



**HAL**  
open science

# Optimal Design of One-sided Exponential EWMA Charts Based on Median Run Length and Expected Median Run Length

Yulong Qiao, Jinsheng Sun, Philippe Castagliola, Xuelong Hu

► **To cite this version:**

Yulong Qiao, Jinsheng Sun, Philippe Castagliola, Xuelong Hu. Optimal Design of One-sided Exponential EWMA Charts Based on Median Run Length and Expected Median Run Length. *Communications in Statistics - Theory and Methods*, 2022, 51 (9), pp.2887-2907. 10.1080/03610926.2020.1782937 . hal-03620915

**HAL Id: hal-03620915**

**<https://hal.science/hal-03620915v1>**

Submitted on 27 Mar 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Optimal Design of One-sided Exponential EWMA Charts Based on Median Run Length and Expected Median Run Length

YuLong Qiao<sup>1</sup>, JinSheng Sun<sup>1</sup>, Philippe Castagliola<sup>2</sup>, and  
XueLong Hu<sup>\*3</sup>

<sup>1</sup>School of Automation, Nanjing University of Science and Technology,  
Nanjing, China

<sup>2</sup>Université de Nantes & LS2N UMR CNRS 6004, Nantes, France

<sup>3</sup>School of Management, Nanjing University of Posts and  
Telecommunications, Nanjing, China, \*Corresponding author,  
hxl0419@hotmail.com

## Abstract

Exponential type charts are useful tools to monitor the time between events in high-quality processes with a low defect rate. Most studies on exponential charts are

designed with the average run length ( $ARL$ ) metric. The only use of  $ARL$  in the design of control charts is sometimes criticized because the shape of the run length ( $RL$ ) distribution of control charts changes with the shift size. In fact, the  $RL$  distribution of the exponential exponentially weighted moving average (EWMA) chart is skewed, especially when the process is in-control. Hence, the median run length ( $MRL$ ) serves as a more meaningful indicator. Moreover, in some situations, the shift size in the process is unknown in advance. Under this case, the expected median run length ( $EMRL$ ) can be used as the metric. In this paper, the  $RL$  properties of both the upper- and lower-sided exponential EWMA charts are studied through a Markov chain approach. Two optimal design procedures are developed for one-sided exponential EWMA charts based on the  $MRL$  and  $EMRL$ , respectively. The choices of reflecting boundaries for one-sided exponential EWMA charts are discussed through many numerical studies. The  $MRL$  and  $EMRL$  performances of the one-sided exponential EWMA charts are investigated.

**Keywords:** Exponential chart; Median run length; Expected median run length; Exponentially weighted moving average chart.

## 1 Introduction

Quality of a product is important for manufacturing or service organizations to improve their competitiveness (see Oakland (2007)). Statistical Process Control (SPC) plays an important role in the quality improvement of a product. As a useful tool in SPC, control charts are usually used to monitor the state of processes.

Control charts are usually designed under the assumption that the quality character-

istic follows a normal distribution. This assumption is not appropriate in the monitoring of some particular processes, such as in monitoring the rate of occurrences of congenital malformations (see Lucas (1985)), in monitoring the consecutive radiation pulses (see Luo et al. (2012)), and in monitoring the medical errors (see Doğu (2012)), where the exponential distribution is usually assumed for the time between events (TBE) data. Lucas (1985) first investigated the properties of the exponential chart. Following Lucas (1985), many research works on exponential charts were performed. Gan (1994) developed an optimal design procedure of the exponential Cumulative Sum (CUSUM) control charts. Based on the average time to signal (*ATS*), Liu et al. (2006) compared the properties of several exponential charts. The Phase I implementation of the exponential chart was further considered by Jones and Champ (2002). More related research works on exponential charts can be found in Santiago and Smith (2013), Yang et al. (2015), Zhang et al. (2005, 2006), to name a few. These researches are mainly focused on two-sided exponential charts. In fact, one-sided charts are more appropriate for processes for which the direction of the potential mean shift is anticipated (see Shu et al. (2007)). Some existing researches focused on one-sided exponential charts can be found in Gan (1998), Ozsan et al. (2010), Pehlivan and Testik (2010). Moreover, if a two-sided chart is needed, a combined scheme of both the lower- and upper-sided charts could be designed instead, and there is practically no difference between the combined scheme and the two-sided case (see Reynolds et al. (1990)).

Exponentially Weighted Moving Average (EWMA) type charts gained extensive attention from many researchers for its advantage in detecting small to moderate shift sizes in a process (see Chandrasekaran et al. (1995), Gan (1991), Waldmann (1986)). Much researches on EWMA type charts can be found in Haq (2019), Knoth (2015), Maravelakis et al. (2019)

and so on. These researches are mainly based on the assumption of normally distributed observations. For non-normally distributed observations, researches have also been studied to investigate the performance of EWMA type charts. In high-quality processes, TBE data are sometimes assumed to be exponentially distributed. Based on this fact, Gan (1998), Gan and Chang (2000) developed the exponential EWMA control chart to monitor the rate of occurrences of rare events and presented a program to determine the average run length ( $ARL$ ) profiles of the exponential EWMA chart. Liu et al. (2007) proposed a new EWMA chart to monitor the exponentially distributed TBE data using the double square root (SQRT) transformation. Pehlivan and Testik (2010) investigated the sensitivity of the exponential EWMA chart in detecting the process shift and they concluded that the exponential EWMA chart could be extremely robust to departures from the assumed exponential distribution. The conditional expected values were employed to monitor small mean shifts in Raza et al. (2016). By transforming the exponential data to approximately normal ones, Khan et al. (2016) proposed an EWMA chart to monitor the exponentially distributed data. The effect of parameter estimation on the performance of the exponential EWMA chart has been analyzed in Ozsan et al. (2010). A survey of the related research works on exponential EWMA charts can be found in Ali et al. (2016).

Generally, the  $ARL$  is used as a criterion to evaluate the performance of control charts. Nevertheless, the only use of  $ARL$  has been criticized by some researchers (see Khoo (2004), Khoo et al. (2011), Lee and Khoo (2017b), Tang et al. (2019), Waldmann (1986) and so on). The  $ARL$  is regarded as the representation of the “central tendency” of the run length ( $RL$ ) distribution, however, it is clearly not suitable as its shape becomes asymmetric. Furthermore, it has been pointed out that the  $RL$  distribution of control charts may be skewed,

especially when the process is in-control. Therefore, the interpretation of the *ARL* is sometimes confusing for practitioners to evaluate a control chart's performance with skewed *RL* distributions. For example (see Section 2, Table 1), a lower-sided exponential EWMA chart with  $(\lambda^-, h^-) = (0.4, 0.2045)$  has an in-control *ARL* of 500, while 50% of in-control *RLs* is smaller than 348. When the process is out-of-control ( $c = 0.6$ ), the *ARL* is 57.1, and 50% of the out-of-control *RLs* is smaller than 41. Instead, the median run length (*MRL*), defined as the median number of samples that are plotted on a chart before an out-of-control signal is issued, has no such interpretation problem. In the above example, the in-control and out-of-control *MRL* values are 348 and 41, respectively, which indicate that half of the in-control and out-of-control *RLs* are smaller than 348 and 41, respectively. For practitioners, the interpretation based on the *MRL* is more intuitive than the one based on the *ARL*. Hence, more and more researchers promote using the *MRL* as a better criterion to evaluate the performance of control charts (see Lee and Khoo (2006, 2017a), Nyau et al. (2017), Teoh et al. (2016)). Moreover, practitioners may not have enough historical knowledge of shift sizes in a process. To circumvent this problem, the expected *MRL* (*EMRL*) is recommended as a measure of a chart's performance when practitioners cannot specify the exact value of the shift size (see Tang et al. (2019), Teoh et al. (2016), You et al. (2016)).

In fact, *RL* distributions of one-sided exponential EWMA control charts are also skewed, especially when the process is in-control (see Section 2). The *MRL* and *EMRL* are good alternatives to evaluate the properties of one-sided exponential EWMA control charts. As far as we know, there is no research conducted on the *MRL* or *EMRL* properties of exponential charts yet. The purpose of this paper is to fill this gap and to develop the optimal design procedures for both the upper- and lower-sided exponential EWMA charts based on *MRL*

and *EMRL*. The rest of this paper is organized as follows: Section 2 introduces the one-sided exponential EWMA charts and briefly discusses problems arising from the only use of *ARL*. Then, an optimal design procedure is developed for the upper- and lower-sided exponential EWMA charts based on *MRL* and *EMRL* in Section 3. The performances of these one-sided exponential EWMA charts are investigated in Section 4 and the effect of parameter estimation is briefly discussed in Section 5. Finally, some conclusions and recommendations are given in Section 6. The *RL* properties of the chart are presented using a Markov chain approach in the Appendix.

## 2 One-Sided Exponential EWMA Charts

As in most reported research works, let us assume that the TBE variable  $X$  follows an exponential distribution with a single parameter  $\theta$ . The probability density function (p.d.f.)  $f(x)$  and cumulative distribution function (c.d.f.)  $F(x)$  of the exponential distribution are given as follows:

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x \geq 0 \quad (1)$$

$$F(x) = 1 - e^{-\frac{x}{\theta}}, \quad x \geq 0 \quad (2)$$

The process is said to be in-control when  $\theta = \theta_0$ , where  $\theta_0$  is the known in-control mean of the process. Otherwise, the process is out-of-control when  $\theta = c\theta_0$  ( $c \neq 1$ ), where  $c = \theta/\theta_0$  is a constant that reflects the shift in  $\theta_0$ .

