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Effective plastic properties of laminates
made of isotropic elastic plastic materials

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Abstract

The effective elastic behavior of laminates is well understood. In this contribution, we make use of the fact that the elastic homogenization can be extended relatively easy to the plastic case (He and Z.-Q. Feng, 2012), which is interesting for bilayer metallic structures. With the aid of stress concentration tensors, the effective yield limit is calculated, and its properties are examined. It turns out to be anisotropic, pressure-dependent, non-smooth, and evolves with the plastic deformation. Using the constitutive equations of von Mises elastic plastic materials, the effective plastic flow behavior is obtained as a set of coupled nonlinear ordinary differential equations with algebraic conditions. A general solution can hardly be given, but it can be integrated numerically. Further, it is possible to examine some properties of its solution without explicitly stating it, such as its poles. The latter allow to extract closed form-expressions for the ultimate loading stress in monotonic stress-driven tests. All analytical results are compared to representative volume element simulations, and found to match very well.

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1 Introduction

Composite materials are of constant interest to engineers and researchers, as they often display desirable properties that can moreover be adjusted by the microstructure design. A nice example is the combination of stiff brittle glass layers with a small fraction of soft ductile glue, which results in a tremendous increase of fracture resistance, while the high effective stiffness of the glass is retained (Fratzl et al., 2007).

The simplest conceivable composites are laminates (Milton, 2002). As such, a large amount of literature is dedicated to them, which is compiled in textbooks such as Reddy (2004). Many contributions focus on the full 3D effective elastic properties (Backus, 1962; Bogetti, Hoppel, and Drysdale, 1995; Chou, Carleone, and Hsu, 1972; Dumontet, 1982; Francfort and Murat, 1986; He and Z.-Q. Feng, 2012; Liu, X. Feng, and Zhang, 2009; Pagano, 1974; Postma, 1955; Salamon, 1968; Tartar, 1985), in some cases restricted to isotropic constituents. A compact summary of the effective stiffness of laminates made up of arbitrary many anisotropic layers is given in (Glüge and Kalisch, 2014), and shown to be a generalization of classical laminate theory (Berthelot, 1999).

The stiffness of laminates often serves as a reference solution for iterated microstructures, either through interface orientation averages (Kalisch and Glüge, 2015) or nested (rank n) laminates (deBotton, 2005; deBotton and Hariton, 2002). Likewise, many contributions deal with material failure and fatigue under cyclic loading, like delamination and fiber fracture in carbon fiber laminates (Bogetti, Hoppel, Harik, et al., 2004; Hu, Lin, and Tu, 2015; Jalalvand, Hosseini-Toudeshky, and Mohammadi, 2013). For such composites, these properties are more important than the plastic properties.

However, there are also laminates that contain metallic layers, which considerably deform plastically before material failure occurs. Such laminates arise naturally as lamellar eutectics (Trepczyńska-Lent, 2010), in forming processes (Sakai et al., 1992) or by martensitic transformations or by twinning (Bhattacharya, 2003). They may also be manufactured by accumulative rolling or explosive welding of different metallic or intermetallic layers (Bataev et al., 2012; Hausöl, Höppel, and Göken, 2011) or by gluing thin metallic layers and fiber reinforced layers (Sinnmazcelik et al., 2011). They often display an increased strength and ductility compared to the pure phases.

There is much less literature on the plastic properties of such laminates. The probably most advanced works in this regard are deBotton and Ponte-Castañeda (1992), El Omri et al. (2000), Ponte-Castañeda and deBotton (1992), Pruchnicki (1998a,b,c), and Sawicki (1981) and He and Z.-Q. Feng (2012). Sawicki (1981) considered the effective yield limit by using the stress concentration tensors, as will be shown later. Ponte-Castañeda and deBotton (1992) approach the post-yield behavior, using the deformation theory of plasticity (i.e. a nonlinear elastic law mimicking plasticity) and a variational homogenization approach. Consequently, their approximations may serve in case of monotonic processes. Pruchnicki (1998c) used the same nonlinear elastic approach with an asymptotic expansion for the homogenization. It was apparently Pruchnicki (1998a,b) who firstly considered the evolution of internal state variables in this context. He derived closed form expressions for the effective elasticity, the effective yield function and the effective hardening rule. Later, El Omri et al. (2000) and He and Z.-Q. Feng (2012) recognized that the analytical homogenization of the elasticity of laminates can be transferred relatively easy to the elastoplastic rate form. A detailed comparison to the works of Pruchnicki, El Omri et al. (2000) and He and Z.-Q. Feng (2012) is postponed to Sec. 4.4.

Outline. After describing the notation and the setting (Sects 2 and 3), we build on the closed-form expressions of the effective stiffness and stress concentration tensors for purely elastic laminates (given in Glüge and Kalisch (2014), Sec. 3.4). Assigning yield limits to the phases, this allows already to examine the properties of the effective yield limit (Sec. 4.1). Further, with the constitutive equations of isotropic von Mises plasticity and some very mild assumptions, it is then possible to derive the ordinary differential equations that govern the
plastic response (Sec. 4.2). Being a coupled nonlinear ODE system, its general solution is not accessible, but its poles can be extracted easily. These allow to examine properties beyond the onset of plastic flow, like the ultimate loading in monotonic tensile tests (Sec. 4.2.1), for which we give the explicit solution. We finally put our results into context with the findings of other researchers (Sec. 4.4).

2 Notation and setting

Notation. A direct notation is preferred. Vectors are denoted as bold minuscules (like $\mathbf{a}$), second-order tensors as bold majuscules or bold greek letters (like $\mathbf{A}, \mathbf{e}, \mathbf{\sigma}$), and fourth-order tensors as blackboard bold majuscules (like $\mathbb{A}$). The dyadic product and single scalar contractions are denoted like $(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) \cdot (\mathbf{d} \otimes \mathbf{e}) = (\mathbf{b} \cdot \mathbf{d}) (\mathbf{c} \cdot \mathbf{e}) \mathbf{a}$, with $\cdot$ being the usual scalar product between vectors. The upper index $^T$ denotes the transpose of a second-order tensor, $(\mathbf{a} \otimes \mathbf{b})^T := \mathbf{b} \otimes \mathbf{a}$. The upper index $^S$ is used to define symmetric parts for second- and fourth-order tensors,

$$\mathbf{A}^S := \frac{1}{2} (\mathbf{A} + \mathbf{A}^T),$$

$$\mathbf{A}^S := \frac{1}{4} (A_{ijkl} + A_{jikl} + A_{ijlk} + A_{jilk}) \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \quad (2)$$

