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MINIMAL VARIATIONAL SURFACES AND QUALITY MESHES

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ABSTRACT

Many physical phenomena in science and engineering can be modelled by partial differential equations (PDEs) and solved using the finite element method (FEM). Such a method uses as computational spatial support a mesh of the domain where the equations are formulated. The mesh quality is a key-point for the accuracy of the numerical solution. This paper describes a methodology to construct a quality mesh of the domain from a given discretization of its boundary. We show that the size map related to such a mesh constitutes a minimal variational surface supported by a given contour. This surface can be constructed, from its boundary, using the finite element method or by the resolution of a simple discrete optimization problem. The quality mesh of the domain is then a mesh conforming to the size map given by this surface. A numerical example is given to demonstrate the method.

Keywords: minimal variational surfaces, quality meshes, equilateral meshes, regular meshes, adaptive meshing, finite elements, convex quadratic optimization

1. INTRODUCTION

The resolution of a physical problem formulated in terms of PDEs using the finite element method (FEM) is based on a spatial discretization, or mesh, of the computational domain. The convergence of such a method, as well as the quality of the resulting solution, is strongly related to the element shape quality (the ideal shape is that of an equilateral or regular element). Many methods (cf. [2]) can be used to construct a mesh of the domain, in general from the data of its boundary. Historically, the first methods have been based on the “local” regularity of the mesh. One can realize that such methods can fail in the case of a significant size variation in the boundary discretization of the domain. The present methods establish a continuous size field in the domain from the size given by the specified boundary discretization, and propose

various algorithms to generate a mesh that conforms to this size field.

This paper discusses what can be an optimal size field which is related to a quality mesh of the domain (i.e. a mesh which is as regular as possible). Such a size field presents a “minimal size variation” among all the size fields verifying the specified size on the boundary of the domain. The size variation can be quantified in two different ways which characterize the gradation of the related meshes. Let h be the size function in the domain, the first approach consists in considering the “usual” gradient ∇h of h . As for the second approach, the variation is measured by the “logarithmic” gradient $\nabla \log h$ of h . Let us denote by $v(h)$ the variation of h irrespective of the type of variation. A surface with minimal variation defined in a domain Ω supported by a given smooth contour Σ is a surface in which the sum of the square of the modulus of

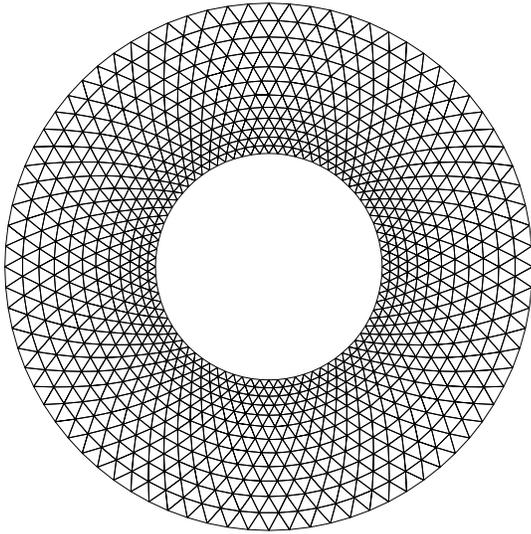


Figure 1. *Equilateral mesh.*

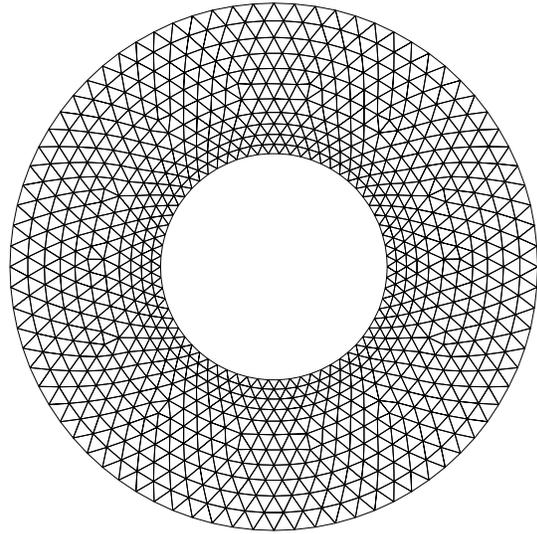


Figure 2. *Almost equilateral mesh.*

the variation on Ω is minimal. Mathematically, if h corresponds to such a surface, it is characterized by:

$$h(\Gamma) = \Sigma \quad \text{and} \quad J(h) = \min_{u \in H_v^1(\Omega)} J(u),$$

$$\text{with } J(u) = \int_{\Omega} \|v(u)\|^2 dX,$$

where Γ is the boundary of Ω and $H_v^1(\Omega) = \{u \in L^2(\Omega) ; \forall i \ v_i(u) \in L^2(\Omega)\}$, $v_i(u)$ denoting the i -th component of $v(u)$. To solve this problem, we propose a collocation method which allows us to construct approximatively the minimal variational surfaces.