Since this paper focuses on one-sided charts, we introduce the following upper- and lower-

sided exponential EWMA charts:

- If the goal is to detect an increase in the process mean ( $c > 1$ ), the upper-sided exponential EWMA chart is used. The charting statistic for the  $t$ th sample is computed as follows:

$$Q_t^+ = \max(B^+, (1 - \lambda^+)Q_{t-1}^+ + \lambda^+ X_t), \quad t = 1, 2, \dots \quad (3)$$

where  $0 < \lambda^+ \leq 1$  and  $B^+$  are the smoothing parameter and the reflecting boundary, respectively. The initial value  $Q_0^+$  is generally constrained by  $B^+ \leq Q_0^+ < h^+$ , where  $UCL = h^+$  is the upper control limit of the chart. An out-of-control signal is issued when  $Q_t^+ > h^+$ .

- If the goal is to detect a decrease in the process mean ( $c < 1$ ), the lower-sided exponential EWMA chart is used. The charting statistic for the  $t$ th sample is computed as follows:

$$Q_t^- = \min(B^-, (1 - \lambda^-)Q_{t-1}^- + \lambda^- X_t), \quad t = 1, 2, \dots \quad (4)$$

where  $0 < \lambda^- \leq 1$  and  $B^-$  are the smoothing constant and the reflecting boundary, respectively. The initial value  $Q_0^-$  is generally constrained by  $h^- < Q_0^- \leq B^-$ , where  $LCL = h^-$  is the lower control limit of the chart. An out-of-control signal is issued when  $Q_t^- < h^-$ .

As in Gan (1998) and Pehlivan and Testik (2010), without loss of generality, we assume that the in-control mean  $\theta_0 = 1$  and the initial values  $Q_0^- = Q_0^+ = \theta_0 = 1$  (other situations with specific initial values can be analyzed similarly). Furthermore, the reflecting boundary  $B^+(B^-)$  is used to ensure that the charting statistic is at most a certain distance away from



the control limit  $h^+(h^-)$ , which can improve the sensitivity of the chart (see Pehlivan and Testik (2010)).

In order to draw attention to the use of the *MRL* metric, some *ARL* and percentile values of the *RL* distribution are presented in Table 1 when  $\lambda^- \in \{0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1\}$  and the in-control *ARL* (denoted as  $ARL_0$ ) roughly equals to 500 for the lower sided exponential EWMA chart. In this article, the *RL* properties of one-sided exponential EWMA charts can be obtained using the Markov chain approach given in the Appendix. Moreover, the values of the reflecting boundaries are set to be  $B^+ = 0.5$  and  $B^- = 2.0$  (discussed in Section 3.1). It is obvious that there is a large difference between  $ARL_0$  and in-control *MRL* (denoted as  $MRL_0$ ) when the process is in-control. However, the difference between *ARL* and *MRL* values decreases with the shift size increasing. For example, if  $(\lambda^-, h^-) = (0.05, 0.6861)$ , when the process is in-control ( $c = 1$ ), the  $ARL_0$  and  $MRL_0$  are 500.1 and 351, respectively. When the process is out-of-control ( $c = 0.2$ ), the *ARL* and the *MRL* values decrease down to 10.2 and 10, respectively. Similar results can also be drawn from other values of  $(\lambda^-, h^-)$  shown in Table 1. It is noted that the shape of the *RL* distribution of the lower-sided exponential EWMA chart changes with the shift size, i.e., from highly right skewed when the process is in-control to nearly symmetric when the process is out-of-control.

On the other hand, for the upper-sided exponential EWMA chart, Table 2 also shows the *ARL* and some percentiles of the *RL* distribution with  $\lambda^+ \in \{0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1\}$  and the  $ARL_0$  roughly equals to 500. It can be seen that the difference between *ARL* and *MRL* also changes with the shift size. For example, if  $(\lambda^+, h^+) = (0.2, 2.2378)$ ,  $ARL_0$  and  $MRL_0$  are 500 and 347, respectively, when the process is in-control ( $c = 1$ ). When the mean shift  $c$  changes to  $c = 5$ , *ARL* and *MRL* values are 3.1 and 3, respectively.

(Please Insert Tables 1-2 to Here)

For a skewed  $RL$  distribution, the interpretation solely based on the  $ARL$  is different from the one with respect to a nearly symmetric distribution. A practitioner should be advised against viewing  $ARL$  as a symbol of “half of the time” because  $ARL$  only provided information about the expected  $RL$  (see Khoo et al. (2011), Nyau et al. (2017)). It is not intuitive enough and can sometimes result in misunderstandings when the  $RL$  distribution changes with the shift size. Fortunately, the percentiles of the  $RL$  distribution provide more information on the performance of a control chart, no matter the  $RL$  distribution is skewed or symmetric. As a good alternative, the  $MRL$  has been suggested by some researchers to evaluate the properties of control charts (see Khoo et al. (2011), Lee and Khoo (2017a), Nyau et al. (2017), Tang et al. (2019), Teoh et al. (2017), to name a few). In this article, for one-sided exponential EWMA charts, the  $MRL$  is also suggested as the criterion to be investigated.

### **3 Optimal Design of One-sided Exponential EWMA Control Charts**

In this section, we mainly focus on the optimal design of one-sided (upper- and lower-sided) exponential EWMA charts based on the  $MRL$  and  $EMRL$ , respectively. The setting of the reflecting boundaries  $B^+$  and  $B^-$  are discussed first, and the procedure for the design of the smoothing parameter and control limits of the chart are developed lately.

### 3.1 Choices of the reflecting boundaries $B^+$ and $B^-$ for the *MRL*-based one-sided exponential EWMA charts

In order to design the *MRL*-based one-sided exponential EWMA control charts, the reflecting boundary  $B^+$  ( $B^-$ ) of the upper-sided (lower-sided) EWMA control chart should be selected first. For the *ARL*-based exponential EWMA charts, by doing many numerical analyses, values of the reflecting boundaries have been suggested to be  $B^+ = 0.5$  and  $B^- = 2.0$  by Gan (1998). These values are also used in other researches conducted on the *ARL*-based exponential EWMA chart (see Ozsan et al. (2010)). Now, we are wondering if these suggested values can also be used for the *MRL*-based one-sided exponential EWMA charts? Considering this issue, many numerical simulations have been performed for different values of  $\lambda^- \in \{0.1, 0.3, 0.5, 0.7\}$  and mean shifts  $c \in \{0.1, 0.2, \dots, 1.0\}$  when  $MRL_0 \in \{100, 500, 1000\}$ , for the lower-sided exponential EWMA chart. The control chart limit  $h^-$  is selected to produce the desired  $MRL_0$  for different combinations of  $\lambda^-$  and  $B^-$ . Owing to the space limitation, only the numerical simulation with  $MRL_0 = 100$  is presented in Table 3. It can be noted that the control chart limit  $h^-$  generally increases as the reflecting boundary  $B^-$  increases. When  $B^-$  is larger than 2.0, the value of  $h^-$  converges to a constant. For example, when  $\lambda^- = 0.3$ , the value of  $h^-$  converges to 0.348 with the increase in  $B^-$ . Moreover, it can also be seen that, for fixed values of  $c$ , the out-of-control *MRL* value (denoted as  $MRL_1$ ) decreases as the reflecting boundary  $B^-$  increases, which implies that the sensitivity of the control chart increases. For values of  $B^-$  larger than 2.0, it can be noted that the detection ability of the lower-sided exponential EWMA chart generally remains the same. For example, when  $\lambda^- = 0.3$  and  $c = 0.8$ , if the value of  $B^-$  increases

from 1.0 up to 1.5, the  $MRL_1$  values decrease from 44 down to 40. While if the value of  $B^-$  increases up to 2.3 or 2.5, the  $MRL_1$  values are both equal to 39. For other values of the desired  $MRL_0$ , similar results can also be drawn. Consequently, as in Gan (1998), a reasonable value of the reflecting boundary  $B^-$  for the  $MRL$ -based lower-sided exponential EWMA chart is 2.0. In the rest of this article, the reflecting boundary  $B^-$  is set to be 2.0.

For the upper-sided exponential EWMA chart, similar numerical simulations are also performed to investigate the choice of the reflecting boundary  $B^+$  for different values of  $MRL_0 \in \{100, 500, 1000\}$ . Owing to the space limitation, these results are not presented here. It can be concluded from the results that a reasonable value of the reflecting boundary  $B^+$  is 0.5, which leads to a good detection ability of the upper-sided exponential EWMA chart for different values of  $\lambda^+$  and  $MRL_0$ . In this article, the value of the reflecting boundary  $B^+$  is set to be 0.5.

