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► **To cite this version:**

Abderrahim El Omri, A. Fennan, François Sidoroff, Abdelaziz Hihi. Elastic-plastic homogenization for layered composites. *European Journal of Mechanics - A/Solids*, 2000, 19 (4), pp.585-601. 10.1016/S0997-7538(00)00182-0 . hal-03620030

**HAL Id: hal-03620030**

**<https://hal.science/hal-03620030v1>**

Submitted on 25 Mar 2022

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# Elastic-plastic homogenization for layered composites

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**Abstract** – A general method for the homogenization of rigid plastic or elastic-plastic layered structures is proposed. This method is based on a hybrid formulation of the constitutive equation for each phase, thus reducing the homogenization procedure to a simple mixture law. This non-classical hybrid formulation is obtained for both the rigid-plastic and elastic-plastic models. The resulting homogenized behaviour is then illustrated by some examples with special emphasis on the resulting anisotropy and in particular on the off-axis tensile behaviour.

**layered composite / rigid plasticity / elasto-plasticity / anisotropy**

## 1. Introduction

It has been shown in previous work that elastic-plastic homogenization provides a proper framework for analysing the influence of microstructural heterogeneities on the mechanical behaviour of metallic materials as it can be observed for instance in cyclic plasticity (Mughrabi, 1978, 1988; Fougères and Sidoroff, 1989). In the latest studies it has been shown that the evolution of local stresses induced by plastic heterogeneities plays an important role in the behaviour of such materials. However, these models are based on a very crude description of the geometrical structure.

Two heterogeneous models based on a more realistic and simple layered description of the plastic heterogeneity have been proposed (El Omri and Sidoroff, 1991a; El Omri et al., 1991). The first is a representation of the material by a two phase layered composite while the second introduces the notion of ‘Polylayered composite’. Although the method presented by (El Omri and Sidoroff, 1991a) was specific, it cannot be naturally extended to the general situations investigated here (elastic heterogeneity, general plastic yield condition, arbitrary number of phase).

To overcome these limitations, the purpose of this work was to develop another method for the elastic-plastic homogenization of ‘Multilayered’ composites. This method is based on the jump conditions at an interface, which suggest a hybrid formulation of the local constitutive equation allowing the constitutive formulation of each phase independently on the others.

This method will be presented, and illustrated first for classical elastic and the rigid plastic cases. The more complex elastoplastic case will then be developed. As an application, the rigid-plastic elastic-plastic case will be exemplified on a discretized model representation of the behaviour of an aluminium alloy.

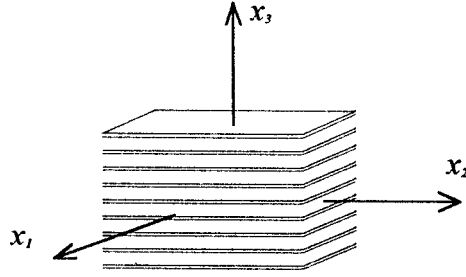


Figure 1. Layered structure.

## 2. Layered homogenization

The homogenization procedure in Mechanics of Materials consists of replacing a heterogeneous medium by an equivalent homogeneous one, which is defined by a homogenized constitutive law relating the macroscopic variables, namely macroscopic stress  $\Sigma$  and strain  $\epsilon$  tensors. For periodic structures these quantities are obtained by averaging the corresponding non-uniform microscopic stress  $\sigma$  and strain  $\varepsilon$

$$\epsilon = \langle \varepsilon \rangle, \quad \Sigma = \langle \sigma \rangle. \quad (1)$$

In elasticity, this micromechanical approach is now classical. However, in the plastic and in the elastic-plastic cases it leads to a difficult problem for which only numerical solutions can be found (Suquet, 1982, 1983, 1985; de Buhan, 1983, 1991; Dvorak and Telpy, 1990). For layered composites, it will be shown here that the corresponding homogenized elastic-plastic behaviour law can be found explicitly and without any limitation. This is provided by an appropriate and so-called layered homogenization procedure.

The composite investigated herein, consists of a periodic layered structure with homogeneous layers perpendicular to the  $\bar{x}_3$  direction (see figure 1).

The local mechanical properties depend only on  $x_3$ . Jump conditions that relate microscopic variables to the macroscopic ones can easily be derived from continuity and equilibrium

$$\begin{aligned} \sigma_{i3}(x_3) &= \Sigma_{i3}, \quad i = 1, 2, 3, \\ \varepsilon_{\alpha\beta}(x_3) &= \epsilon_{\alpha\beta}, \quad \alpha, \beta = 1, 2. \end{aligned} \quad (2)$$

This leads to a uniform stress  $\sigma$  and strain  $\varepsilon$  in each layer. The average operator (equation (1)) is then

$$\langle \varphi \rangle = \frac{1}{h} \int \varphi(x_3) dx_3, \quad (3)$$

where  $h$  is the size of the representative unit cell.

For elastic, rigid-plastic and elastic-plastic composites, the first step of this procedure consists of expressing local constitutive laws in a hybrid form. The macroscopic behaviour of such a composite is then obtained only by applying the averaging operator as for mixture laws.

### 2.1. Symmetric tensors decomposition

The layered homogenization procedure is based on a particular tensorial decomposition. The jump condition (equation (2)), which is the starting point of this procedure, suggests a decomposition of any symmetrical

second order tensor  $\mathbf{t}$  into its plane  $\mathbf{t}_P$  and antiplane  $\mathbf{t}_A$  components

$$\mathbf{t} = \mathbf{t}_P + \mathbf{t}_A, \quad (4)$$

where

$$\mathbf{t}_P = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{t}_A = \begin{pmatrix} 0 & 0 & t_{13} \\ 0 & 0 & t_{23} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}.$$

This introduces a decomposition of the space  $S$  of symmetrical tensors into a direct sum of two orthogonal subspaces

$$S = S_P \oplus S_A, \quad (5)$$

such that the localization condition (equation (2)) can be written as

$$\sigma_A = \Sigma_A, \quad \varepsilon_P = \in_P. \quad (6)$$

From a practical point of view this decomposition will be most easily dealt with in an appropriate orthonormal basis  $\{\mathbf{f}_I, \mathbf{f}_{II}, \mathbf{f}_{III}, \mathbf{f}_{IV}, \mathbf{f}_V, \mathbf{f}_{VI}\}$  of  $S$ , defined by

$$\begin{aligned} \mathbf{f}_I &= \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbf{f}_{II} &= \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbf{f}_{III} &= \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{f}_{IV} &= \begin{pmatrix} 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \end{pmatrix}, & \mathbf{f}_V &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}, & \mathbf{f}_{VI} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (7)$$

Then, any tensor  $\mathbf{t}$  belonging to  $S$  can be written as

$$\mathbf{t} = \sum_{J=I..VI} t_J \mathbf{f}_J, \quad (8)$$

where the scalar  $t_J$  are given by

$$\begin{aligned} t_I &= \frac{t_{11} + t_{22}}{\sqrt{2}}, & t_{II} &= \frac{t_{11} - t_{22}}{\sqrt{2}}, & t_{III} &= \sqrt{2}t_{12}, \\ t_{IV} &= \sqrt{2}t_{13}, & t_V &= \sqrt{2}t_{23}, & t_{VI} &= t_{33}. \end{aligned}$$

The plane and antiplane components of a symmetrical second order tensor belong respectively to  $S_P$  and  $S_A$ , which are defined by the following sets:

$$S_P: \{\mathbf{f}_I, \mathbf{f}_{II}, \mathbf{f}_{III}\}, \quad S_A: \{\mathbf{f}_{IV}, \mathbf{f}_V, \mathbf{f}_{VI}\}. \quad (9)$$

In the isotropic behaviour case, considered in the further illustrations, it is often necessary to decompose a symmetrical tensor  $\mathbf{f}$  into its spherical and deviatoric parts, denoted, respectively, by  $\mathbf{t}^I$  and  $\mathbf{t}^D$  ( $\mathbf{t} = \mathbf{t}^I + \mathbf{t}^D$ ). So, it remains to decompose the above tensors into their plane and antiplane components.

