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# Pontryagin calculus in Riemannian geometry

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**Abstract.** In this contribution, we study systems with a finite number of degrees of freedom as in robotics. A key idea is to consider the mass tensor associated to the kinetic energy as a metric in a Riemannian configuration space. We apply Pontryagin's framework to derive an optimal evolution of the control forces and torques applied to the mechanical system. This equation under covariant form uses explicitly the Riemann curvature tensor.

This contribution is dedicated to the memory of Claude Vallée (1945-2014).

**Keywords:** robotics, Euler-Lagrange, Riemann curvature tensor.

**AMS classification:** 49S05, 51P05, 53A35.

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## Introduction

As part of studies based on the calculus of variations, the choice of a Lagrangian or a Hamiltonian is essential. When we study the dynamics of articulated systems, the choice of the Lagrangian is directly linked to the conservation of energy. The Euler-Lagrange methodology is applied to the kinetic and potential energies. This establishes a system of second order ordinary differential equations for the movement. These equations are identical to those deduced from the fundamental principle of dynamics. The choice of configuration parameters does not affect the energy value. Because the kinetic energy is a positive definite quadratic form with respect to the configuration parameters derivatives, its coefficients are ideal candidates to define and create a Riemannian metric structure on the configuration space. The Euler Lagrange equations have a contravariant tensorial nature and highlight the covariant derivatives with respect to time with the introduction of the Christoffel symbols.

- For the control of articulated robot choosing a Hamiltonian and a cost function are delicate. Here the presence of the Riemann structure is sound. It enables a cost function invariant when coordinates change. The application of the Pontryagin method from the optimal Hamiltonian leads to a system of second order differential equations for the control variables. Its tensorial nature is covariant and the Riemann-Christoffel curvature tensor is naturally revealed. In this development, the adjoint variables are directly interpreted and have a physical sense.

### 1) Pontryagin framework for differential equations

We study a dynamical system, where the state vector  $y(t; \lambda(\bullet))$  is a function of time. This system is controlled by a set of variables  $\lambda(t)$  and satisfies a first order ordinary differential equation:

$$(1) \quad \frac{dy}{dt} = f(y(t), \lambda(t), t).$$

We suppose also given an initial condition:

$$(2) \quad y(0; \lambda(\bullet)) = x.$$

We search an optimal solution associated to the optimal control  $t \mapsto \lambda(t)$  in order to minimize the following cost function  $J$ :

$$(3) \quad J(\lambda(\bullet)) \equiv \int_0^T g(y(t), \lambda(t), t) dt,$$

where  $g(\bullet, \bullet, \bullet)$  is a given real valued function.

- Pontryagin's main idea [3] is to consider the differential equation (1) as a *constraint* satisfied by the variable  $y$ . Then he introduces a Lagrange multiplier  $p$  associated to the constraint (1). This new variable, due to the continuous nature of the constraint (1), is a covariant vector function of time:  $p = p(t)$ . A global Lagrangian functional can be considered:

$$\mathcal{L}(y, \lambda, p) \equiv \int_0^T g(y, \lambda, t) dt + \int_0^T p \left( \frac{dy}{dt} - f(y, \lambda, t) \right) dt.$$

• **Proposition 1. Adjoint equations.**

If the Lagrange multiplier  $p(t)$  satisfies the so-called *adjoint equations*,

$$(4) \quad \frac{dp}{dt} + p \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} = 0$$

and the so-called *final condition*,

$$(5) \quad p(T) = 0,$$

then the variation of the cost function for a given variation  $\delta\lambda$  of the parameter is given by the simple relation

$$\delta J = \int_0^T \left[ \frac{\partial g}{\partial \lambda} - p \frac{\partial f}{\partial \lambda} \right] \delta\lambda(t) dt.$$

At the optimum this variation is identically null and we find the so-called Pontryagin optimality condition:

$$(6) \quad \frac{\partial g}{\partial \lambda} - p \frac{\partial f}{\partial \lambda} = 0.$$