(Please Insert Table 3 to Here)

### **3.2 Design procedure of one-sided exponential EWMA charts based on the $MRL$**

When the process is in-control ( $c = 1$ ), it is expected that the control chart runs with a small false alarm rate, i.e., a large  $MRL_0$ . While at the same time, the chart is also expected to detect a mean shift quickly, i.e., a small  $MRL_1$ , when the process is out-of-control ( $c \neq 1$ ). The optimal design of the one-sided exponential EWMA charts based on the  $MRL$  consists in finding the smallest  $MRL_1$  value for a specified mean shift  $c$  (denoted as  $c_{opt}$ ) with the

constraint on the desired  $MRL_0$ , i.e.,

$$(\lambda^*, h^*) = \underset{(\lambda, h)}{\operatorname{argmin}} MRL_1(\lambda, h, c_{\text{opt}}), \quad (5)$$

subject to

$$MRL(\lambda^*, h^*, c = 1) = MRL_0, \quad (6)$$

where  $\lambda$  and  $h$  are the smoothing parameter and the control chart limit, respectively. It is noted that  $\lambda$  stands for  $\lambda^+$  or  $\lambda^-$  for the upper- or the lower-sided exponential EWMA charts, respectively. Similarly,  $h$  stands for  $h^+$  or  $h^-$  for the upper- or lower-sided exponential EWMA charts, respectively. The upper-sided exponential EWMA chart is used to detect an increase in the mean shift ( $c_{\text{opt}} > 1$ ). On the contrary, the lower-sided exponential EWMA chart is employed if the specified mean shift decreases ( $c_{\text{opt}} < 1$ ).

For a specified value of  $MRL_0$  and the interested mean shift  $c_{\text{opt}}$ , one can obtain the optimal parameter combinations of the one-sided exponential EWMA chart using Eqs. (5) and (6). The procedure is summarized as follows:

- (1) Specify the desired  $MRL_0$  and the interested mean shift  $c_{\text{opt}}$ .
- (2) Initialize the smoothing constant  $\lambda = 0.05$ .
- (3) Determine the control limit  $h$  for the corresponding  $\lambda$ , based on the desired  $MRL_0$  specified in Step (1).
- (4) Compute the  $MRL_1$  for the corresponding combinations  $(\lambda, h)$  specified in Step (3) and

$c_{\text{opt}}$ .

- (5) Repeat Steps (3) to (4) for all values of  $\lambda$  varying from 0.05 to 1 with a step size 0.001.
- (6) Record the combinations  $(\lambda, h)$  that produce the minimum  $MRL_1$  for the specified  $c_{\text{opt}}$  as the optimal parameter combinations.

Since the  $MRL$  is a discrete quantity, there may exist several values of control limit  $h$  for each smoothing constant  $\lambda$ , based on the constraint on the specified  $MRL_0$ . Without loss of generality, the median of all the  $h$  values is selected for the corresponding  $\lambda$ . It is noted that there may exist several optimal parameter combinations  $(\lambda, h)$  that produce the same minimum value of  $MRL_1$  for a specified mean shift  $c_{\text{opt}}$ . In this situation, the median of all the optimal  $\lambda$  values and the corresponding  $h$  are selected in this article.

### **3.3 Design procedure of one-sided exponential EWMA charts based on the $EMRL$**

The optimal  $MRL$ -based chart proposed in this paper is effective in detecting a shift of particular size and the computation of the  $MRL$  requires the exact value of the mean shift  $c$ . However, the practitioners are not always able to specify the exact shift size in advance, but the range of shift is roughly known in practice (see Celano et al. (2013)). In this case, the shift size  $c$  should be considered as a random quantity that may vary according to a certain distribution and the  $EMRL$  can be used instead of the  $MRL$ , i.e.,

$$EMRL = \int_{\Omega} MRL(\lambda, h, c) \times f_c(c) dc, \quad (7)$$

where  $f_c(c)$  is the p.d.f. of the mean shift  $c$  over the range  $\Omega$ . It is generally assumed explicitly (see Domangue and Patch (1991), Sparks (2000)) or implicitly (see Reynolds and Stoumbos (2004)) that all process shifts occur with an equal probability, for example, the shift in the mechanical parts produced by multi-purpose Computer Numerical Control (CNC) machines (see Celano et al. (2013)), thus a uniform distribution for  $c$  is recommended. Whereas, other p.d.f. of the mean shift  $c$ , like for instance, truncated normal distribution (see Ryu et al. (2010)), Rayleigh distribution (see Wu et al. (2004)), to name a few, can be considered according to the historic data of the out-of-control cases or the practitioner's knowledge about the process. Following Castagliola et al. (2011), Tang et al. (2019), Teoh et al. (2017), a uniform distribution is assumed in this article.

The optimal parameter combinations  $(\lambda^*, h^*)$  of the one-sided exponential EWMA chart based on the *EMRL* can be obtained as follows:

$$(\lambda^*, h^*) = \underset{(\lambda, h)}{\operatorname{argmin}} \quad EMRL_1(\lambda, h, \Omega), \quad (8)$$

subject to

$$MRL(\lambda^*, h^*, c = 1) = MRL_0, \quad (9)$$

where  $EMRL_1$  is the out-of-control *EMRL* value computed using Eq. (7). The optimal design procedure of the *EMRL*-based one-sided exponential EWMA charts is similar to the steps in Section 3.2 except that the interested mean shift  $c_{\text{opt}}$  and the  $MRL_1$  are replaced by the interested mean shift range  $\Omega$  and the  $EMRL_1$ , respectively.

## 4 Performance Evaluation Based on $MRL$

In this section, the performances of both the upper- and lower-sided exponential EWMA charts are studied. The optimal charting parameter combinations used here are obtained following the procedures described in Sections 3.2 and 3.3.

### 4.1 Performance evaluation for the mean shift of known size

The  $MRL$  profiles for the lower-sided exponential EWMA chart and the lower-sided exponential Shewhart chart are displayed in Table 4, for different values of the specified mean shifts  $c_{opt} \in \{0.2, 0.4, 0.6, 0.8\}$  and the desired  $MRL_0 \in \{100, 200, 300, 500\}$ . The optimal parameter combinations  $(\lambda^-, h^-)$  of the lower-sided exponential EWMA chart and the lower control limit ( $LCL$ ) of the corresponding lower-sided exponential Shewhart chart are given in rows 4 and 5.

It is shown that the lower-sided exponential EWMA chart with a small optimal  $\lambda^-$  value has better performance for small shift sizes. At the same time, the chart with a large optimal  $\lambda^-$  value performs better in the cases of large shift sizes. For example, considering the case of  $MRL_0 = 300$ , the  $MRL_1$  values increases from 54 up to 77 when  $\lambda^-$  increases from 0.052 up to 0.196 when the mean shift  $c = 0.8$ , which implies that the control chart with  $\lambda^- = 0.052$  detects the mean shift ( $c = 0.8$ ) more quickly than the chart with  $\lambda^- = 0.196$ . While, if the mean shift  $c$  decreases down to 0.2, the  $MRL_1$  values are 10 and 7 for  $\lambda^- = 0.052$  and  $\lambda^- = 0.196$ , respectively, which implies that the control chart with  $\lambda^- = 0.196$  detects the mean shift ( $c = 0.2$ ) more quickly than the chart with  $\lambda^- = 0.052$ .

By comparing the performances of the lower-sided exponential EWMA chart with the



lower-sided exponential Shewhart chart, it is also noted that the EWMA-type chart is superior to the Shewhart-type chart in most cases. The lower-sided exponential Shewhart chart only performs slightly better than the lower-sided exponential EWMA chart when the mean shift is large ( $c < 0.05$ ). For instance, when  $MRL_0 = 200$ , the  $MRL_1$  values of the lower-sided exponential EWMA chart with optimal parameter combinations designed for  $c_{\text{opt}} = 0.4$  are all smaller than the ones of the lower-sided exponential Shewhart chart when  $c \geq 0.03$ . When  $c = 0.01$ , the  $MRL_1 = 2$  for the lower-sided exponential Shewhart chart is a little smaller than the  $MRL_1 = 4$  for the lower-sided exponential EWMA chart.

Moreover, the  $MRL$  profiles of the upper-sided exponential EWMA and Shewhart charts are also displayed in Table 5 for  $MRL_0 \in \{100, 200, 300, 500\}$ . In order to observe the effect of the shift size comprehensively, we chose several representative values from Gan (1998), i.e.,  $c_{\text{opt}} \in \{1.04, 1.4, 2, 5\}$ . The optimal parameter combinations  $(\lambda^+, h^+)$  of the upper-sided exponential EWMA chart and the upper control limit ( $UCL$ ) of the upper-sided exponential Shewhart chart are displayed in rows 4 and 5. Similar to the lower-sided exponential EWMA chart, the upper-sided exponential EWMA chart with a small value of  $\lambda^+$  gives better performance for smaller shift sizes compared with the upper-sided exponential Shewhart chart. From Table 5, it is also clear that the upper-sided exponential EWMA chart consistently performs better than the upper-sided exponential Shewhart chart, for small to moderate shift sizes ( $1 < c < 3$ ). The upper-sided exponential Shewhart chart only outperforms the upper-sided exponential EWMA chart when the mean shift is large ( $c > 3$ ). Furthermore, as in Kumar and Baranwal (2019), a comparative study is carried out to compare the charts designed for a fixed nominal  $ARL_0$  and the corresponding  $MRL_0$ . Table 6 presents the  $MRL$  and  $ARL$  values of the chart for different parameter combinations

of  $(\lambda^-, h^-)$  and  $c$  when  $MRL_0 = 350$  and  $ARL_0 = 500$ . It is noted that the parameter combinations  $(\lambda^-, h^-)$  are obtained by optimizing the *MRL*-based chart (the upper part in Table 6) and the *ARL*-based chart (the lower part in Table 6) for different values of  $c_{\text{opt}}$ . The *MRL* and *ARL* values corresponding to  $c = c_{\text{opt}}$  are also bolded in the table. It can be concluded from the table that for the specific  $c = c_{\text{opt}}$  in column 1, the *MRL* of the *MRL*-based chart performs a little better than the *ARL*-based chart and the *ARL* of the *ARL*-based chart performs a little better than the *MRL*-based chart. For example, when  $c_{\text{opt}} = 0.6$ , the  $MRL = 23$  for the *MRL*-based chart performs a little better than the  $MRL = 24$  for the *ARL*-based chart when  $c = 0.6$ . At the same time, the  $ARL = 26.6$  for the *ARL*-based chart performs a little better than the  $ARL = 27.1$  for the *MRL*-based chart when  $c = 0.6$  (see the bolded values in columns 4 and 5).