Components are given w.r.t. orthonormalized bases $\{\mathbf{e}_i\}$ or $\{\mathbf{E}_I\}$, where

$$\mathbf{E}_1 := \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{E}_4 := \sqrt{2} (\mathbf{e}_2 \otimes \mathbf{e}_3)^S, \quad (3)$$

$$\mathbf{E}_2 := \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{E}_5 := \sqrt{2} (\mathbf{e}_1 \otimes \mathbf{e}_3)^S, \quad (4)$$

$$\mathbf{E}_3 := \mathbf{e}_3 \otimes \mathbf{e}_3, \quad \mathbf{E}_6 := \sqrt{2} (\mathbf{e}_1 \otimes \mathbf{e}_2)^S. \quad (5)$$

$\mathbf{E}_{1...6}$ form a basis is for symmetric second order tensors. These can be further decomposed into their deviatoric and dilatropic parts $\mathbf{A}'$ and $\mathbf{A}^o$,

$$\mathbf{A} = \mathbf{A}' - \frac{\text{tr}(\mathbf{A})}{3} \mathbf{I} + \frac{\text{tr}(\mathbf{A})}{3} \mathbf{I}^o. \quad (6)$$

We make use of Einstein’s summation convention with implicit summation from 1 to 3 over minuscule and 1 to 6 over majuscule indices that appear pairwise in a product. For example, the second order identity tensor and the identity on symmetric order tensors are given by

$$\mathbf{I} = \mathbf{e}_i \otimes \mathbf{e}_i = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3, \quad I = \frac{1}{2} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i) = \mathbf{E}_I \otimes \mathbf{E}_I. \quad (7)$$

Setting. We consider laminates that consist of two isotropic elastic, isotropic von Mises-plastic layers without hardening in the small strain setting. The two materials are distinguished by the upper indices $^+$ and $^-$. Further, without loss of generality, we take the laminate normal $\mathbf{n}$ parallel to the $\mathbf{e}_3$-direction, see Fig. 1.

3 Basic equations

In the following subsections, we collect the basic equations that we need. While jump conditions (Sec. 3.1), mixture rule (Sec. 3.2) and von Mises plasticity (Sec. 3.3) are well known, the analysis of the stress concentration tensors in case of isotropic phases (Sec. 3.4) appears to be not documented elsewhere\textsuperscript{1}. The latter play an important role by relating the effective stresses to the partial stresses, which enter the individual constitutive laws. It turns out that the stress concentration tensors affect largely the effective plastic behavior.

\textsuperscript{1}Similar but more involved analysis exist for cracks and bounded inclusions, see Barthélémy (2009) and Nazarov (2009). In the present laminate case the analysis is particularly simple, as the phases remain homogeneous, see Sec. 4.2.
3.1 Jump conditions at the interface

The well known kinematic and dynamic compatibility conditions at the interface are

\[ \mathbf{E}^+ - \mathbf{E}^- = (\mathbf{a} \otimes \mathbf{n})^s, \]  
\[ (\mathbf{T}^+ - \mathbf{T}^-) \cdot \mathbf{n} = 0, \]  

respectively, with the strains \( \mathbf{E}^\pm \) and stresses \( \mathbf{T}^\pm \) on either side of the laminate. One can decompose both \( \mathbf{E} \) and \( \mathbf{T} \) into its continuous and its discontinuous parts. With our choice \( \mathbf{e}_3 = \mathbf{n} \), this decomposition is

\[ \mathbf{E}^\pm = \begin{bmatrix} E_{11}^\pm & E_{12}^\pm & E_{13}^\pm \\ E_{12}^\pm & E_{22}^\pm & E_{23}^\pm \\ E_{13}^\pm & E_{23}^\pm & E_{33}^\pm \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j, \]
\[ \mathbf{T}^\pm = \begin{bmatrix} T_{11}^\pm & T_{12}^\pm & T_{13}^\pm \\ T_{12}^\pm & T_{22}^\pm & T_{23}^\pm \\ T_{13}^\pm & T_{23}^\pm & T_{33}^\pm \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j, \]

In terms of block matrices we have

\[ \mathbf{E}^\pm = \begin{bmatrix} E_{11}^\pm & E_{12}^\pm \\ E_{12}^\pm & E_{22}^\pm \\ E_{13}^\pm & E_{23}^\pm \end{bmatrix}, \]
\[ \mathbf{T}^\pm = \begin{bmatrix} T_{11}^\pm & T_{12}^\pm & T_{13}^\pm \\ T_{12}^\pm & T_{22}^\pm & T_{23}^\pm \\ T_{13}^\pm & T_{23}^\pm & T_{33}^\pm \end{bmatrix}, \]

where the shaded, continuous parts of the component matrices are common to \( \mathbf{E}^\pm \) and \( \mathbf{T}^\pm \). Hence, we need not to distinguish the continuous parts on either side of the phases. The continuous and discontinuous parts of \( \mathbf{E}^\pm \) and \( \mathbf{T}^\pm \) are perpendicular to each other (Laws, 1975), i.e.,

\[ \mathbf{T}_{\text{cont}}^\pm \cdot \mathbf{E}_{\text{cont}}^\pm = 0, \]
\[ \mathbf{T}_{\text{discont}}^\pm \cdot \mathbf{E}_{\text{discont}}^\pm = 0. \]

One can define fourth order interface projection tensors that filter these parts from \( \mathbf{E}^\pm \) and \( \mathbf{T}^\pm \) (He and Z.-Q. Feng, 2012; Hill, 1983).

3.2 Rules of mixture

Presuming homogeneity in each layer, the effective stresses and strains are related to the partial stresses and strains by

\[ \mathbf{T} = v^- \mathbf{T}^- + v^+ \mathbf{T}^+, \]
\[ \mathbf{E} = v^- \mathbf{E}^- + v^+ \mathbf{E}^+, \]

where \( v^+ + v^- = 1 \) are the volume fractions. This holds actually only in absence of localization, which we presume throughout the text. The mathematical requirements for this to hold are certain convexity requirements on the
(incremental) material potentials (Schröder, 2010). For linear elasticity in the small strain setting this is not an issue, but for plasticity, some minimum hardening is needed to suppress localization (Bartels, Mielke, and Roubíček, 2012).

