A self-consistent mechanics of composite materials

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A SELF-CONSISTENT MECHANICS OF COMPOSITE MATERIALS

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THE MACROSCOPIC elastic moduli of two-phase composites are estimated by a method that takes account of the inhomogeneity of stress and strain in a way similar to the Hershey–Kröner theory of crystalline aggregates. The phases may be arbitrarily aeolotropic and in any concentrations, but are required to have the character of a matrix and effectively ellipsoidal inclusions. Detailed results are given for an isotropic dispersion of spheres.

1. INTRODUCTION

PREDICTIONS of macroscopic properties of two-phase solid composites have mostly been restricted to stating universal bounds on various overall elastic moduli (HASHIN 1964; 1965; HILL 1963). Such bounds depend only on the relative volumes and do not reflect any particular geometry, except when one phase consists of continuous aligned fibres (HASHIN and ROSEN 1964; HILL 1964). However, when one phase is a dispersion of ellipsoidal inclusions, not necessarily dilute, a much more direct approach is available†. This is the 'self-consistent method' of HERSHEY (1954) and KRÖNER (1958), originally proposed for aggregates of crystals. In that connexion it has recently been reviewed and elaborated by the writer (1965a).

The method draws on the familiar solution to an auxiliary elastic problem, namely a uniformly loaded infinite mass containing an ellipsoidal inhomogeneity. In applying this solution the properties and orientation of a typical crystal are assigned to the inclusion, and the macroscopic properties of the polycrystal to the matrix. For self-consistency the orientation average of the inclusion stress or strain is set equal to the overall stress or strain. The result is an implicit tensor formula for the macroscopic moduli.

The analysis for the composite proceeds in similar spirit but necessarily differs in an important respect : only the particulate phase can reasonably be treated on this footing. However, as is well known (*op. cit.* 1963; § 2 (iii)), a knowledge of average stress or strain in this one phase suffices to determine the overall properties when the matrix is homogeneous. As a matter of fact, notwithstanding this difference in viewpoint, the entire analysis is found to remain structurally close to that for a crystal aggregate (as given in *op. cit.* 1965a, §§ 3 and 4).

†No mention of it in this context has been traced in the literature. But Professor B. Budiansky recently informed me that he tried the approach in 1961; his conclusions appear elsewhere in this issue of the Journal. My own investigation dates from March 1962, when preliminary results were given in a letter to Dr. J. D. Eshelby.

2. Symbolic Notation

For brevity cartesian tensors of second order are denoted simply by their kernel letter, $\boldsymbol{u}$ say, set in lower case bold face as if for a vector. Correspondingly, their tensor components are considered to be arranged in some definite sequence as a 9×1 column. Tensors of fourth order are denoted by an ordinary capital, A say, and are regarded as 9×9 matrices. More precisely, the leading pair of indices is set in correspondence with rows, and the terminal pair with columns (both in the chosen sequence), so that the second-order inner product of tensors A and $\boldsymbol{u}$ can be written as the matrix product $A\boldsymbol{u}$. Similarly, AB can stand for the fourth-order inner product of A and B.

We shall only be concerned with fourth-order tensors that are symmetric with respect to interchange of the leading pair of indices and also of the terminal pair. The representative matrices are consequently singular, with rank $\leqslant 6$. Nevertheless, equations of type $\boldsymbol{u} = A\boldsymbol{v}$ are compatible when $\boldsymbol{u}$ and $\boldsymbol{v}$ are any *symmetric* second-order tensors and matrix A has rank 6. In this sense we can define a unique inverse A^{-1} as the solution of

$$AA^{-1} = I \text{ or } A^{-1}A = I$$

where I is the suitably symmetric 'unit' tensor

$$I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$$

formed with the Kronecker delta. One can then verify that

$$A^{-1}\boldsymbol{u} = A^{-1}A\boldsymbol{v} = I\boldsymbol{v} = \boldsymbol{v}$$

as required, for any A, $\boldsymbol{u}$ and $\boldsymbol{v}$ with the stated properties.