**Proof of Proposition 1.**

We write in a general way the variation of the Lagrangian  $\mathcal{L}(y, \lambda, p)$  in a variation  $\delta y$ ,  $\delta\lambda$  and  $\delta p$  of the variables  $y$ ,  $\lambda$ , and  $p$  respectively. We use classical calculus rules as

$$\delta \left( \int_0^T g dt \right) = \int_0^T \delta g dt, \quad \delta \left( \frac{dy}{dt} \right) = \frac{d}{dt}(\delta y),$$

and we integrate by parts. We get

$$\begin{aligned} \delta \mathcal{L} &= \int_0^T \left[ \frac{\partial g}{\partial y} \delta y + \frac{\partial g}{\partial \lambda} \delta \lambda \right] dt + \int_0^T p \left( \frac{d\delta y}{dt} - \frac{\partial f}{\partial y} \delta y - \frac{\partial f}{\partial \lambda} \delta \lambda \right) dt \\ &\quad + \int_0^T \delta p \left( \frac{dy}{dt} - f(y, \lambda, t) \right) dt \\ &= \int_0^T \left[ \frac{\partial g}{\partial y} - p \frac{\partial f}{\partial y} \right] \delta y dt + \int_0^T \left[ \frac{\partial g}{\partial \lambda} - p \frac{\partial f}{\partial \lambda} \right] \delta \lambda dt + \left[ p \delta y \right]_0^T - \int_0^T \frac{dp}{dt} \delta y dt \\ &= p(T) \delta y(T) - \int_0^T \left[ \frac{dp}{dt} + p \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} \right] \delta y dt + \int_0^T \left[ \frac{\partial g}{\partial \lambda} - p \frac{\partial f}{\partial \lambda} \right] \delta \lambda dt \end{aligned}$$

because  $\delta y(0) = 0$  taking fixed the initial condition (2). By canceling the first two terms of the right hand side of the previous relation, we find the adjoint equation (4) giving the evolution of the Lagrange multiplier and the associated final condition (5). The third term allows to calculate the change in the functional  $J(\bullet)$  for a variation  $\delta\lambda$  of control.  $\square$

## 2) Pontryagin hamiltonian

We introduce the Hamiltonian

$$(7) \quad \mathcal{H}(p, y, \lambda) \equiv p f - g$$

and the optimal Hamiltonian

$$H(p, y) \equiv \mathcal{H}(p, y, \lambda^*)$$

for  $\lambda(t) = \lambda^*(t)$  equal to the optimal value associated to the optimal condition (6).

• **Proposition 2. Symplectic form of the dynamic equations.**

With the notations introduced previously, the “forward” differential equation (1) and the “backward” adjoint differential equation (4) take the following symplectic form:

$$(8) \quad \frac{dy}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial y}.$$

**Proof of Proposition 2.**

Since  $\frac{\partial \mathcal{H}}{\partial \lambda} = 0$  at the optimum, we have  $\frac{\partial H}{\partial p} = \frac{\partial \mathcal{H}}{\partial p} = f$  and the first relation of (8) is proven. On the other hand,  $\frac{\partial H}{\partial y} = \frac{\partial \mathcal{H}}{\partial y} = p \frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}$  and the property is established.  $\square$

### 3) Riemanian metric

We consider now a dynamical system parameterized by a finite number of functions  $q^j(t)$  like a poly-articulated system for robotics applications, as developed previously in [1, 4, 5, 7]. The set of all states  $q \equiv \{q^j\}$  is denoted by  $Q$ . The kinetic energy  $K$  is a positive definite quadratic form of the time derivatives  $\dot{q}^j$  for each state  $q \in Q$ . The coefficients of this quadratic form define a so-called *mass tensor*  $M(q)$ . The mass tensor is composed by *a priori* a nonlinear regular function of the state  $q \in Q$ . We have

$$(9) \quad K(q, \dot{q}) \equiv \frac{1}{2} \sum_{k\ell} M_{k\ell}(q) \dot{q}^k \dot{q}^\ell.$$

The mass tensor  $M(q)$  in (9) is symmetric and positive definite for each state  $q$ . It contains the mechanical characteristics of mass, inertia of the articulated system. With Lazrak and Vallée [1] and Siebert [6], we consider the Riemannian metric  $g$  defined by the mass tensor  $M$ . We set:

$$g_{k\ell}(q) \equiv M_{k\ell}(q).$$

- With this framework, the space of states  $Q$  has now a structure of *Riemannian manifold*. Therefore all classical geometrical tools of Riemannian geometry can be used (see *e.g.* the book [2]):

covariant space derivation  $\partial_j \equiv \frac{\partial}{\partial q^j}$

contravariant space derivation  $\partial^j: \langle \partial^j, \partial_k \rangle = \delta_k^j$

component  $j, \ell$  of the inverse mass tensor  $M^{-1}$ :  $M^{j\ell}, M_{ij} M^{j\ell} = \delta_i^\ell$