3.3 Material law

The equations of isotropic, von Mises elastic plastic material behavior are well known, and therefore reproduced only in due brevity:

\[ \text{strain decomposition:} \quad E = E_e + E_p \]  
\[ \text{elastic law:} \quad T = 3K E_e^\circ + 2G E_e' \]  
\[ \text{yield criterion:} \quad \phi(T) = \sqrt{\frac{3}{2}} \|T'\| - \sigma_F \]  
\[ \text{loading condition:} \quad \phi(T) = 0, \quad \dot{\phi}(T)|_{E_p=\text{const.}} > 0 \]  
\[ \text{associative flow rule:} \quad \dot{E}_p = \lambda N, \quad N = \phi'(T) = \frac{1}{\|T'\|} T' \]  
\[ \text{consistency condition:} \quad \ddot{\phi}(T) = 0 \quad \text{during plastic flow:} \quad \lambda = N \cdot \dot{E} \]

They can be found in textbooks on continuum mechanics, like Bertram and Gluge (2015). The simple equation for the consistency parameter \( \lambda \) is due to the elastic isotropy and the lack of hardening. The flow stress \( \sigma_F \) is the stress at which plastic deformations occur in a tensile test. \( K \) and \( G \) are the elastic compression and shear moduli, respectively. Here, we abbreviate the elastic law with the more general linear mapping

\[ T = C \cdot E, \]  
\[ C = 3K P^\circ + 2G P', \]

with the isotropic projectors \( P^\circ = \frac{1}{3} I \otimes I, P' = I - P^\circ \), which extract the dilatonic and deviatoric parts of a second order tensor,

\[ E^\circ = P^\circ \cdot E, \]  
\[ E' = P' \cdot E. \]

The key ingredient that allows to extend the analytical homogenization from elasticity to plasticity is the similarity of the elastic law (Eq. 23) to the stress rate in case of plastic yielding. This has been recognised by El Omri et al. (2000) He and Z.-Q. Feng (2012) (Sec. 3.1 in He and Z.-Q. Feng (2012)). Upon inserting the strain decomposition (Eq. 17) and then the flow rule (Eq. 21) into the rate form of the elastic law we obtain

\[ \dot{T} = C \cdot (\dot{E} - \dot{E}_p) \]  
\[ = C \cdot (\dot{E} - \frac{3}{2\sigma_F} T' \otimes T' \cdot \dot{E}) \]  
\[ = C \cdot (I - \frac{3}{2\sigma_F} T' \otimes T') \cdot \dot{E} \]  
\[ = C(T') \cdot \dot{E}. \]

The tangential modulus \( \tilde{C} \) depends on \( T \), which eventually gives a system of ordinary differential equations, but the algebraic structure is the same as in case of total form elasticity and rate-form plasticity (see Sec. 4.2).
3.4 Stress concentration tensors

In linear elasticity, the jump conditions, mixture rules and elastic laws (Eqs 8, 9, 15, 16, 23) constitute a linear system for $\mathbf{E}$ when $\mathbf{T}$ is given, or vice versa, the coefficients of which being the effective stiffness tetrad $\mathbf{C}$,

$$\mathbf{T} = \mathbf{C} \cdot \mathbf{E},$$

$$\mathbf{C} = v^+ \mathbf{C}^+ + v^- \mathbf{C}^- - v^+ v^- \Delta \mathbf{C} \cdot \mathbf{Z} \cdot \Delta \mathbf{C},$$

$$\Delta \mathbf{C} = \mathbf{C}^+ - \mathbf{C}^-,$$

$$\mathbf{Z} = \mathbf{n} \otimes \mathbf{Z} \otimes \mathbf{n},$$

$$\mathbf{Z} = (v^+ \mathbf{A}^- + v^- \mathbf{A}^+)^{-1},$$

$$\mathbf{A}^\pm = \mathbf{n} \cdot \mathbf{C}^\pm \cdot \mathbf{n}.$$ (36)

Further, one can derive stress and strain concentration tensors $\mathbf{L}^\pm$ that relate the effective stresses and strains to the partial stresses and strains,

$$\mathbf{E}^\pm = (\mathbf{I} \mp v^+ \mathbf{Z}^S \cdot \Delta \mathbf{C}) \cdot \mathbf{E},$$

$$\mathbf{T}^\pm = \mathbf{C}^\pm \cdot \left(\mathbf{I} \mp v^+ \mathbf{Z}^S \cdot \Delta \mathbf{C}\right) \cdot \mathbf{E}^{-1} \cdot \mathbf{T}.$$ (38)

A derivation of these tensors is given in Glüge and Kalisch (2014), but the concept is rather old. They have been derived and used by Sawicki (1978, 1981). Here, we want to discuss their properties in more detail. For isotropic materials and the laminate normal $\mathbf{e}_3$, the partial stresses in terms of the effective stresses and the stress concentration tensors take the form

$$\mathbf{T}^\pm = \mathbf{L}^\pm \cdot \mathbf{T},$$

$$\mathbf{L}^\pm = \begin{bmatrix} L_{11}^\pm & L_{12}^\pm & L_{13}^\pm \\ L_{12}^\pm & L_{12}^\pm & L_{13}^\pm \\ 1 & 1 & 1 \\ \end{bmatrix} \mathbf{E}_I \otimes \mathbf{E}_J,$$ (40)

$$L_{11}^\pm = \frac{G^\pm}{G^V} (1 + X^\pm),$$

$$L_{12}^\pm = \frac{G^\pm}{G^V} X^\pm,$$ (42)

$$L_{13}^\pm = -v^+ \frac{(A^\pm)^2 B^\mp - (A^\mp)^2 B^\pm + B^\pm B^\mp (\Delta A - 2 \Delta B)}{6 N^\pm},$$ (43)

$$L_{66}^\pm = L_{11}^\pm - L_{12}^\pm = \frac{G^\pm}{G^V},$$

$$X^\pm = \frac{v^+ G^V (A^\mp B^\mp - A^\pm B^\pm)}{3 N^\pm}$$

$$N^\pm = v^+ G^V K^\mp A^\pm + v^+ G^\mp K^\pm A^\mp,$$ (46)

where the abbreviations are

$$\Delta (\cdot) = (\cdot)^+ - (\cdot)^-,$$

$$v^\pm = v^+ (\cdot)^+ + v^- (\cdot)^-,$$

$$A^\pm = K^\pm + \frac{4}{3} G^\pm,$$ (49)

$$B^\pm = K^\pm - \frac{2}{3} G^\pm.$$ (50)

The inverse concentration tensors are also referred to as polarization tensors (Hill, 1963).
The $L^\pm$-tensors have been derived for purely elastic isotropic phases, i.e. they are valid as long as $E^+_p = E^-_p = 0$ holds (Glüge and Kalisch, 2014; Sawicki, 1978). They take different values otherwise. Thus, the effective yield surface that is discussed later holds only for the plastically undeformed material.

