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Explicit Dead-Time Compensation in Linear Parameter Varying Model Predictive Control^{*}

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Abstract: Model Predictive Control (MPC) is able to directly deal with dead-time (DT) phenomena. Nevertheless, implicit delay compensation heavily affects computational aspects of such control algorithm. The corresponding stability and feasibility analyses also become numerically tougher. Recently, the Linear Parameter Varying (LPV) toolkit has become a popular representation for complex, nonlinear dynamics, with corresponding MPC algorithms being developed over the last few years. In this context, we propose a novel MPC scheme for DT LPV systems, which is able to ensure input-to-state stability, recursive feasibility, and constraint satisfaction. The method operates under a DT compensation framework and uses a DT-free model. Thereby, the DT is explicitly accounted for, and thus, augmented state-space models are avoided. The scheme also offers robustness properties, which are synthesised w.r.t. model-mismatches and delay estimation uncertainties. A solar collector benchmark example is used to illustrate the advantages of the method, which is compared against a regular LPV MPC algorithm (with standard implicit DT compensation).

Keywords: Dead-time, Model Predictive Control, Linear Parameter Varying Systems.

1. INTRODUCTION

Model Predictive Control (MPC) is very well recognised and widely applied for constrained processes. Over the last decade, there has been astounding progress on enhancing the real-time capabilities of MPC algorithms (Gros et al., 2020). In parallel to this advance, the Linear Parameter Varying (LPV) toolkit has been widely popularised (Sename et al., 2013). For many nonlinear and complex systems, LPV realisations stand for an elegant modelling strategy, with no nonlinear state transitions but rather linear maps scheduled by (known, bounded) parameters ρ . Accordingly, recent progress has been established on LPV MPC algorithms, refer to (Morato et al., 2020a) and references therein.

Dead-time (DT) phenomena are observed in many real-life situations, such as when time is required to transport energy, mass, or information, or due to the accumulation of lags when a series of low-order systems are connected in series, as argues Normey-Rico (2007). With regard to this matter, we note that DT compensation is one of the inherent advantages of MPC algorithms (Santos et al., 2012). In these controllers, DTs are taken into account by systematically augmenting the prediction model, using additional state variables that linearly relate to the magnitude of the DT (da Silva et al., 2020). Nevertheless, recur-

sive feasibility and stability properties of the closed-loop system are typically demonstrated using terminal ingredients based on delay-free (nominal order) models (Köhler et al., 2021) or via complex delay-dependent Lyapunov-Krasovskii maps (Kharitonov and Zhabko, 2003). Therefore, it is reasonable to conclude that implicit delay compensation in MPC is not very practical in some cases, especially since: (i) the resulting algorithm has an increase on numerical demand, which complicates real-time applications; and (ii) stability-related tools, such as terminal invariant sets, also become harder to determine (Normey-Rico et al., 2015).

A widely used solution for the control of delayed processes is the use of DT compensators (DTCs). Based on the seminal work of Smith (Smith, 1957), these schemes are used to eliminate the DT from the closed-loop characteristic equation. DTCs have been studied and diversified over the last six decades, with recent extensions for nonlinear (Lima et al., 2015) and time-varying DT (Morato and Normey-Rico, 2021) cases. Recent debate on the topic clearly indicates that these scheme are still relevant (Skogestad, 2018; da Silva et al., 2020).

Taking into account the previous discussion, our goal in this paper is to assess on how MPC algorithms can handle DT LPV systems by explicitly compensating the effect of DT under a corresponding DTC. Specifically, we build upon the ideas of (Santos et al., 2012), generalising

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the prior result for the LPV case, while exploiting the parameter-dependent stability conditions recently developed by the authors (Morato et al., 2020b, 2021a). The contributions are as follows:

- An explicit DTC scheme is presented for the case of delayed LPV processes (Sec. 2).
- A corresponding LPV MPC scheme is proposed with parameter-dependent terminal ingredients (Sec. 3).
- Robustness analyses regarding DT estimation errors are given (Sec. 4).
- A solar collector benchmark simulation is given in order to illustrate and discuss the advantages of the proposed method when compared to a standard LPV MPC algorithm with an implicit DT model (Sec. 5).

Notation. The index set $\mathbb{N}_{[a,b]}$ represents $\{i \in \mathbb{N} \mid a \leq i \leq b\}$, with $0 \leq a \leq b$. The identity matrix of size j is denoted as I_j ; $I_{\{j\}}$ denotes the j -th row of I_j ; $\text{col}\{\cdot\}$ denotes the vectorization of the entries and $\text{diag}\{v\}$ denotes the diagonal matrix generated with the line vector v . The prediction of a given variable $v(k)$ at time instant $k+i$, computed based on the information available at instant k , is denoted as $v(k+i|k)$. \mathcal{K}_∞ -class scalar functions are positive, increasing, and pass through the origin. A *real* \mathcal{C}^1 function $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is such that it is differentiable and holds continuous first derivatives. $\Pi_{i=a}^b(A(\rho(i)))$ denotes the left-side matrix product from $i = a$ to $i = b$, i.e. $A(\rho(a))A(\rho(a+1)) \dots A(\rho(b))$. The Minkowski set addition is defined by $A \oplus B := \{a + b \mid a \in A, b \in B\}$, while the Pontryagin set difference is defined by $A \ominus B := \{a \mid a \oplus B \subseteq A\}$.

2. PRELIMINARIES

2.1 Process model

Consider the following discrete-time LPV system:

$$\begin{cases} x(k+1) = A(\rho(k))x(k) + B(\rho(k))u(k-d), \\ \rho(k) \in \mathcal{P}, \end{cases} \quad (1)$$

being $k \in \mathbb{N}$ the sampling instant, $x(k) \in \mathbb{R}^{n_x}$ the vector of the current state variables, $u(k) \in \mathbb{R}^{n_u}$ the vector of the current control inputs, and $\rho(k) \in \mathbb{R}^{n_\rho}$ the current scheduling variables. This LPV system is prone to DT, since the control action $u(k)$ takes d samples to have an effect on the state dynamics.