3. The Auxiliary Problem

A single inclusion, arbitrarily ellipsoidal in shape, is imagined to be embedded in a homogeneous mass of some different material. The tensors of elastic moduli, not necessarily isotropic, are denoted by L_1 and L, respectively, and their inverse compliances by M_1 and M. In addition to the symmetries mentioned already in § 2, the representative matrices have full diagonal symmetry so that all cross-moduli and compliances are pairwise equal.

The displacement at infinity is prescribed to correspond to a uniform overall strain $\bar{\boldsymbol{\epsilon}}$. Across the phase interface both displacement and traction are required to be continuous. The solution, certainly unique when the tensors of moduli are positive definite, has the character of a uniform field locally perturbed in the neighbourhood of the inclusion. In particular the overall average, or macroscopic, stress $\bar{\boldsymbol{\sigma}}$ is equal to $L\bar{\boldsymbol{\epsilon}}$, since the contribution from the inclusion is vanishingly small; furthermore, $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\epsilon}}$ are also the local field values at infinity. The principal feature of the solution is that the inclusion is strained uniformly, though not necessarily coaxially (Eshelby 1957; 1961).

This property prompts the introduction of an 'overall constraint' tensor L^* for the L phase, with inverse M^*, in respect of loading over the interface by

any distribution of traction-rate compatible with a uniform field of stress, $\boldsymbol{\sigma}^*$ say. That is, if $\boldsymbol{\epsilon}^*$ is the accompanying uniform strain of the ellipsoid,

$$\boldsymbol{\sigma}^* = -\boldsymbol{L}^* \boldsymbol{\epsilon}^*, \quad \boldsymbol{\epsilon}^* = -\boldsymbol{M}^* \boldsymbol{\sigma}^*. \tag{1}$$

The corresponding matrices naturally have diagonal symmetry, as may be shown by Betti's reciprocal theorem, and are functions of $\boldsymbol{L}$ or $\boldsymbol{M}$ and the aspect ratios of the ellipsoid. Once $\boldsymbol{L}^*$ and $\boldsymbol{M}^*$ have been determined, the solution of the auxiliary problem follows by superimposing the uniform fields $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\epsilon}}$, and identifying $\boldsymbol{\sigma}^*$ with $\boldsymbol{\sigma}_1 - \bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\epsilon}^*$ with $\boldsymbol{\epsilon}_1 - \bar{\boldsymbol{\epsilon}}$ where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\epsilon}_1$ are the actual fields in the inclusion. Then

$$\boldsymbol{\sigma}_1 - \bar{\boldsymbol{\sigma}} = \boldsymbol{L}^* (\bar{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}_1), \quad \boldsymbol{\epsilon}_1 - \bar{\boldsymbol{\epsilon}} = \boldsymbol{M}^* (\bar{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_1), \tag{2}$$

and so

$$(\boldsymbol{L}^* + \boldsymbol{L}_1) \boldsymbol{\epsilon}_1 = (\boldsymbol{L}^* + \boldsymbol{L}) \bar{\boldsymbol{\epsilon}}, \quad (\boldsymbol{M}^* + \boldsymbol{M}_1) \boldsymbol{\sigma}_1 = (\boldsymbol{M}^* + \boldsymbol{M}) \bar{\boldsymbol{\sigma}}, \tag{3}$$

which furnish the required stress and strain in the inclusion in terms of the macroscopic quantities (HERSHEY 1954).

In an alternative approach (ESHELBY 1957), seemingly adopted by all later writers, attention is focussed first on a certain transformation problem for an infinite homogeneous elastic continuum with stiffness tensor $\boldsymbol{L}$. In this, an ellipsoidal region would undergo a transformation strain $\boldsymbol{e}$ if free, but attains only the strain $\boldsymbol{Se}$ *in situ*. The components of tensor $\boldsymbol{S}$, being dimensionless, are functions of the moduli ratios and of the aspect ratios of the ellipsoid and its orientation in the frame of reference. When $\boldsymbol{L}$ is isotropic, explicit formulae for the components on the principal axes have been given by Eshelby (op. cit.). When $\boldsymbol{L}$ is orthotropic and the transformed region is an elliptic cylinder whose axes coincide with the material axes, explicit formulae have been given by BHARGAVA and RADHAKRISHNA (1964); when $\boldsymbol{L}$ has cubic symmetry equivalent results have also been given by WILLIS (1964).