Let us now introduce the following tensors

$$\mathbf{f}^i = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{f}^d = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (10)$$

They are easily related to the basis (equation (7)) by:

$$\begin{aligned}\mathbf{f}^i &= \frac{\sqrt{2}}{\sqrt{3}}\mathbf{f}_I + \frac{1}{\sqrt{3}}\mathbf{f}_{VI}, \\ \mathbf{f}^d &= \frac{-1}{\sqrt{3}}\mathbf{f}_I + \frac{\sqrt{2}}{\sqrt{3}}\mathbf{f}_{VI}.\end{aligned}\quad (11)$$

Then the spherical and the deviatoric parts of any symmetrical second order tensor  $\mathbf{t}$  can be written in the following forms:

$$\begin{aligned}t^I &= t^i f^i, \quad t^i = \frac{1}{\sqrt{3}}(\sqrt{2}t_I + t_{VI}), \\ t^D &= t^d f^d + \sum_{k=II\dots V} t_k f_k, \quad t^d = \frac{1}{\sqrt{3}}(-t_I + \sqrt{2}t_{VI}),\end{aligned}\quad (12)$$

since  $\mathbf{f}^d$  and  $\mathbf{f}^i$  are orthogonal to  $\{\mathbf{f}_{II}, \mathbf{f}_{III}, \mathbf{f}_{IV}, \mathbf{f}_V\}$ .

Finally, the plane and antiplane components of the spherical and the deviatoric parts of a symmetrical tensor are obtained by substituting equation (11) in equation (12). This can be written in a usual matricial form:

$$\begin{pmatrix} \mathbf{t}_P^I \\ \mathbf{t}_A^I \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{PP} & \mathbf{S}_{PA} \\ \mathbf{S}_{AP} & \mathbf{S}_{AA} \end{pmatrix} \begin{pmatrix} \mathbf{t}_P \\ \mathbf{t}_A \end{pmatrix}, \quad \begin{pmatrix} \mathbf{t}_P^D \\ \mathbf{t}_A^D \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{PP} & \mathbf{D}_{PA} \\ \mathbf{D}_{AP} & \mathbf{D}_{AA} \end{pmatrix} \begin{pmatrix} \mathbf{t}_P \\ \mathbf{t}_A \end{pmatrix}, \quad (13)$$

$$\begin{aligned}\mathbf{D}_{PP} &= \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathbf{D}_{AA} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}, & \mathbf{D}_{PA} = \mathbf{D}_{AP}^T &= \begin{pmatrix} 0 & 0 & \frac{-\sqrt{2}}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{S}_{PP} &= \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbf{S}_{AA} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, & \mathbf{S}_{PA} = \mathbf{S}_{AP}^T &= \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

where  $\mathbf{X}^T$  denotes the transposed matrix of  $\mathbf{X}$ .

## 2.2. Elastic homogenization

As a first example, and to illustrate the layered homogenization procedure, let us consider the case where all of the constituents are elastic

$$\sigma_{ij} = \mathbf{a}_{ijkl}\varepsilon_{kl},$$

where fourth-order tensor  $\mathbf{a}$  denotes the elastic stiffness tensor.

Using the tensorial decomposition (equation (5)) and the jump condition (equation (6)), the elastic law can be written as

$$\begin{pmatrix} \sigma_P \\ \Sigma_A \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{PP} & \mathbf{a}_{PA} \\ \mathbf{a}_{AP} & \mathbf{a}_{AA} \end{pmatrix} \begin{pmatrix} \varepsilon_P \\ \varepsilon_A \end{pmatrix}. \quad (14)$$

The general method for layered homogenization follows simply from averaging a mixed form of the constitutive equation relating microscopic variables  $\sigma_P$  and  $\varepsilon_A$  to macroscopic ones  $\varepsilon_P$  and  $\Sigma_A$ . This hybrid

form is obtained by partial inversion of equation (13).

$$\begin{pmatrix} \sigma_P \\ \varepsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{k}_{PP}^e & \mathbf{k}_{PA}^e \\ \mathbf{k}_{AP}^e & \mathbf{k}_{AA}^e \end{pmatrix} \begin{pmatrix} \varepsilon_P \\ \Sigma_A \end{pmatrix}, \quad \mathbf{k}_{PP}^e = \mathbf{a}_{PP} - \mathbf{a}_{PA} \mathbf{a}_{AA}^{-1} \mathbf{a}_{AP}, \quad \mathbf{k}_{AA}^e = \mathbf{a}_{AA}^{-1}, \quad \mathbf{k}_{PA}^e = -\mathbf{k}_{AP}^{eT} = \mathbf{a}_{PA} \mathbf{a}_{AA}^{-1}. \quad (15)$$

The homogenized elastic behaviour is simply obtained by averaging this relation:

$$\begin{pmatrix} \Sigma_A \\ \varepsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{PP}^e & \mathbf{K}_{PA}^e \\ \mathbf{K}_{AP}^e & \mathbf{K}_{AA}^e \end{pmatrix} \begin{pmatrix} \varepsilon_P \\ \Sigma_A \end{pmatrix}, \quad \text{with } \mathbf{K}^e = \langle \mathbf{k}^e \rangle. \quad (16)$$

It can also be written in a more usual form

$$\begin{pmatrix} \Sigma_P \\ \Sigma_A \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{PP} & \mathbf{A}_{PA} \\ \mathbf{A}_{AP} & \mathbf{A}_{PP} \end{pmatrix} \begin{pmatrix} \varepsilon_P \\ \varepsilon_A \end{pmatrix}, \quad \mathbf{A}_{PP} = \mathbf{K}_{PP}^e - \mathbf{K}_{PA}^e \mathbf{K}_{AA}^{e-1} \mathbf{K}_{AP}^e, \quad \mathbf{A}_{AA} = \mathbf{K}_{AA}^e, \quad \mathbf{A}_{PA} = \mathbf{K}_{PA}^e \mathbf{K}_{AA}^{e-1} = \mathbf{A}_{AP}^T, \quad (17)$$

where the fourth-order tensor  $\mathbf{A}$  denotes the effective stiffness of the layered composite.

More specifically, let us consider the isotropic case where each layer obeys a local Hooke's law

$$\sigma = 3k\varepsilon^I + 2G\varepsilon^D, \quad (18)$$

with

$$3k = \frac{E}{1-2\nu} \quad \text{and} \quad 2G = \frac{E}{1+\nu},$$

where Young's modulus  $E$  and Poisson's ratio  $\nu$  are functions of  $x_3$ .