### 3.4.1 Properties of the concentration tensors.

A key ingredient to the proper analysis of the effective flow criterion are the stress concentration tensors $L^\pm$. They satisfy

$$ I = v^+ L^+ + v^- L^- . $$

Further, they are because of $G^\pm > 0$ and $K^\pm > 0$ strictly positive definite. Four out of the six eigenvalues are found directly on the principle diagonal in Eq. (40). The remaining two eigenvalues satisfy

$$ \lambda_1 \lambda_2 = \left( \frac{G_\pm}{G^\mp} \right)^2 (1 + 2X^\pm) , $$

$$ \lambda_1 + \lambda_2 = \frac{G_\pm}{G^\mp} (2 + 2X^\pm) . $$

Positivity of $\lambda_{1,2}$ is ensured by $X^\pm > -1/2$. Due to the positivity of $K^\pm$ and $G^\pm$, also $A^\pm > 0$ and $N^\pm > 0$ hold. Taking the limit $K^\pm \to 0$ from above leads to the smallest possible value for $X^\pm$, and yields $X^\pm \to -1/2$ from above. The eigenvalues of $L^\pm$ are

$$ \lambda_{1,6}^\pm = \frac{G^\pm}{G^\mp} , $$

$$ \lambda_2^\pm = \frac{G^\pm}{G^\mp} (1 + 2X^\pm) , $$

$$ \lambda_{3,4,5} = 1 . $$

Thus, there are only three different eigenvalues and eigenspaces. The numbering corresponds (up to $\lambda_{1,2}$) to the position on the principle diagonal in $L^\pm$ in Eq. (40). The 2nd-order right eigentensors of $L^\pm$ are

$$ V_1 = \frac{1}{\sqrt{2}} (E_1 - E_2) , $$

$$ V_2 = \frac{1}{\sqrt{2}} (E_1 + E_2) , $$

$$ V_3^\pm = \frac{\kappa^\pm - y^\pm}{\kappa^\pm + 2y^\pm} (E_1 + E_2) + E_3 , $$

$$ V_4 = E_4 , $$

$$ V_5 = E_5 , $$

$$ V_6 = E_6 , $$

$$ \kappa^\pm = 3K^\pm K^\mp (G^\mp - G^\pm) , $$

$$ y^\pm = 2G^\pm G^\mp (K^\mp - K^\pm) . $$

From the eigentensors, only $V_3^\pm$ depends on the elasticity constants. Further, the eigentensors do not depend on the volume fractions, unlike the eigenvalues. Note that $V_3^+$ is not normalized for the sake of simplicity. Also, all eigentensors are orthogonal to each other except $V_2$ and $V_3^\pm$. We can therefore give easily the left eigentensors $V_{Li}$ which are the dual vectors to the right eigentensors, i.e. $V_{Li} \cdot V_j = \delta_{ij}$. Of these, only $V_{L2,3}$ differ from the right eigentensors,

$$ V_{L2}^\pm = \frac{1}{\sqrt{2}} (E_1 + E_2) - \sqrt{2} \frac{\kappa^\pm - y^\pm}{\kappa^\pm + 2y^\pm} E_3 , $$

$$ V_{L3} = E_3 . $$

The right eigentensors have the following interpretations in terms of stresses:
\( V_2 \) to the eigenvalue \( \lambda_2^\pm \): isotropic stress inside the laminate plane (2D hydrostatic stress)

\( V_1, V_6 \) with the eigenvalue \( G^\pm / G^V \): plane stress inside the laminate plane in absence of in-plane isotropic stress (2D deviatoric stress)

\( V_3^\pm, V_4, V_5 \) with the eigenvalue 1: shear stress parallel or perpendicular to the laminate plane (2D hydrostatic stress)

\( V_1, V_6 \) with the eigenvalue \( G^\pm / G^V \): plane stress inside the laminate plane in absence of in-plane isotropic stress (2D deviatoric stress)

It is due to the latter mixing of modes (in-plane hydrostatic stress with deviatoric stress states) that the effective yield surface is pressure dependent, as will be seen in Sec. 4.1.

### 3.4.2 Special choices for \( K^\pm \) and \( G^\pm \)

In case of \( T \) being parallel to one of \( V_{1,2,4,5,6} \), all three stress tensors \( T, T^+ \) and \( T^- \) are coaxial. For \( T \) parallel to \( V_3^\pm \), coaxiality between the three (which implies \( V_3^+ = V_3^- \)) occurs for special material combinations:

\[ \kappa^\pm = y^\pm: \text{This constraint can be written as} \]

\[ \frac{3\Delta G}{G^+ G^-} = \frac{2\Delta K}{K^+ K^-}, \quad (67) \]

or in terms of Young’s modulus and Poisson’s ratio

\[ \frac{\nu^+}{E^+} = \frac{\nu^-}{E^-}. \quad (68) \]

For such material combinations, \( V_3 \) becomes \( E_3 \), i.e. an effective tension/compression along the laminate normal is coaxial to both stress concentrations. There are still three different eigenvalues, but the eigenspaces are orthogonal (hence \( L^\pm \) are symmetric): We observe a decoupling of in-plane hydrostatic tension/compression and tension/compression along the laminate normal.

\[ y^\pm = 0 \leftrightarrow K^+ = K^-: \text{For such material combinations,} \quad V_3 \text{ becomes} \quad I. \text{ There are still three different eigenvalues.} \]

An effective hydrostatic pressure is coaxial to both stress concentrations, i.e. it does not enter the local yield criteria, and therefore the effective yield criterion is independent of the effective hydrostatic pressure in this case. However, an effective deviatoric stress can affect the hydrostatic stresses in \( T^\pm \). This is due to the non-orthogonality of the eigenbasis.

\[ \kappa^\pm = 0 \leftrightarrow G^+ = G^-: \text{For such material combinations, all eigenvalues except} \lambda_2 \text{ become} \lambda_1, \quad \text{and} \quad V_3 \text{ becomes} \quad -\frac{1}{2}E_1 - \frac{1}{2}E_2 + E_3. \quad \text{Thus,} \quad V_{1,3,4,5,6} \text{ form a deviator basis, i.e. all deviatoric stress states are conducted directly to the stress concentrations.} \]

Again, due to the non-orthogonality of the eigenbasis, a hydrostatic stress state in the effective stresses \( T \) affects the deviatoric parts of the stress concentration tensors.

## 4 The effective plastic properties

### 4.1 The effective yield limit \( \bar{\phi} \)

We proceed similar to Sawicki (1981). We have the elastic range limits

\[ \phi^\pm(T^\pm) \leq 0, \quad (69) \]

in which we can express \( T^\pm \) by the concentration tensors and the effective stresses (see Eq. 39). Plastic deformations occur if one of the two phases yields, thus we join \( \phi^\pm \) to an effective elastic range limit by

\[ \bar{\phi}(\bar{T}) = - \phi^+(T^+)(\phi^{-}(T^-)) \]

\[ = - \phi^+(L^+ \cdot T) \phi^-(L^- \cdot T). \quad (71) \]

Effective plastic properties of laminates made of isotropic elastic plastic materials
Figure 2: Sketch of the parametrization of a uniaxial stress state in a transversely isotropic laminate by one angle \( \theta \) and the absolute stress \( \sigma \) in direction of \( v \).

The minus is introduced to keep the usual orientation of \( \varphi \) such that \( \varphi \leq 0 \) indicates the elastic range. A similar approach has been used, e.g., by Goda and Ganghoffer (2015), who approached the effective brittle fracture criterion of bone as the intersection of local fracture criteria of oriented trusses.