In Section 1, the concept of a regular mesh is recalled. In Section 2, the minimal variational surfaces are introduced and their relation with regular meshes is indicated. In Section 3, an algorithm which creates approximatively minimal variational surfaces is given. In Section 4, the method is applied to build regular meshes of a domain whose boundary discretization is prescribed.

2. REGULAR MESH

Let Ω be a closed bounded domain in R^2 defined by its boundary. A quality triangular mesh, *equilateral* or *regular*, of Ω is a mesh whose elements are equilateral or regular. The existence of such a mesh is not guaranteed in general. Indeed, it depends, to some degree, on the domain boundary discretization. Therefore, we will call a *regular* triangular mesh the “best” triangular mesh which can be completed. As the issue of

constructing a regular mesh for an arbitrary domain is still an open problem, there are various methods which allow the construction of “almost” open regular meshes.

As an illustration, let us consider the domain defined by a circular ring whose outer boundary is a circle with radius $R = 1$ centered at the origin, and the inner boundary is a circle with radius $r = (1 - \frac{\sqrt{3}}{80} \pi)^{12}$ also centered at the origin. First, the outer (resp. inner) boundary is discretized with a constant step which equals $\frac{2\pi R}{80}$ (resp. $\frac{2\pi r}{80}$), so that the discretization of each boundary has 80 segments. For this discretization, let us notice that the value r of the inner radius is calculated in order to identify the equilateral mesh of the domain (in the sense of the best possible mesh). This mesh, illustrated by Figure 1, is actually made up of 12 layers of nearly equilateral triangles. Then, a slightly different discretization of the inner boundary is considered, taking a step which equals $\frac{2\pi r}{80} + 0.007$. Figure 2 shows an almost equilateral mesh of the domain corresponding to this new discretization. As shown in this figure, non regular triangles appear around the fifth and sixth layers.

In a classical context, two main types of discretization of the domain boundary can be considered. The first one involves uniform discretizations with a constant stepsize. The advantage of this type of discretization is its possibility, in general, to construct equilateral meshes. However, it cannot guarantee a good representation of the domain boundary for a given stepsize. Figures 3 and 4 show respectively two uniform

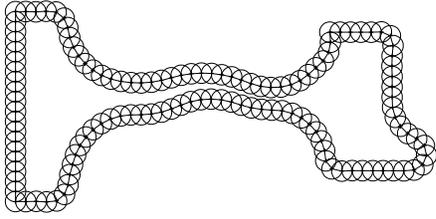


Figure 3. *Uniform discretization.*

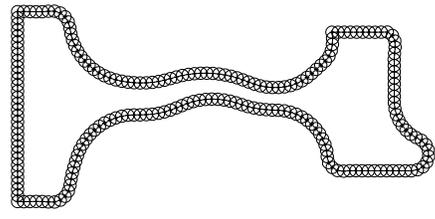


Figure 4. *Another uniform discretization.*

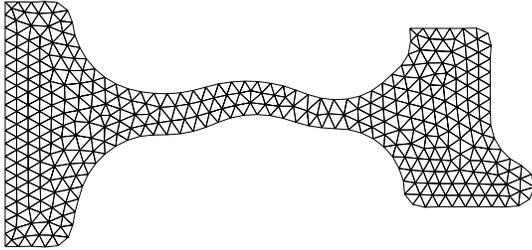


Figure 5. *Uniform mesh.*

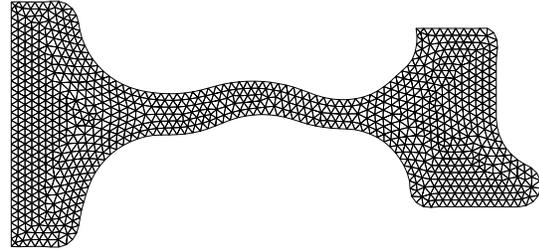


Figure 6. *Another uniform mesh.*

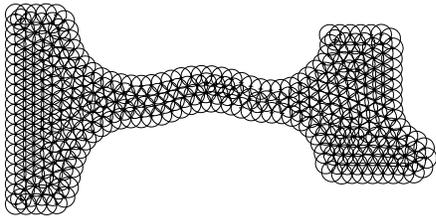


Figure 7. *Uniform sizes.*

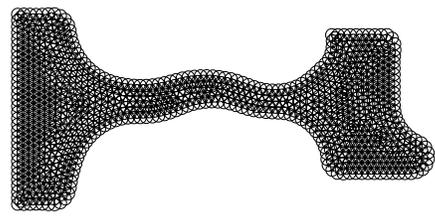


Figure 8. *Other uniform sizes.*

discretizations of the contour of a wheel cross-section (French railway company), each circle representing the stepsize along any direction.