The system in Eq. (1) is subject to state and input constraints: $x \in \mathcal{X} := \{x \in \mathbb{R}^{n_x} : H_x x \leq g_x\}$ and $u \in \mathcal{U} := \{u \in \mathbb{R}^{n_u} : H_u u \leq g_u\}$, which imply $\|x_j\| \leq \bar{x}_j, \forall j \in \mathbb{N}_{[1, n_x]}, \|u_i\| \leq \bar{u}_i, \forall i \in \mathbb{N}_{[1, n_u]}$, respectively. Slew-rate constraints are implied over the control increments, this is $\delta u \in \delta \mathcal{U} := \{\delta u \in \mathbb{R}^{n_u} : H_{\delta u} \delta u \leq g_{\delta u}\}$, where $\delta u(k+1) = u(k+1) - u(k)$. Without loss of generality, the origin is considered an equilibrium point of Eq. (1). Moreover, the state transition map is of class \mathcal{C}^1 over $\mathcal{Z} := \mathcal{X} \times \mathcal{U}, \forall \rho \in \mathcal{P}$.

Assumption 1. The scheduling variables exhibit bounded rates of variation, i.e. $\delta \rho(k+1) = (\rho(k+1) - \rho(k)) \in \delta \mathcal{P}$, with: $\delta \mathcal{P} := \{\delta \rho_j \in \mathbb{R} : \underline{\delta \rho}_j \leq \delta \rho_j \leq \bar{\delta \rho}_j, \forall j \in \mathbb{N}_{[1, n_\rho]}\}$.

Remark 1. Bounded rates of scheduling parameter variations is standard in LPV control (Mohammadpour and

Scherer, 2012). If $\delta \rho$ is unbounded, the following discussions can still be used, but rather using $\rho(k+1) \in \mathcal{P}$ (which implies in further conservatism).

2.2 Problem statement

The states x and scheduling variables ρ are measurable for all sampling instants. Therefore, we use a parameter-dependent state-feedback

$$u(k) := \kappa(\rho(k))x(k) \quad (2)$$

to ensure that the closed-loop dynamics adheres to desired specifications.

W.r.t. this matter, the main objective is, thus, to derive an MPC control law $u(k) := \kappa(\rho(k))x(k)$, based on a prediction model **without** DT, that is able to steer x to the origin (regulation), while guaranteeing input-to-state stability (ISS), recursive feasibility of the optimisation, and constraint satisfaction.

2.3 Implicit DT Compensation in LPV MPC

As exposed in (Morato et al., 2020a), the stability of an LPV system, as in Eq. (1), under state-feedback MPC laws, as in Eq. (2), boils down to the existence of terminal ingredients (a terminal offset cost, a terminal state constraint, and a local stabilising feedback). Nevertheless, in systems with DT, the following state measure $x(k+1)$ cannot be inferred from $(x(k), u(k))$. As of this, Eq. (1) cannot be used directly in the synthesis of the MPC scheme; it requires one to consider all control inputs (from $u(k-d)$ to $u(k-1)$) in order to represent the complete set of state dynamics and, thus, derive the stabilising terminal ingredients. As argued in (Lima et al., 2015), one can use an implicit dead-time model to find a DT-free realisation of Eq. (1), which is given as follows:

$$\xi(k+1) = A_d(\rho(k))\xi(k) + B_d u(k), \quad (3)$$

where $\xi(k) := [x(k)' u(k-d)' \dots u(k-1)']$ and¹:

$$A_d(\rho) := \begin{bmatrix} A(\rho) & B(\rho) & 0 & 0 & \dots & 0 \\ 0 & 0 & I_{n_u} & 0 & \dots & 0 \\ 0 & 0 & 0 & I_{n_u} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I_{n_u} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B_d := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_{n_u} \end{bmatrix}.$$

The underlying concept in using implicit DT compensation, via the augmented state-space representation in Eq. 3, is that a regular LPV model is used to conceive the stabilising terminal ingredients, with the same tools as in DT-free LPV MPC, e.g. (Köhler et al., 2019; Morato et al., 2019). Nevertheless, since ξ stores all past control signals, the extended state representation grows linearly with the length of the DT, which causes an undesirable increase of the resulting numerical demand of the MPC (in terms of numerical cost during the online implementation and also in the sense of the offline synthesis of the stabilising components).

¹ In the case of models with DTs in the scheduling parameter dependence, i.e. $B(\rho(k-d))u(k-d)$, one can use an extended scheduling variable $\Theta(k) := [\rho(k-d)', \dots, \rho(k)']$ in a similar fashion.

2.4 Explicit DT Compensation

In this paper, we follow the idea of explicit DT compensation in MPC design, as proposed in (Santos et al., 2012). The idea is to obtain a prediction for $x(k+d)$ based on the data available at the current sampling instant k . This prediction, namely $x(k+d|k)$, is directly obtained from the DT model in Eq. (1), since the current control input $u(k)$ has no effect over $x(k+j|k)$, $\forall j \in \mathbb{N}_{[1,d]}$, due to the DT phenomena. Due to this fact, $x(k+d|k)$ depends only on the prior state and scheduling parameter trajectory, and, thus, we can express this prediction as in Eq. (4). We stress that Eq. (4) can be presented in a compact form, for when the scheduling parameters do not vary, i.e. $\rho(k+j|k) = \rho$, given by:

$$x(k+d|k) = (A(\rho))^d x(k) + \sum_{j=1}^d (A(\rho))^{j-1} B(\rho) u(k-j).$$

We note, nevertheless, that the future scheduling variables $\rho(k+j|k)$, $\forall j \in \mathbb{N}_{[1,d-1]}$ are unknown, from the viewpoint of the current sample, by definition. These future scheduling variables could be estimated under a variety of strategies, as those presented in (Morato et al., 2021b; Cisneros and Werner, 2017). For ease of notation, we use the simpler formulation from (Morato et al., 2019), using $\rho(k+j|k) = \rho(k)$, $\forall j \in \mathbb{N}_{[1,d-1]}$. In Sec. 4, we assess on how this "frozen" scheduling trajectory guess affects the robustness of the controller.

Taking into account the previous discussions, we cast Eq. (4) into the explicit compact form from Eq. (5), where $\check{x}(k) := x(k+d|k)$. Note that this prediction model can be considered to generate the desired MPC feedback policy. The advantage is that, due to the DTC structure, the derived model is free from DT and does not have any additional state variable, as in the implicit DT model from Eq. (3). Thus, from this fact, a traditional LPV MPC synthesis procedure can be followed, using terminal ingredients based on this same DT-compensated model.

$$\check{x}(k+1) = A(\rho(k))\check{x}(k) + B(\rho(k))u(k) + \tilde{w}(k). \quad (5)$$

We note that the offset disturbance term $\tilde{w}(k) \in \tilde{\mathcal{W}}$ in Eq. (5) is used to bound the differences between the real system prediction for the first sample ahead of the delay, i.e. $x(k+d+1|k) = A(\rho(k))x(k+d|k) + B(\rho(k))u(k)$, and the nominal DT-compensated prediction made at the following instant, i.e. $x(k+d+1|k+1) = \check{x}(k+1)$. By including $\tilde{w}(k) = x(k+d+1|k+1) - x(k+d+1|k)$ in Eq. (5), we are able to limit the error propagation between these predictions and, thus, ensure recursive feasibility of the corresponding MPC. The explicit representation of $\tilde{w}(k)$ is given in Sec. 4, where we discuss properties of the corresponding closed-loop.