Properties of the effective yield limit \( \bar{\varphi} \).

Given elastic ranges \( \phi^\pm \) that are convex in the stress space, the effective elastic range \( \bar{\phi} \) is as well convex in the stress space. This follows from the fact that the transformations \( L^\pm \) are linear and therefore affine mappings on the stress space, which inherit the convexity from \( \phi^\pm \). Considering \( \phi^\pm(T) < 0 \) as two convex sets, the intersection of both gives a new convex set \( \bar{\phi}(T) \).

The effective yield surface \( \bar{\phi}(T) \) has an edge where the two smooth yield surfaces \( \phi^\pm(T) \) coincide, i.e. we have a loss of smoothness compared to the partial yield surfaces.

The effective yield surface depends on the effective hydrostatic pressure unless \( K^+ = K^- \), see the second bullet point in Sec. 3.4.2.

If \( K^+ \neq K^- \), an effective hydrostatic pressure may cause yielding in both phases, but in opposite flow directions (see the analysis of the stress concentration tensors in Sec. 3.4). Therefore, the plastic deformations remain very small, and should be considered as eigenstrains that cancel out on average. This has also been noted by He and Z.-Q. Feng (2012).

Illustrating example. Consider a laminate made of steel (+) and aluminum (−), with the material parameters \( E^+ = 200 \text{GPa}, \nu^+ = 0.3, E^- = 70 \text{GPa}, \nu^- = 0.34, \sigma_F^+ = 200 \text{MPa}, \sigma_F^- = 100 \text{MPa}, \) and \( v^+ = v^- = 0.5 \).

Due to the transverse isotropy around the laminate normal \( e_3 \), we need only one angle \( 0 \leq \theta \leq \pi/2 \) to represent all relevant uniaxial stress states,

\[
T = \sigma v \otimes v, \quad v = \cos \theta e_1 + \sin \theta e_3,
\]

see Fig. 2. The elastic ranges are depicted in Fig. 3. Although the effective flow criterion appears not convex in the \( e_1-e_3 \)-plane, it is convex in the stress space. If we superimpose this tensile test with a hydrostatic pressure \(-pI\), the partial yield surfaces and hence the effective yield limit change considerably, as depicted in Fig. 4.

4.2 The effective flow behavior

As outlined in Sec. 3.3, we can go from the purely elastic to the elastic-plastic case by considering the stress and strain rates and the tangential stiffnesses \( \bar{C}^\pm \),

\[
\dot{T}^\pm = \bar{C}^\pm \cdot (\dot{E}^\pm - \dot{E}_p^\pm) = \bar{C}^\pm \cdot (\dot{E}^\pm - \frac{3}{2\sigma_F^\pm} T^\pm \otimes T^\pm \cdot \dot{E}^\pm) = \bar{C}^\pm \cdot (I - \frac{3}{2(\sigma_F^\pm)^2} T^\pm \otimes T^\pm) \cdot \dot{E}^\pm.
\]
Figure 3: Elastic range for uniaxial stress states for an example material combination. The vertical axis corresponds to the laminate normal, the horizontal axis is parallel to the laminate. For each point in the plane, the direction of tension is given by its direction from the origin, the distance from the origin is the stress magnitude $\sigma$. The two left plots indicate the change of the elastic range of the steel ($\sigma_F = 200\text{MPa}$) and aluminum ($\sigma_F = 100\text{MPa}$). The circular sections are the original isotropic von Mises yield limits $\phi(\mathbf{T})$, the shaded regions are the yield limits $\phi(\mathbf{T})$. The right graph contains the range in which both phases deform elastically, which is the intersection of the shaded regions from the first two plots. The circular sections in the right plot are the arithmetic and the harmonic averages of the von Mises yield limits $\sigma_F$. Unlike in elasticity, they do not necessarily correspond to energetic bounds. The approximate equivalence of the effective yield limit and the harmonic mean for $\theta = 0$ and $\theta = \pi/2$ is a coincidence, due to the specific elasticity parameters. The dots in the right plot represent the elastic range limit as found in representative volume element simulations, see Sec. 4.3.

With the yield conditions $\|\mathbf{T}^\pm\| = \sqrt{3}/2 \sigma_F^\pm$, the tensors $\mathbf{I} - \mathbf{N}^\pm \otimes \mathbf{N}^\pm$, $\mathbf{N}^\pm = \mathbf{T}^\pm/\|\mathbf{T}^\pm\|$ project all stresses into their parts perpendicular to the current yield surface normals $\phi^\pm(T)$. Moreover, the time derivatives of the jump conditions and mixture rules are

$$
\dot{\mathbf{E}} - \dot{\mathbf{E}} = (\dot{\mathbf{a}} \otimes \mathbf{n})^S, \quad \text{(76)}
$$

$$
(\mathbf{T}^+ - \mathbf{T}^-) \cdot \mathbf{n} = \mathbf{o}, \quad \text{(77)}
$$

$$
\mathbf{T} = v^- \mathbf{T}^- + v^+ \mathbf{T}^+, \quad \text{(78)}
$$

$$
\mathbf{E} = v^- \mathbf{E}^- + v^+ \mathbf{E}^+, \quad \text{(79)}
$$

with the auxiliary quantity $\mathbf{a}$ that does not appear in $\mathbf{L}^\pm$ or $\mathbf{C}^\pm$. The latter equations imply the following assumptions:

The orientation of $\mathbf{n}$ is constant, as already implied by using the small strain setting.

The volume fractions are constant.

Homogeneity in each layer is preserved during plastic flow.

The last point is probably most critical, as it is known that a substantial amount of hardening is needed to suppress the formation of microstructures in plasticity, see, e.g., Carstensen, Hackl, and Mielke (2002). However, this is mostly an issue in finite plasticity and when boundary conditions allow for localization (Gligic, 2013). Presuming the persistance of the plane interface together with isochoric plastic deformations already reduces possible localization modes to a sub-lamination by shearing parallel to the interface. This quasi-1D-localization can be excluded by assuming weak hardening. There is a large amount of literature that deals with complex microstructures due to strain localization, phase transitions and the like. A good entry point is Bhattacharya (2003).

Accepting these fairly reasonable presumptions, we observe that the elastic solution is conducted to the plastic case, but all involved fourth order tensors depend now on $\mathbf{T}^\pm$ through the tangent moduli $\tilde{\mathbf{C}}^\pm(T^\pm)$. 