Given a uniform discretization of the domain boundary, a regular mesh of the domain can only be a mesh whose element size equals to the boundary stepsize. So, the desired size of the elements is *a priori* known at each vertex of the regular mesh of the domain. Figures 5 and 6 illustrate the two uniform (with constant stepsize), almost equilateral, meshes of the wheel corresponding to the two given discretizations of its contour. The desired size (here constant) along all the directions at each vertex of these meshes are drawn on Figures 7 and 8. We can notice that, at each vertex of these meshes, the circle representing the desired size of the elements almost passes through the adjacent vertices. This is a characteristic, among others, of almost equilateral meshes. Let us also notice that the domain geometry is better represented by the second mesh, conceding that the number of elements is greater.

Let us consider the case where the domain boundary is

made up of several connected components, discretized with different stepsizes (this is a frequent case for domains involved in fluid dynamics problems). A regular mesh of such a domain is a mesh whose element size in the neighborhood of each boundary is close to the stepsize of the boundary discretization. As for the element size at other points of the domain, it must remain locally almost constant. Figure 9 shows a domain whose boundary contains five connected components, one representing the infinite, and the four others different parts of an aircraft wing. The infinite component is discretized with a relatively large stepsize, and the others with much smaller and different stepsizes. Figures 9 and 11 show almost equilateral mesh of the domain. Figures 10 and 12 show respective enlargements of these meshes around the wing body, thus illustrating the boundary discretizations and the corresponding meshes.

The second type of discretization is called “geometric”, for it is adapted to the geometry of the domain boundary. In that case, it can be shown that the step-

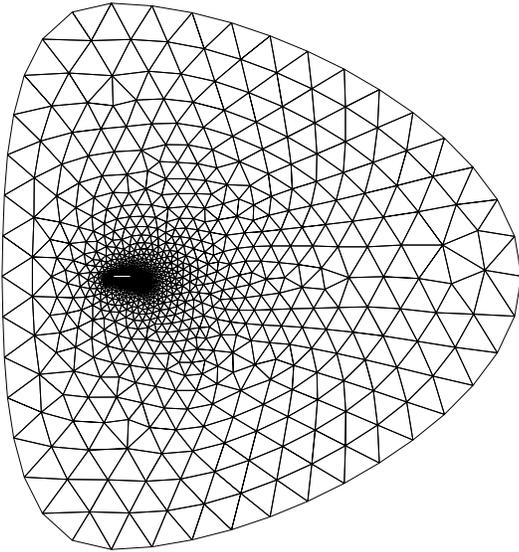


Figure 9. *Almost equilateral mesh of a domain in computational fluid dynamics (CFD).*