The overall control structure is given in Fig. 1. This scheme is very close to those seen in traditional DTC literature, i.e. (Normey-Rico, 2007; Normey-Rico and Camacho, 2009; Morato and Normey-Rico, 2021). In this paper, due to brevity, we do not include input disturbances, which will be dealt with in future works. We stress that corroborating analyses on the robustness properties of LPV MPC schemes against the frozen scheduling trajectory estimate

($\rho(k+j|k) = \rho(k)$, $\forall j \in \mathbb{N}_{[1,d-1]}$) are brought to focus in (Morato et al., 2021a).

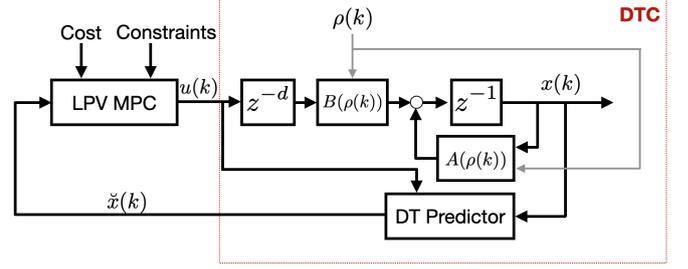


Fig. 1. Proposed LPV MPC with explicit dead-time compensation.

3. LPV MPC WITH EXPLICIT DT COMPENSATION

The LPV system from Eq. (1), now represented under the explicit DT compensated model in Eq. (5) must be regulated in such way that the state trajectories are steered to the origin, in an admissible manner (with all constraints respected).

As discussed in the prequel, we apply a state-feedback control policy, generated under the LPV MPC scheme from Fig. 1. Therefore, we now use $u(k) := \kappa(\rho(k))\check{x}(k)$, since the state feedback measure, from the viewpoint of the MPC, is given in terms of \check{x} .

Accordingly, at each sampling instant k , we measure the DT-compensated state $\check{x}(k)$ and the scheduling parameter $\rho(k)$, and solve the following optimization problem, which embeds all process constraints given in Sec. 2:

$$\begin{aligned} \min_{U_k} & V(\check{x}(k+N_p|k)) + \sum_{j=0}^{N_p-1} \ell(\check{x}(k+j|k), u(k+j|k)) \quad (6) \\ \text{s.t.} & \check{x}(k+j+1|k) = A(\rho(k))\check{x}(k+j|k) \\ & + B(\rho(k))u(k+j|k), \forall j \in \mathbb{N}_{[0, N_p-1]}, \\ & \check{x}(k+j|k) \in \mathcal{X}_{\mathcal{W}}(j), \forall j \in \mathbb{N}_{[0, N_p-1]}, \\ & u(k+j|k) \in \mathcal{U}, \forall j \in \mathbb{N}_{[0, N_p-1]}, \\ & \delta u(k+j|k) \in \delta \mathcal{U}, \forall j \in \mathbb{N}_{[1, N_p]}, \\ & \check{x}(k+N_p|k) \in \mathbf{X}_f, \end{aligned}$$

where $\ell(\check{x}, u) = \|\check{x}\|_Q^2 + \|u\|_R^2$ is a quadratic stage cost, $V(\check{x})$ is a terminal cost and \mathbf{X}_f is a terminal set. $V(\check{x})$ and \mathbf{X}_f are together named the terminal ingredients, which are used to ensure ISS. The MPC considers a moving-window horizon of N_p steps. The first entry of solution of this optimisation, namely $U_k^* = \text{col}\{u^*(k+j)\}$, $\forall j \in \mathbb{N}_{[0, N_p-1]}$ is applied to the process under state-feedback form. We note that the prediction model in this optimisation problem does not include the \tilde{w} term from Eq. (5). Thus, sets $\mathcal{X}_{\mathcal{W}}(j)$ are related to the problem uncertainties, used to tighten the state constraints over the prediction horizon. These sets are presented in Sec. 4.

We note that the prediction model in Eq. (6), used to describe the behaviour of the system along the horizon window of N_p steps is simply the forward expansion of Eq. (5). Accordingly, the horizon does not have to comprise

$$x(k+d|k) := \left(\prod_{j=0}^{d-1} A(\rho(k+j|k)) \right) x(k) + \sum_{j=1}^{d-1} \left(\prod_{m=d-1}^{d-j} A(\rho(k+m|k)) \right) B(\rho(k+d-1-j|k)) u(k-j-1) + B(\rho(k+d-1|k)) u(k-1). \quad (4)$$

the size of the DT, as in implicit strategies, which require $N_p > d$.

3.1 Terminal Ingredients

In order to ensure input-to-state stability of the closed-loop and recursive feasibility of the MPC optimisation, the cost functions must satisfy the following assumptions (Mayne et al., 2000): $\ell(x, v) \geq \beta_\ell(\|x\|), \forall x \in \mathcal{X}, \beta_\ell(\cdot) \in \mathcal{K}_\infty$ -class, $0 \leq V(x) \leq \beta_V(\|x\|), \forall x \in \mathcal{X}, \beta_V(\cdot) \in \mathcal{K}_\infty$ -class. Accordingly, in order to tacitly ensure these requirements, we use parameter-dependent terminal ingredients: an ellipsoidal terminal set $\mathbf{X}_f(\rho) := \{\check{x} \mid \check{x}' P(\rho) \check{x} \leq 1\}$ and a sub-level cost $V(\check{x}, \rho) = \check{x}' P(\rho) \check{x}$. Likewise, we consider a terminal local stabilising feedback gain $K(\rho)$, which derives from $P(\rho)$.

The terminal set \mathbf{X}_f must be a positively invariant set for the DT-compensated dynamics from Eq. (5), as provides the following Theorem.