The general connexion with $\boldsymbol{L}^*$ or $\boldsymbol{M}^*$ is most easily obtained by imagining the transformation problem solved from the viewpoint of (1). That is, we substitute

$$\boldsymbol{\epsilon}^* = \boldsymbol{Se}, \quad \boldsymbol{\sigma}^* = \boldsymbol{L} (\boldsymbol{\epsilon}^* - \boldsymbol{e}) \text{ in } \boldsymbol{\sigma}^* = -\boldsymbol{L}^* \boldsymbol{\epsilon}^*.$$

Then, since these hold for all $\boldsymbol{e}$,

$$\boldsymbol{L}^* \boldsymbol{S} = \boldsymbol{L} (\boldsymbol{I} - \boldsymbol{S}), \quad (\boldsymbol{I} - \boldsymbol{S}) \boldsymbol{M}^* = \boldsymbol{SM}, \tag{4}$$

where $\boldsymbol{I}$ is the unit tensor defined in § 2. These are equivalent formulae for $\boldsymbol{L}^*$ or $\boldsymbol{M}^*$ in terms of $\boldsymbol{S}$. Or they can be put inversely as

$$\boldsymbol{S} = (\boldsymbol{L}^* + \boldsymbol{L})^{-1} \boldsymbol{L} = \boldsymbol{M}^* (\boldsymbol{M}^* + \boldsymbol{M})^{-1}$$

for $\boldsymbol{S}$ in terms of $\boldsymbol{L}^*$ or $\boldsymbol{M}^*$.

Another dimensionless tensor $\boldsymbol{T}$, the dual of $\boldsymbol{S}$, could just as well be admitted on this footing. Set

$$\left.\begin{aligned} \boldsymbol{M}^* \boldsymbol{T} &= \boldsymbol{SM} = \boldsymbol{P}, \text{ say,} \\ \boldsymbol{TL} &= \boldsymbol{L}^* \boldsymbol{S} = \boldsymbol{Q}, \quad \text{say,} \end{aligned}\right\} \tag{5}$$

so that

$$\left.\begin{aligned} \boldsymbol{M}^* \boldsymbol{T} &= \boldsymbol{M} (\boldsymbol{I} - \boldsymbol{T}), \quad (\boldsymbol{I} - \boldsymbol{T}) \boldsymbol{L}^* = \boldsymbol{TL}, \\ \text{and} \qquad \boldsymbol{T} &= \boldsymbol{L}^* (\boldsymbol{L}^* + \boldsymbol{L})^{-1} = (\boldsymbol{M}^* + \boldsymbol{M})^{-1} \boldsymbol{M}. \end{aligned}\right\} \tag{6}$$

The significance of $\boldsymbol{T}$ is that the stress $\boldsymbol{\sigma}^*$ in the transformed region can be written as $\boldsymbol{Ts}$, where $\boldsymbol{s}$ is the stress that would remove the strain $\boldsymbol{e}$. Separate symbols $\boldsymbol{P}$ and $\boldsymbol{Q}$ have been introduced for the products in (5) since these appear frequently hereafter. We note the further connexions

$$\left.\begin{aligned} \boldsymbol{PL} + \boldsymbol{MQ} &= \boldsymbol{I}, \\ \boldsymbol{P} = \boldsymbol{M}(\boldsymbol{I} - \boldsymbol{T}), &\qquad \boldsymbol{Q} = \boldsymbol{L}(\boldsymbol{I} - \boldsymbol{S}), \\ \text{and} \qquad \boldsymbol{P}^{-1} = \boldsymbol{L}^* + \boldsymbol{L}, &\qquad \boldsymbol{Q}^{-1} = \boldsymbol{M}^* + \boldsymbol{M}. \end{aligned}\right\} \tag{7}$$

From the latter pair one sees that matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ have the diagonal symmetry stipulated for the moduli and compliances (while $\boldsymbol{S}$ and $\boldsymbol{T}$ generally do not). This can of course also be established purely within the context of the transformation problem by means of Betti's reciprocal theorem. The interpretation of $\boldsymbol{Q}$ is that an ellipsoidal cavity in a medium under stress $\boldsymbol{Q\epsilon}$ at infinity would deform by amount $\boldsymbol{\epsilon}$; a dual interpretation may be given for $\boldsymbol{P}$.

