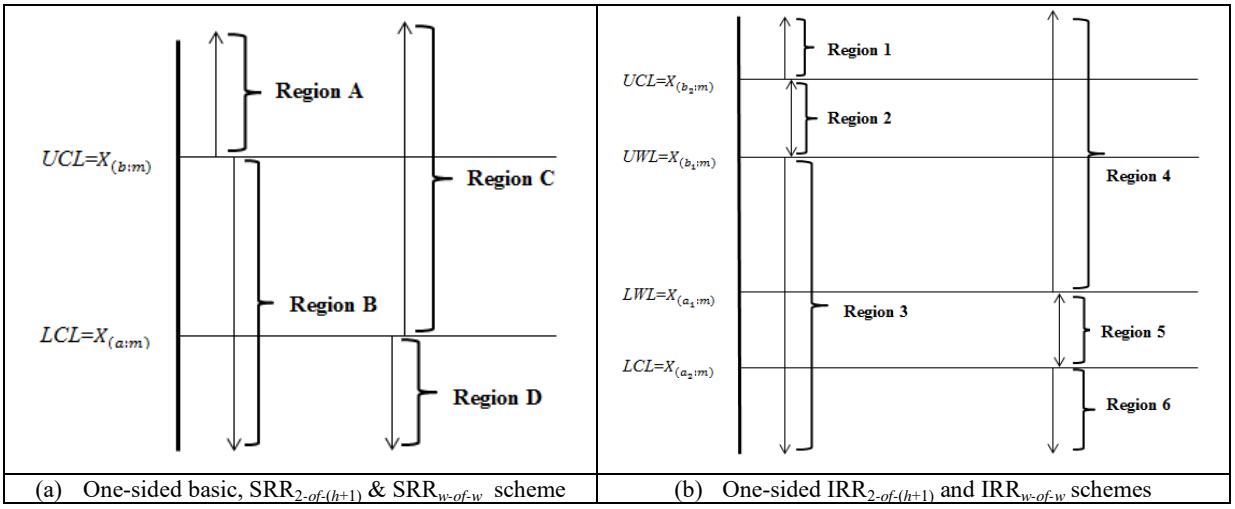
Table 8 The unconditional zero-state (and steady-state, in brackets) *AEQL* values of the upper one-sided $IRR_{2\text{-of-}(h+1)}$, $SRR_{2\text{-of-}(h+1)}$ and basic precedence schemes for the $N(0,1)$, $t(5)$ and $GAM(1,1)$ distributions under different types shifts when $w \in \{2,5,10,15\}$

		$N(0,1)$			$t(5)$			$GAM(1,1)$		
[#]**Shifts**	***w***	**$IRR_{w\text{-of-}w}$**	**$SRR_{w\text{-of-}w}$**	***Basic***	**$IRR_{w\text{-of-}w}$**	**$SRR_{w\text{-of-}w}$**	***Basic***	**$IRR_{w\text{-of-}w}$**	**$SRR_{w\text{-of-}w}$**	***Basic***
Small only	2	**78.33 (78.31)**	79.85 (79.83)	80.44	**75.57 (75.56)**	76.23 (76.20)	79.02	**68.71 (68.68)**	79.56 (79.49)	70.08
Small to moderate		**67.01 (67.00)**	68.09 (68.06)	69.44	**56.55 (56.54)**	57.09 (57.05)	59.93	**85.03 (85.01)**	85.72 (85.64)	87.11
Small to large		**61.15 (61.14)**	64.53 (64.50)	62.78	**52.10 (52.09)**	55.21 (55.18)	54.19	**102.67 (102.64)**	105.53 (105.42)	104.97
Small only	5	**71.94 (71.91)**	72.17 (72.14)	80.44	**59.91 (59.89)**	62.12 (62.09)	79.02	**67.07 (67.05)**	69.31 (69.26)	70.08
Small to moderate		**61.43 (61.42)**	63.16 (63.13)	69.44	**47.05 (47.04)**	48.44 (48.36)	59.93	**82.68 (82.67)**	83.47 (83.41)	87.11
Small to large		**57.92 (57.79)**	86.11 (86.10)	62.78	**46.45 (46.32)**	72.83 (72.79)	54.19	**100.96 (100.74)**	108.49 (108.42)	104.97
Small only	10	**68.24 (67.93)**	69.09 (69.06)	80.44	**54.64 (54.62)**	56.94 (56.87)	79.02	**67.05 (67.03)**	67.41 (67.33)	70.08
Small to moderate		**60.90 (60.49)**	62.04 (62.00)	69.44	**46.25 (46.24)**	47.89 (47.78)	59.93	**83.73 (83.72)**	84.33 (84.27)	87.11
Small to large		**57.66 (57.41)**	91.53 (91.46)	62.78	**45.98 (45.73)**	84.18 (84.07)	54.19	**102.59 (102.30)**	120.20 (120.13)	104.97
Small only	15	**67.64 (67.12)**	69.14 (69.09)	80.44	**55.34 (54.63)**	55.72 (55.61)	79.02	**67.37 (67.19)**	67.18 (67.15)	70.08
Small to moderate		**61.65 (61.14)**	61.61 (61.54)	69.44	**47.52 (45.72)**	47.17 (47.12)	59.93	**84.66 (84.34)**	83.62 (83.53)	87.11
Small to large		**58.11 (57.81)**	96.11 (96.01)	62.78	**46.74 (46.40)**	89.23 (89.12)	54.19	**103.41 (103.16)**	136.76 (136.61)	104.97