(Please Insert Tables 4–6 to Here)

## 4.2 Performance evaluation for the mean shift of unknown size

When the mean shift is unknown, but the range of the shift size can be roughly specified, the optimal parameter combinations and properties of the lower-sided exponential EWMA chart and the lower-sided exponential Shewhart chart are provided in Table 7 for several uniform distributions of shift size  $\Omega \in \{[0.1, 0.3], [0.3, 0.5], [0.5, 0.7], [0.7, 0.9]\}$  when  $MRL_0 \in \{100, 200, 300, 500\}$ . For example, if  $MRL_0 = 200$ , the optimal parameter combinations  $(\lambda^-, h^-)$  and the corresponding  $EMRL_1$  value of the lower-sided exponential EWMA chart over the shift size range  $\Omega = [0.3, 0.5]$  are  $(0.204, 0.406)$  and 10.3, respectively. Using the optimal parameter combinations  $(\lambda^-, h^-)$ , the out-of-control *MRL* value of the exponential

EWMA chart is 10 when  $c = 0.4$ .

From Tables 4 and 7, it is noticed that the performance of the lower-sided exponential EWMA chart designed based on *EMRL* is close to the one designed based on *MRL*, if the specified mean shift  $c$  is in the mean shift range  $\Omega$ . For example, when  $MRL_0 = 200$  and  $c_{\text{opt}} = 0.6$  in Table 4, the  $MRL_1$  values of the lower-sided exponential EWMA chart are 28, 19, 14 for shift sizes of 0.7, 0.6, 0.5, respectively. While in Table 7, the  $MRL_1$  values are 28, 19, 14, respectively, when using the parameter combinations ( $\lambda^- = 0.087, h^- = 0.610$ ) designed based on  $\Omega = [0.5, 0.7]$ . This implies that the optimal parameter combinations obtained based on *EMRL* can be considered as a robust alternative to the parameter combinations obtained based on *MRL* if the specified mean shift  $c$  of the optimal *MRL*-based chart is in the range  $\Omega$  of the optimal *EMRL*-based chart. Furthermore, Table 7 also shows the superiority of the lower-sided exponential EWMA chart over the lower-sided exponential Shewhart chart. For example, when  $MRL_0 = 200$ , the  $MRL_1$  values of the lower-sided exponential EWMA chart are smaller than the ones of the lower-sided exponential Shewhart chart when  $c \in [0.05, 0.9]$ . The lower-sided exponential Shewhart chart is more sensitive than the EWMA type chart only when  $c < 0.05$ . Similar findings are also obtained for the upper-sided exponential EWMA chart. Due to the space limitation, these results are not presented here.

(Please Insert Table 7 to Here)

## 5 The Effect of Parameter Estimation on the One-Sided Exponential EWMA Charts

Parameter estimation is an important research issue in control charts because the selection of Phase I samples influences the performance of control charts via the degree of accuracy in the estimator. Suppose that  $\{Z_1, Z_2, \dots, Z_m\}$  is a sample from an in-control Phase I sample, then a common estimator for  $\theta_0$  is the maximum likelihood estimator  $\hat{\theta}_0 = \frac{1}{m} \sum_{i=1}^m Z_i$ . In order to investigate the effect of parameter estimation on the  $RL$  distribution of the chart, Tables 8 and 9 show the effect of the amount of Phase I data (i.e.,  $m$ ) on the Phase II performance of the optimal lower-sided charts. Table 8 presents the in-control  $MRL$  values for the lower-sided exponential EWMA chart when  $m \in \{5, 10, 20, 30, 50, 100, +\infty\}$ . Note that the parameter combinations of  $(\lambda^-, h^-)$  are the optimal settings for the specific  $MRL_0 \in \{100, 300\}$  and  $c_{\text{opt}} \in \{0.2, 0.4, 0.6, 0.8\}$  when the process parameter is known ( $m = +\infty$ ). It is found that there is a difference between the in-control  $MRL$  values corresponding to the known parameter cases and the ones corresponding to the estimated parameter cases, especially for small values of  $\lambda^-$  and  $m$ . For example, when  $MRL_0 = 100$  and  $\lambda^- = 0.487$ , the in-control  $MRL$  of the chart are 122 and 100 corresponding to  $m = 5$  and  $m = +\infty$ , respectively. If  $\lambda^- = 0.06$ , the in-control  $MRL$  value of the chart are 174 and 100 corresponding to  $m = 5$  and  $m = +\infty$ , respectively.

Moreover, when  $MRL_0 = 300$ , for the same parameter combinations  $(\lambda^-, h^-)$  presented in Table 8, the out-of-control 5th percentile of  $RL$  distribution,  $MRL$  and 95th percentile of  $RL$  distribution calculated using Eqs. (A.11) and (A.12) in the Appendix are also presented in Table 9. It is shown that the difference of  $MRL_1$  between the known and estimated

parameters cases becomes large when  $m$  decreases. For example, when  $c_{\text{opt}} = 0.8$ , the  $MRL_1$  value equals to 78 when  $m = 5$ . If  $m$  increases from 5 up to 100, the  $MRL_1$  value decreases from 78 down to 55, which is more close to the  $MRL_1 = 54$  corresponding to the known parameter case. It is obvious that the  $MRL_1$  value of the chart with estimated parameter tends to be close to their known parameter counterpart, as  $m$  increases. Furthermore, it can also be noted that as the  $c_{\text{opt}}$  decreases, the difference of  $MRL_1$  between the known and estimated parameters cases decreases. This fact implies that the effect of the parameter estimation becomes small when a large shift happens in the process. In addition, the 5th percentile of the  $RL$  distribution increases and the 95th percentile of the  $RL$  distribution decreases when  $m$  increases.

Due to the space limitation, the results for the upper-sided exponential EWMA chart are not presented here. Based on the results presented above, the  $RL$  distribution of the chart is obviously affected by the estimated parameter. To obtain a close performance to the chart with known parameter, the number  $m$  of Phase I samples should be large enough to minimize the effect of parameter estimation.

(Please Insert Tables 8–9 to Here)

## 6 Conclusions and Recommendations

Since the shape of the  $RL$  distributions of control charts changes with shift size, this paper studied the properties of the exponential EWMA chart based on  $MRL$  and  $EMRL$  when the process shift sizes are either known or unknown, respectively. For an increase or a decrease in the process mean, the optimal design procedures of the upper- and lower-sided

exponential EWMA charts based on  $MRL$  have been presented. In practice, when the shift size is unknown in advance, based on  $EMRL$ , the optimal design procedures for both the upper- and lower-sided charts are also presented. Moreover, through the comparisons of the  $MRL$ -based one-sided exponential EWMA chart and the Shewhart chart, it is noted that the EWMA type chart generally performs better than the Shewhart type chart, except for large shift sizes in the process. As this work is based on the univariate variable, future research works may be extended to the chart for monitoring the multivariate exponential distributed processes.

## Acknowledgments

The authors are grateful to the anonymous reviewers for their useful and detailed comments which contributed to the improvement of this article. This work are supported by National Natural Science Foundation of China No. 71802110, Natural Science Foundation of JiangSu Province No. BK20170894, Humanity and Social Science Foundation of Ministry of Education of China No. 19YJA630061, Key Research Base of Philosophy and Social Sciences in Jiangsu-Information Industry Integration Innovation and Emergency Management Research Center.