4. Self-Consistent Theory

We consider statistically homogeneous dispersions in which the inclusions can be treated, on average, either as variously-sized spheres or as similar ellipsoids with corresponding axes aligned†. Each phase may be arbitrarily anisotropic but is assumed homogeneous *in situ*. Consequently, in a common frame of reference, every tensor in the generic auxiliary problem has the same components for all inclusions.

Let the respective phase properties be distinguished by subscripts 1 and 2, and let c_1 and c_2 be the fractional concentrations by volume, such that $c_1 + c_2 = 1$. The elementary relations between the phase and overall averages of stress and strain are

$$\left.\begin{aligned} c_1(\bar{\boldsymbol{\sigma}}_1 - \bar{\boldsymbol{\sigma}}) + c_2(\bar{\boldsymbol{\sigma}}_2 - \bar{\boldsymbol{\sigma}}) &= \boldsymbol{0}, \\ c_1(\bar{\boldsymbol{\epsilon}}_1 - \bar{\boldsymbol{\epsilon}}) + c_2(\bar{\boldsymbol{\epsilon}}_2 - \bar{\boldsymbol{\epsilon}}) &= \boldsymbol{0}. \end{aligned}\right\} \tag{8}$$

These incidentally imply the vanishing of the averages of the ' polarization ' stress or strain :

$$\left.\begin{aligned} c_1(\bar{\boldsymbol{\sigma}}_1 - \boldsymbol{L}\bar{\boldsymbol{\epsilon}}_1) + c_2(\bar{\boldsymbol{\sigma}}_2 - \boldsymbol{L}\bar{\boldsymbol{\epsilon}}_2) &= \boldsymbol{0}, \\ c_1(\bar{\boldsymbol{\epsilon}}_1 - \boldsymbol{M}\bar{\boldsymbol{\sigma}}_1) + c_2(\bar{\boldsymbol{\epsilon}}_2 - \boldsymbol{M}\bar{\boldsymbol{\sigma}}_2) &= \boldsymbol{0}, \end{aligned}\right\} \tag{9}$$

since $\bar{\boldsymbol{\sigma}} = \boldsymbol{L}\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\epsilon}} = \boldsymbol{M}\bar{\boldsymbol{\sigma}}$.

Now, according to the basic postulate of the self-consistent method,

$$\bar{\boldsymbol{\sigma}}_1 - \bar{\boldsymbol{\sigma}} = \boldsymbol{L}^*(\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}_1), \tag{10}$$

from the leading equation (2). It follows automatically from (8) that

$$\bar{\boldsymbol{\sigma}}_2 - \bar{\boldsymbol{\sigma}} = \boldsymbol{L}^*(\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}_2), \tag{11}$$

and *vice versa*. Thus, right at the outset, it is evident that both phases will enter subsequent formulae on the same footing. However, this does not imply that the

†Fibres of elliptic section may be envisaged as a limiting case in which one principal axis becomes infinite. A direct analysis is given elsewhere (Hill 1965b).

matrix phase also is treated as particulate in the theory, through a kind of conceptual fragmentation. It simply means that the same overall moduli are predicted for another composite in which the roles of the phases are reversed : that is, where the first phase forms a coherent matrix and the second phase is distributed as inclusions shaped and oriented as before, both in their original concentrations.

It is also obvious that either of (8) would imply the other, and then (9), if *both* (10) and (11) were postulated. This, indeed, is the standpoint in the polycrystal theory, where an equation corresponding to (2) is assumed for grains of all orientations. But, as already remarked, such an *a priori* standpoint for a dispersion would seem unconvincing.