[#]Note that $\delta \in [\delta_{min}, \delta_{max}]$ with δ_{min}=0; δ_{max}=0.7 for 'small only', δ_{max}=1.5 for 'small to moderate', and δ_{max}=2.5 for 'small to large'.

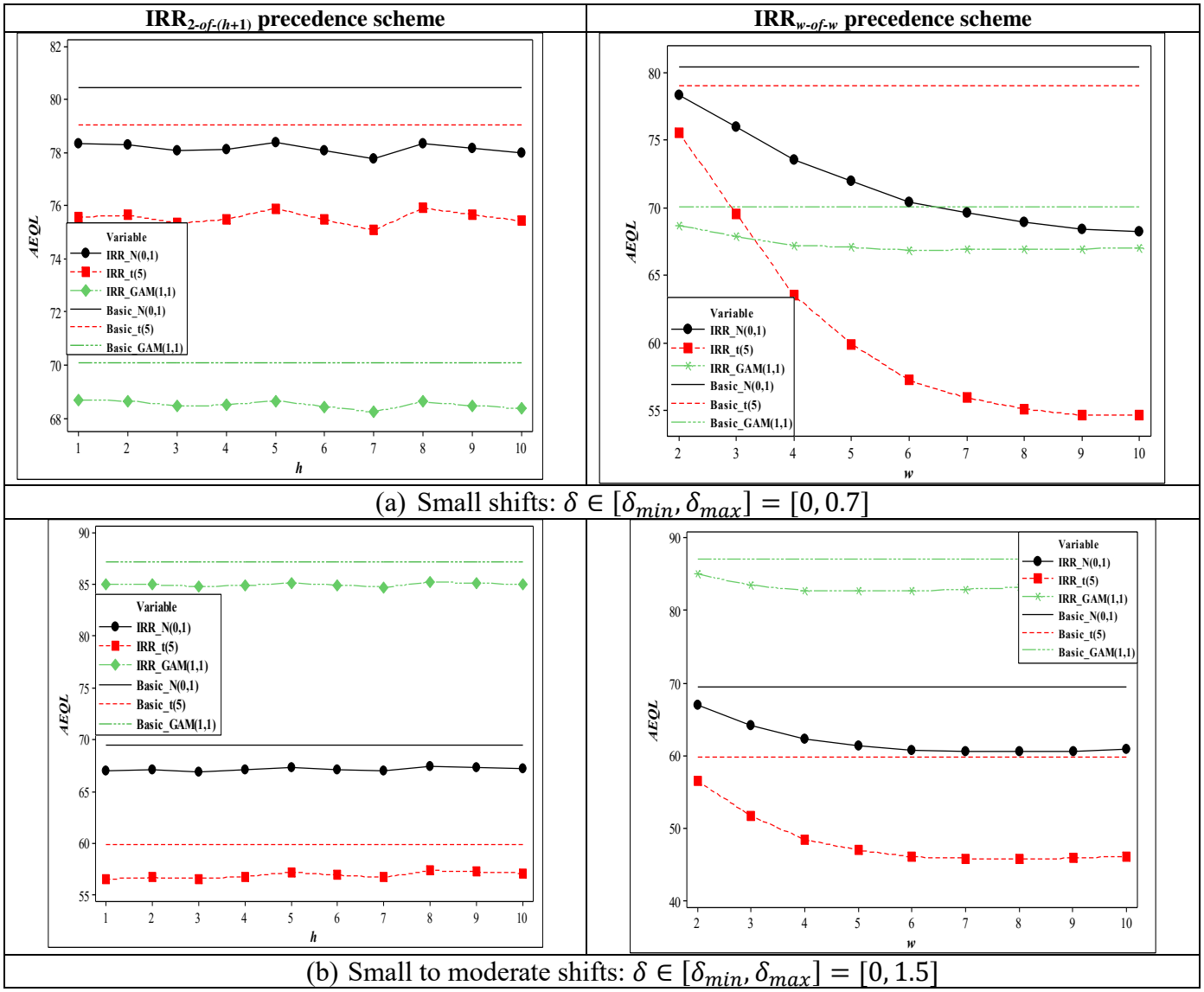
Table 9 The unconditional *ZSARL* profiles of the two-sided side-sensitive $IRR_{2\text{-of-}(h+1)}$ and $IRR_{w\text{-of-}w}$ precedence schemes for $h \in \{1,5\}$, $w \in \{2,5\}$ and $(\delta_{min},\delta_{max})=(0,2)$ when m = 500 and n = 5 for a nominal $ZSARL_0$ of 500