## Appendix

The  $RL$  properties of one-sided exponential EWMA charts can be obtained using the Markov chain approach. The states of the Markov chain are defined by partitioning the control

interval into a finite set of sub-intervals and by considering the midpoint value within each sub-interval to approximate the value of the corresponding statistic. Assuming a discrete-time Markov chain with  $N$  transient states, the transition probability matrix  $\mathbf{Q}$  is given as follows:

$$\mathbf{Q} = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ p_{N,1} & p_{N,2} & \cdots & p_{N,N} \end{pmatrix}, \quad (\text{A.1})$$

where  $p_{i,j}$  is the probability of a transition from state  $i$  to state  $j$  in one step. Let  $\mathbf{q}$  be the initial probabilities associated with the  $N$  transient states, i.e.,  $\mathbf{q} = (q_1, q_2, \dots, q_N)^\top$ . As stated in Latouche and Ramaswami (1999) or Neuts (1981), the  $RL$  of the one-sided exponential EWMA chart is a Discrete Phase-type (DPH) random variable of parameters  $(\mathbf{Q}, \mathbf{q})$ , the probability mass function (p.m.f.)  $f_{RL}(\ell|\mathbf{Q}, \mathbf{q})$  and the c.d.f.  $F_{RL}(\ell|\mathbf{Q}, \mathbf{q})$  of the  $RL$  distributions are given as follows:

$$f_{RL}(\ell|\mathbf{Q}, \mathbf{q}) = \mathbf{q}^\top \mathbf{Q}^{\ell-1} \mathbf{r}, \quad (\text{A.2})$$

$$F_{RL}(\ell|\mathbf{Q}, \mathbf{q}) = 1 - \mathbf{q}^\top \mathbf{Q}^\ell \mathbf{1}. \quad (\text{A.3})$$

where  $\mathbf{r} = (\mathbf{1} - \mathbf{Q}\mathbf{1})$  with  $\mathbf{1} = (1, 1, \dots, 1)^\top$ . The  $ARL$  of the one-sided exponential EWMA chart is

$$ARL = \mathbf{q}^\top (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}. \quad (\text{A.4})$$

Following Gan (1993), the  $100\gamma$ th ( $0 < \gamma < 1$ ) percentile of the  $RL$  distribution, denoted by  $\ell_\gamma$  is defined as

$$F_{RL}(\ell_\gamma - 1 | \mathbf{Q}, \mathbf{q}) \leq \gamma \quad \text{and} \quad F_{RL}(\ell_\gamma | \mathbf{Q}, \mathbf{q}) > \gamma, \quad (\text{A.5})$$

Moreover, the  $MRL$  of the one-sided exponential EWMA chart can be computed by setting  $\gamma = 0.5$ ,

$$F_{RL}(MRL - 1 | \mathbf{Q}, \mathbf{q}) \leq 0.5 \quad \text{and} \quad F_{RL}(MRL | \mathbf{Q}, \mathbf{q}) > 0.5. \quad (\text{A.6})$$

For the upper-sided exponential EWMA chart, the interval between the reflecting boundary  $B^+$  and the control limit  $h^+$  is divided into  $N$  subintervals, each with width  $\Delta^+ = \frac{h^+ - B^+}{N}$ . When the number  $N$  of subintervals is sufficiently large ( $N = 500$  in this paper), this finite approach provides an effective method that allows the  $RL$  properties to be accurately evaluated. The statistic  $Q_t^+$  is said to be in transient state  $j$  at time  $t$  if  $H_j^+ - \frac{\Delta^+}{2} \leq Q_t^+ < H_j^+ + \frac{\Delta^+}{2}$ , where  $H_j^+ = B^+ + (j - \frac{1}{2})\Delta^+$  ( $j = 1, 2, \dots, N$ ) represents the midpoint of the  $j$ th sub-interval. The statistic  $Q_t^+$  is in the absorbing state if  $Q_t^+$  falls outside the control limits. Then, the transition probability  $p_{i,j}$  is given as:

$$p_{i,j} = \begin{cases} \Pr(Q_t^+ < H_1^+ + \frac{\Delta^+}{2} | Q_{t-1}^+ = H_i^+) & \text{for } j = 1 \\ \Pr(H_j^+ - \frac{\Delta^+}{2} \leq Q_t^+ < H_j^+ + \frac{\Delta^+}{2} | Q_{t-1}^+ = H_i^+) & \text{for } j = 2, \dots, N \end{cases} \quad (\text{A.7})$$

After some algebraic calculations, we can obtain

$$p_{i,j} = \begin{cases} F\left(\frac{B^+ + \Delta^+ - (1-\lambda^+)H_i^+}{c\lambda^+}\right) & \text{for } j = 1 \\ F\left(\frac{H_j^+ + \frac{\Delta^+}{2} - (1-\lambda^+)H_i^+}{c\lambda^+}\right) - F\left(\frac{H_j^+ - \frac{\Delta^+}{2} - (1-\lambda^+)H_i^+}{c\lambda^+}\right) & \text{for } j = 2, \dots, N \end{cases} \quad (\text{A.8})$$



where  $F(\cdot)$  is c.d.f. of the exponential distribution (see Eq. (2)).

Similarly, the transition probability  $p_{i,j}$  of the lower-sided exponential EWMA chart can be obtained:

$$p_{i,j} = \begin{cases} 1 - F\left(\frac{B^- - \Delta^- - (1-\lambda^-)H_i^-}{c\lambda^-}\right) & \text{for } j = 1 \\ F\left(\frac{H_j^- + \frac{\Delta^-}{2} - (1-\lambda^-)H_i^-}{c\lambda^-}\right) - F\left(\frac{H_j^- - \frac{\Delta^-}{2} - (1-\lambda^-)H_i^-}{c\lambda^-}\right) & \text{for } j = 2, \dots, N \end{cases} \quad (\text{A.9})$$

where  $H_j^- = B^- - (j - \frac{1}{2})\Delta^-$  ( $j = 1, 2, \dots, N$ ) represents the midpoint of the  $j$ th sub-interval and  $\Delta^- = \frac{B^- - h^-}{N}$ . By using Eq. (A.6), one can easily compute the *MRL* of the upper- and lower-sided exponential EWMA charts.

If the process parameter is estimated, let  $w = \theta_0/\hat{\theta}_0$  represents the ratio of the actual in-control mean to its estimator. It is known that  $w$  is distributed according to an inverse gamma distribution, denoted as  $\text{IG}(m, m)$ , and the p.d.f. of  $w$  is,

$$f_W(w) = \frac{m^m}{(m-1)!} w^{-m-1} e^{-\frac{m}{w}}. \quad (\text{A.10})$$

Since  $w$  is an independent random variable, the unconditional c.d.f.  $F_{RL}(\ell|\hat{\mathbf{Q}}, \mathbf{q})$  and the *MRL* of the *RL* distribution can be obtained as follows:

$$F_{RL}(\ell|\hat{\mathbf{Q}}, \mathbf{q}) = \int_0^{+\infty} (1 - \mathbf{q}^\top \hat{\mathbf{Q}}^\ell \mathbf{1}) f_W(w) dw. \quad (\text{A.11})$$

$$F_{RL}(MRL - 1|\hat{\mathbf{Q}}, \mathbf{q}) \leq 0.5 \quad \text{and} \quad F_{RL}(MRL|\hat{\mathbf{Q}}, \mathbf{q}) > 0.5. \quad (\text{A.12})$$

where the *conditional* transition probability matrix  $\hat{\mathbf{Q}} = [\hat{p}_{i,j}]_{N \times N}$  is computed as follows:

- For the upper-sided exponential EWMA chart with estimated parameters

$$\hat{p}_{i,j} = \begin{cases} F\left(\frac{B^+ + \Delta^+ - (1-\lambda^+)H_i^+}{wc\lambda^+}\right) & \text{for } j = 1 \\ F\left(\frac{H_j^+ + \frac{\Delta^+}{2} - (1-\lambda^+)H_i^+}{wc\lambda^+}\right) - F\left(\frac{H_j^+ - \frac{\Delta^+}{2} - (1-\lambda^+)H_i^+}{wc\lambda^+}\right) & \text{for } j = 2, \dots, N \end{cases} \quad (\text{A.13})$$

- For the lower-sided exponential EWMA chart with estimated parameters

$$\hat{p}_{i,j} = \begin{cases} 1 - F\left(\frac{B^- - \Delta^- - (1-\lambda^-)H_i^-}{wc\lambda^-}\right) & \text{for } j = 1 \\ F\left(\frac{H_j^- + \frac{\Delta^-}{2} - (1-\lambda^-)H_i^-}{wc\lambda^-}\right) - F\left(\frac{H_j^- - \frac{\Delta^-}{2} - (1-\lambda^-)H_i^-}{wc\lambda^-}\right) & \text{for } j = 2, \dots, N \end{cases} \quad (\text{A.14})$$

## References

- Ali, S., Pievatolo, A., and Göb, R. (2016). An overview of control charts for high-quality processes. *Quality and Reliability Engineering International*, 32(7):2171–2189.
- Castagliola, P., Celano, G., and Psarakis, S. (2011). Monitoring the coefficient of variation using EWMA charts. *Journal of Quality Technology*, 43(3):249–265.
- Celano, G., Castagliola, P., Fichera, S., and Nenes, G. (2013). Performance of t control charts in short runs with unknown shift sizes. *Computers & Industrial Engineering*, 64(1):56–68.
- Chandrasekaran, S., English, J. R., and Disney, R. L. (1995). Modeling and analy-

- sis of EWMA control schemes with variance-adjusted control limits. *IIE Transactions*, 27(3):282–290.
- Doğu, E. (2012). Monitoring time between medical errors to improve health-care quality. *International Journal for Quality Research*, 6(2):151–157.
- Domangue, R. and Patch, S. C. (1991). Some omnibus exponentially weighted moving average statistical process monitoring schemes. *Technometrics*, 33(3):299–313.
- Gan, F. F. (1991). Computing percentage points of run length of an exponentially weighted moving average control chart. *Journal of Quality Technology*, 23(4):359–365.
- Gan, F. F. (1993). An optimal design of EWMA control charts based on median run length. *Journal of Statistical Computation and Simulation*, 45(3-4):169–184.
- Gan, F. F. (1994). Design of optimal exponential CUSUM control charts. *Journal of Quality Technology*, 26(2):109–124.
- Gan, F. F. (1998). Designs of one- and two-sided exponential EWMA charts. *Journal of Quality Technology*, 30(1):55–69.
- Gan, F. F. and Chang, T. C. (2000). Computing average run lengths of exponential EWMA charts. *Journal of Quality Technology*, 32(2):183–187.
- Haq, A. (2019). A new nonparametric synthetic EWMA control chart for monitoring process mean. *Communications in Statistics-Simulation and Computation*, 48(6):1665–1676.
- Jones, L. A. and Champ, C. W. (2002). Phase I control charts for times between events. *Quality and Reliability Engineering International*, 18(6):479–488.