Theorem 1. (Mayne et al., 2000)

Suppose $\exists u = K(\rho)\check{x}$. Consider that the MPC is given by Eq. (6), with a terminal state set given by $\mathbf{X}_f(\rho)$ and a terminal cost $V(\check{x}, \rho)$. Then, input-to-state stability is ensured if the following conditions hold $\forall \rho \in \mathcal{P}$:

- (C1) The origin lies in the interior of $\mathbf{X}_f(\rho)$;
- (C2) Any consecutive state to \check{x} , in closed-loop given by $(A(\rho) + B(\rho)K(\rho))\check{x} + \tilde{w}$ lies within $\mathbf{X}_f(\rho + \delta\rho), \forall \delta\rho \in \delta\mathcal{P}$ and $\tilde{w} \in \tilde{\mathcal{W}}$;
- (C3) The discrete Lyapunov equation is verified within this invariant set, this is, $\forall \check{x} \in \mathbf{X}_f(\rho)$ and $\forall \rho \in \mathcal{P}$ and $\forall \delta\rho \in \delta\mathcal{P}$: $V((A(\rho) + B(\rho)K(\rho))\check{x}, \rho + \delta\rho) - V(\check{x}, \rho) \leq -\check{x}' Q \check{x} - x'(K(\rho)' R K(\rho))\check{x}$.
- (C4) The image of the nominal feedback lies within the admissible control domain: $K(\rho)\check{x} \in \mathcal{U}, \forall \rho \in \mathcal{P}$.
- (C5) The terminal set $\mathbf{X}_f(\rho)$ is a subset of $\mathcal{X}, \forall \rho \in \mathcal{P}$.

Assuming that the initial solution of the MPC problem U_k^* is feasible, then, the MPC is recursively feasible, stabilising the state \check{x} to the origin.

Proof 1. Refer to (Mayne et al., 2000). \square

Taking into account the requirements for ISS and recursive feasibility presented in Theorem 1, the following Lemma provides a Linear Matrix Inequality (LMI) solution that generates the corresponding terminal ingredients.

Lemma 1. Terminal Ingredients (Morato et al., 2021a) The conditions (C1)-(C5) of Theorem 1 are satisfied if there exist a symmetric parameter-dependent positive definite matrix $P(\rho) : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_x \times n_x}$, a parameter-dependent rectangular matrix $W(\rho) : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_u \times n_x}$, and a scalar $0 < \lambda \in \mathbb{R}$, with $Y(\rho) = (P(\rho))^{-1} > 0$, and $W(\rho) = K(\rho)Y(\rho)$, s.t. (7)-(10) hold for all $(\rho, \delta\rho) \in \mathcal{P} \times \delta\mathcal{P}$, under $\min \log \det\{Y(\rho)\}$. Use $(\cdot)_* = (\cdot)^{-1}$.

$$\left[\begin{array}{c|c} \lambda Y(\rho) & \star \\ \hline 0 & (1-\lambda) \tilde{w} \end{array} \middle| \begin{array}{c} \star \\ \star \end{array} \right] \begin{array}{c} \star \\ \star \end{array} \geq 0, \quad (7)$$

$$\left[\begin{array}{c|c} Y(\rho) & \star \\ \hline (A(\rho)Y(\rho) + B(\rho)W(\rho)) & Y(\rho + \delta\rho) \end{array} \middle| \begin{array}{c} \star \\ \star \end{array} \right] \begin{array}{c} \star \\ \star \end{array} \geq 0, \quad (8)$$

$$\left[\begin{array}{c|c} \tilde{u}_i^2 & I_{\{i\}} W(\rho) \\ \hline \star & Y(\rho) \end{array} \right] \geq 0, \quad i \in \mathbb{N}_{[1, n_u]}, \quad (9)$$

$$\left[\begin{array}{c|c} \tilde{x}_j^2 & I_{\{j\}} Y(\rho) \\ \hline I_{\{j\}}' & Y(\rho) \end{array} \right] \geq 0, \quad j \in \mathbb{N}_{[1, n_x]}. \quad (10)$$

Proof 2. This Theorem generates a positive definite parameter dependent matrix $P(\rho)$ which is used to compute the MPC terminal ingredients $V(\cdot)$ and \mathbf{X}_f such that the conditions from Theorem 1 are fulfilled. The full demonstration is enable through consecutive Schur complements, refer to (Morato et al., 2021a). The BMI (7) ensures the positive invariance of \mathcal{X}_f w.r.t. the offset \tilde{w} in Eq. (5). LMI (8) ensures energy dissipation of the MPC cost, while LMIs (9) and (10) imply in admissible input and state trajectories, respectively. The corresponding result is that for any initial condition $\check{x}(0) \in \mathbf{X}_f$, the MPC is recursively feasible for all $k > 0$ and ensures that the closed-loop dynamics are ISS.

Remark 2. LMIs (9)-(10) in Theorem 1 require box-type constraints on each u_i and x_j . Nonetheless, these can be converted into more generic polyhedral constraints, if necessary. The BMI (7) can be converted into an LMI using simple bisection search over the optimisation plane.

Remark 3. Theorem 1 provides infinite-dimensional inequalities, which must hold $\forall \rho \in \mathcal{P}$ and $\forall \delta\rho \in \delta\mathcal{P}$. In practice, the solution can be found by enforcing the inequalities over a sufficiently dense grid of points $(\rho, \delta\rho)$ along the $\mathcal{P} \times \delta\mathcal{P}$ plane. Then, the solution can be verified over a denser grid. The parameter-dependency of P may be dropped if the system is quadratically stabilisable, but this may result in quite conservative performances.

Remark 4. Tracking objectives are out of the scope of this paper, and thus we reference to (Köchler et al., 2019; Morato and Normey-Rico, 2021). Nevertheless, it is worth mentioning that the proposed scheme can be used for tracking under slight modifications. Specifically, instead of steering \check{x} to the origin, the MPC should be set to steer (\check{x}, u) to some steady-state target $(x_r, u_r) \in \mathcal{Z}$. In this case, an offline reference selector can be used to determine such possible steady-state pairs. Moreover, the MPC stage cost and terminal ingredients should be stated in terms of the tracking error dynamics $\check{x}(k+1) - x_r$.

4. ROBUSTNESS ANALYSIS

In this Section, we analyse the stability of the corresponding closed-loop system, as well as the robustness of the controller against uncertainties on the delay. First, we demonstrate the model uncertainty that arises when

prediction the state trajectories after the DT at sampling instants $k + j$ and $k + j + 1$. Then, we discuss on how the "frozen" scheduling trajectory guess affects the prediction error of the system. Finally, robustness, and recursive feasibility of the optimisation are demonstrated by the means of constraints tightening.