	***h* =1 (i.e. *w* = 2)**			***h*=5**			***w*=5**		
δ	***N*(0,1)**	***t*(5)**	***GAM*(1,1)**	***N*(0,1)**	***t*(5)**	***GAM*(1,1)**	***N*(0,1)**	***t*(5)**	***GAM*(1,1)**
0.0	503.16	503.16	503.16	517.51	517.51	517.51	499.55	499.55	499.55
0.2	231.30	**211.94**	418.48	217.16	**215.24**	458.66	120.33	**84.52**	223.71
0.4	64.50	**50.79**	145.55	53.38	**49.13**	160.23	14.44	**7.33**	22.17
0.6	21.18	**14.60**	52.85	16.53	**13.63**	59.45	3.21	**1.97**	3.01
0.8	8.48	**5.47**	20.47	6.73	**5.18**	23.76	1.62	**1.30**	1.40
1.0	4.28	**2.73**	8.40	3.53	**2.71**	10.33	1.26	1.12	**1.10**
1.2	2.51	**1.71**	3.87	2.25	**1.77**	5.19	1.13	1.06	**1.02**
1.4	1.74	**1.31**	2.03	1.67	**1.36**	3.01	1.06	1.02	**1.01**
1.6	1.35	**1.12**	1.28	1.36	**1.16**	1.95	1.03	**1.00**	**1.00**
1.8	1.17	1.05	**1.04**	1.19	**1.07**	1.41	1.01	**1.00**	**1.00**
2.0	1.08	1.02	**1.00**	1.09	**1.03**	1.13	**1.00**	**1.00**	**1.00**
AEQL	27.75	**21.74**	50.38	24.61	**21.59**	59.19	12.10	**10.27**	14.41
Charting constant	$a_2 = 74$, $a_1 = 80$, $b_1 = 421$, $b_2 = 427$			$a_2 = 53$, $a_1 = 60$, $b_1 = 441$, $b_2 = 448$			$a_2 = 145$, $a_1 = 163$, $b_1 = 338$, $b_2 = 356$		