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Figure 4: Yield surface for the example material in tension superimposed with a hydrostatic pressure. The bold lines mark the effective yield limit that results from the intersection of the individual elastic ranges.
We denote this distinction by a tilde. The former constant stress concentrations become a system of coupled ordinary differential equations with the partial yield limits as two algebraic conditions,

\[
\begin{align*}
\dot{T}^\pm &= \bar{L}^\pm(T^\pm) \cdot \bar{T}, \\
L^\pm &= \bar{C}^\pm \cdot (I \mp v^\pm \bar{Z} \cdot \Delta \bar{C}) \cdot \bar{T}^{-1}, \\
\bar{C} &= v^+\bar{C}^+ + v^-\bar{C}^- - v^+v^-\Delta \bar{C} \cdot \bar{Z} \cdot \Delta \bar{C}, \\
\Delta \bar{C} &= \bar{C}^+ - \bar{C}^-, \\
\bar{Z} &= n \otimes \bar{Z} \otimes n, \\
\bar{A}^\pm &= n \cdot \bar{C}^\pm \cdot n, \\
\bar{C}^\pm &= \begin{cases} 
C^\pm & \text{if } \phi^\pm(T^\pm) < 0 \\
C^\pm \cdot (I - T^\pm \otimes T^\pm' \cdot T^\pm |T^\pm|^2) & \text{if } \phi^\pm(T^\pm') = 0 \text{ and } \phi^\pm(T^\pm)|_{\bar{C} = \bar{C}^\pm} > 0
\end{cases}
\end{align*}
\]

Eqs (80) to (87) give 12 evolution equations for the partial stresses \( T^\pm \), which are either trivial when both phases deform elastically, or somewhat complicated in case of plastic yielding. In the latter case, we are confronted with nonlinear coupled ODEs that are homogeneous of degree 0 in \( T^\pm \), as one expects from rate-independent material behavior.

**4.2.1 Ultimate loading**

A closed form solution to the latter system of nonlinear ODEs is rather hard to find, even when \( \dot{T} \) is constant. Nevertheless, we can examine equilibrium points without giving explicit solutions by examining the effective tangent modulus \( \bar{C} \). A characteristic point is the ultimate loading, which is attained when the strain increment does not result in a stress increment, i.e.

\[
\bar{C}(T^\pm) \cdot \dot{E} = 0.
\]

The latter equation implies that \( \bar{C}(T^\pm) \) has a zero eigenvalue, which is most easily identified by examining the determinant of \( \bar{C} \). Two cases can be distinguished:

Yielding takes place in both phases, i.e. both \( \bar{C}^\pm \) depend on \( T^\pm \), and in both phases \( \phi^\pm(T^\pm) = 0 \) holds. It turns out that in this case the determinant of \( \bar{C} \) is always zero.

Yielding takes place in one phase, where we take the + phase, see Fig. 5. The denominator of the determinant of \( \bar{C} \) is a quadratic form in the entire \( T^\pm' \). With \( T^\pm \) on the yield surface, the denominator cannot become zero. However, the numerator of the determinant of \( \bar{C} \) is

\[
\begin{align*}
\min(\det(\bar{C})) &= -96G^{-3}G^{2}v^{-G}G \cdot (3G^{2}K^{2}K^{2}T_{11}^{2}T_{12}^{2}T_{12}^{2} + T_{11}^{2}T_{22}^{2} + T_{22}^{2}T_{22}^{2}) + \\
&K^{2}G^{2}G^{2} + (4T_{12}^{2}T_{12}^{2} + (T_{11}^{2} - T_{22}^{2})^{2})).
\end{align*}
\]

This is a negative semidefinite quadratic form in \( T_{11}^{2}, T_{22}^{2}, T_{12}^{2} \). Thus, \( \bar{C} \) is singular whenever

\[
T_{11}^{2} = 0, \quad T_{22}^{2} = 0, \quad T_{12}^{2} = 0.
\]

We know from the jump conditions for \( T^\pm \) that \( T^+ \) and \( T^- \) have these components in common (Eq. 9). Since we look at the deviator, only the shear stress components \( T_{13}^{2} \) and \( T_{23}^{2} \) are non-zero. This means just that \( \bar{C} \) becomes singular when the deviatoric part of the partial stresses of the plastically deforming phase is a pure shear parallel to the laminate.
Conclusion. Yielding will in general occur firstly in one phase, say the + phase. Then, two cases need to be distinguished:

1. \( T^+ \) evolves into a pure shear parallel to the laminate, while \( T^- \) remains inside the elastic range \( \phi^- \).
2. After some plastic flow in the + phase, the − phase also reaches its yield limit.

We refer to these cases of ultimate loading as asymptotic yielding in one phase and simultaneous yielding in both phases.

4.2.2 Ultimate loading in uniaxial tensile tests

We will use the results of the latter section to determine the ultimate loading in tensile tests, which are as before parametrized by the angle \( \theta \). The general problem is that we search points of the solution of the DAE system without considering the complete evolution of the system. Therefore, we are restricted to simple cases of monotonic loading with a constant loading direction. Further, we cannot use the kinematic compatibility condition, as it gives only three additional equations, but the jump of the plastic strains has 5 degrees of freedom. Therefore, the elastic strains and hence the elasticity parameters do not appear in the ultimate loading limit. In the remainder, examples use the same parameters as in Sec. 4.1.

Asymptotic yielding in one phase. The necessary condition is that one partial stress is on the partial yield limit (say \( \phi^+(T^+) = 0 \)). Further, it was found that \( T^+ \) must be a pure shear parallel to the laminate (Eq. 91), such that only \( T^+_1 \) and \( T^+_3 \) can be nonzero. Additional symmetry arguments render this sufficient to determine the ultimate loading: Due to the effective tension direction being constant inside the \( e_1-e_3 \) plane (see Fig. 2) and the transversal isotropy, we can presume \( T^+_2 = 0 \) throughout the test. We can determine \( T^+_1 \) directly:

\[
\sigma^+_1 = \frac{\sqrt{3}}{2} ||T^+_1||
\]

\[
= \frac{\sqrt{3}}{2} ||T^+_1(e_1 \otimes e_3 + e_3 \otimes e_1)||
\]

\[
= \sqrt{3} T^+_1^3.
\]

We obtain \( T^+_1 = \sigma^+_1 / \sqrt{3} \) MPa, while all other components of \( T^+ \) are zero. With \( T = \sigma v \otimes v \) (see Eq. 72), we have \( T^+_1 = \sigma \cos \theta \sin \theta \). Further, the mixture rule with \( v^+ = v^- = 1/2 \) together with the compatibility of the

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which allows to write $\sigma$ in terms of $\theta$. It can be seen that this solution contributes considerably to the quality of the ultimate loading estimate (Fig. 6). It is easy to see that in case of compression the same magnitude for the ultimate loading is attained.