Equations (10) and (11), which may as well now be taken together, can be re-arranged as

$$\left.\begin{aligned} &(L^* + L_1)\,\bar{\epsilon}_1 = (L^* + L_2)\,\bar{\epsilon}_2 = (L^* + L)\,\bar{\epsilon} \\ \text{or dually as}\quad &(M^* + M_1)\,\bar{\sigma}_1 = (M^* + M_2)\,\bar{\sigma}_2 = (M^* + M)\,\bar{\sigma} \end{aligned}\right\} \qquad (12)$$

as in (3). Combining these with (8) yields a pair of equivalent formulae for the overall stiffness and compliance tensors L and M :

$$\left.\begin{aligned} &c_1\,(L^* + L_1)^{-1} + c_2\,(L^* + L_2)^{-1} = (L^* + L)^{-1} = P, \\ &c_1\,(M^* + M_1)^{-1} + c_2\,(M^* + M_2)^{-1} = (M^* + M)^{-1} = Q. \end{aligned}\right\} \qquad (13)$$

Since the constraint tensor L^* and its inverse M^* are themselves functions of L and M, these formulae are actually quite complex. Variants obtainable with the help of the last pair in (7) are

$$\left.\begin{aligned} &c_1\left[(L_1 - L)^{-1} + P\right]^{-1} + c_2\left[(L_2 - L)^{-1} + P\right]^{-1} = 0, \\ &c_1\left[(M_1 - M)^{-1} + Q\right]^{-1} + c_2\left[(M_2 - M)^{-1} + Q\right]^{-1} = 0, \end{aligned}\right\} \qquad (14)$$

which are essentially in the form (9). An inversion immediately produces

$$\left.\begin{aligned} &c_1\,(L - L_2)^{-1} + c_2\,(L - L_1)^{-1} = P, \\ &c_1\,(M - M_2)^{-1} + c_2\,(M - M_1)^{-1} = Q, \end{aligned}\right\} \qquad (15)$$

which seem to be the simplest obtainable, superficially at least.

Finally, we can read off from (12) the phase ' concentration-factor ' tensors, A_1 and A_2 for strain, B_1 and B_2 for stress, which are defined by

$$A_1^{-1}\,\bar{\epsilon}_1 = A_2^{-1}\,\bar{\epsilon}_2 = \bar{\epsilon}, \qquad B_1^{-1}\,\bar{\sigma}_1 = B_2^{-1}\,\bar{\sigma}_2 = \bar{\sigma}.$$

Thus :

$$\begin{aligned} A_1^{-1} &= P\,(L^* + L_1) &&= I + P\,(L_1 - L), \\ A_2^{-1} &= P\,(L^* + L_2) &&= I + P\,(L_2 - L), \\ B_1^{-1} &= Q\,(M^* + M_1) &&= I + Q\,(M_1 - M), \\ B_2^{-1} &= Q\,(M^* + M_2) &&= I + Q\,(M_2 - M). \end{aligned}$$

Equations (13) are of course an expression of the basic connexions

$$c_1\,A_1 + c_2\,A_2 = I = c_1\,B_1 + c_2\,B_2.$$

When the dispersion is dilute, with c_1 small, (14) reduces to

$$\left.\begin{aligned} L - L_2 &\simeq c_1 (L_1 - L_2)\,[I + P_2 (L_1 - L_2)]^{-1}, \\ M - M_2 &\simeq c_1 (M_1 - M_2)\,[I + Q_2 (M_1 - M_2)]^{-1}, \end{aligned}\right\} \tag{16}$$

correct to first order. These can alternatively be obtained (HILL 1962, § 7) by substituting the zeroth order approximation for the concentration factors in

$$L - L_2 = c_1 (L_1 - L_2)\, A_1, \quad M - M_2 = c_1 (M_1 - M_2)\, B_1,$$

which are exact relations.