- Khan, N., Aslam, M., and Jun, C. H. (2016). A EWMA control chart for exponential distributed quality based on moving average statistics. *Quality and Reliability Engineering International*, 32(3):1179–1190.
- Khoo, M. B. C. (2004). Performance measures for the Shewhart control chart. *Quality Engineering*, 16(4):585–590.
- Khoo, M. B. C., Wong, V. H., Wu, Z., and Castagliola, P. (2011). Optimal designs of the multivariate synthetic chart for monitoring the process mean vector based on median run length. *Quality and Reliability Engineering International*, 27(8):981–997.
- Knoth, S. (2015). Run length quantiles of EWMA control charts monitoring normal mean or/and variance. *International Journal of Production Research*, 53(15):4629–4647.
- Kumar, N. and Baranwal, A. (2019). Design and implementation of  $q$  th quantile-unbiased  $tr$ -chart for monitoring times between events. *Quality and Reliability Engineering International*.
- Latouche, G. and Ramaswami, V. (1999). *Introduction to matrix analytic methods in stochastic modeling*. ASA-SIAM, Philadelphia.
- Lee, M. H. and Khoo, M. B. C. (2006). Optimal statistical design of a multivariate EWMA chart based on ARL and MRL. *Communication in Statistics-Simulation and Computation*, 35(3):831–847.
- Lee, M. H. and Khoo, M. B. C. (2017a). Optimal design of synthetic  $np$  control chart based on median run length. *Communication in Statistics-Theory and Methods*, 46(17):8544–8556.

- Lee, M. H. and Khoo, M. B. C. (2017b). Optimal designs of multivariate synthetic  $|s|$  control chart based on median run length. *Communications in Statistics-Theory and Methods*, 46(6):3034–3053.
- Liu, J. Y., Xie, M., Goh, T. N., and Chan, L. Y. (2007). A study of EWMA chart with transformed exponential data. *International Journal of Production Research*, 45(3):743–763.
- Liu, J. Y., Xie, M., Goh, T. N., and Sharma, P. R. (2006). A comparative study of exponential time between events charts. *Quality Technology & Quantitative Management*, 3(3):347–359.
- Lucas, J. M. (1985). Counted data CUSUM's. *Technometrics*, 27(2):129–144.
- Luo, P., Devol, T. A., and Sharp, J. L. (2012). CUSUM analyses of time-interval data for online radiation monitoring. *Health Physics*, 102(6):637–645.
- Maravelakis, P. E., Castagliola, P., and Khoo, M. B. C. (2019). Run length properties of run rules EWMA chart using integral equations. *Quality Technology & Quantitative Management*, 16(2):129–139.
- Neuts, M. F. (1981). *Matrix-geometric solutions in stochastic models: an algorithmic approach*. Dover Publications Inc, New York.
- Nyau, S. Y., Lee, M. H., and Wong, M. L. D. (2017). Optimal statistical design of variable sample size multivariate exponentially weighted moving average control chart based on median run-length. *Quality Technology & Quantitative Management*, 14(4):478–495.

- Oakland, J. S. (2007). *Statistical Process Control*. Routledge, London, 6th edition.
- Ozsan, G., Testik, M. C., and Weiß, C. H. (2010). Properties of the exponential EWMA chart with parameter estimation. *Quality and Reliability Engineering International*, 26(6):555–569.
- Pehlivan, C. and Testik, M. C. (2010). Impact of model misspecification on the exponential EWMA charts: a robustness study when the time-between-events are not exponential. *Quality and Reliability Engineering International*, 26(2):177–190.
- Raza, S. M. M., Riaz, M., and Ali, S. (2016). EWMA control chart for poisson-exponential lifetime distribution under type I censoring. *Quality and Reliability Engineering International*, 32(3):995–1005.
- Reynolds, M. R., Amin, R. W., and Arnold, J. C. (1990). CUSUM charts with variable sampling intervals. *Technometrics*, 32(4):371–384.
- Reynolds, M. R. and Stoumbos, Z. G. (2004). Should observations be grouped for effective process monitoring? *Journal of Quality Technology*, 36(4):343–366.
- Ryu, J. H., Wan, G., and Kim, S. (2010). Optimal design of a CUSUM chart for a mean shift of unknown size. *Journal of Quality Technology*, 42(3):311–326.
- Santiago, E. and Smith, J. (2013). Control charts based on the exponential distribution: adapting runs rules for the  $t$  chart. *Quality Engineering*, 25(2):85–96.
- Shu, L. J., Jiang, W., and Wu, S. J. (2007). A one-sided EWMA control chart for monitoring process means. *Communications in Statistics-Simulation and Computation*, 36(4):901–920.

- Sparks, R. S. (2000). CUSUM charts for signalling varying location shifts. *Journal of Quality Technology*, 32(2):157–171.
- Tang, A. A., Castagliola, P., Sun, J. S., and Hu, X. L. (2019). Optimal design of the adaptive EWMA chart for the mean based on median run length and expected median run length. *Quality Technology & Quantitative Management*, 16(4):439–458.
- Teoh, W. L., Chong, J. K., Khoo, M. B. C., Castagliola, P., and Yeong, W. C. (2017). Optimal designs of the variable sample size  $\bar{X}$  chart based on median run length and expected median run length. *Quality and Reliability Engineering International*, 33(1):121–134.
- Teoh, W. L., Khoo, M. B. C., Castagliola, P., and Lee, M. H. (2016). The exact run length distribution and design of the Shewhart chart with estimated parameters based on median run length. *Communications in Statistics-Simulation and Computation*, 45(6):2081–2103.
- Waldmann, K. H. (1986). Bounds for the distribution of the run length of geometric moving average charts. *Journal of the Royal Statistical Society*, 35(2):151–158.
- Wu, Z., Shamsuzzaman, M., and Pan, E. S. (2004). Optimization design of control charts based on Taguchi’s loss function and random process shifts. *International Journal of Production Research*, 42(2):379–390.
- Yang, J., Yu, H., Cheng, Y., and Xie, M. (2015). Design of exponential control charts based on average time to signal using a sequential sampling scheme. *International Journal of Production Research*, 53(7):2131–2145.

- You, H. W., Khoo, M. B. C., Castagliola, P., and Qu, L. (2016). Optimal exponentially weighted moving average  $\bar{X}$  charts with estimated parameters based on median run length and expected median run length. *International Journal of Production Research*, 54(17):1–22.
- Zhang, C. W., Xie, M., and Goh, T. N. (2005). Economic design of exponential charts for time between events monitoring. *International Journal of Production Research*, 43(23):5019–5032.
- Zhang, C. W., Xie, M., and Goh, T. N. (2006). Design of exponential control charts using a sequential sampling scheme. *IIE Transactions*, 38(12):1105–1116.



Table 1: The percentiles of the  $RL$  distribution and  $ARL$  for the lower-sided exponential EWMA chart when  $ARL_0 = 500$

$(\lambda^-, h^-)$	$c$	$ARL$	5th	10th	20th	30th	40th	50th	60th	70th	80th	90th
(0.05,0.6861)	1.0	500.1	39	65	122	187	262	351	459	599	797	1134
	0.8	74.8	19	23	31	39	48	58	71	87	110	149
	0.6	26.6	13	15	17	19	21	24	26	30	35	43
	0.4	14.9	10	11	12	13	13	14	15	16	17	20
	0.2	10.2	9	9	9	10	10	10	10	10	11	12
(0.10,0.5450)	1.0	500.2	35	62	119	185	260	350	459	600	799	1139
	0.8	91.0	16	21	30	41	53	67	85	107	139	193
	0.6	27.8	11	12	15	17	20	23	27	32	38	49
	0.4	13.5	8	9	10	11	12	12	13	15	16	19
	0.2	8.5	7	7	8	8	8	8	9	9	9	10
(0.20,0.3793)	1.0	500.2	32	59	117	183	259	349	459	601	801	1143
	0.8	122.1	14	20	34	49	66	87	113	145	192	271
	0.6	35.0	9	11	14	18	22	27	33	41	52	70
	0.4	13.6	7	7	8	9	11	12	13	15	18	22
	0.2	7.2	5	6	6	6	7	7	7	8	8	9
(0.40,0.2045)	1.0	500.0	30	57	115	181	258	348	459	601	802	1146
	0.8	175.4	14	23	43	66	92	123	161	210	279	398
	0.6	57.1	8	10	17	24	32	41	53	68	89	125
	0.4	18.5	5	6	8	10	12	14	17	21	27	36
	0.2	6.9	4	4	5	5	6	6	7	8	9	10
(0.60,0.1091)	1.0	499.6	29	55	114	180	257	347	458	601	802	1146
	0.8	221.2	15	26	52	81	115	154	203	266	354	505
	0.6	86.0	8	12	22	33	46	61	79	103	136	194
	0.4	28.8	5	6	9	13	16	21	27	34	44	62
	0.2	8.5	3	4	4	5	6	7	8	10	12	15
(0.80,0.0492)	1.0	500.6	28	55	114	180	257	348	459	602	804	1150
	0.8	267.5	16	30	62	97	138	186	245	322	429	613
	0.6	125.3	9	15	30	46	65	88	115	150	200	285
	0.4	48.2	5	7	13	19	26	34	44	57	76	108
	0.2	13.4	3	4	5	6	8	10	13	16	20	28
(1.00,0.0020)	1.0	500.5	26	53	112	179	256	347	459	602	805	1152
	0.8	400.5	21	43	90	143	205	278	367	482	644	922
	0.6	300.5	16	32	67	108	154	208	275	362	483	691
	0.4	200.5	11	22	45	72	103	139	184	241	322	461
	0.2	100.5	6	11	23	36	52	70	92	121	161	231