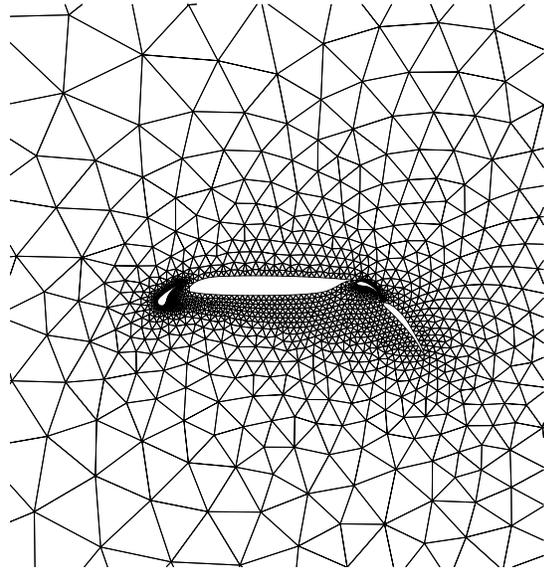


Figure 10. *Partial enlargement around the wing body.*

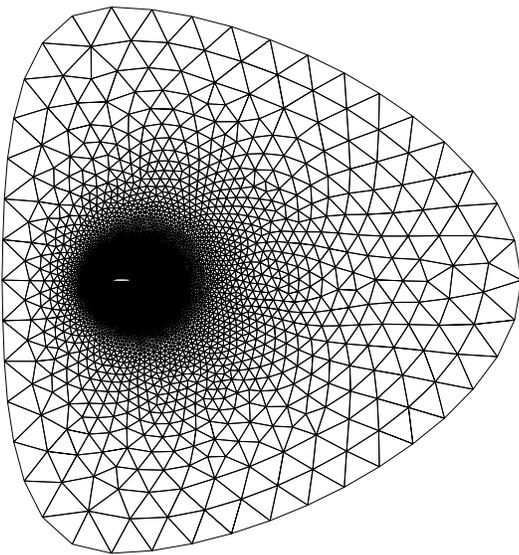


Figure 11. *Another almost equilateral mesh of the domain.*

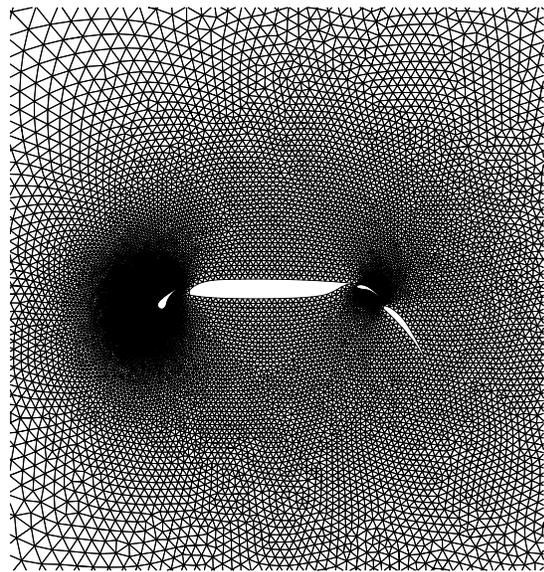


Figure 12. *Partial enlargement around the wing body.*

size must be proportional to the radius of curvature. The drawback of this type of discretization is that it can give rise to large variations of the stepsize. To reduce this phenomenon, different methods of stepsize smoothing can be used.

Figures 13 and 14 show, respectively, two geometric discretizations of the wheel contour. The first one is defined so that any edge of the discretization is sufficiently close to the contour (the stepsizes are locally

proportional to the radii of curvature). This condition guarantees, among others, that the angle between each edge and the tangents at its extremities remains smaller than a given threshold value (here 4°). Regarding the second discretization, the stepsizes of the first one are smoothed so that the ratio between the lengths of two consecutive edges is bounded. As can be seen on the figures, the circles representing the desired sizes violate in the first case, and respect (more or less) in the second case, the discretization obtained.