Figure 6: Elastic range and ultimate loading in tensile tests with the yield limits $\sigma^+ = 100\text{MPa}$ and $\sigma^- = 200\text{MPa}$. The vertical axis corresponds to the laminate normal, the horizontal axis is parallel to the laminate. For each point in the plane, the direction of tension is given by its direction from the origin, the distance from the origin is the stress magnitude $\sigma$. The dots are obtained through representative volume element simulations (see Sec. 4.3), the shaded region is the effective elastic range. The circular section is the arithmetic mean (1) of both yield limits $\sigma^+_\ell$, the hyperbolic curve (3) is the ultimate loading in case of asymptotic yielding in one phase (Eq. 95) and the dotted curve (4) is ultimate loading in case of simultaneous yielding in both phases (Eq. 106). The arrow (2) indicates the intersection of the partial yield limits $\phi^+(L^+ \cdot \mathbf{T}) = \phi^-(L^- \cdot \mathbf{T})$, which lies on the ultimate loading limit, since it is equivalent to simultaneous yielding in both phases.

**Simultaneous yielding in both phases.** We now search for partial stresses $\mathbf{T}^\pm$ that

- are on the yield surface in both phases,
- are dynamically compatible,
- result in a uniaxial effective stress state,
- do not break the test symmetry.

In equations we have respectively

\[
\frac{3}{2} T^{+\prime} \cdot T^{+\prime} = \sigma^+ F^2 
\]
\[
\frac{3}{2} T^{-\prime} \cdot T^{-\prime} = \sigma^- F^2 
\]
\[
T^{+}_{13} = T_{13}^{-} 
\]
\[
T^{+}_{23} = T_{23}^{-} 
\]
\[
T^{+}_{33} = T_{33}^{-} 
\]
\[
v^+ \mathbf{T}^+ + v^- \mathbf{T}^- = 2 \sigma \mathbf{v} \otimes \mathbf{v}, \quad \mathbf{v} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_3 
\]
\[
T^{+}_{12} = T_{12}^+ = 0 
\]
By the latter equations, all partial stress components are determined as functions of \( \sigma \) and \( \theta \). For a given tension direction \( \theta \) but varying \( \sigma \) we find infinitely many points \( T^\pm \) that satisfy the above constraints. For example, let \( \theta = 0 \), \( \sigma_F^+ = 100\text{MPa} \), \( \sigma_v^+ = 200\text{MPa} \) and \( v^+ = v^- = 1/2 \). Then we find solutions for \( 50\text{MPa} \leq \sigma \leq 150\text{MPa} \), namely

\[
\begin{align*}
\sigma = 50\text{MPa} & \rightarrow T^+ = -100\text{MPa} \, e_1 \otimes e_1, \quad T^- = 200\text{MPa} \, e_1 \otimes e_1 \quad (103) \\
\sigma = 150\text{MPa} & \rightarrow T^+ = 100\text{MPa} \, e_1 \otimes e_1, \quad T^- = 200\text{MPa} \, e_1 \otimes e_1. \quad (104)
\end{align*}
\]

The first case corresponds to phases under eigenstresses that partly cancel out, the second case corresponds to partial stresses that sum up to the largest possible external loading. We are interested in the latter case, hence we search to maximize \( \sigma \) for each angle \( \theta \) over the range of admissible partial stresses. The general solution for \( T^\pm \) of Eqs (96) to (102) is lengthy. It contains a square root term

\[
\sqrt{(-3(\sigma_F^{-2} - \sigma_v^{-2})^2 + 15(\sigma_F^{-2} + \sigma_v^{-2})\sigma^2 - 30\sigma^4 + 9\sigma^2(\sigma_F^{-2} + \sigma_v^{-2}) - 2\sigma^2)\cos^2 \theta}. \quad (105)
\]

The extremal cases are separating the real from the complex solutions, hence we can determine it by setting the argument of the latter root to zero and solve for \( \sigma \). The resulting biquadratic equation has four solutions. These correspond to maximal and minimal external loading (maximal eigenstresses cancellation) in tension and compression. Being interested in the ultimate loading in tension we consider the largest value for \( \sigma \), i.e. we take plus signs in front of the two involved square roots

\[
\sigma = \frac{1}{2} \sqrt{\sigma_F^{-2} + \sigma_v^{-2} + \sqrt{-3\sigma_F^{-4} + 26\sigma_F^{-2} \sigma_v^{-2} - 30\sigma_v^{-4} + 3(\sigma_F^{-2} + \sigma_v^{-2})^2 \cos^4 \theta}}. \quad (106)
\]

As before, in a compression test (a minus in front of the outer square root and a plus in front of the inner square root), the ultimate loading has the same magnitude as in the tensile test, i.e. there is no strength differential effect. For \( \theta = 0 \) and \( \theta = \pi/2 \) we find the ultimate loading to be the arithmetic mean of the yield limits \( \sigma_F^+ \). For angles around \( \theta = \pi/4 \), the solution becomes complex. This is exactly when the case of asymptotic yielding needs to be considered, see Figs. 6 and 8. Consequently, we divide the stress-strain plots into three families, given in Fig. 7.

To summarize, we can estimate the ultimate loading stress by considering

1. the arithmetic mean of the partial yield stresses in tests that lead to coaxial partial stresses, like tension parallel or perpendicular to the laminate,
2. stress states that lead to partial stresses such that both partial yield limits are met,
3. the equilibrium points in case of yielding in only one phase (asymptotic yielding),
4. the maximal effective loading in case of yielding in both phases,

see Fig. 6. (3) and (4) completely define the ultimate loading curve in tensile and compression tests, where the case (4) contains (1) and (2) as special cases. Naturally, the range of (3) is larger for phases with very different yield limits, while the converse is true for the range of (4).

To conclude, we determine the angle \( \theta \) at which the curves (3) and (4) meet. From the preceding analysis we have for the ultimate loading in case of asymptotic yielding in one phase

\[
\begin{align*}
T^- &= \frac{1}{v^-}(T - v^+ T^+) \quad (107) \\
T^+ &= \sigma_F^+ / \sqrt{3} \text{MPa} \, (e_1 \otimes e_3 + e_3 \otimes e_1) \quad (108) \\
T &= \sigma v \otimes v, \quad v = \cos \theta e_1 + \sin \theta e_3 \quad (109) \\
\sigma &= \sigma_F^+ / \sqrt{3} \text{MPa} / \sin \theta / \cos \theta. \quad (110)
\end{align*}
\]

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Figure 7: Stress-strain curves in the plastic range for uniaxial tensile tests in direction of $\theta$, with the discretization $\theta = n\pi/40$, with an integer $n$. The left plot covers $n = 0 \ldots 6$, the center plot covers $n = 7 \ldots 13$, the right one covers $n = 14 \ldots 20$. The arrows connecting the points of ultimate loading indicate the parametrization with $n$.

We just have to put $T^-$ on the yield surface to determine the angles $\theta$ that separate the range of asymptotic yielding in one phase from the ranges of eventual yielding in both phases,

$$\frac{3}{2} T^- \cdot T^- = (\sigma^-)^2 \rightarrow \theta = \pi/4 \pm 0.2549.$$  \hspace{1cm} (111)

This corresponds basically to considering the intersection of the two different yield limits. The corresponding section has been highlighted in Fig. 8, where the elongation to ultimate loading is depicted. In case of yielding in one phase, it is infinite, since the ultimate loading stress is approached asymptotically. In case of one phase yielding until the second phase yields, one has to track the evolution of the internal variables $E^\pm$.