connection  $\Gamma_{ik}^j = \frac{1}{2} M^{j\ell} (\partial_i M_{\ell k} + \partial_k M_{\ell i} - \partial_\ell M_{ik})$ ,  $\Gamma_{ki}^j = \Gamma_{ik}^j$ ,

$$d\partial_j = \Gamma_{jk}^\ell dq^k \partial_\ell, \quad d\partial^j = -\Gamma_{k\ell}^j dq^k \partial^\ell,$$

relations between covariant components  $\varphi_j$  and contravariant components  $\varphi^k$

$$\text{of a vector field: } \varphi_j = M_{jk} \varphi^k, \quad \varphi^k = M^{kj} \varphi_j$$

covariant derivation of a vector field  $\varphi \equiv \varphi^j \partial_j$ :  $d\varphi = (\partial_\ell \varphi^j + \Gamma_{\ell k}^j \varphi^k) dq^\ell \partial_j$

covariant derivation of a covector field  $\varphi \equiv \varphi_\ell \partial^\ell$ :

$$d\varphi = (\partial_k \varphi_\ell - \Gamma_{k\ell}^j \varphi_j) dq^k \partial^\ell$$

Ricci identities:

$$(10) \quad \begin{cases} \partial_j M_{k\ell} = \Gamma_{jk}^p M_{\ell p} + \Gamma_{j\ell}^p M_{kp}, \\ \partial_j M^{k\ell} = -\Gamma_{jp}^k M^{p\ell} - \Gamma_{jp}^\ell M^{pk} \end{cases}$$

gradient of a scalar field:  $dV = \partial_\ell V dq^\ell = \langle \nabla V, dq^j \partial_j \rangle$  and

$$\nabla V = \partial_\ell V \partial^\ell$$

gradient of a covector field  $\varphi = \varphi_\ell \partial^\ell$ :  $d\varphi \equiv \langle \nabla \varphi, dq^j \partial_j \rangle$  and

$$\nabla \varphi = (\partial_k \varphi_\ell - \Gamma_{k\ell}^j \varphi_j) \partial^k \partial^\ell$$

second order gradient of a scalar field  $V$ :  $\nabla^2 V = \nabla(\nabla V)$  and

$$(11) \quad \nabla^2 V = (\partial_k \partial_\ell V - \Gamma_{k\ell}^j \partial_j V) \partial^k \partial^\ell$$

components  $R_{ik\ell}^j$  of the Riemann tensor:

$$(12) \quad R_{ik\ell}^j \equiv \partial_\ell \Gamma_{ik}^j - \partial_k \Gamma_{i\ell}^j + \Gamma_{ik}^p \Gamma_{p\ell}^j - \Gamma_{i\ell}^p \Gamma_{pk}^j$$

anti-symmetry of the Riemann tensor:  $R_{ik\ell}^j = -R_{i\ell k}^j$ .

• **Proposition 3. Riemanian form of the Euler-Lagrange equations.**

With the previous framework, in the presence of an external potential  $V = V(q)$ , the Lagrangian  $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$  allows to write the equations of motion in the classical Euler-Lagrange form:

$$(13) \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i}$$

These equations take also the Riemannian form:

$$(14) \quad M_{k\ell} (\ddot{q}^\ell + \Gamma_{ij}^\ell \dot{q}^i \dot{q}^j) + \partial_k V = 0.$$

**Proof of Proposition 3.**

The proof is presented in the references [5] and [7]. We detail it to be complete. We have, due to (9),

$$\frac{\partial K}{\partial \dot{q}^k} = M_{k\ell} \dot{q}^\ell.$$

We have also the following calculus:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^k} \right) - \frac{\partial L}{\partial q^k} &= \frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}^k} \right) - \partial_k (K - V(q)) \\ &= \frac{d}{dt} (M_{k\ell} \dot{q}^\ell) - \partial_k \left( \frac{1}{2} M_{ij} \dot{q}^i \dot{q}^j \right) + \partial_k V \\ &= (\partial_j M_{k\ell}) \dot{q}^j \dot{q}^\ell + M_{k\ell} \ddot{q}^\ell - \frac{1}{2} (\partial_k M_{ij}) \dot{q}^i \dot{q}^j + \partial_k V \\ &= (M_{ks} \Gamma_{j\ell}^s + M_{\ell s} \Gamma_{jk}^s) \dot{q}^j \dot{q}^\ell - \frac{1}{2} (M_{is} \Gamma_{kj}^s + M_{js} \Gamma_{ki}^s) \dot{q}^i \dot{q}^j + M_{k\ell} \ddot{q}^\ell + \partial_k V \\ &\quad \text{due to the first Ricci identity (10)} \\ &= \left( M_{ks} \Gamma_{ji}^s + M_{is} \Gamma_{jk}^s - \frac{1}{2} M_{is} \Gamma_{kj}^s - \frac{1}{2} M_{js} \Gamma_{ki}^s \right) \dot{q}^i \dot{q}^j + M_{k\ell} \ddot{q}^\ell + \partial_k V \\ &= M_{k\ell} (\ddot{q}^\ell + \Gamma_{ij}^\ell \dot{q}^i \dot{q}^j) + \partial_k V \end{aligned}$$