Using the decomposition (equation (13)), the elastic operator (equation (18)) can then be written in the new basis of  $S_P \oplus S_A$  as:

$$\begin{aligned} \mathbf{a}_{AA} &= 3k\mathbf{S}_{AA} + 2G\mathbf{D}_{AA}, & \mathbf{a}_{PA} &= \mathbf{a}_{AP}^T = 3k\mathbf{S}_{PA} + 2G\mathbf{D}_{PA}, \\ \mathbf{a}_{PP} &= 3k\mathbf{S}_{PP} + 2G\mathbf{D}_{PP}. \end{aligned} \quad (19)$$

The homogenized elastic law (equation (17)) is obtained by using equations (13), (16), (15) and (19):

$$\mathbf{K}^e = \langle \mathbf{k}^e \rangle, \quad (20)$$

with

$$\begin{aligned} \mathbf{k}_{PP}^e &= \begin{pmatrix} \frac{E}{1-\nu} & 0 & 0 \\ 0 & \frac{E}{1+\nu} & 0 \\ 0 & 0 & \frac{E}{1+\nu} \end{pmatrix}, & \mathbf{k}_{PA}^e &= -\mathbf{k}_{AP}^{eT} = \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}\nu}{1-\nu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{k}_{AA}^e &= \begin{pmatrix} \frac{1+\nu}{E} & 0 & 0 \\ 0 & \frac{1+\nu}{E} & 0 \\ 0 & 0 & \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} \end{pmatrix}. \end{aligned}$$

To find the effective homogenized elastic properties of the layered composite, one can proceed by identification. Let us now write the resulting macroscopic transverse isotropic law in the same basis. By using standard notations, one has

$$\begin{pmatrix} \epsilon_P \\ \epsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{PP} & \mathbf{C}_{PA} \\ \mathbf{C}_{AP} & \mathbf{C}_{AA} \end{pmatrix} \begin{pmatrix} \Sigma_P \\ \Sigma_A \end{pmatrix} \quad (21)$$

with

$$\mathbf{C}_{PP} = \begin{pmatrix} \frac{1-\nu_T}{E_T} & 0 & 0 \\ 0 & \frac{1+\nu_T}{E_T} & 0 \\ 0 & 0 & \frac{1+\nu_T}{E_T} \end{pmatrix}, \quad \mathbf{C}_{AA} = \begin{pmatrix} \frac{1}{2G_L} & 0 & 0 \\ 0 & \frac{1}{2G_L} & 0 \\ 0 & 0 & \frac{1}{E_L} \end{pmatrix},$$

$$\mathbf{C}_{PA} = \mathbf{C}_{AP}^T = \begin{pmatrix} 0 & 0 & \frac{-\sqrt{2}\nu_L}{E_L} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The hybrid formulation (equation (16)) is obtained again by a partial inversion:

$$\begin{pmatrix} \Sigma_P \\ \Sigma_A \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{PP}^e & \mathbf{K}_{PA}^e \\ \mathbf{K}_{AP}^e & \mathbf{K}_{AA}^e \end{pmatrix} \begin{pmatrix} \epsilon_P \\ \epsilon_A \end{pmatrix}, \quad \mathbf{K}_{PP}^e = \mathbf{C}_{PP}^{-1}, \mathbf{K}_{AA}^e = \mathbf{C}_{AA} - \mathbf{C}_{AP}\mathbf{C}_{PP}^{-1}\mathbf{C}_{PA}, \mathbf{K}_{PA}^e = -\mathbf{K}_{AP}^{eT} = -\mathbf{C}_{PP}^{-1}\mathbf{C}_{PA}, \quad (22)$$

$$\mathbf{K}_{PP}^e = \begin{pmatrix} \frac{E_T}{1-\nu_T} & 0 & 0 \\ 0 & \frac{E_T}{1+\nu_T} & 0 \\ 0 & 0 & \frac{E_T}{1+\nu_T} \end{pmatrix}, \quad \mathbf{K}_{AA}^e = \begin{pmatrix} \frac{1}{2G_L} & 0 & 0 \\ 0 & \frac{1}{2G_L} & 0 \\ 0 & 0 & \frac{1-2E_T\nu_T^2}{E_L(1-\nu_T)} \end{pmatrix},$$

$$\mathbf{K}_{PA}^e = -\mathbf{K}_{AP}^{eT} = \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}\nu_L E_T}{E_L(1-\nu_T)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, by identifying equation (20) and equation (22), the effective elastic properties  $E_L$ ,  $E_T$ ,  $\nu_L$ ,  $\nu_T$  and  $G_L$  are

$$\frac{1}{E_T} = \frac{1}{2} \left( \frac{1}{\langle E/(1-\nu) \rangle} + \frac{1}{\langle E/(1+\nu) \rangle} \right),$$

$$\nu_T = \frac{E_T}{2} \left( \frac{1}{\langle E/(1+\nu) \rangle} - \frac{1}{\langle E/(1-\nu) \rangle} \right),$$

$$\frac{1}{E_L} = \left\langle \frac{(1-2\nu)1+\nu}{E(1-\nu)} \right\rangle + \frac{2\langle \nu/(1-\nu) \rangle^2}{\langle E/(1-\nu) \rangle},$$

$$\nu_L = E_L \frac{\langle \nu / (1 - \nu) \rangle}{\langle E / (1 - \nu) \rangle}, \quad \frac{1}{2G_L} = \left\langle \frac{1 + \nu}{E} \right\rangle. \quad (23)$$

This can be considered as a classical result, but it is expressed here in the form of a generalised mixture law.

### 3. Rigid plasticity

The second example to be analysed here is the rigid perfectly plastic model with a quadratic plastic criterion. Using standard notation, the local constitutive equations read

$$d\varepsilon = \begin{cases} 0, & \text{if } f(\sigma^D) \leq 0, \\ d\lambda \frac{\partial f(\sigma^D)}{\partial \sigma^D}, & \text{if } f(\sigma^D) = 0, \quad d\lambda \geq 0, \end{cases} \quad (24)$$

$$f(\sigma^D) = \bar{\sigma} - y; \quad \bar{\sigma} = \sqrt{\sigma^D \mathbf{H}' \sigma^D}.$$

The yield condition  $f(\sigma^D)$  introduces an equivalent stress  $\bar{\sigma}$  which is defined in each layer by a normalised fourth order tensor  $\mathbf{H}'$  and by a scalar yield stress  $y$ .

#### 3.1. Behaviour law in $S_P \oplus S_A$

As in the elastic case, the first step of homogenization procedure is the decomposition of the local constitutive equation in  $S_P \oplus S_A$ . The yield function  $f$  and plastic evolution law (equation (24)) are therefore written in decomposed form by using the tensor  $\mathbf{D}$  introduced in equation (13):

$$\begin{pmatrix} \sigma_P \\ \Sigma_A \end{pmatrix} \begin{pmatrix} \mathbf{H}_{PP} & \mathbf{H}_{PA} \\ \mathbf{H}_{AP} & \mathbf{H}_{AA} \end{pmatrix} \begin{pmatrix} \sigma_P \\ \Sigma_A \end{pmatrix} = y^2 \quad \text{with } \mathbf{H} = \mathbf{D}\mathbf{H}'\mathbf{D}, \quad (25)$$

$$d\varepsilon_P = \frac{d\lambda}{y} (\mathbf{H}_{PP} \sigma_P + \mathbf{H}_{PA} \Sigma_A),$$

$$d\varepsilon_A = \frac{d\lambda}{y} (\mathbf{H}_{AP} \sigma_P + \mathbf{H}_{AA} \Sigma_A). \quad (26)$$

The appropriate local constitutive equation will be written in hybrid form by expressing  $\sigma_P$  and  $d\varepsilon_A$  in terms of  $d\varepsilon_P$  and  $\Sigma_A$

$$\sigma_P = \frac{y}{d\lambda} \mathbf{H}_{PP}^{-1} d\varepsilon_P - \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA} \Sigma_A,$$

$$d\varepsilon_A = \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} d\varepsilon_P + \frac{d\lambda}{y} (\mathbf{H}_{AA} - \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA}) \Sigma_A. \quad (27)$$

Substituting this expression of  $\sigma_P$  in the plastic flow condition (equation (25)) gives

$$\left( \frac{y}{d\lambda} \right)^2 d\varepsilon_P \mathbf{H}_{PP}^{-1} d\varepsilon_P + \Sigma_A (\mathbf{H}_{AA} - \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA}) \Sigma_A = y^2. \quad (28)$$

So that, when plastic flows occur, the plastic multiplier  $d\lambda$  can be obtained from

$$\frac{y}{d\lambda} = \left( \frac{y^2 - \Sigma_A (\mathbf{H}_{AA} - \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA}) \Sigma_A}{d\varepsilon_P \mathbf{H}_{PP}^{-1} d\varepsilon_P} \right)^{1/2}. \quad (29)$$



The homogenized behaviour is then obtained by averaging equation (29)

$$\begin{cases} \Sigma_P = \left\langle \frac{y}{d\lambda} \mathbf{H}_{PP}^{-1} \right\rangle d \in_P - \langle \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA} \rangle \Sigma_A, \\ d \in_A = \langle \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} \rangle d \in_P + \left\langle \frac{d\lambda}{y} (\mathbf{H}_{AA} - \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA}) \right\rangle \Sigma_A \end{cases} \quad (30)$$

with  $d\lambda/y$  given in each layer by equation (29) in terms of  $d \in_P$  and  $\Sigma_A$ . This provides a complete but somewhat unusual plastic constitutive equation, which in the general case requires a numerical resolution.