4.1 Model Uncertainty

As previously detailed, we include $\tilde{w}(k) = x(k + d + 1|k + 1) - x(k + d + 1|k)$ in Eq. (5) in order to bound the error propagation between these predictions at different samples. Using $\rho(k + j|k) = \rho(k)$ and $\rho(k + j|k + 1) = \rho(k + 1)$, we obtain $x(k + d + 1|k + 1) = A(\rho(k + 1))^d x(k + 1) + \sum_{j=1}^d A(\rho(k + 1))^{j-1} B(\rho(k + 1)) u(k + 1 - j)$, while $x(k + d + 1|k) = A(\rho(k))^{d+1} x(k) + \sum_{j=1}^{d+1} A(\rho(k))^{j-1} B(\rho(k)) u(k + 1 - j)$. By subtracting these expressions, and using $x(k + 1)$ from Eq. (1), we are able to obtain:

$$\begin{aligned} \tilde{w}(k) := & [A(\rho(k + 1))^d - A(\rho(k))^d] A(\rho(k)) x(k) \quad (11) \\ & + [A(\rho(k + 1))^d - A(\rho(k))^d] B(\rho(k)) u(k - d) \\ & + \sum_{j=1}^d A(\rho(k + 1))^{j-1} B(\rho(k + 1)) u(k + 1 - j) \\ & - \sum_{j=1}^d A(\rho(k))^{j-1} B(\rho(k)) u(k + 1 - j). \end{aligned}$$

Since all the involved variables in Eq. (11) are bounded by definition, we get $\tilde{w}(k) \in \tilde{\mathcal{W}}, \forall k \geq 1$, where $\tilde{\mathcal{W}}$ can be generated by the means of zonotopes or interval algebra. In App. A, we provide the corresponding full development.

4.2 Prediction Error

One has not only to take into account the prediction bias of the nominal system trajectory, but also the error that arises due to the mismatches between the nominal model \tilde{x} and the real system trajectories x . We consider two factors: (a) the error that arises due to the differences between the real scheduling parameter trajectory $\rho(k + j)$ and the frozen trajectory estimate $\rho(k + j|k) = \rho(k)$; and (b) badly estimated DTs in the nominal model, which often occurs in practice. Thus, we consider the error dynamics as follows $e(k + d) = x(k + d) - \tilde{x}(k) = x(k + d) - x(k + d|k)$. This error variable is split as follows: $e(k + d) = e_\rho(k + d) + e_d(k + d)$, where the first term relates to (a), and the latter to (b).

Firstly, we assess (a): $e_\rho(k + d)$ derives from the differences caused by the scheduling trajectory estimate. Thus, when spanning Eq. (5) forward, we get $x(k + d + j + 1|k) = A(\rho(k + j + d|k)) x(k + d + j|k) + B(\rho(k + j + d|k)) u(k + j)$. Thus, for $\rho(k + d + j|k) = \rho(k)$, we obtain:

$$\begin{aligned} e_\rho(k + d + j) := & [A(\rho(k + d + j)) - A(\rho(k))] x(k + d + j|k) \\ & + [B(\rho(k + d + j)) - B(\rho(k))] u(k + j). \quad (12) \end{aligned}$$

Due to the nature of the considered LPV system, we have it ensured by design that $e_\rho(k) \in \mathbb{E}_\rho, \forall k \geq 0$, where \mathbb{E}_ρ is a compact and convex set that contains the origin.

Remark 5. We stress that e_ρ is directly related to the strategy used to describe the values of the scheduling parameter along the horizon which comprises the whole prediction horizon, i.e. $\rho(k + j|k), \forall j \in \mathbb{N}_{[0, N_p - 1]}$. When scheduling trajectory extrapolation strategies are used, such as those presented in (Cisneros and Werner, 2017; Hanema et al., 2017; Morato et al., 2021b), we are able to accurate descriptions for $\rho(k + j|k)$, which does reduce the magnitude of e_ρ over the horizon. Through the sequel, we consider the more generic case, for when only $\rho(k)$ is known and thus one uses $\rho(k + j|k) = \rho(k)$.

Remark 6. For quasi-LPV (qLPV) systems, $\rho(k + j), \forall j \in \mathbb{N}_{[0, d]}$ depends on the system variables ($x(k + j)$ and $u(k + j)$). We note that in some of these cases, when $\rho(k + j) = f_\rho(u(k + j))$, the corresponding DT-compensated model from Eq. (5) becomes much simpler. For these, $\rho(k + j)$ becomes known variables, since the control inputs $u(k + j)$ are inherently available from $j = 0$ to $j = d$. Thence, no trajectory estimate for the future scheduling parameters along the DT becomes necessary, and e_ρ naturally becomes null.

With regard to (b): Consider that the nominal DT is given by d , while the real DT is $d_r = d + \delta d$, for $\delta d \geq 1$. Then, taking this into account, we can express the real state trajectories by $x(k + 1) = A(\rho(k)) x(k) + B(\rho(k)) u(k - d_r)$, whereas the DT-compensated predictions are given by $\tilde{x}(k + 1)$. Thereof, using $u(k - d_r) = u(k - d) - \sum_{j=1}^{\delta d} \delta u(k - d - j)$, we obtain, for $e_\rho = 0$, and for any $\delta d > 1$:

$$e_d(k) := -B(\rho(k)) \left(\sum_{j=0}^{\delta d - 1} \delta u(k - d - j) \right). \quad (13)$$

Taking into account the previous discussions, we stress that the under-estimated DT leads to a cumulative effect upon the prediction error $e_d(k)$ due to the control input variations over samples. Accordingly we obtain $e_d(k) \in \mathbb{E}_d, \forall k \geq 0$, where:

$$\mathbb{E}_d := \overbrace{(\bar{B} \delta \mathcal{U}) \oplus \dots \oplus (\bar{B} \delta \mathcal{U})}^{\delta d \text{ times}}, \quad (14)$$

where \bar{B} stands for $B(\rho)$ evaluated for the maximal energy transfer due to $\rho \in \mathcal{P}$. Synthetically, we use $e(k) = e_\rho(k) + e_d(k) \in \mathbb{E}, \forall k \geq 0$, where $\mathbb{E} = \mathbb{E}_\rho \oplus \mathbb{E}_d$.

4.3 Properties

Now, we pay special attention to the robust constraint satisfaction and ISS over the real state trajectories x (instead of \tilde{x}). For this reason, we must take into account how the prediction error $e(k)$ affects the closed-loop.