5. Isotropic Dispersion of Spheres

Suppose that the inclusions are spheres distributed in any way such that the composite is statistically isotropic overall. The first equation (15) then reduces to a pair of scalar formulae for the bulk and shear moduli, κ and μ :

$$\frac{c_1}{\kappa - \kappa_2} + \frac{c_2}{\kappa - \kappa_1} = \frac{\alpha}{\kappa}, \tag{17}$$

$$\frac{c_1}{\mu - \mu_2} + \frac{c_2}{\mu - \mu_1} = \frac{\beta}{\mu}, \tag{18}$$

where

$$\alpha = 3 - 5\beta = \kappa/(\kappa + \tfrac{4}{3}\mu). \tag{19}$$

The dimensionless quantities α and β are those that appear in the specific form of Eshelby's S tensor in the auxiliary problem for a sphere (cf. HILL 1965a, § 4 (ii)):

$$S_{ijkl} = \tfrac{1}{3}(\alpha - \beta)\,\delta_{ij}\,\delta_{kl} + \tfrac{1}{2}\beta\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}).$$

After substituting for α, (17) can be solved for κ parametrically in terms of μ, for instance in the form

$$\frac{1}{\kappa + \frac{4}{3}\mu} = \frac{c_1}{\kappa_1 + \frac{4}{3}\mu} + \frac{c_2}{\kappa_2 + \frac{4}{3}\mu}. \tag{20}$$

It is noteworthy that this is identical with the known *exact* solution for composites with arbitrary geometry, when the phases have equal shear moduli (HILL 1963, § 4; 1964, § 6), and also with the solution for a spherical composite element whose shell has rigidity μ.

To discuss (18) in general terms one may retain β as a parameter in view of its restricted range, namely

$$\tfrac{2}{5} \leqslant \beta < \tfrac{3}{5} \text{ when } \kappa, \mu \geqslant 0.$$

Then, clearing fractions,

$$(1 - \beta)\,\mu^2 + \{\beta\,(\mu_1 + \mu_2) - (c_1\,\mu_1 + c_2\,\mu_2)\}\,\mu - \beta\,\mu_1\,\mu_2 = 0.$$

The left side is found to be positive or negative respectively when μ is put equal in turn to the so-called Voigt and Reuss estimates :

$$\mu_V = c_1\,\mu_1 + c_2\,\mu_2, \quad \mu_R = \left(\frac{c_1}{\mu_1} + \frac{c_2}{\mu_2}\right)^{-1}.$$

Consequently, the required root lies between these limits. It follows that κ is certainly in the interval obtained by substituting μ_R and μ_V in the monotonic relation (20), and hence *a fortiori* between the rigorous best-possible bounds for arbitrary geometry, which are known to correspond to μ_1 and μ_2 in (20) (HILL 1963, § 5). These are further satisfactory features of the theory.

To derive the explicit equation for μ in its most convenient form, however, we express both sides of the first of (19) in terms of μ with the help of (18) and (20). The result is

$$\left(\frac{c_1\,\kappa_1}{\kappa_1 + \frac{4}{3}\mu} + \frac{c_2\,\kappa_2}{\kappa_2 + \frac{4}{3}\mu}\right) + 5\left(\frac{c_1\,\mu_2}{\mu - \mu_2} + \frac{c_2\,\mu_1}{\mu - \mu_1}\right) + 2 = 0. \tag{21}$$

[This could be multiplied out as a quartic but is far better left as it stands for iterative or graphical solution, by tabulating c_1 or c_2 as a function of μ between μ_1 and μ_2]. As μ increases from 0 to ∞, the first bracketed function decreases monotonically to zero from 1 if $\kappa_1\,\kappa_2 \neq 0$, from c_1 if $\kappa_2 = 0$, from c_2 if $\kappa_1 = 0$, and vanishes if both κ_1 and κ_2 are 0. If $\mu_1\,\mu_2 \neq 0$, with $\mu_1 > \mu_2$ say, the second bracketed function decreases monotonically from -1 to $-\infty$ in the range $(0, \mu_2)$; from $+\infty$ to $-\infty$ in (μ_2, μ_1), with values 0 and -1 at μ_R and μ_V; and from $+\infty$ to 0 in (μ_1, ∞). It is thereby confirmed again, provided neither phase rigidity vanishes, that there is precisely one positive root and that it lies between the Reuss and Voigt estimates.