### 3.2. Plastic similarity

The homogenized constitutive law (equation (30)) can be transformed in a classical plastic form in the special case where all layers have the same plastic shape tensor  $\mathbf{H}$ . So, they differ only by the yield stress  $y$ . The average operator  $\langle \cdot \rangle$  performed in equation (3) on  $x_3$  can be performed now on the variable  $y$ :

$$\langle \varphi \rangle = \int_0^{+\infty} \varphi(y) g(y) dy \quad (31)$$

by introducing the distribution function  $g(y)$  for yield stress  $y$ .

Substituting equation (29) for  $y/d\lambda$  in the average relation (equation (30)) results in

$$\Sigma_P + \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA} \Sigma_A = Y(\Sigma_A) \frac{\mathbf{H}_{PP}^{-1} d \in_P}{\sqrt{d \in_P \mathbf{H}_{PP}^{-1} d \in_P}}, \quad (32)$$

with

$$Y(\Sigma_A) = \int_0^{+\infty} \sqrt{y^2 - \Sigma_A (\mathbf{H}_{AA} - \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA} \Sigma_A)} g(y) dy.$$

The homogenized criterion is obtained by multiplying equation (32) by

$$\mathbf{H}_{PP} (\Sigma_P + \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA} \Sigma_A).$$

It provides finally an explicit rigid plastic homogenized constitutive model:

$$\Sigma_P \mathbf{H}_{PP} \Sigma_P + \Sigma_P \mathbf{H}_{PA} \Sigma_A + \Sigma_A \mathbf{H}_{AP} \Sigma_P + \Sigma_A \mathbf{H}_{AP} \mathbf{H}_{PP}^{-1} \mathbf{H}_{PA} \Sigma_A = Y^2(\Sigma_A). \quad (33)$$

An important special case consists of a composite material resulting from the superposition of isotropic layers obeying von Mises yield condition

$$f(\sigma^D) = \sqrt{\frac{3}{2} |\sigma^D|^2} - y, \quad (34)$$

$$\mathbf{H}' = \frac{3}{2} \mathbf{I}, \quad \mathbf{H} = \frac{3}{2} \mathbf{D}, \quad (\mathbf{D}^2 = \mathbf{D}),$$

$$\mathbf{H}_{PP} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix}, \quad \mathbf{H}_{AA} = \begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{H}_{PA} = \mathbf{H}_{AP}^T = \begin{pmatrix} 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The function  $Y$  and the homogenized constitutive equation for  $\Sigma_P$ , given in equation (32), are in this case

$$\left\{ \begin{array}{l} \Sigma_I = \frac{Y(\Sigma_{IV}, \Sigma_V)}{\sqrt{\frac{2}{3}y^2 - \Sigma_{IV}^2 - \Sigma_V^2}} d_{\in I} + \sqrt{2}\Sigma_{VI}, \\ \Sigma_{II} = \frac{Y(\Sigma_{IV}, \Sigma_V)}{\sqrt{\frac{2}{3}y^2 - \Sigma_{IV}^2 - \Sigma_V^2}} d_{\in II}, \quad \Sigma_{III} = \frac{Y(\Sigma_{IV}, \Sigma_V)}{\sqrt{\frac{2}{3}y^2 - \Sigma_{IV}^2 - \Sigma_V^2}} d_{\in III}, \end{array} \right. \quad (35)$$

$$Y(\Sigma_{IV}, \Sigma_V) = \int_0^{+\infty} \sqrt{\frac{2}{3}y^2 - \Sigma_{IV}^2 - \Sigma_V^2} g(y) dy.$$

Finally the homogenized yield criterion (equation (33)) can be written as

$$\frac{1}{3}(\Sigma_I - \sqrt{2}\Sigma_{VI})^2 + \Sigma_{II}^2 + \Sigma_{III}^2 = \left( \int_0^{+\infty} \sqrt{\frac{2}{3}y^2 - \Sigma_{IV}^2 - \Sigma_V^2} g(y) dy \right)^2. \quad (36)$$

### 3.3. Illustration

To illustrate this anisotropic yield criterion let us consider a plane stress state:

$$\Sigma = \begin{pmatrix} \Sigma_{11} & 0 & \Sigma_{13} \\ 0 & 0 & 0 \\ \Sigma_{13} & 0 & \Sigma_{33} \end{pmatrix}, \quad \begin{array}{ll} \Sigma_I = \Sigma_{11}/\sqrt{2}, & \Sigma_{IV} = \sqrt{2}\Sigma_{13}, \\ \Sigma_{II} = \Sigma_{11}/\sqrt{2}, & \Sigma_V = 0, \\ \Sigma_{III} = 0, & \Sigma_{VI} = \Sigma_{33}. \end{array} \quad (37)$$

Substituting this special form in equation (36) provides the plane stress yield condition as

$$\Sigma_{11}^2 + \Sigma_{33}^2 - \Sigma_{11}\Sigma_{33} = Y^2(\Sigma_{13}) = \left( \int_0^{+\infty} \sqrt{y^2 - 3\Sigma_{13}^2} g(y) dy \right)^2. \quad (38)$$

In the general case  $\Sigma_{13} \neq 0$ , the corresponding yield boundary in the  $(\Sigma_{11}, \Sigma_{33})$  plane is still von Mises ellipses with varying size. The plane stress yield condition therefore is entirely defined by its shape in the  $(\Sigma_{11}, \Sigma_{13})$  plane as

$$\Sigma_{11}^2 = Y^2(\Sigma_{13}),$$

which differs from the von Mises condition  $\Sigma_{11}^2 + 3\Sigma_{13}^2 = Y^2$  by the form of the function  $Y^2(\Sigma_{13})$ .

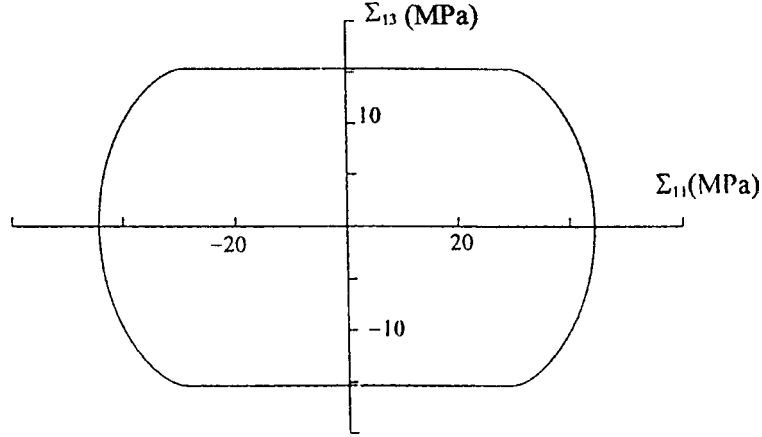
To exemplify this, let us consider for instance a layered composite consisting of a discrete distribution function of yield limit relative to eight layers (*table I*). This discrete distribution is representative of a typical aluminium alloy (it has been obtained practically by identification of an experimental test).

The corresponding yield surface in the  $(\Sigma_{11}, \Sigma_{13})$  plane is given in *figure 2*.