Table 2: The percentiles of the  $RL$  distribution and  $ARL$  for the upper-sided exponential EWMA chart when  $ARL_0 = 500$

$(\lambda^+, h^+)$	$c$	$ARL$	5th	10th	20th	30th	40th	50th	60th	70th	80th	90th
(0.05,1.4167)	1.00	500.0	31	58	116	182	258	348	459	601	801	1144
	1.04	306.7	23	39	74	114	160	215	282	368	489	696
	1.40	35.5	7	9	14	18	23	28	34	42	53	72
	5.00	3.4	1	1	2	2	3	3	4	4	5	6
	10.00	2.1	1	1	1	1	2	2	2	2	3	4
(0.10,1.7141)	1.00	500.0	29	56	114	181	257	348	458	601	803	1147
	1.04	329.5	21	39	77	120	170	230	302	396	528	753
	1.40	39.6	5	8	13	18	23	30	37	47	61	84
	5.00	3.2	1	1	2	2	2	3	3	4	5	6
	10.00	1.9	1	1	1	1	1	2	2	2	3	3
(0.20,2.2378)	1.00	500.0	27	54	113	180	256	347	458	602	804	1149
	1.04	353.1	20	39	81	127	181	245	324	425	567	810
	1.40	48.4	5	8	13	19	26	35	45	58	76	108
	5.00	3.1	1	1	1	2	2	3	3	4	4	6
	10.00	1.8	1	1	1	1	1	2	2	2	3	3
(0.40,3.2213)	1.00	500.0	27	54	112	179	256	347	458	602	804	1150
	1.04	374.7	20	40	84	134	192	260	343	451	602	861
	1.40	63.0	4	8	15	23	33	44	58	76	101	143
	5.00	3.1	1	1	1	2	2	2	3	4	4	6
	10.00	1.8	1	1	1	1	1	1	2	2	2	3
(0.60,4.2015)	1.00	500.0	26	53	112	179	256	347	458	602	804	1150
	1.04	384.8	20	41	86	138	197	267	353	463	619	885
	1.40	73.2	5	8	17	27	38	51	67	88	117	167
	5.00	3.2	1	1	1	2	2	2	3	4	5	6
	10.00	1.8	1	1	1	1	1	1	2	2	3	3
(0.80,5.1986)	1.00	500.0	26	53	112	179	256	347	458	602	804	1151
	1.04	390.5	21	42	88	140	200	271	358	470	628	898
	1.40	80.1	5	9	18	29	41	56	73	96	129	184
	5.00	3.3	1	1	1	2	2	2	3	4	5	7
	10.00	1.8	1	1	1	1	1	1	2	2	3	3
(1.00,6.2147)	1.00	500.0	26	53	112	179	256	347	458	602	804	1151
	1.04	393.7	21	42	88	141	201	273	361	474	633	906
	1.40	84.7	5	9	19	31	44	59	78	102	136	194
	5.00	3.5	1	1	1	2	2	3	3	4	5	7
	10.00	1.9	1	1	1	1	1	1	2	2	3	3

Table 3: The  $MRL$  of the lower-sided exponential EWMA chart when  $MRL_0 = 100$

		$\lambda^- = 0.1$						$\lambda^- = 0.3$					
$c$	$B^-$ $h^-$	1.0	1.5	2.0	2.3	2.5	5.0	1.0	1.5	2.0	2.3	2.5	5.0
		0.591	0.629	0.631	0.631	0.631	0.630	0.323	0.344	0.348	0.348	0.348	0.348
1.0		100	100	100	100	100	100	100	100	100	100	100	100
0.9		59	53	52	52	52	52	67	63	62	62	62	62
0.8		37	31	31	31	31	30	44	40	39	39	39	39
0.7		24	20	20	20	20	20	29	26	25	25	25	25
0.6		17	15	14	14	14	14	19	17	16	16	16	16
0.5		13	11	11	11	11	11	13	12	11	11	11	11
0.4		10	9	9	9	9	9	9	8	8	8	8	8
0.3		8	7	7	7	7	7	7	6	6	6	6	6
0.2		7	6	6	6	6	6	5	5	5	5	5	5
0.1		6	6	5	5	5	5	4	4	4	4	4	4
		$\lambda^- = 0.5$						$\lambda^- = 0.7$					
$c$	$B^-$ $h^-$	1.0	1.5	2.0	2.3	2.5	5.0	1.0	1.5	2.0	2.3	2.5	5.0
		0.190	0.201	0.204	0.205	0.205	0.205	0.106	0.110	0.112	0.113	0.113	0.113
1.0		100	100	100	100	100	100	100	100	100	100	100	100
0.9		72	70	69	69	69	68	77	75	75	74	74	74
0.8		51	48	47	47	46	46	57	55	54	54	54	54
0.7		35	32	31	31	31	31	42	40	39	38	38	38
0.6		24	22	21	21	21	21	29	28	27	27	27	26
0.5		16	14	14	14	14	14	20	19	18	18	18	18
0.4		11	10	9	9	9	9	13	12	12	12	12	12
0.3		7	7	6	6	6	6	9	8	8	8	8	8
0.2		5	5	5	5	5	5	5	5	5	5	5	5
0.1		4	3	3	3	3	3	3	3	3	3	3	3

Table 4: The  $MRL$  of the lower-sided exponential EWMA chart and the lower-sided exponential Shewhart chart when  $MRL_0 \in \{100, 200, 300, 500\}$

		$MRL_0 = 100$				$MRL_0 = 200$					
		EWMA				Shewhart	EWMA				Shewhart
$c_{opt}$		0.8	0.6	0.4	0.2		0.8	0.6	0.4	0.2	
$\lambda^-$		0.060	0.101	0.258	0.487	$LCL =$	0.051	0.100	0.220	0.435	$LCL =$
$c$	$h^-$	0.736	0.629	0.390	0.211	0.0069	0.716	0.579	0.386	0.207	0.0035
1.00		100	100	100	100	100	200	200	200	200	200
0.90		49	52	60	69	90	82	91	108	127	180
0.80		29	31	37	46	80	43	48	60	80	160
0.70		20	20	24	31	70	28	28	35	49	140
0.60		14	14	16	21	60	20	19	21	30	120
0.50		11	11	11	14	50	15	14	14	19	100
0.40		9	9	8	9	40	12	11	10	12	80
0.30		8	7	6	6	30	10	9	8	8	60
0.20		7	6	5	5	20	9	7	6	5	40
0.10		6	5	4	3	10	8	6	5	4	20
0.05		6	5	4	3	5	7	6	5	4	10
0.03		6	5	4	3	3	7	6	5	3	6
0.01		5	5	4	3	1	7	6	4	3	2
		$MRL_0 = 300$				$MRL_0 = 500$					
		EWMA				Shewhart	EWMA				Shewhart
$c_{opt}$		0.8	0.6	0.4	0.2		0.8	0.6	0.4	0.2	
$\lambda^-$		0.052	0.089	0.196	0.407	$LCL =$	0.051	0.079	0.180	0.375	$LCL =$
$c$	$h^-$	0.688	0.580	0.392	0.206	0.0023	0.664	0.577	0.387	0.206	0.0014
1.00		300	300	300	300	300	500	500	500	500	500
0.90		111	124	150	183	270	161	179	229	288	450
0.80		54	59	77	109	240	70	78	108	162	400
0.70		33	34	42	64	210	40	41	55	90	350
0.60		23	22	25	38	180	27	26	30	49	300
0.50		17	16	16	22	150	20	18	19	28	250
0.40		14	12	11	14	120	16	14	13	16	200
0.30		11	10	9	9	90	13	11	10	10	150
0.20		10	8	7	6	60	11	10	8	7	100
0.10		8	7	6	4	30	9	8	6	5	50
0.05		8	7	5	4	15	9	8	6	4	25
0.03		8	7	5	4	9	9	7	6	4	15
0.01		8	6	5	4	3	8	7	5	4	5

Table 5: The  $MRL$  of the upper-sided exponential EWMA chart and the upper-sided exponential Shewhart chart when  $MRL_0 \in \{100, 200, 300, 500\}$