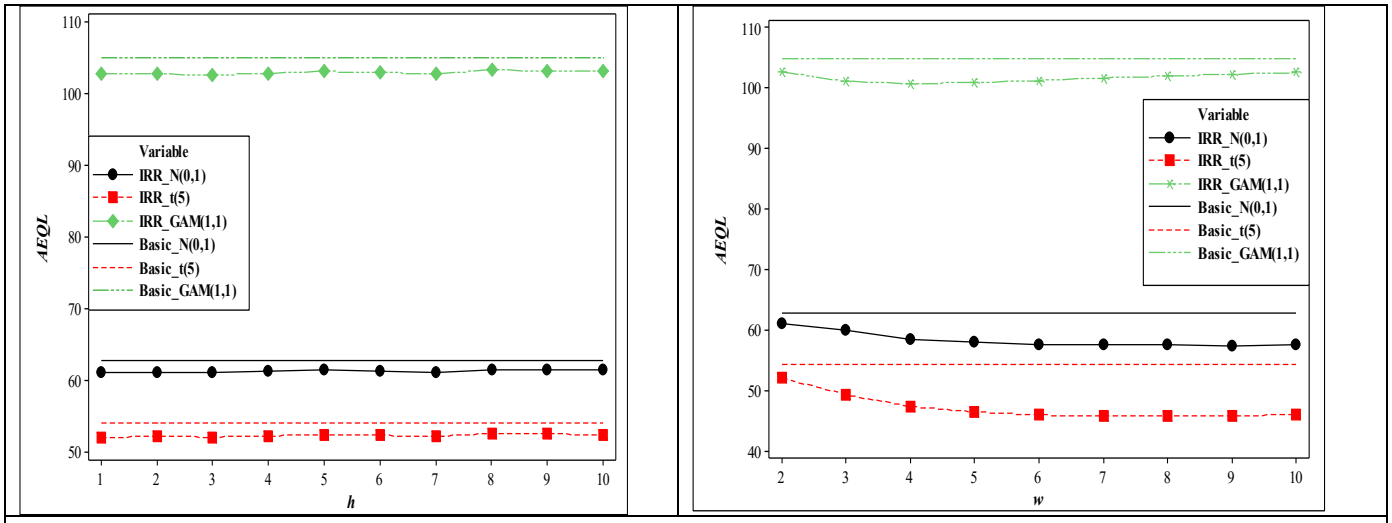
(a) One-sided basic, $SRR_{2\text{-}of\text{-}(h+1)}$ & $SRR_{w\text{-}of\text{-}w}$ scheme (b) One-sided $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ schemes

Figure 1. Different charting regions of the basic, $SRR_{2\text{-}of\text{-}(h+1)}$, $SRR_{w\text{-}of\text{-}w}$, $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ one-sided precedence schemes



(a) Small shifts: $\delta \in [\delta_{min}, \delta_{max}] = [0, 0.7]$

(b) Small to moderate shifts: $\delta \in [\delta_{min}, \delta_{max}] = [0, 1.5]$



(c) Small to large shifts: $\delta \in [\delta_{min}, \delta_{max}] = [0, 2.5]$

Figure 2. Performance comparison of the upper one-sided $\text{IRR}_{2\text{-}of\text{-}(h+1)}$ (left panel) and $\text{IRR}_{w\text{-}of\text{-}w}$ (right panel) precedence schemes versus the basic precedence scheme under various distributions for $h \in \{1,2,\ldots,10\}$ and $w \in \{2,3,\ldots,10\}$