The strong anisotropy of the homogenized criterion (equation (38)) can be seen for example in an off-axis tensile test:

$$\Sigma = \Sigma(\vec{n} \otimes \vec{n}), \quad (39)$$

where  $\vec{n}$  is the direction of the applied stress and the symbol  $\otimes$  denotes product tensor. By defining  $\theta$  the angle between  $\vec{n}$  and the  $\vec{x}_3$  normal to the layers, the vector  $\vec{n}$  and the stress tensor read in the Cartesian coordinate



**Figure 2.** Homogenized criterion in  $(\Sigma_{11}, \Sigma_{13})$  plane stresses.

basis  $(x_1, x_2, x_3)$  as:

$$\vec{n} = \sin \theta \vec{x}_1 + \cos \theta \vec{x}_3,$$

$$\Sigma = \begin{pmatrix} \Sigma \sin^2 \theta & 0 & \Sigma \sin \theta \cos \theta \\ 0 & 0 & 0 \\ \Sigma \sin \theta \cos \theta & 0 & \Sigma \cos^2 \theta \end{pmatrix}. \quad (40)$$

The macroscopic criterion (equation (38)) reads in this case:

$$\Sigma^2 (1 - 3 \cos^2 \theta \sin^2 \theta) = \left( \int_0^{+\infty} \sqrt{y^2 - 3 \Sigma^2 \cos^2 \theta \sin^2 \theta} g(y) dy \right)^2. \quad (41)$$

*Figure 3* shows the evolution of the macroscopic yielding  $\Sigma$  for different values of  $\theta$ . If the composite is submitted to simple shear

$$\Sigma = \tau (\vec{n} \otimes \vec{m}) = \begin{pmatrix} \tau \sin 2\theta & 0 & \tau \cos 2\theta \\ 0 & 0 & 0 \\ \tau \cos 2\theta & 0 & \tau \sin 2\theta \end{pmatrix}, \quad (42)$$

where  $\vec{m}$  is perpendicular to  $\vec{n}$ , the macroscopic criterion reads:

$$(3\tau \sin \theta)^2 = \left( \int_0^{+\infty} \sqrt{y^2 + 3\tau^2 \cos^2 2\theta} g(y) dy \right)^2. \quad (43)$$

The evolution of the corresponding macroscopic yielding  $\tau$  is reported in *figure 4*.

#### 4. Elastic plastic layered homogenization

Each layer will now be considered elastic perfectly plastic. The constitutive equation is then obtained by combining the elastic law

$$\sigma = \mathbf{a} \varepsilon^e = \mathbf{a} (\varepsilon - \varepsilon^p) \quad (44)$$

with the plastic evolution law given by equation (24).

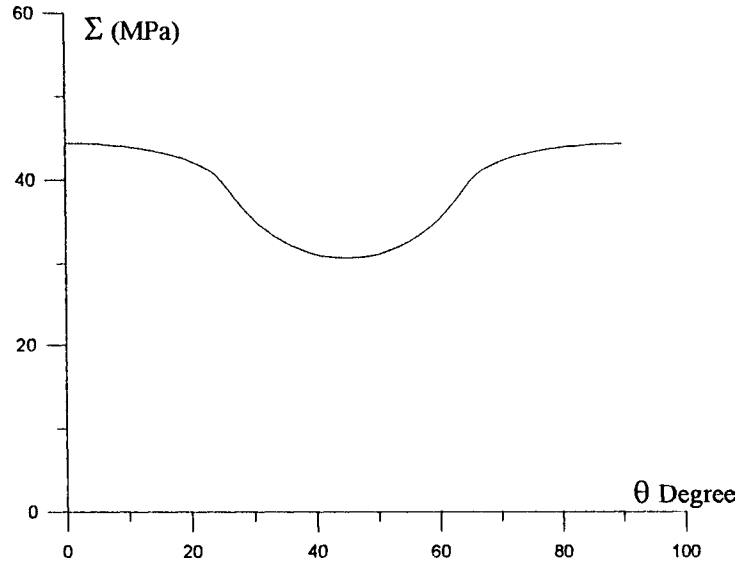


Figure 3. Off-axis tensile test.

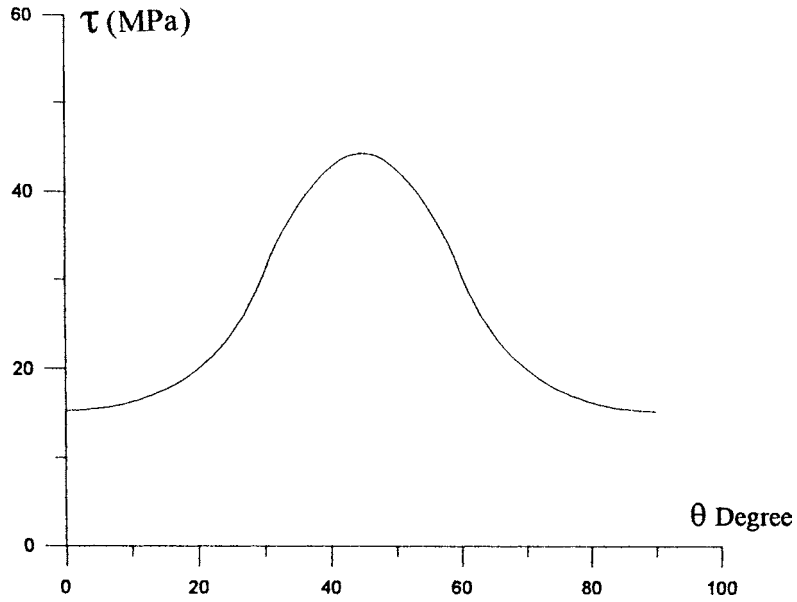


Figure 4. Shear test.

#### 4.1. Elastic-plastic localization

The decomposition of the elastic law given by equation (44) in  $S_P \oplus S_A$  gives:

$$\begin{pmatrix} \sigma_P \\ \varepsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{k}_{PP}^e & \mathbf{k}_{PA}^e \\ \mathbf{k}_{AP}^e & \mathbf{k}_{AA}^e \end{pmatrix} \begin{pmatrix} \varepsilon_P \\ \Sigma_A \end{pmatrix} + \begin{pmatrix} -\mathbf{k}_{PP}^e \varepsilon_P^p \\ \varepsilon_A^p - \mathbf{k}_{AP}^e \varepsilon_P^p \end{pmatrix}. \quad (45)$$

The homogenized elastic law is obtained by averaging equation (45):

$$\begin{pmatrix} \Sigma_P \\ \epsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{PP}^e & \mathbf{K}_{PA}^e \\ \mathbf{K}_{AP}^e & \mathbf{K}_{AA}^e \end{pmatrix} \begin{pmatrix} \epsilon_P \\ \Sigma_A \end{pmatrix} + \begin{pmatrix} -\langle \mathbf{k}_{PP}^e \epsilon_P^p \rangle \\ \epsilon_A^p - \mathbf{k}_{AP}^e \epsilon_P^p \end{pmatrix}, \quad (46)$$

with

$$\mathbf{K}^e = \langle \mathbf{k}^e \rangle.$$

A partial inversion of (46) gives:

$$\begin{pmatrix} \Sigma_P \\ \Sigma_A \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{PP} & \mathbf{A}_{PA} \\ \mathbf{A}_{AP} & \mathbf{A}_{AA} \end{pmatrix} \begin{pmatrix} \epsilon_P \\ \epsilon_A \end{pmatrix} - \begin{pmatrix} -\langle \mathbf{k}_{PP}^e \epsilon_P^p \rangle + \mathbf{K}_{PA}^e \mathbf{K}_{AA}^{e-1} \langle \epsilon_A^p - \mathbf{k}_{AP}^e \epsilon_P^p \rangle \\ \mathbf{K}_{AA}^e \langle \epsilon_A^p - \mathbf{k}_{AP}^e \epsilon_P^p \rangle \end{pmatrix}, \quad (47)$$

with the homogenized elastic operator  $\mathbf{A}$  obtained earlier by equation (16).