		$MRL_0 = 100$					$MRL_0 = 200$				
		EWMA				Shewhart	EWMA				Shewhart
$c$	$c_{opt}$	1.04	1.40	2.00	5.00	$UCL =$	1.04	1.40	2.00	5.00	$UCL =$
	$\lambda^+$	0.054	0.086	0.318	0.525	4.9651	0.051	0.064	0.160	0.552	5.6616
	$h^+$	1.297	1.448	2.376	3.154		1.363	1.437	1.917	3.653	
1.00		100	100	100	100	100	200	200	200	200	200
1.02		84	85	88	90	90	161	163	169	177	179
1.04		71	73	79	81	82	132	134	144	157	160
1.10		47	49	57	60	63	80	82	94	113	119
1.20		29	30	37	40	44	45	46	53	70	78
1.40		16	16	19	21	24	22	22	24	34	40
2.00		7	7	7	7	8	9	9	8	10	12
3.00		4	4	3	3	4	5	5	4	4	5
5.00		2	2	2	2	2	3	3	2	2	2
10.00		1	1	1	1	1	2	2	1	1	1
		$MRL_0 = 300$					$MRL_0 = 500$				
		EWMA				Shewhart	EWMA				Shewhart
$c$	$c_{opt}$	1.0	1.4	2.0	5.0	$UCL =$	1.0	1.4	2.0	5.0	$UCL =$
	$\lambda^+$	0.051	0.055	0.141	0.596	6.0681	0.050	0.054	0.146	0.525	6.5798
	$h^+$	1.407	1.432	1.907	4.094		1.454	1.481	2.031	4.029	
1.00		300	300	300	300	300	500	500	500	500	500
1.02		235	237	248	263	266	380	382	407	432	439
1.04		189	190	207	232	237	296	298	334	376	388
1.10		108	110	129	164	173	157	160	198	255	275
1.20		56	57	68	100	109	75	76	98	147	167
1.40		26	26	29	46	53	32	32	38	62	76
2.00		10	10	9	12	15	11	11	11	15	19
3.00		5	5	5	5	5	6	6	5	5	6
5.00		3	3	3	2	2	3	3	3	3	3
10.00		2	2	2	1	1	2	2	2	1	1

Table 6: The  $MRL$  and  $ARL$  of the lower-sided exponential EWMA chart when  $MRL_0 = 350$  and  $ARL_0 = 500$

<i>MRL</i> -based with $MRL_0 = 350$									
	$c_{opt}$	0.8		0.6		0.4		0.2	
	$\lambda^-$	0.0510		0.0830		0.1480		0.3950	
	$h^-$	0.6826		0.5857		0.4535		0.2074	
$c$		<i>MRL</i>	<i>ARL</i>	<i>MRL</i>	<i>ARL</i>	<i>MRL</i>	<i>ARL</i>	<i>MRL</i>	<i>ARL</i>
1.0		350	499.2	350	500.4	350	501.6	350	502.6
0.9		124	170.5	137	191.9	159	225.0	209	299.8
0.8		<b>58</b>	<b>75.0</b>	64	85.4	77	106.6	123	175.0
0.7		34	41.2	36	44.7	41	54.7	71	100.2
0.6		24	26.6	<b>23</b>	<b>27.1</b>	25	30.9	41	56.6
0.5		18	19.2	17	18.5	16	19.3	24	32.0
0.4		14	14.8	13	13.7	<b>12</b>	<b>13.3</b>	14	18.3
0.3		12	12.0	10	10.8	9	9.8	9	10.9
0.2		10	10.1	9	8.9	7	7.7	<b>6</b>	<b>6.9</b>
0.1		9	8.8	8	7.6	6	6.4	5	4.8
<i>ARL</i> -based with $ARL_0 = 500.0$									
	$c_{opt}$	0.8		0.6		0.4		0.2	
	$\lambda^-$	0.0501		0.0530		0.1446		0.3519	
	$h^-$	0.6859		0.6755		0.4592		0.2363	
$c$		<i>MRL</i>	<i>ARL</i>	<i>MRL</i>	<i>ARL</i>	<i>MRL</i>	<i>ARL</i>	<i>MRL</i>	<i>ARL</i>
1.0		350	499.6	350	499.5	349	499.9	348	500.0
0.9		123	169.9	125	172.0	157	222.8	202	288.6
0.8		<b>58</b>	<b>74.8</b>	59	75.7	76	105.2	115	163.5
0.7		34	41.1	34	41.4	41	53.9	65	91.6
0.6		24	26.6	<b>24</b>	<b>26.6</b>	24	30.5	37	51.1
0.5		18	19.2	18	19.1	16	19.2	22	28.9
0.4		14	14.9	14	14.7	<b>12</b>	<b>13.2</b>	13	16.9
0.3		12	12.1	12	11.9	9	9.8	9	10.4
0.2		10	10.2	10	10.0	7	7.7	<b>6</b>	<b>6.9</b>
0.1		9	8.8	9	8.7	6	6.4	5	4.9

Table 7: The  $MRL$  and  $EMRL$  of the lower-sided exponential EWMA chart and the lower-sided exponential Shewhart chart when  $MRL_0 \in \{100, 200, 300, 500\}$

		$MRL_0 = 100$				$MRL_0 = 200$					
		EWMA				Shewhart	EWMA				Shewhart
		[0.7,0.9]	[0.5,0.7]	[0.3,0.5]	[0.1,0.3]	$LCL =$	[0.7,0.9]	[0.5,0.7]	[0.3,0.5]	[0.1,0.3]	$LCL =$
$c$	$\Omega$ $\lambda^-$ $h^-$	0.050	0.093	0.244	0.510	0.0069	0.050	0.087	0.204	0.414	0.0035
	$EMRL_1$	30.3	14.7	8.3	4.7	-	46.7	19.6	10.3	5.6	-
0.90		48	52	60	69	90	82	89	106	126	180
0.80		29	30	37	47	80	43	46	58	78	160
0.70		19	20	23	32	70	27	28	34	48	140
0.60		15	14	16	21	60	19	19	21	29	120
0.50		12	11	11	14	50	15	14	14	18	100
0.40		10	9	8	9	40	12	11	10	12	80
0.30		8	7	6	6	30	10	9	8	8	60
0.20		7	6	5	5	20	9	8	6	5	40
0.10		6	6	4	3	10	8	7	5	4	20
0.05		6	5	4	3	5	7	6	5	4	10
0.03		6	5	4	3	3	7	6	5	4	6
0.01		6	5	4	3	1	7	6	5	3	2
		$MRL_0 = 300$				$MRL_0 = 500$					
		EWMA				Shewhart	EWMA				Shewhart
		[0.7,0.9]	[0.5,0.7]	[0.3,0.5]	[0.1,0.3]	$LCL =$	[0.7,0.9]	[0.5,0.7]	[0.3,0.5]	[0.1,0.3]	$LCL =$
$c$	$\Omega$ $\lambda^-$ $h^-$	0.050	0.085	0.179	0.370	0.0023	0.050	0.070	0.157	0.330	0.0014
	$EMRL_1$	59.5	22.8	11.6	6.2	-	79.5	27.1	13.3	6.9	-
0.90		111	122	146	178	270	160	173	219	276	450
0.80		54	59	74	104	240	70	75	101	150	400
0.70		33	33	41	60	210	40	40	51	81	350
0.60		23	22	24	35	180	27	26	29	44	300
0.50		17	16	16	21	150	20	19	18	25	250
0.40		14	12	11	13	120	16	14	13	15	200
0.30		11	10	9	9	90	13	12	10	10	150
0.20		10	8	7	6	60	11	10	8	7	100
0.10		9	7	6	5	30	9	8	6	5	50
0.05		8	7	5	4	15	9	8	6	4	25
0.03		8	7	5	4	9	9	8	6	4	15
0.01		8	6	5	4	3	8	8	6	4	5

Table 8: The in-control  $MRL$  values for the lower-sided exponential EWMA chart with estimated parameters when  $m \in \{5, 10, 20, 30, 50, 100, +\infty\}$  and  $MRL_0 \in \{100, 300\}$

$m$	$c_{opt}$ $\lambda^-$ $h^-$	$MRL_0 = 100$				$MRL_0 = 300$			
		0.8	0.6	0.4	0.2	0.8	0.6	0.4	0.2
		0.060	0.101	0.258	0.487	0.052	0.089	0.196	0.407
0.736	0.629	0.390	0.211	0.688	0.580	0.392	0.206		
5		174	159	135	122	650	554	453	387
10		127	123	114	109	421	392	356	332
20		111	109	105	104	346	334	319	311
30		106	105	103	102	326	318	310	305
50		103	103	101	101	312	307	303	302
100		101	101	101	100	303	301	300	300
$+\infty$		100	100	100	100	300	300	300	300

Table 9: The out-of-control 5th, 50th and 95th percentiles of  $RL$  distribution for the lower-sided exponential EWMA chart with estimated parameters when  $m \in \{5, 10, 20, 30, 50, 100, +\infty\}$  and  $MRL_0 = 300$

$m$	$c_{opt}$	0.8			0.6			0.4			0.2		
		5th	$MRL$	95th	5th	$MRL$	95th	5th	$MRL$	95th	5th	$MRL$	95th
	$\lambda^-$	0.052			0.089			0.196			0.407		
	$h^-$	0.688			0.580			0.392			0.206		
5		13	78	$> 10^4$	9	26	87787	6	12	392	4	6	32
10		14	64	$> 10^4$	10	24	994	6	12	74	4	6	18
20		15	58	5566	10	23	181	6	12	40	4	6	14
30		16	57	1524	10	23	115	6	11	34	4	6	13
50		16	56	595	10	22	83	6	11	30	4	6	13
100		17	55	312	10	22	67	6	11	27	4	6	12
$+\infty$		17	54	174	11	22	55	6	11	25	4	6	12