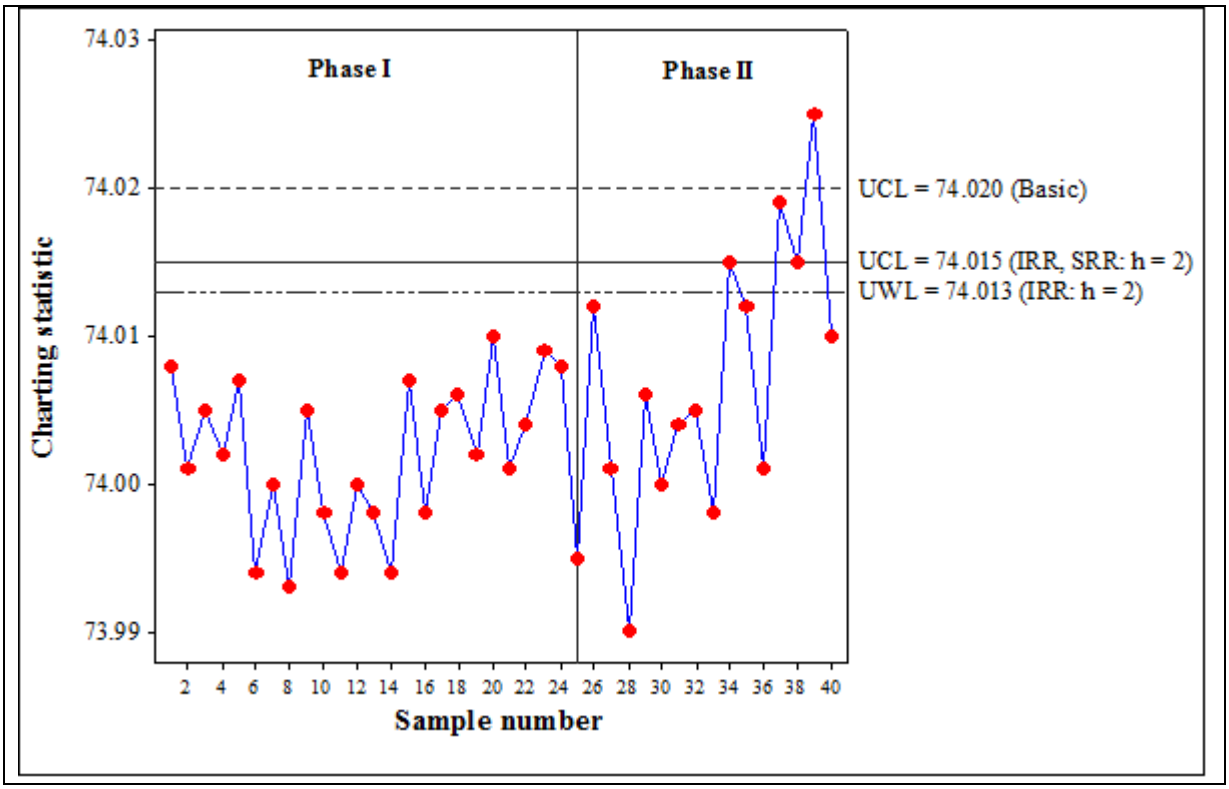


Figure 3. The $\text{IRR}_{2\text{-}of\text{-}(h+1)}$ scheme versus the $\text{SRR}_{2\text{-}of\text{-}(h+1)}$ and basic precedence schemes for the Montgomery (2005) piston ring data when $(m, n) = (125, 5)$ and $h = 2$