Identification with the classical elastic law

$$\begin{pmatrix} \Sigma_P \\ \Sigma_A \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{PP} & \mathbf{A}_{PA} \\ \mathbf{A}_{AP} & \mathbf{A}_{AA} \end{pmatrix} \begin{pmatrix} \epsilon_P - \epsilon_P^p \\ \epsilon_A - \epsilon_A^p \end{pmatrix} \quad (48)$$

show that the macroscopic plastic strain  $\epsilon_P^p$  and  $\epsilon_A^p$  are solutions of the following linear system:

$$\begin{aligned} \mathbf{A}_{PP} \epsilon_P^p + \mathbf{A}_{PA} \epsilon_A^p &= \langle \mathbf{k}_{PP}^e \epsilon_P^p \rangle + \mathbf{K}_{PA}^e \mathbf{K}_{AA}^{e-1} \langle \epsilon_A^p - \mathbf{k}_{AP}^e \epsilon_P^p \rangle, \\ \mathbf{A}_{AP} \epsilon_P^p + \mathbf{A}_{AA} \epsilon_A^p &= \mathbf{K}_{AA}^e \langle \epsilon_A^p - \mathbf{k}_{AP}^e \epsilon_P^p \rangle. \end{aligned} \quad (49)$$

This is easily solved as:

$$\begin{aligned} \epsilon_P^p &= \mathbf{K}_{PP}^{e-1} \langle \mathbf{k}_{PP}^e \epsilon_P^p \rangle \\ \epsilon_A^p &= \langle \epsilon_A^p - \mathbf{k}_{AP}^e \epsilon_P^p \rangle + \mathbf{K}_{AP}^e \epsilon_P^p. \end{aligned} \quad (50)$$

From eqns. (45) and (46) the local stress  $\sigma_P$  can be expressed as a function of macroscopic stress  $\Sigma$ , plastic strains  $\epsilon^p$  and the microscopic plastic strain  $\epsilon^p$ . This gives the stress condition:

$$\sigma_P = \mathbf{k}_{PP}^e \mathbf{K}_{PP}^{e-1} \Sigma_P + (\mathbf{k}_{PA}^e - \mathbf{k}_{PP}^e \mathbf{K}_{PP}^{e-1} \mathbf{k}_{PA}^e) \Sigma_A + \mathbf{k}_{PP}^e (\epsilon_P^p - \epsilon_P^p). \quad (51)$$

The strain condition can be found in the same way:

$$\sigma_A = (\mathbf{k}_{AP}^e - \mathbf{k}_{AA}^e \mathbf{K}_{AA}^{e-1} \mathbf{k}_{AP}^e) (\epsilon_P - \epsilon_P^p) + \mathbf{k}_{AA}^e \mathbf{K}_{AA}^{e-1} (\epsilon_A - \epsilon_A^p) + \mathbf{k}_{AP}^e (\epsilon_P^p - \epsilon_P^p) + \epsilon_A^p. \quad (52)$$

#### 4.2. Plastic evolution

As in the previous cases,  $d\epsilon_P^p$  and  $d\epsilon_A^p$  must be obtained as a function of  $d\Sigma_A$  and  $d\epsilon_P$ . This requires the determination of  $d\lambda$  which, as usually, will be derived from the consistency condition ( $df(\sigma) = 0$ ) (equation (24)):

$$\begin{pmatrix} d\sigma_P \\ d\sigma_A \end{pmatrix} \begin{pmatrix} \mathbf{H}_{PP} & \mathbf{H}_{PA} \\ \mathbf{H}_{AP} & \mathbf{H}_{AA} \end{pmatrix} \begin{pmatrix} \sigma_P \\ \Sigma_A \end{pmatrix} = 0. \quad (53)$$

Starting from equation (45), one has:

$$d\sigma_P = \mathbf{k}_{PP}^e d\epsilon_P + \mathbf{k}_{PA}^e d\Sigma_A - \mathbf{k}_{PP}^e d\epsilon_P^p. \quad (54)$$

Using equations (53) and (24), the plastic multiplier is finally written in the hybrid form:

$$\frac{d\lambda}{\sigma} = \frac{\sigma'_p \mathbf{k}_{PP}^e d\epsilon_P + (\sigma'_p \mathbf{k}_{PA}^e + \sigma'_A) d\Sigma_A}{\sigma'_p \mathbf{k}_{PP}^e \sigma'_P}, \quad (56)$$

where  $\sigma'$  denotes

$$\sigma' = \mathbf{H}\sigma.$$

After some computations, the plastic evolution laws (equation (24)) can then be written in  $S_P \oplus S_A$  as:

$$\begin{pmatrix} d\epsilon_P^p \\ d\epsilon_A^p \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{PP} & \mathbf{P}_{PA} \\ \mathbf{P}_{AP} & \mathbf{P}_{AA} \end{pmatrix} \begin{pmatrix} d\epsilon_P \\ d\Sigma_A \end{pmatrix}, \quad (57)$$

where  $\mathbf{P}_{PP}$ ,  $\mathbf{P}_{PA}$ ,  $\mathbf{P}_{AP}$  and  $\mathbf{P}_{AA}$  denote respectively:

$$\begin{aligned} \mathbf{P}_{PP} &= \frac{(\mathbf{k}_{PP}^e \sigma'_P)^T \otimes \sigma'_P}{\sigma'_p \mathbf{k}_{PP}^e \sigma'_P}, & \mathbf{P}_{PA} &= \frac{(\mathbf{k}_{PA}^e \sigma'_P + \sigma'_A)^T \otimes \sigma'_P}{\sigma'_p \mathbf{k}_{PP}^e \sigma'_P}, \\ \mathbf{P}_{AP} &= \frac{(\mathbf{k}_{PP}^e \sigma'_P)^T \otimes \sigma'_A}{\sigma'_p \mathbf{k}_{PP}^e \sigma'_P}, & \mathbf{P}_{AA} &= \frac{(\mathbf{k}_{PA}^e \sigma'_P + \sigma'_A)^T \otimes \sigma'_A}{\sigma'_p \mathbf{k}_{PP}^e \sigma'_P}. \end{aligned} \quad (58)$$

It is clear in equation (58) that local plastic evolution depends on macroscopic stress  $\Sigma$ , local plastic strain  $\epsilon^p$  and the elastic property  $\mathbf{k}^e$  of the considered layer. Hence, the decomposition in  $S_P \oplus S_A$  allows a formulation of the plastic evolution of any layer independently from the mechanical properties of the others.