4.3 Representative volume element simulations

To check the analytical results, representative volume element simulations have been devised. A cubical section of the laminate with one interface dividing the $+$ and the $-$ phase has been subjected to periodic boundary conditions and monotonic tensile tests, see Fig. 9. We used a regular hexahedral mesh with quadratic shape functions. To capture potential localizations or pattern formation, we used fine meshes, but did not observe such events. Thus, the presumptions made in Sec. 4.2 appear feasible. The stress-strain curves (Fig. 7) have been extracted, and on each the effective stresses and strains at the yield point and the ultimate loading have been determined (Figs. 3, 6, 8).

4.4 Comparison to other works

Pruchnicki (1998a,b). The main differences to the works of Pruchnicki are that

he carried out the scale transition by the asymptotic expansion method, where laminates are treated as a special case, and

he considers tangential slip (Pruchnicki, 1998a) and boundary layers (Pruchnicki, 1998b).

Conversely, we (and El Omri et al. (2000) and He and Z.-Q. Feng (2012)) consider perfectly bonded interfaces. Consequently, Pruchnickis results depend on additional material parameters that do not appear here. Nevertheless, the following common features are observed:

Symmetry of the ultimate loading limit w. r. t. the $45^\circ$ tensile direction, while this symmetry is not displayed in the yield limit, compare Fig. 6 here and Fig. 11 in Pruchnicki (1998b).

Asymptotic yielding in tension close to the $45^\circ$ tensile tests and finite ultimate elongation due to yielding in both phases when tension is applied approximately parallel or perpendicular to the laminate, compare
Figure 8: Elongation to yielding (inner curve) and elongation to ultimate loading (outer curve) for uniaxial tensile tests in direction of $\theta$, with the discretization $\theta = n\pi/40$, with an integer $n$. The shaded region indicates the range of $\theta$ in which the ultimate loading is approached asymptotically. It is larger when both phases have markedly different yield limits. The gray circle sections serve as markings. The dashed lines indicate the transition to asymptotic behavior.

Figure 9: Representative volume element subjected to elongation in direction $\theta = 36$ (black arrow) at the ultimate loading. The displacement is scaled by a factor of 35 to make the small deformation visible. The coloring indicates the norm of the plastic strain tensors $E_p^{\pm}$ (0 and $8.543 \times 10^{-3}$).
El Omri et al. (2000) and He and Z.-Q. Feng (2012). He and Z.-Q. Feng (2012) derived the general ordinary differential equations that govern the evolution of the partial plastic variables, and thus the effective plastic response, as we did (Eqs (80) to (87)). They show that the effective material law can be cast in the framework of generalized standard materials with a finite number of internal variables, which they exploit in their derivation. However, it appears that they miss to point out that the requirement of homogeneity of the phases during plastic flow is only given in absence of softening. Further differences are that

we use a different method to derive the governing differential equations and that

He and Z.-Q. Feng (2012) derive the effective flow rule for the general anisotropic case, while we focus here on examining the system of evolution equations, which finally leads to analytical expressions for the effective ultimate loading in uniaxial tests in laminate of isotropic, von Mises plastic phases.

The latter result is in line with deBotton and Ponte-Castañeda (1992) and Ponte-Castañeda and deBotton (1992) and He and Z.-Q. Feng (2012): deBotton and Ponte-Castañeda (1992) and Ponte-Castañeda and deBotton (1992) found that the ultimate loading surface cannot be given as a single quadratic form. He and Z.-Q. Feng (2012) state that the ultimate loading surface is a piecewise quadratic form. Our examination of laminates of isotropic von Mises plastic phases in monotonous tensile tests, for which we gave the ultimate loading explicitly (Eqs 95,106), confirms these findings. Additionally, we have discussed the source of this behavior, which is the case distinction into asymptotic yielding in one phase and simultaneous yielding in both phases. This arises naturally when looking for states for which the effective tangent modulus has a zero eigenvalue. Further, we could show that in these uniaxial monotonic tests, there is no strength differential effect on the macroscale, neither for the yield nor the ultimate loading limit.

The work of El Omri et al. (2000) is somewhat similar to He and Z.-Q. Feng (2012) and the present work. They consider tensile and shear tests, where the tensile tests show similar stress-strain curves as determined by us, namely anisotropic hardening and kinks or asymptotic behavior (Fig. 5 in El Omri et al. (2000)). However, they restrict their examples to phases with equal elasticities.

5 Summary

We have examined the plastic properties of laminates of isotropic, elastic plastic layers. It has been found that most of the desirable properties of the von Mises yield limit are lost in the homogenization. The effective yield limit is given as the intersection of the partial yield limits, which is therefore non-smooth. Due to the properties of the stress concentration tensors, which map the effective stress state to the partial stresses, the effective yield limit depends on the effective hydrostatic pressure and is anisotropic. However, it remains convex in the effective stresses. The yield surface has been analyzed for the virgin material. The results have been confirmed numerically by representative volume element simulations.

Further, with some mild restrictions on the plastic deformations (basically the persistence of the laminate structure and small strains), the homogenization of the rate-form plastic equations is similar to the homogenization of the total-form elastic equations. Instead of constant stress concentration tensors one has then stress rate concentration tensors that depend on the partial stresses, i.e. one obtains a set of nonlinear ordinary differential equations for the partial stresses, which characterize the internal state of the microstructure. For example, one observes effective hardening behavior while only one phase yields plastically, unlike in pure von Mises material without hardening. The ODE system is homogeneous of degree zero in the partial stresses. A closed form solution can hardly be attained. Nevertheless, it is possible to extract the ultimate loading (i.e. the plastic reserve) in tensile tests as a function of the tension direction. Two fundamental cases have to be distinguished, namely.
yielding in one phase until a shear stress parallel to the laminate is approached asymptotically in the yielding phase, while the other phase remains elastic, and

yielding in one phase until the second phase starts yielding.

For both cases, one can give closed form expressions for the effective ultimate loading. It contains special cases, namely the arithmetic mean of the yield stresses in tension parallel and perpendicular to the laminate and the intersection of the yield surface, which confirms the plausibility of the result. The analytical findings match exactly to results from representative volume element simulations. Further, the effective ultimate loading does not depend on the elasticity parameters, unlike the effective yield limit. For the monotonic uniaxial tests, we can exclude a strength differential effect for the yield and the ultimate loading stress.

In case of a complex loading path or when hardening is included, no closed form expressions can be given. Still, the time integration of the partial stresses and possibly some hardening variables is much more convenient than considering representative volume elements (Nezamabadi et al., 2015).

Literature