and the relation (14) is established.  $\square$

• When a mechanical forcing control  $u$  is present (forces and torques typically), the equations of motion can be formulated as follows:

$$(15) \quad \ddot{q}^j + \Gamma_{k\ell}^j \dot{q}^k \dot{q}^\ell + M^{j\ell} \partial_\ell V = u^j.$$

We observe that with this form (15) of the equations of motion, the contravariant components of the control  $u$  have to be considered in the right hand side of the dynamical equations.

#### 4) Optimal dynamics

In this section, we follow the ideas proposed in [4, 5, 7]. The space of states  $Q$  has a natural Riemannian structure. Therefore, it is natural to choose a cost function that is intrinsic and invariant, and in consequence non sensible to the change of coordinates. Following Rojas Quinteros's thesis [4], we introduce a particular cost function to control the dynamics (15):

$$(16) \quad J(u) = \frac{1}{2} \int_0^T M_{k\ell}(q) u^k u^\ell dt.$$

The controlled system (15) (16) is of type (1) (3) with

$$(17) \quad \begin{cases} Y = \{q^j, \dot{q}^j\}, & f = \{Y_2^j, -\Gamma_{k\ell}^j \dot{q}^k \dot{q}^\ell - M^{j\ell} \partial_\ell V + u^j\}, \\ \lambda = \{u^k\}, & g = \frac{1}{2} M_{k\ell}(Y_1) u^k u^\ell. \end{cases}$$

The Pontryagin method introduces Lagrange multipliers (or adjoint states)  $p_j$  and  $\xi_j$  to form the Hamiltonian  $\mathcal{H}(Y, P, \lambda)$  function of state  $Y$  defined in (17) and adjoint  $P$  obtained by combining the two adjoint states:

$$(18) \quad P = \{p_j, \xi_j\}$$

and  $\lambda = \{u^k\}$  as proposed in (17). Taking into account (7), (17) and (18), we have:

$$(19) \quad \mathcal{H}(Y, P, \lambda) = p_j \dot{q}^j + \xi_j [-\Gamma_{k\ell}^j \dot{q}^k \dot{q}^\ell - M^{j\ell} \partial_\ell V + u^j] - \frac{1}{2} M_{k\ell}(Y_1) u^k u^\ell.$$

• **Proposition 4. Interpretation of one adjoint state.**

When the cost function  $J$  defined in (16) is stationary, the adjoint state  $\xi^j$  is exactly equal to the applied force (and torque!)  $u^j$  in the right hand side of the dynamic equation (15):

$$(20) \quad \xi^j = u^j.$$

**Proof of Proposition 4.**

Due to the expression (19) of the Hamiltonian function, the optimality condition  $\frac{\partial \mathcal{H}}{\partial \lambda} = 0$  takes the simple form

$$\xi_j = M_{j\ell} \xi^\ell.$$

This relation is equivalent to the condition (20). □

• The reduced Hamiltonian  $H(Y, P)$  at the optimum can be explicitated without difficulty. We just replace the control force  $u_j$  by the adjoint state  $\xi_j$ :

$$H(Y, P) = p_j \dot{q}^j + \xi_j [-\Gamma_{k\ell}^j \dot{q}^k \dot{q}^\ell - M^{j\ell}(Y_1) \partial_\ell V] + \frac{1}{2} M^{k\ell}(Y_1) \xi_k \xi_\ell.$$

The symplectic dynamics (8) can be written simply:

$$(21) \quad \dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \ddot{q}^j = \frac{\partial H}{\partial \xi_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \quad \dot{\xi}_j = -\frac{\partial H}{\partial \dot{q}^j}.$$

The two first equations of (21) give the initial controlled dynamics (15). We have also