This root can be stated explicitly when the dispersion is dilute. Thus, if $c_1 \ll 1$, we find $\mu \simeq \mu_2\,(1 + \lambda\,c_1)$ where

$$\frac{1}{\lambda} = \frac{\mu_1}{\mu_1 - \mu_2} - (2 + \alpha_2)$$

from (21), correct to zeroth order. That is,

$$\frac{\mu - \mu_2}{\mu_1 - \mu_2} \simeq \left[1 + \beta_2\left(\frac{\mu_1}{\mu_2} - 1\right)\right]^{-1} c_1, \tag{22}$$

which is a special case of (16). This coincides with formulae of OLDROYD (1956, equation (40)) and ESHELBY (1957, § 5; 1961, equation (6.10)).

When one phase is vacuous, say κ_2 and $\mu_2 \to 0$, equation (21) has a positive root when and only when the concentration of this phase is less than $\frac{1}{2}$; for instance the root is $\mu_1\,(1 - 2c_2)/(1 - \frac{1}{3}\,c_2)$ when $\kappa_1 \to \infty$. On the other hand, when both phases are incompressible ($\kappa_1, \kappa_2 \to \infty$), equation (21) can be reduced to

$$\left(\frac{3\mu + 2\mu_2}{\mu - \mu_2}\right) c_1 = \left(\frac{3\mu + 2\mu_1}{\mu_1 - \mu}\right) c_2$$

which gives either volume fraction explicitly in terms of μ, or to

$$3\mu^2 + \{(2 - 5c_1)\,\mu_1 + (2 - 5c_2)\,\mu_2\}\,\mu - 2\mu_1\,\mu_2 = 0$$

giving μ as a function of concentration. In particular, when $\mu_1/\mu_2 \to \infty$, the relevant root of this quadratic behaves asymptotically like

$$\mu \sim \begin{cases} 2\mu_2/(2 - 5c_1), & c_1 < \frac{2}{5} \\ \mu_1\,(5c_1 - 2)/3, & c_1 > \frac{2}{5}. \end{cases}$$

A study of these examples makes it plain that the theory is unreliable under extreme conditions, *except when the dispersed phase is sufficiently dilute.* Some such restriction on the range of validity was already to be expected : the general formulae in § 4 do not distinguish between the phases and yet the actual overall properties are totally different according to which of two disparate materials is the matrix.

With this proviso it appears that, in practice, the theory should be useful when rigorous bounds are either not known or are too far apart for empirical interpolation. This conclusion, already indicated for the bulk modulus, is reinforced by consideration of the only non-trivial bounds presently available for the rigidity modulus. These have been given by HASHIN and SHTRIKMAN (1963). It is convenient to re-arrange their formulae so as to isolate the volume fractions :

$$\left.\begin{aligned} \frac{\mu' - \mu_2}{\mu_1 - \mu_2} &= \left[1 + \beta_1 \left(\frac{\mu'}{\mu_1} - 1\right)\right] c_1, \\ \frac{\mu'' - \mu_1}{\mu_2 - \mu_1} &= \left[1 + \beta_2 \left(\frac{\mu''}{\mu_2} - 1\right)\right] c_2, \end{aligned}\right\} \tag{23}$$

where μ' is the upper and μ'' the lower bound when the materials are numbered so that $\mu_1 > \mu_2$, and where β_1 and β_2 denote the phase values of β in (19). The coefficients of the volume fractions are the shear-strain concentration factors in the respective phases. These bounds were derived under the apparently essential restriction $\kappa_1 \geqslant \kappa_2$.