Lemma 2. Constraints Tightening

Take $\mathcal{X}_W(j)$ in the MPC optimisation (Eq. (6)) as follows: $\mathcal{X}_W(0) = \mathcal{X} \ominus \mathbb{E}$ and $\mathcal{X}_W(j) := \mathcal{X}_W(j - 1) \ominus \mathcal{S}(j), \forall j \in \mathbb{N}_{[1, N_p]}$. Moreover, consider $\mathcal{S}(0) = \tilde{\mathcal{W}}$ and $\mathcal{S}(j) := [A(\rho(k + j|k + 1)) - A(\rho(k + j|k))] \mathcal{S}(j - 1), \forall j \in \mathbb{N}_{[1, N_p]}$. These sets ensure the open-loop propagation of \tilde{w} over the prediction model and the according compensation in the state constraints.

Proof 3. Refer to (Morato et al., 2021a).

Remark 7. Note that a nominal static feedback can be used to attenuate the propagation of \tilde{w} over the state predictions. Nevertheless, in such case, the input constraints must also be accordingly tightened.

We stress that \mathbb{E} grows with the size of the DT estimation error δd , while $\mathcal{S}(j), \forall j \geq 1$ depends on the scheduling parameter changes between samples. In consonance with (Mayne et al., 2006), Lemma 2 provides conditions such that the MPC is able to ensure that $\tilde{x}(k+j) \in \mathcal{X}_{\mathcal{W}}(j), \forall j \in \mathbb{N}_{[1, N_p]}$, which conversely implies that $x(k) - \tilde{x}(k)$ belong to some subset of $\mathcal{X} \oplus \mathbb{E}$, thus ensuring that $x(k) \in \mathcal{X}, \forall k \geq d$. We stress that \mathbb{E} is a compact polytope with contains the origin, but grows in size w.r.t. the size of the DT estimation error δd . The following Lemma provides ISS conditions for the closed-loop under the action of the constraint-tightened MPC algorithm.

Lemma 3. ISS under DT Estimation Error

Let $u(k) = \kappa(\rho(k))z(k)$ be a control law such that $z(k+1) = A(\rho(k))z(k) + B(\rho(k))u(k)$ is ISS. Moreover, let $x(k+1) = A(\rho(k))x(k) + B(\rho(k))u(k-d) + \tilde{w}(k)$, with $\tilde{w}(k) \in \tilde{\mathcal{W}}$ be controlled by $u(k) = \kappa(\rho(k))\tilde{x}(k)$, where $\tilde{x}(k)$ follows from Eq. (4). Let the prediction error bound be \mathbb{E} . Then: (a) The dynamics of z are ISS for all $\tilde{w}(k) \in \tilde{\mathcal{W}}$; (b) If $\tilde{x}(k) \in \mathcal{X} \ominus \mathbb{E}, \forall k \geq 0$, then $x(k) \in \mathcal{X}, \forall k \geq d$.

Proof 4. Follows directly from (Santos et al., 2012; Morato et al., 2021a), using $\tilde{w}(k)$ as gives Eq. (11).

Remark 8. We stress that the convex set \mathbb{E} defines the admissible values for the prediction error $e(k) = x(k+d) - \tilde{x}(k)$, i.e. differences between the real system dynamics, Eq. (1), and the DT-compensated ones, Eq (5), thus encompassing the error from DT estimation (when $\delta d \neq 0$) and from the scheduling trajectory guess (when $\rho(k+j|k) \neq \rho(k+j)$). The disturbance propagation set $\tilde{\mathcal{W}}$ bounds the step-ahead nominal prediction discrepancy, which is non-null due to the variance between $\rho(k+j+1)$ and $\rho(k+j)$.

Remark 9. In the case of time-varying delay estimation errors $d_r = d + \delta d(t)$, we recommend using the unified method for time-varying delay compensation recently proposed in (Morato and Normey-Rico, 2021).

5. CASE STUDY: A SOLAR COLLECTOR BENCHMARK

In order to demonstrate the performances achieved with the proposed control system, we consider a solar collector benchmark, as illustrated in Fig. 2. In this plant, the temperature of the heated fluid must be regulated, while the input flow and temperature of the plates must stay within predefined admissible sets. The solar plates are heated due to solar irradiance, but also affected by the external temperature. In accordance with (Pipino et al., 2020), we use a qLPV model to represent this process:

$$x(k+1) = A(\rho(k))x(k) + B(\rho(k))u(k-d) + B_w w(k),$$

$$\rho_1(k) = \alpha_1(1 - e^{-\tau_1 x_1(k)}), \quad \rho_2(k) = \alpha_2(1 - e^{-\tau_2 x_2(k)}).$$

This model has two states x , which describe the deviation of the plate and fluid temperatures to a given constant set-point. The control input u corresponds to flow of the fluid that circulates the collector, while w stands

for the two known disturbances which affect this system (solar irradiance and external temperature). We note that this process exhibit a constant input dead-time d , which corresponds to the time needed for the flow to pass through the system, since the fluid temperature is only measured at the exit of the collector. The model matrices and sets are presented in Pipino et al., 2020, Tables 1,2, Eqs. (12)-(14), whereas not shown herein due to lack of space. The process constraints are $x \in \mathcal{X}, u \in \mathcal{U}$, and $\rho \in \mathcal{P}$, which are all convex and compact sets. The system operates under a sampling period of $T_s = 0.5$ s; the dead-beat is $d = 20$ discrete-time samples.

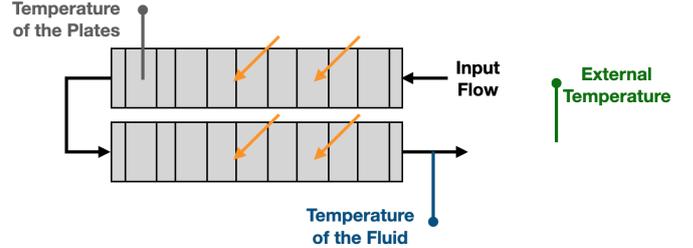


Fig. 2. Solar Collector Field.

We consider a simulation scenario² with the following temperature set-points: 120 °C (plates) and 100 °C (fluid). The initial conditions are: 100 °C (plates) and 90 °C (fluid). The disturbances $w(k)$ vary over time, as shown in Fig. 3. These variables are measured at each sampling instant k , whereas assumed constant for prediction purposes, which means that its effect over the state dynamics can be analytically compensated, converting the previous model into the likes of Eq. (1).

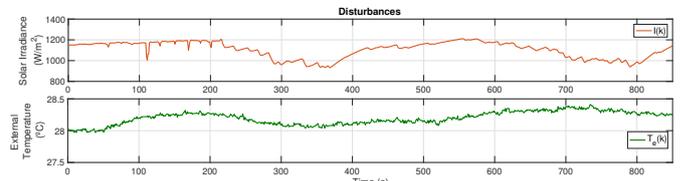


Fig. 3. Simulation Scenario.