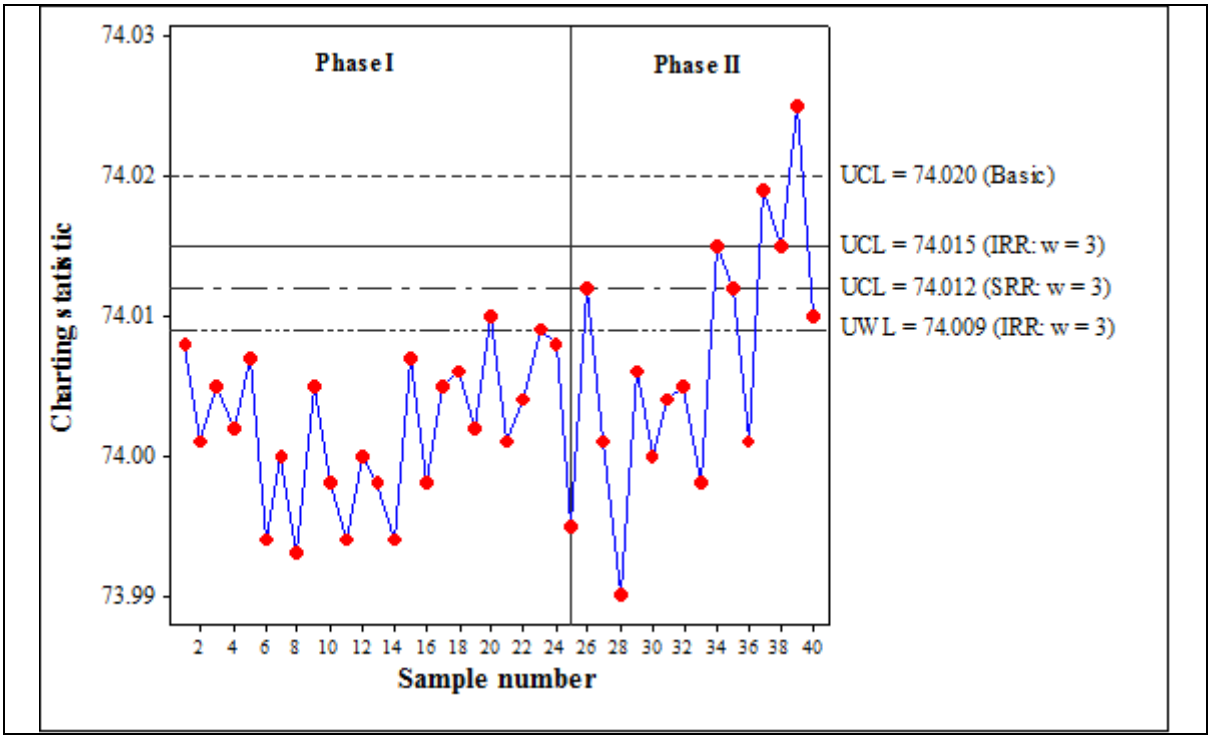


Figure 4. The $IRR_{w\text{-}of\text{-}w}$ scheme versus the $SRR_{w\text{-}of\text{-}w}$ and basic precedence schemes for the Montgomery (2005) piston ring data when $(m, n) = (125, 5)$ and $w = 3$

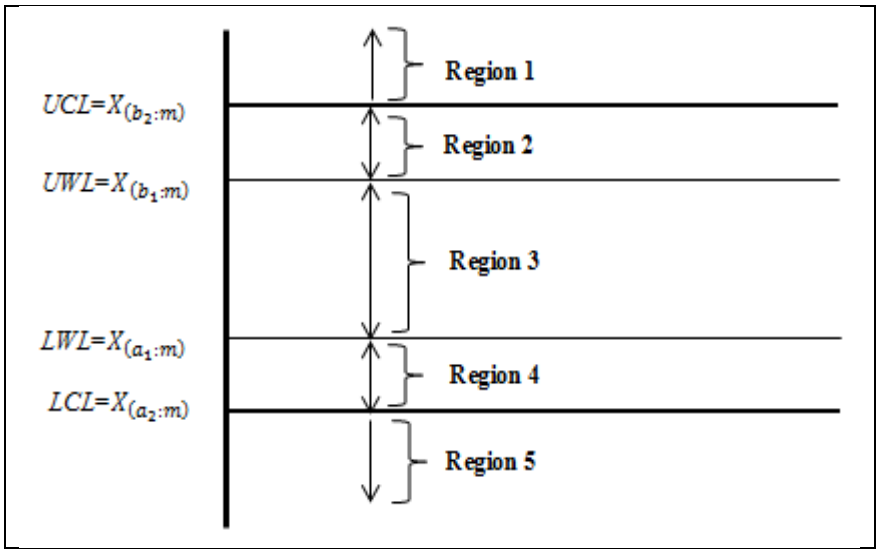


Figure 5: Charting regions of the $IRR_{2\text{-}of\text{-}(h+1)}$ and $IRR_{w\text{-}of\text{-}w}$ two-sided precedence schemes