We now recast (18) similarly in the alternative ways

$$\left.\begin{aligned} \frac{\mu - \mu_2}{\mu_1 - \mu_2} &= \left[1 + \beta \left(\frac{\mu_1}{\mu} - 1\right)\right]^{-1} c_1, \\ \frac{\mu - \mu_1}{\mu_2 - \mu_1} &= \left[1 + \beta \left(\frac{\mu_2}{\mu} - 1\right)\right]^{-1} c_2. \end{aligned}\right\} \tag{24}$$

These, of course, are scalar counterparts of the matrix formulae in § 4. Our plan is to compare the respective values of c_1 and c_2 defined by (23) and (24) when μ' and μ'' are formally set equal to any chosen value of μ between μ_1 and μ_2. That is, we take a horizontal section of the respective (modulus *v.* concentration) relations, in preference to a vertical section which would here be unprofitable. In preparation we note the identity

$$\left[1 + \beta \left(\frac{\mu_1}{\mu} - 1\right)\right]\left[1 + \beta_1 \left(\frac{\mu}{\mu_1} - 1\right)\right] \equiv$$

$$1 + \beta_1 \beta \frac{(\mu_1 - \mu)}{\mu_1 \mu}\left[\frac{(1 - \beta_1)}{\beta_1}\mu_1 - \frac{(1 - \beta)}{\beta}\mu\right],$$

together with a similar one in the second subscript. Now

$$\frac{2(1 - \beta)}{3\beta}\mu = \left(\frac{1}{\mu} + \frac{10}{9\kappa + 8\mu}\right)^{-1} \tag{25}$$

is a monotonically increasing function of both κ and μ. Hence the right-hand bracket in the identity is positive when $\kappa_1 \geqslant \kappa$ (as is the case when $\kappa_1 \geqslant \kappa_2$, by

what was proved before). It follows that each bracketed factor on the left of the identity exceeds the reciprocal of the other, and thus that the value of c_1 defined in (24) is greater than that defined in (23) when $\mu' = \mu$. Similarly, the value of c_2 in (24) is greater than that in (23) when $\mu'' = \mu$.

It may be concluded that, when $(\mu_1 - \mu_2)(\kappa_1 - \kappa_2) > 0$, *the theoretical rigidity lies between the Hashin–Sktrikman bounds at any concentration.* More especially, for a dilute dispersion, (22) coincides with the first-order approximation to one of the bounds (namely μ'' when c_1 is small). On the other hand, when $(\mu_1 - \mu_2)(\kappa_1 - \kappa_2) < 0$, nothing can be concluded from the identity by this line of argument, just as the status of the Hashin–Shtrikman expressions themselves is then also undecided. Indeed, their difference is given by

$$\frac{\mu' - \mu''}{(\mu_1 - \mu_2)^2} = \frac{\left[\dfrac{(1-\beta_1)}{\beta_1}\mu_1 - \dfrac{(1-\beta_2)}{\beta_2}\mu_2\right]\beta_1\beta_2 c_1 c_2}{[\mu_1 + \beta_1 c_1(\mu_2 - \mu_1)][\mu_2 + \beta_2 c_2(\mu_1 - \mu_2)]}, \tag{26}$$

its sign being controlled by a precisely similar factor in the right-hand numerator.

Finally, it should be observed that the modulus (25) has a precise mechanical significance. The overall constraint tensor of an isotropic continuum with a spherical cavity is

$$L^*_{ijkl} = \tfrac{4}{3}\mu\,\delta_{ij}\,\delta_{kl} + \frac{(1-\beta)}{\beta}\,\mu\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk} - \tfrac{2}{3}\,\delta_{ij}\,\delta_{kl}),$$

from the first of (4) and the components of S given in § 5 (cf. Hill 1965a, equation (20)). That is, by the definition (1),

$$-\sigma^*_{ij} = \tfrac{4}{3}\mu\,\epsilon^*_{kk}\,\delta_{ij} + \frac{2(1-\beta)}{\beta}\,\mu\,(\epsilon^*_{ij} - \tfrac{1}{3}\,\epsilon^*_{kk}\,\delta_{ij}).$$

Consequently, a unit fractional increase in radius calls for an internal pressure 4μ, while a unit shear of the cavity calls for tractions corresponding to an internal field of shear stress $2\mu(1-\beta)/\beta$.

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