Two control systems are used³: an LPV MPC algorithm with explicit DT compensation (denoted EDTC MPC), as proposed in Sec. 3, and an LPV MPC algorithm with implicit DT compensation (denoted IDTC MPC), with the implicit DT model in the form of Eq. (3). Since $d = 20$, we stress that the latter approach uses a state-space description with 22 states encompassed within ξ , whilst the proposed method uses the DT-compensated description from Eq. (5), with only 2 states. From this, as previously discussed, we can already expect the proposed MPC to run much fast during the implementation. Accordingly, this phenomenon is observed in Fig. 4, which depicts the average computational time (t_c) required to solve the optimisation corresponding to each approach. The proposed method is over twice faster than the IDTC MPC, which is a significant advantage. We also stress that the terminal ingredients from Sec. 4 are (numerically) much simpler to synthesise for the proposed method.

² Results were obtained in a 2.4 GHz, 8 GB RAM Macintosh computer, using Matlab, yalmip, and SDPT3.

³ Both MPC are tuned with unitary stage cost weights Q and R .

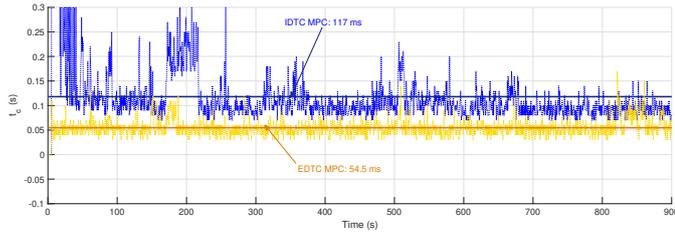


Fig. 4. Results: Computational Time.

Next, we show the system trajectories under two different situations: (a) for the case of perfectly known DT, considering both controllers and (b) for the case of different DT estimation errors, considering only the EDTC controller. We note that scenario (b) does not include the IDTC controller since it embeds no robustness ingredients against badly estimated DTs.

For the first scenario, Fig. 5 shows the state trajectories and the fluid temperature behaviour under the action of both controllers. As one can see, both methods ensure very similar performances, with relatively small offsets to the steady-state temperature goals (we stress that both MPCs do not analytically include integral action, for simplicity, see Remark 4). The results indicate that the proposed EDTC MPC method can indeed well control the considered solar collector process, despite using the DT-compensated model from Eq. (5). We recall that this nominal DT-compensated model is exposed to more prediction errors than the implicit DT model from Eq. (3), since it includes the presence of the scheduling parameter prediction discrepancies, compacted in e_ρ . Even though e_ρ turns null in steady-state, this model-mismatch variable is not measured or estimated, but rather accounted for robustly, since the proposed MPC takes only the upper bound of $e_\rho \in \mathbb{E}_\rho$ into account. The reason for this resides on the use of the tightened constraints mechanism (Lemma 2), which evens out the prediction error caused by e_ρ .

Considering more abrupt solar irradiance variation (twice the size of those from Fig. 3), we show how the magnitude of the delay estimation error δd affects the system dynamics (plate temperature, e.g. Fig. 6). As δd increases, the size of the corresponding uncertainty propagation set \mathcal{E}_d grows, as gives Eq. (14). Thereof, larger offsets toward the steady-state goal are observed. Nevertheless, we stress ISS is ensured (as of Lemma 3), which is an elegant feature of the proposed LPV MPC DTC scheme, ensuring stability and robustness towards DT estimation errors.

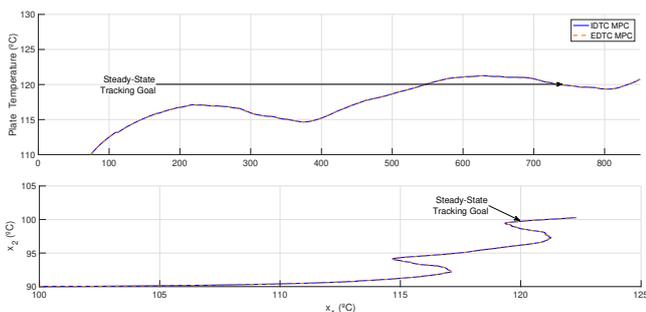


Fig. 5. Results: State Trajectories under nominal DT.

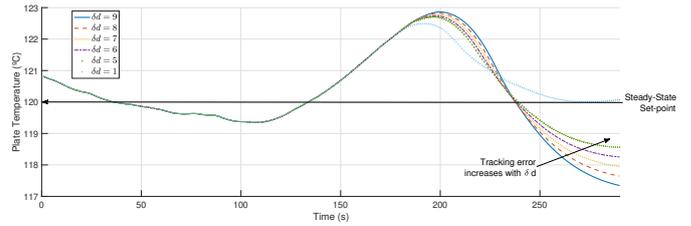


Fig. 6. Results: State Trajectories under uncertain DT.

6. CONCLUSIONS

In this paper, we briefly studied how explicit dead-time compensation can be incorporated to the design of LPV MPC algorithms. Accordingly, we provided a synthesis procedure which ensures input-to-state feasibility of the closed-loop system, as well as recursive feasibility of the MPC optimisation. The method used parameter-dependent terminal ingredients, offered via standard LMI synthesis. We also debate the issue of uncertain delay estimation and how this can be counter-acted directly in the MPC formulation. A solar collector benchmark example is used to illustrate the features of the proposed method, which is compared against a nominal LPV MPC with implicit DT compensation. The proposed method is significantly faster, while providing similar regulation performances. We recall the main features of the proposed method:

- (1) It does not require any augmented state representation, thus maintaining the order of the system from a prediction viewpoint.
- (2) It offers reduced numerical burden in the sense of on-line implementation of the MPC optimisation as well as in terms of computing the terminal ingredients.
- (3) It allows the computation of a scheduled response of the free state dynamics $\check{x}(k) = x(k + d|k)$. These responses can be expressed linearly in an LPV setting.
- (4) It offers simple robustness tools to directly tackle delay estimation uncertainty.

For future works, we aim at generalising the proposed scheme for the case of time-varying delays. We also pretend to assess on how the uncertainty propagation (due to delay estimation errors) can be better attenuated in closed-loop, as well as how qLPV DT predictors can be used to enhance the DT-compensated predictions (reducing mismatches).