### 4.3. Homogenized incremental behaviour

The incremental behaviour law, relating the microscopic variables ( $d\sigma_P$  and  $d\epsilon_A$ ) of each layer and the macroscopic ones ( $d\epsilon_P$  and  $d\Sigma_A$ ), is determined directly by using equations (45), (57) and (58):

$$\begin{pmatrix} d\sigma_P \\ d\epsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{k}_{PP}^{ep} & \mathbf{k}_{AP}^{ep} \\ \mathbf{k}_{AP}^{ep} & \mathbf{k}_{AA}^{ep} \end{pmatrix} \begin{pmatrix} d\epsilon_P \\ d\Sigma_A \end{pmatrix},$$

$$\mathbf{k}_{PP}^{ep} = \mathbf{k}_{PP}^e - \mathbf{k}_{PP}^e \mathbf{P}_{PP}, \quad \mathbf{k}_{AA}^{ep} = \mathbf{k}_{AA}^e - \mathbf{P}_{AA} - \mathbf{k}_{AP}^e \mathbf{P}_{PA}, \quad \mathbf{k}_{PA}^{ep} = -\mathbf{k}_{PA}^{epT} = \mathbf{k}_{PA}^e - \mathbf{k}_{PP}^e \mathbf{P}_{PA}. \quad (59)$$

The homogenized incremental behaviour directly follows by averaging this relation:

$$\begin{pmatrix} d\Sigma_P \\ d\epsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{PP}^{ep} & \mathbf{K}_{AP}^{ep} \\ \mathbf{K}_{AP}^{ep} & \mathbf{K}_{AA}^{ep-1} \end{pmatrix} \begin{pmatrix} d\epsilon_P \\ d\Sigma_A \end{pmatrix}, \quad \mathbf{K}^{ep} = \langle \mathbf{k}^{ep} \rangle. \quad (60)$$

Considering for instance a loading history with prescribed stress, this incremental behaviour law becomes:

$$\begin{pmatrix} d\epsilon_P \\ d\epsilon_A \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{PP}^{ep-1} & -\mathbf{K}_{PP}^{ep-1} \mathbf{K}_{AA}^{ep} \\ \mathbf{K}_{AP}^{ep} \mathbf{K}_{PP}^{ep-1} & \mathbf{K}_{AA}^{ep} - \mathbf{K}_{AP}^{ep} \mathbf{K}_{PP}^{ep-1} \mathbf{K}_{PA}^{ep} \end{pmatrix} \begin{pmatrix} d\Sigma_P \\ d\Sigma_A \end{pmatrix}, \quad (61)$$

which has to be completed by the plastic evolution law in each layer:

$$\begin{pmatrix} d\epsilon_P^p \\ d\epsilon_A^p \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{PP} \mathbf{K}_{PP}^{ep-1} & \mathbf{P}_{PA} - \mathbf{P}_{PP} \mathbf{K}_{PP}^{ep-1} \mathbf{K}_{PA}^{ep} \\ \mathbf{P}_{AP} \mathbf{K}_{PP}^{ep-1} & \mathbf{P}_{AA} - \mathbf{P}_{AP} \mathbf{K}_{PP}^{ep-1} \mathbf{K}_{PA}^{ep} \end{pmatrix} \begin{pmatrix} d\Sigma_P \\ d\Sigma_A \end{pmatrix}. \quad (62)$$

A loading history with prescribed strain can be analysed in the same way.

## 5. Application

### 5.1. Specialisation of the model

As an example, we shall consider the case of isotropic layers with homogeneous elasticity (Young's modulus  $E$ , Poisson's ratio  $\nu$ ) and heterogeneous plasticity. So, the layers differ only by their yield stress.

Homogeneous elasticity ( $\mathbf{A} = \mathbf{a}$  or  $\mathbf{K}^e = \mathbf{k}^e$ ) results in

$$\begin{pmatrix} \Sigma_P \\ \Sigma_A \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{PP} & \mathbf{A}_{PA} \\ \mathbf{A}_{AP} & \mathbf{A}_{AA} \end{pmatrix} \begin{pmatrix} \epsilon_P - \epsilon_P^p \\ \epsilon_A - \epsilon_A^p \end{pmatrix}, \quad (63)$$

so that macroscopic plastic strain given in equation (50) becomes:

$$\epsilon^p = \langle \epsilon^p \rangle \quad (64)$$

and the local stress (equation (51)) and strain (equation (52)) are:

$$\sigma_P = \Sigma_P + \mathbf{k}_{PP}^e (\epsilon_P^p - \epsilon_P^p), \quad (65)$$

$$\epsilon_A = (\epsilon_A - \epsilon_A^p) + \mathbf{k}_{AP}^e (\epsilon_P^p - \epsilon_P^p) + \epsilon_A^p. \quad (66)$$

Using equations (20) and (34), the plastic evolution law (equations (58)) becomes:

$$\mathbf{P}_{PP} = \frac{1}{\Delta} \begin{pmatrix} \frac{1}{4} \frac{E}{1-\nu} s^2 & \frac{3}{4} \frac{E}{1+\nu} s \sigma_{II} & \frac{3}{4} \frac{E}{1+\nu} s \sigma_{III} \\ \frac{3}{4} \frac{E}{1+\nu} s \sigma_{II} & \frac{9}{4} \frac{E}{1+\nu} \sigma_{II}^2 & \frac{9}{4} \frac{E}{1+\nu} \sigma_{III} \sigma_{II} \\ \frac{3}{4} \frac{E}{1+\nu} s \sigma_{III} & \frac{9}{4} \frac{E}{1+\nu} \sigma_{III} \sigma_{II} & \frac{9}{4} \frac{E}{1+\nu} \sigma_{III}^2 \end{pmatrix},$$

$$\mathbf{P}_{PA} = \frac{1}{\Delta} \begin{pmatrix} \frac{3}{4} s \Sigma_{IV} & \frac{3}{4} s \Sigma_V & \frac{1}{2\sqrt{2}} \frac{1-2\nu}{1-\nu} s^2 \\ \frac{9}{4} \Sigma_{IV} \sigma_{II} & \frac{9}{4} \Sigma_V \sigma_{II} & \frac{3}{2\sqrt{2}} \frac{1-2\nu}{1-\nu} s \sigma_{II} \\ \frac{9}{4} \Sigma_{IV} \sigma_{III} & \frac{9}{4} \Sigma_V \sigma_{III} & \frac{3}{2\sqrt{2}} \frac{1-2\nu}{1-\nu} s \sigma_{III} \end{pmatrix},$$

$$\mathbf{P}_{AP} = \frac{1}{\Delta} \begin{pmatrix} \frac{3}{4} \frac{E}{1-\nu} s \Sigma_{IV} & \frac{9}{4} \frac{E}{1+\nu} \sigma_{II} \Sigma_{IV} & \frac{9}{4} \frac{E}{1+\nu} \sigma_{III} \Sigma_{IV} \\ \frac{3}{4} \frac{E}{1-\nu} s \Sigma_V & \frac{9}{4} \frac{E}{1+\nu} \sigma_{II} \Sigma_V & \frac{9}{4} \frac{E}{1+\nu} \sigma_{III} \Sigma_V \\ \frac{-1}{2\sqrt{2}} \frac{E}{1-\nu} s^2 & \frac{-3}{2\sqrt{2}} \frac{E}{1+\nu} s \sigma_{II} & \frac{9}{4} \frac{E}{1+\nu} \sigma_{III} s \end{pmatrix}, \quad (67)$$

$$\mathbf{P}_{AA} = \frac{1}{\Delta} \begin{pmatrix} \frac{9}{4} \Sigma_{IV}^2 & \frac{9}{4} \Sigma_{IV} \Sigma_V & \frac{3}{2\sqrt{2}} \frac{1-2\nu}{1-\nu} \Sigma_{IV} \\ \frac{9}{4} \Sigma_{IV} \sigma_{II} & \frac{9}{4} \Sigma_V \sigma_{II} & \frac{3}{2\sqrt{2}} \frac{1-2\nu}{1-\nu} s \Sigma_V \\ \frac{-3}{2\sqrt{2}} \Sigma_{IV} s & \frac{-3}{2\sqrt{2}} \Sigma_V s & \frac{-1}{\sqrt{2}} \frac{1-2\nu}{1-\nu} s^2 \end{pmatrix}$$

with

$$s = (\sigma_1 - \sqrt{2} \Sigma_{IV}), \quad \Delta = \sigma_P^D \mathbf{k}_{PP}^e \sigma_P^D = \frac{1}{4} \frac{E}{1-\nu} s^2 + \frac{9}{4} \frac{E}{1+\nu} (\sigma_{II}^2 + \sigma_{III}^2).$$

Substitution of these relations in the formulation derived in section 4.3 will allow the determination of the mechanical response for loading history under prescribed stress or strain.