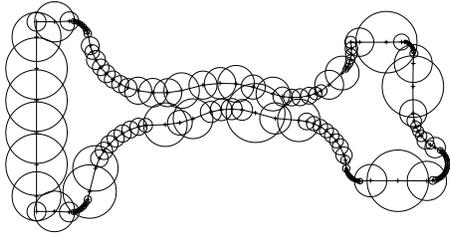


Figure 13. *Geometric discretization.*

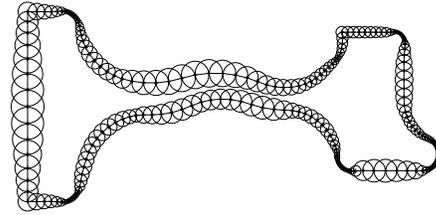


Figure 14. *Controlled geometric discretization.*

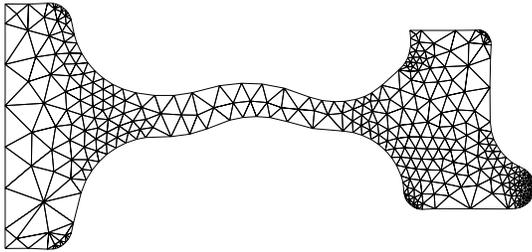


Figure 15. *Geometric mesh.*

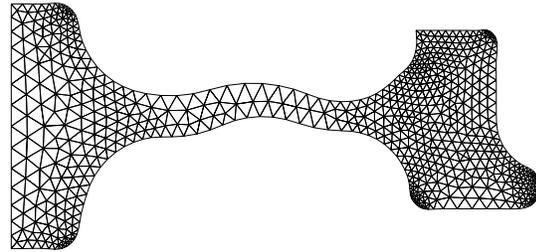


Figure 16. *Controlled geometric mesh.*

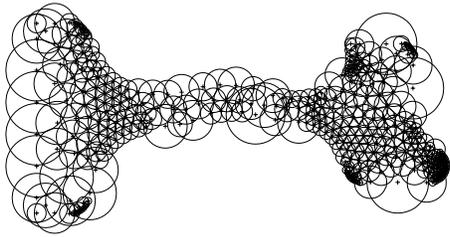


Figure 17. *Geometric sizes.*

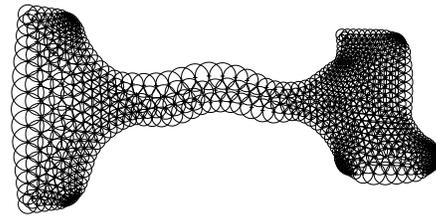


Figure 18. *Controlled geometric sizes.*

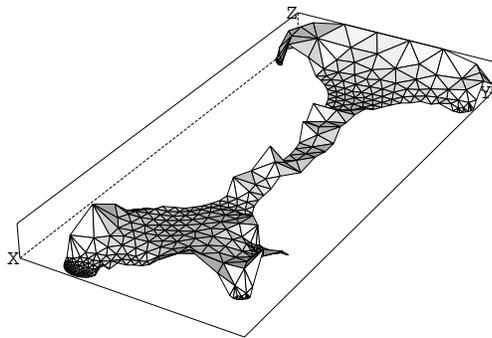


Figure 19. *Cartesian function of the sizes.*

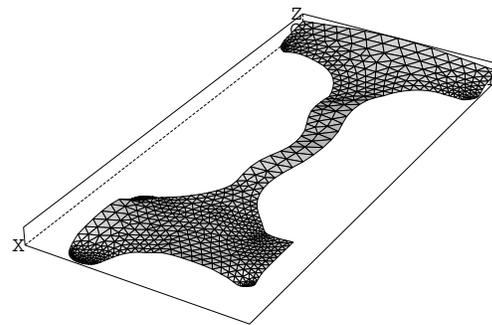


Figure 20. *Cartesian function of the controlled sizes.*

Figures 15 and 16 show two almost equilateral meshes corresponding to the two previous geometric discretizations (in the first case, the stepsizes are locally proportional to the radii of curvature, while in the second case the stepsizes are locally averaged, starting from the first discretization). As can be noticed on these figures, the second mesh is more regular. This

observation can be confirmed by examining Figures 17 and 18, which show the sizes at the vertices of the meshes.

As an indication, Figures 19 and 20 show the surfaces defined by the Cartesian functions of the sizes associated with these two meshes.

Remark. Apart from the discretization of the domain boundary, the geometric characteristics can have an effect on the generation of the equilateral mesh of the domain. These may be of different natures, amongst which are angular corners and very close boundary segments (or thin domains).

In the case of a geometric discretization (generally non uniform) of the domain boundary, it is not obvious to *a priori* establish the desired size of the elements at the vertices of the regular mesh of the domain. Let us recall that a regular mesh is a mesh whose element size variation is locally almost null. Accordingly, the idea consists in finding, among all the continuous size functions which conform to the prescribed sizes on the boundary, the least varying one. Therefore, this function represents a kind of “minimal variational surface”. The size variation can be quantified in two different ways which characterize the gradation of the related meshes. Let h be the size function in the domain, the first approach consists in considering the “usual” gradient ∇h of h . As for the second approach, the variation is measured by the “logarithmic” gradient $\nabla \log h$ of h . In the next sections, we define the minimal variational surfaces based on these two types of variations and we give a simple algorithm to construct these surfaces.

3. MINIMAL VARIATIONAL SURFACES

Let us denote by $v(h)$ the variation of h irrespective of the type of variation considered. A surface with minimal usual variation defined in a closed bounded domain Ω of R^2 , supported by a given contour Σ (sufficiently smooth), is a surface in which the sum of the square of the modulus of the variation on Ω is minimal. Mathematically, if h corresponds to such a surface, it is characterized by:

$$h(\Gamma) = \Sigma \quad \text{and} \quad J(h) = \min_{u \in H_v^1(\Omega)} J(u),$$

$$\text{with} \quad J(u) = \int_{\Omega} \|v(u)\|^2 dX,$$

where Γ is the boundary of Ω and $H_v^1(\Omega) = \{u \in L^2(\Omega) ; \forall i \ v_i(u) \in L^2(\Omega)\}$, L^2 denoting the space of functions having a summable square on Ω , and $v_i(u)$ the i -th component of $v(u)$.

In the case where $v(h) = \nabla h$ and assuming that Γ is regular enough, this surface is nothing other than the solution of the non homogeneous Dirichlet problem:

$$h(\Gamma) = \Sigma \quad \text{and} \quad \Delta h = 0 \quad \text{in } \Omega,$$

which characterizes harmonic functions. Using a classical result of numerical analysis, it can be proved that h is a solution of the problem:

$$h(\Gamma) = \Sigma \quad \text{and} \quad \forall v \in H_0^1(\Omega),$$

$$\int_{\Omega} \langle \nabla h, \nabla v \rangle dX = 0,$$

where $H_0^1(\Omega)$ is the set of functions in the Sobolev space of order 1, $H^1(\Omega)$, having a null value on the boundary Γ of domain Ω . Likewise, if $v(h) = \nabla \log h$, it can be shown that h is a solution of the problem:

$$h(\Gamma) = \Sigma \quad \text{and} \quad \forall v \in H_0^1(\Omega),$$

$$\int_{\Omega} \left(-\frac{2}{h^3} \|\nabla h\|^2 v + \frac{1}{h^2} \langle \nabla h, \nabla v \rangle \right) dX = 0.$$

These two problems can be solved using the finite element method. Considering piecewise P_1 solutions, in the first case, the problem reduces to solving a linear system and, in the second case, a non linear system. Although these solutions give the desired exact surfaces, we find it simpler to use a collocation method, as will be seen in the following, to approximate these minimal variational surfaces.