REFERENCES

- Cisneros, P.G. and Werner, H. (2017). Fast nonlinear MPC for reference tracking subject to nonlinear constraints via quasi-LPV representations. *IFAC-PapersOnLine*, 50(1), 11601–11606.
- da Silva, L.R., Flesch, R.C.C., and Normey-Rico, J.E. (2020). Controlling industrial dead-time systems: When to use a pid or an advanced controller. *ISA transactions*, 99, 339–350.
- Gros, S., Zanon, M., Quirynen, R., Bemporad, A., and Diehl, M. (2020). From Linear to Nonlinear MPC: Bridging the Gap via the Real-time Iteration. *International Journal of Control*, 93(1), 62–80.
- Hanema, J., Tóth, R., and Lazar, M. (2017). Stabilizing Non-linear MPC using Linear Parameter-varying Rep-

- resentations. In *56th Annual Conference on Decision and Control*, 3582–3587. IEEE.
- Kharitonov, V. and Zhabko, A. (2003). Lyapunov-krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39(1), 15–20.
- Köhler, J., Müller, M.A., and Allgöwer, F. (2019). A nonlinear model predictive control framework using reference generic terminal ingredients. *IEEE Transactions on Automatic Control*.
- Köhler, J., Zeilinger, M., and Grüne, L. (2021). Stability and performance analysis of NMPC: Detectable stage costs and general terminal costs. URL <https://arxiv.org/abs/2110.11021>. (Unpublished), Pre-print.
- Lima, D.M., Santos, T.L.M., and Normey-Rico, J.E. (2015). Robust nonlinear predictor for dead-time systems with input nonlinearities. *Journal of Process Control*, 27, 1–14.
- Mayne, D.Q., Raković, S.V., Findeisen, R., and Allgöwer, F. (2006). Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42(7), 1217–1222.
- Mayne, D.Q., Rawlings, J.B., Rao, C.V., and Sokaert, P.O. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6), 789–814.
- Mohammadpour, J. and Scherer, C.W. (2012). *Control of linear parameter varying systems with applications*. Springer Science & Business Media.
- Morato, M.M., Cunha, V.M., Santos, T.L., Normey-Rico, J.E., and Sename, O. (2021a). Robust nonlinear predictive control through qLPV embedding and Zonotope uncertainty propagation. *IFAC-PapersOnLine*, 54(8), 33–38. doi:<https://doi.org/10.1016/j.ifacol.2021.08.577>. 4th IFAC Workshop on Linear Parameter Varying Systems LPVS 2021.
- Morato, M.M. and Normey-Rico, J.E. (2021). A novel unified method for time-varying dead-time compensation. *ISA transactions*, 108, 78–95.
- Morato, M.M., Normey-Rico, J.E., and Sename, O. (2020a). Model predictive control design for linear parameter varying systems: A survey. *Annual Reviews in Control*, 49, 64 – 80.
- Morato, M.M., Normey-Rico, J.E., and Sename, O. (2019). Novel qLPV MPC design with least-squares scheduling prediction. *IFAC-PapersOnLine*, 52(28), 158–163.
- Morato, M.M., Normey-Rico, J.E., and Sename, O. (2021b). A fast dissipative robust nonlinear model predictive control procedure via quasi-linear parameter varying embedding and parameter extrapolation. *International Journal of Robust and Nonlinear Control*.
- Morato, M.M., Normey-Rico, J.E., and Sename, O. (2020b). Sub-optimal Recursively Feasible Linear Parameter-Varying Predictive Algorithm for Semi-active Suspension Control. *IET Control Theory & Applications*, 14(18), 2764–2775.
- Normey-Rico, J. (2007). *Control of dead-time processes*. Springer Science & Business Media.
- Normey-Rico, J.E. and Camacho, E.F. (2009). Unified approach for robust dead-time compensator design. *Journal of Process Control*, 19(1), 38–47.
- Normey-Rico, J.E., Lima, D.M., and Santos, T.L. (2015). Robustness of nonlinear MPC for dead-time processes. *IFAC-PapersOnLine*, 48(23), 332–341.
- Pipino, H.A., Morato, M.M., Bernardi, E., Adam, E.J., and Normey-Rico, J.E. (2020). Nonlinear temperature regulation of solar collectors with a fast adaptive polytopic LPV MPC formulation. *Solar Energy*, 209, 214–225.
- Santos, T.L., Limon, D., Normey-Rico, J.E., and Alamo, T. (2012). On the explicit dead-time compensation for robust model predictive control. *Journal of Process Control*, 22(1), 236–246.
- Sename, O., Gaspar, P., and Bokor, J. (2013). *Robust control and linear parameter varying approaches: application to vehicle dynamics*, volume 437. Springer.
- Skogestad, C.G.S. (2018). Should we forget the smith predictor? *IFAC-PapersOnLine*, 51(4), 769–774.
- Smith, O. (1957). Close control of loops with dead time. *Chemical Engineering progress*, 53(5), 217–219.

Appendix A. COMPLEMENTARY DEVELOPMENT

Take the nominal DT-compensated model using different samplings, i.e. at k and at $k+1$, using a prediction horizon of n samples ahead:

$$x(k+n|k) = (A(\rho(k)))^n x(k) + \sum_{j=1}^n (A(\rho(k)))^{j-1} B(\rho(k)) u(k-d+n-j),$$

$$x(k+n|k+1) = (A(\rho(k+1)))^{n-1} x(k+1) + \sum_{j=1}^{n-1} (A(\rho(k+1)))^{j-1} B(\rho(k+1)) u(k-d+n-j),$$

Thus, using the real system trajectories $x(k+1) = A(\rho(k))x(k) + B(\rho(k))u(k-d)$, we obtain:

$$\begin{aligned} x(k+d+1|k+1) &= (A(\rho(k+1)))^d x(k+1) \\ &+ \sum_{j=1}^d (A(\rho(k+1)))^{j-1} B(\rho(k+1)) u(k+1-j) \\ &= (A(\rho(k+1)))^d A(\rho(k)) x(k) \\ &+ (A(\rho(k+1)))^d B(\rho(k)) u(k-d) \\ &+ \sum_{j=1}^d (A(\rho(k+1)))^{j-1} B(\rho(k+1)) u(k+1-j). \end{aligned} \tag{A.1}$$

Complementary, we have:

$$\begin{aligned} x(k+d+1|k) &= (A(\rho(k)))^{d+1} x(k) \\ &+ (A(\rho(k)))^d B(\rho(k)) u(k-d) \\ &+ \sum_{j=1}^d (A(\rho(k)))^{j-1} B(\rho(k)) u(k+1-j). \end{aligned} \tag{A.2}$$

Using $\tilde{w}(k) = x(k+d+1|k+1) - x(k+d+1|k)$, we can subtract Eq. (A.2) from (A.1) in order to get Eq. (11).