## 5.2. Anisotropic behaviour

As an example we shall consider an elastic-plastic layered composite representation of an aluminium alloy. The homogeneous elastic properties are taken as

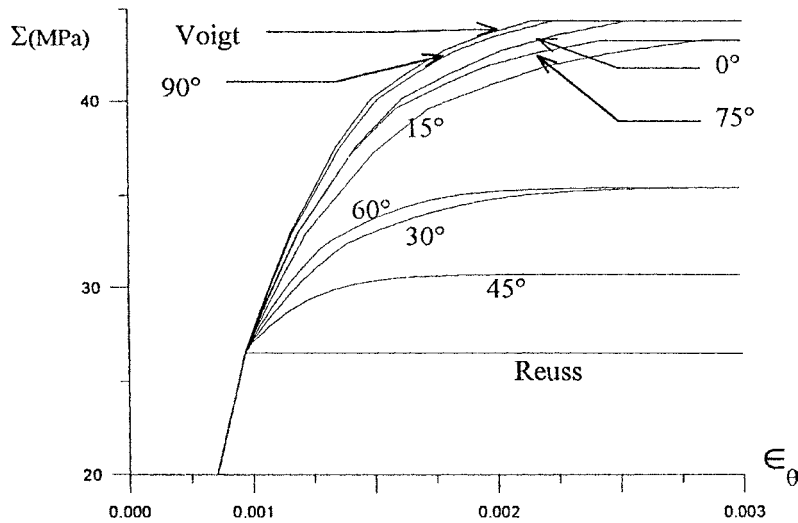
$$E = 56700 \text{ MPa}; \quad \nu = 0.35.$$

While the heterogeneous plasticity consists of eight layers as described in section 3 (*table I*).

The corresponding anisotropic behaviour is described in off-axis tensile tests. Different mechanical responses are given in *figure 5* for different values of  $\theta$ . Voigt's and Reuss's approximations have also been represented

**Table I.** Yield limit distribution.

$y$ (MPa)	26.59	37.68	48.95	57.78	67.16	75.96	85.90	98.77
$g(y)$	0.4170	0.1832	0.1065	0.1413	0.0160	0.0505	0.0250	0.0600



**Figure 5.** Tensile test ( $\epsilon_\theta = \epsilon_{ij} n_i n_j$  vs  $\Sigma$ ).



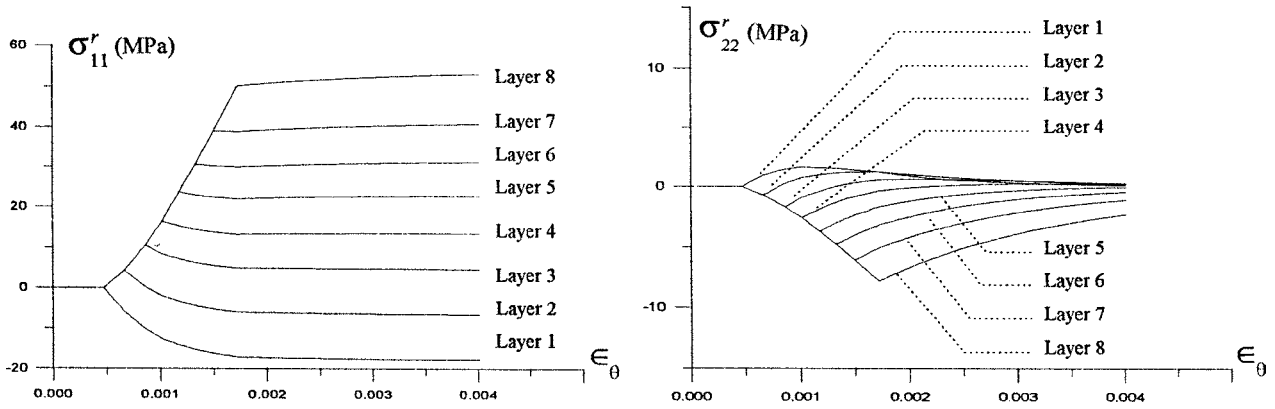


Figure 6. Residual stresses  $\sigma_{11}^r = \sigma_{22}^r$  in the case of  $\theta = 0$ .

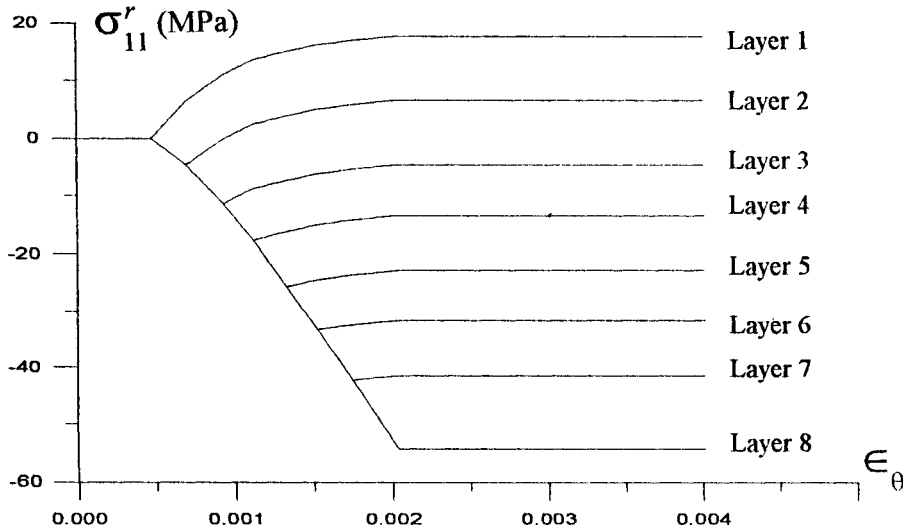


Figure 7. Residual stresses in the case of  $\theta = \pi/2$ .

for comparison. As already observed in the case of a two-phase composite (El Omri and Sidoroff, 1991), strong anisotropy and non-linear hardening are observed.

The obtained behaviour in the layer direction ( $\theta = 90^\circ$ ) and in the normal direction ( $\theta = 0^\circ$ ) are rather close from Voigt's approximation (close in fact for  $\theta = 90^\circ$ ). More generally, close behaviours are obtained for symmetrical directions  $\theta$  and  $\pi/2 - \theta$ . This is related to similar symmetry as obtained in rigid plasticity.

Also, it should be noted that for  $\theta = 0^\circ$  and  $\theta = 90^\circ$ , plasticity is obtained in each layer, in agreement with Voigt's model. On the contrary, for  $\theta = 45^\circ$  plasticity is localized in the two weakest layers. In fact the number of plastified layers decreases as  $\theta$  decreases from  $90^\circ$  to  $45^\circ$  or increases from  $0^\circ$  to  $45^\circ$ . The Reuss's model that corresponds to one plastified layer therefore is not good approximation. In fact, as already observed in (El Omri and Sidoroff, 1991), the resulting anisotropic behaviour is rather complex.

It may also be interesting to describe evolution of the residual stress  $\sigma^r$  after unloading. This is done in figure 6 and figure 7 for the directions  $\theta = 0^\circ$  ( $\sigma_{11}^r = \sigma_{22}^r$ ) and  $\theta = 90^\circ$  ( $\sigma_{11}^r \neq \sigma_{22}^r$ ). The residual stresses in each layer are found to tend to a fixed value for large deformations.

## 6. Conclusion

The layered geometry allows a complete analysis of the homogenized behaviour. This is due to the fact that the localization condition allows to homogenized behaviour to be obtained by averaging the local behaviour using a hybrid mixture law. This approach could be extended to other types of behaviour (visco-elastic, viscoplastic and so on).

From a practical point of view we can observe that, even with this very simple layered geometrical structure, the anisotropic behaviour of a heterogeneous elastic plastic material may be very complex. The Voigt and Reuss's models frequently used in discussing dislocation microstructures should therefore be used with care.

## Acknowledgements

This work was supported by the program P.A.R.S.–S.P.I. 101.

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