4. APPROXIMATE CONSTRUCTION ALGORITHM

Like in the finite element method, we search for minimal variational surfaces by using a given mesh of Ω . Then, the problem consists in determining the elevations at the vertices of this mesh. Let \mathcal{T} be a uniform mesh of domain Ω , including N vertices, the M first vertices being on the boundary. Knowing the contour Σ (which represents the sizes associated with the discretization of the boundary Γ of Ω), we have the elevations at the M first vertices of \mathcal{T} . We must now determine the elevations at the other vertices.

Let $v_D(h, e)$ be a discrete measure of the variation of h along the edge e of the mesh, we consider the following optimization problem:

$$\text{Minimize} \quad J(h) \quad \text{with} \quad J(h) = \frac{1}{2} \sum_{e \in \mathcal{T}} v_D^2(h, e),$$

where e sweeps all the edges of the mesh. This problem consists in minimizing the size variation along the directions given by the mesh considered. We find that this simpler formulation is more efficient in practice to approximate the minimal variational surfaces.

Let us denote by h_i the elevation (or the desired size) at vertex i of the mesh, and by d_{ij} the (“theoretically” constant) size of edge ij of the mesh. Considering the first type of variation, we have:

$$v_D(h, ij) = \frac{h_i - h_j}{d_{ij}} \quad \text{and}$$

$$J(h) = \frac{1}{2} \sum_{M+1 \leq i, j \leq N} \left(\frac{h_i - h_j}{d_{ij}} \right)^2,$$

and, in this case, we obtain a simple problem of convex quadratic optimization which reduces to solving a linear system whose matrix is definite positive. It can be shown that this system of equations represents, indeed, the discretization at order 2 of a non homogeneous and non isotropic transport-diffusion equation, of the type:

$$\alpha(X) \frac{\partial h}{\partial x} + \beta(X) \frac{\partial h}{\partial y} - a(X) \frac{\partial^2 h}{\partial^2 x} - b(X) \frac{\partial^2 h}{\partial x \partial y} - c(X) \frac{\partial^2 h}{\partial^2 y} = 0.$$

with $h(\Gamma) = \Sigma$ ($X = (x, y)$ representing the generic point of the domain).

Similarly, analyzing the second type of variation, we can consider:

$$v_D(h, ij) = \frac{1}{d_{ij}} \frac{h_i - h_j}{h_i + h_j} \quad \text{and}$$

$$J(h) = \frac{1}{2} \sum_{M+1 \leq i, j \leq N} \left(\frac{1}{d_{ij}} \frac{h_i - h_j}{h_i + h_j} \right)^2,$$

which corresponds to a more complex optimization problem which can be solved by an approach based on a quasi-Newton method (algorithm "BFGS", cf. [3]). Analyzing this new discrete criterion would exhibit a non linear partial differential equation, having the same anisotropic feature as the first type of variation. Considering a uniform mesh of the domain would partly avoid this shortcoming.

5. EXAMPLES

In this section, we present the academic example of a circular ring which is centered at the origin and whose outer boundary has a radius equal to 1. We suppose that the boundary of this domain is discretized conforming to a size function with a sinusoidal variation. This discretization is illustrated in figure 21. Besides, we consider a uniform mesh of the domain (Figure 22), on which we will construct the minimal surfaces supported by the curve defined by the size function on the domain boundary. Figures 23 and 24 show, respectively, the two types of minimal variational surfaces, usual and logarithmic, obtained by solving the optimization problem introduced in the previous section. These two surfaces can be differentiated by examining the isovalues corresponding to constant elevations. To solve both problems, the iterative algorithm BFGS is used. In the first case, the searched function h is initialized on the domain boundary by the specified size

function and, for each vertex inside the domain, by $h_{min} + (h_{max} - h_{min}) \frac{i}{N}$. In the latter expression, h_{min} and h_{max} are respectively the smallest and the largest size prescribed on the boundary, i is the vertex number, and N is the number of vertices of the uniform mesh considered (this initialization voluntarily brings a perturbation of $\nabla J(h)$ in the domain). The initial values obtained are $J(h) = 4.43e04$ and $\|\nabla J(h)\| = 4.9e04$. The convergence of the method is obtained after 119 iterations in 6 seconds (on a HP9000-J7000-440Mhz) and corresponds to $J(h) = 9.12e02$ and $\|\nabla J(h)\| = 1.2e-02$. In the second case, the searched function h is initialized by the size function obtained in the first case, and the initial values are $J(h) = 1.21e05$ and $\|\nabla J(h)\| = 2.9e05$. The convergence of the method is obtained after 129 iterations in 8 seconds and corresponds to $J(h) = 1.07e05$ and $\|\nabla J(h)\| = 1.7$.

The meshes on figures 25 and 26 represent respectively the meshes conforming to the size fields given by the minimal variational surfaces obtained, and are made up of respectively 1638 and 2460 elements. The average shape quality of these meshes (using a chosen monotonous measure varying from 0 (flat triangle) to 1 (regular triangle)) is in the order of 0.96 and 0.97. These meshes are generated by a method based on a combined frontal-Delaunay approach [1], [4].

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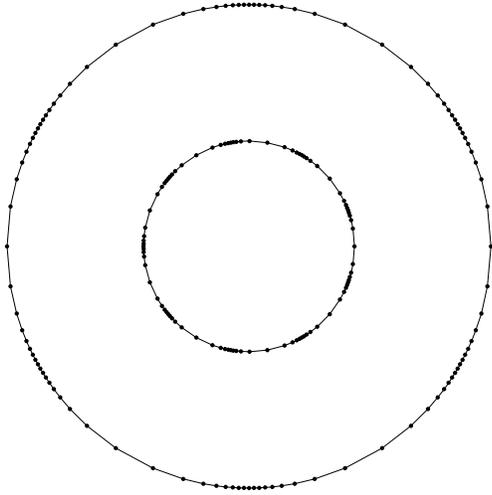


Figure 21. *Specified boundary discretization.*

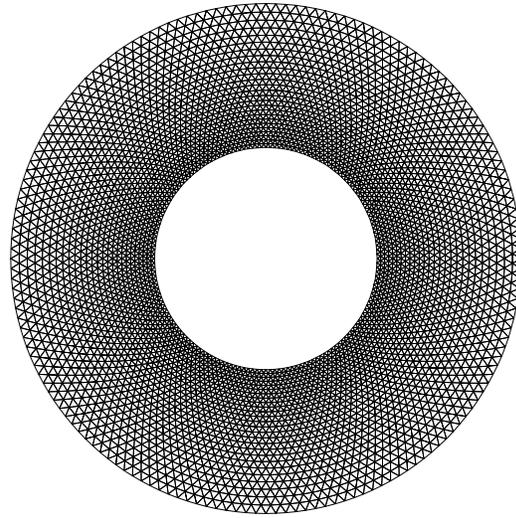


Figure 22. *Uniform mesh.*

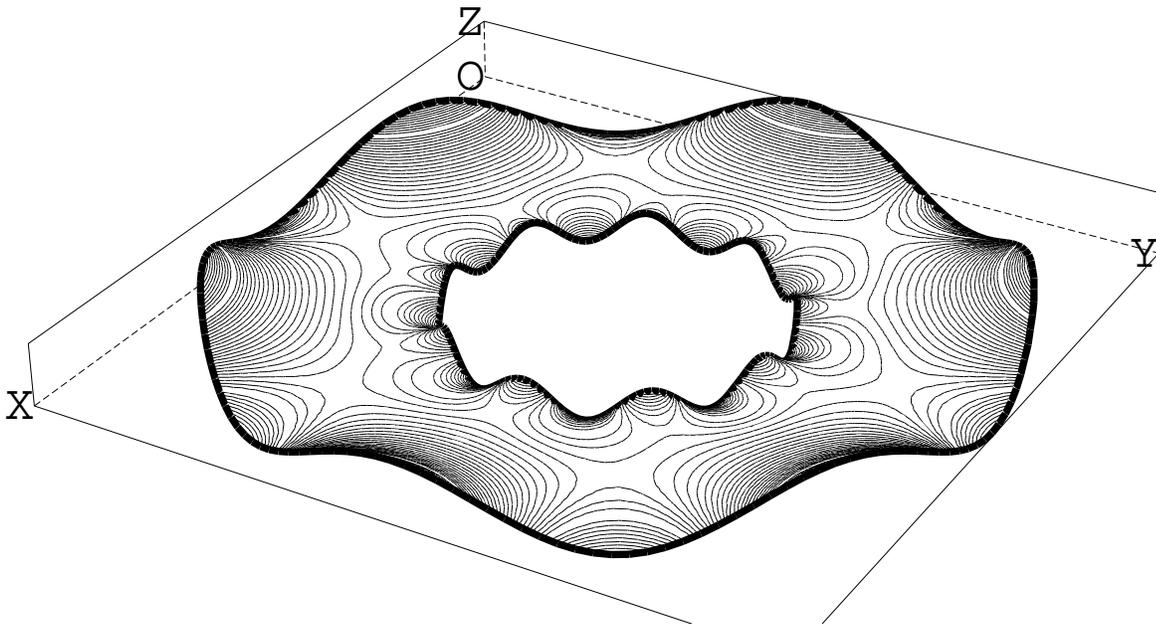


Figure 23. *Minimal usual variational surface.*

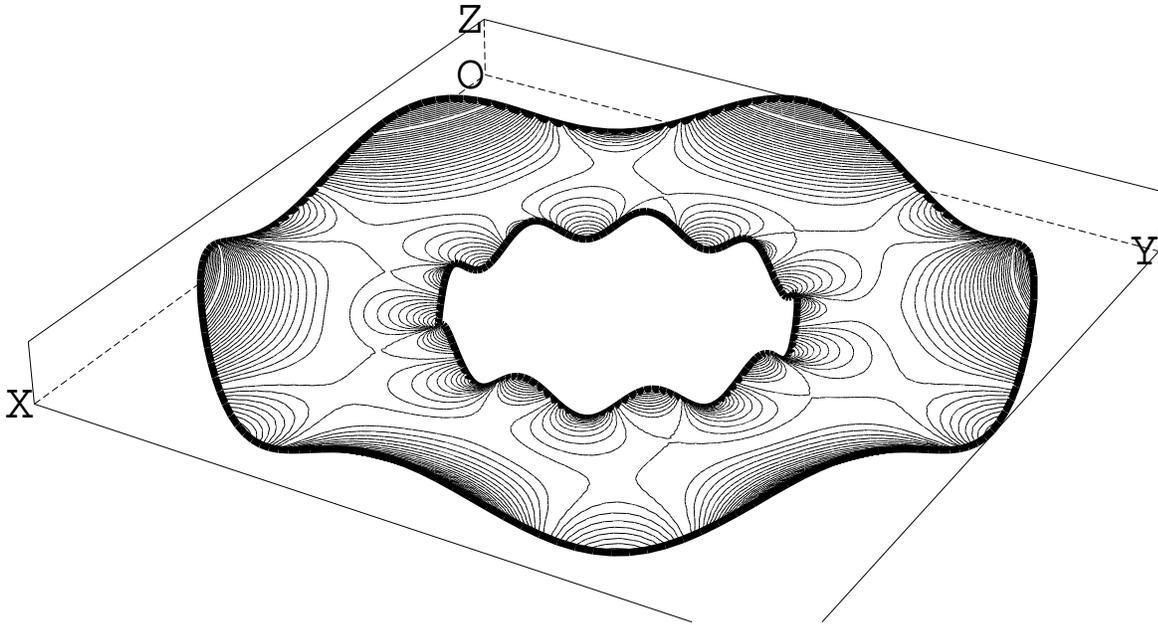


Figure 24. *Minimal Logarithmic variational surface.*

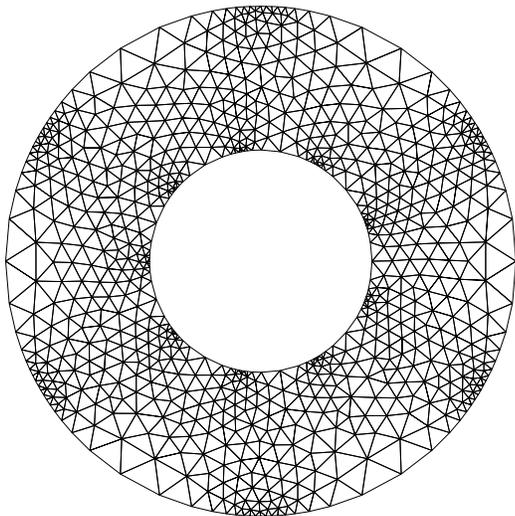


Figure 25. *Usual quality mesh.*

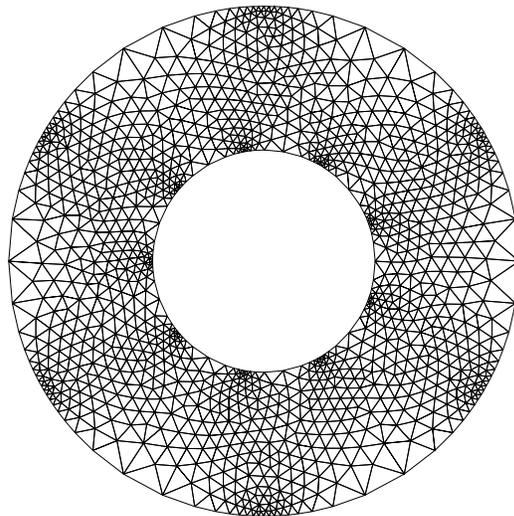


Figure 26. *Logarithmic quality mesh.*