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial q^j} = -(\partial_j \Gamma_{k\ell}^i) \dot{q}^k \dot{q}^\ell \xi_i - \partial_j (M^{i\ell} \partial_\ell V) \xi_i + \frac{1}{2} (\partial_j M^{k\ell}) \xi_k \xi_\ell \\ \frac{\partial H}{\partial \dot{q}^j} = p_j - 2 \Gamma_{kj}^i \dot{q}^k \xi_i. \end{array} \right.$$

We deduce the developed form of the two last equations of (21):

$$(22) \quad \dot{p}_j = (\partial_j \Gamma_{k\ell}^i) \dot{q}^k \dot{q}^\ell \xi_i + \partial_j (M^{i\ell} \partial_\ell V) \xi_i - \frac{1}{2} (\partial_j M^{k\ell}) \xi_k \xi_\ell$$

$$(23) \quad \dot{\xi}_j = 2 \Gamma_{kj}^i \dot{q}^k \xi_i - p_j.$$

## 5) Intrinsic evolution of the generalized force

We introduce the covector  $\xi$  according to its covariant coordinates:  $\xi = \xi_j \partial^j$ . We have the following result, first established in [4] and [7]:

### • Proposition 5. Covariant evolution equation of the optimal force.

With the above notations and hypotheses, the forces and torques  $u$  satisfy the following time evolution:

$$(24) \quad \left( \frac{d^2 u}{dt^2} \right)_j + R_{k\ell j}^i \dot{q}^k \dot{q}^\ell u_i + (\nabla_{jk}^2 V) u^k = 0.$$

### Proof of Proposition 5.

The time covariant derivative of the covector  $\xi$  is given by

$$\frac{d\xi}{dt} = (\dot{\xi}_j - \Gamma_{jk}^i \dot{q}^k \xi_i) \partial^j$$

that is  $\left( \frac{d\xi}{dt} \right)_j = \dot{\xi}_j - \Gamma_{jk}^i \dot{q}^k \xi_i$ . We report this expression in (23):

$$(25) \quad p_j = \Gamma_{jk}^i \dot{q}^k \xi_i - \left( \frac{d\xi}{dt} \right)_j.$$

We wish to differentiate relative to time the expression  $p_j$  given in (25). The covariant derivatives of the covector  $p$  can be evaluated as follows:

$$\left( \frac{dp}{dt} \right)_j = \dot{p}_j - \Gamma_{jk}^\ell \dot{q}^k p_\ell.$$

Then we have, taking into account again the relation (25):

$$\begin{aligned} \dot{p}_j &= \frac{d}{dt} \left[ \Gamma_{jk}^i \dot{q}^k \xi_i - \left( \frac{d\xi}{dt} \right)_j \right] + \Gamma_{jk}^\ell \dot{q}^k \left[ \Gamma_{s\ell}^i \dot{q}^s \xi_i - \left( \frac{d\xi}{dt} \right)_\ell \right] \\ &= \partial_\ell (\Gamma_{jk}^i) \dot{q}^k \dot{q}^\ell \xi_i + \Gamma_{jk}^i \ddot{q}^k \xi_i + \Gamma_{jk}^i \dot{q}^k \left( \frac{d\xi}{dt} \right)_i - \left( \frac{d^2 \xi}{dt^2} \right)_j \\ &\quad + \Gamma_{jk}^\ell \Gamma_{s\ell}^i \dot{q}^k \dot{q}^s \xi_i - \Gamma_{jk}^\ell \dot{q}^k \left( \frac{d\xi}{dt} \right)_\ell \\ &= \partial_\ell (\Gamma_{jk}^i) \dot{q}^k \dot{q}^\ell \xi_i + \Gamma_{kj}^s \Gamma_{s\ell}^i \dot{q}^k \dot{q}^\ell \xi_i - \left( \frac{d^2 \xi}{dt^2} \right)_j + \Gamma_{kj}^i \ddot{q}^k \xi_i \end{aligned}$$



due to the simplification of two terms

$$\begin{aligned}
&= \partial_\ell (\Gamma_{jk}^i) \dot{q}^k \dot{q}^\ell \xi_i + \Gamma_{kj}^s \Gamma_{sl}^i \dot{q}^k \dot{q}^\ell \xi_i - \left( \frac{d^2 \xi}{dt^2} \right)_j \\
&\quad + \Gamma_{kj}^i \left( -\Gamma_{ls}^k \dot{q}^s \dot{q}^\ell - M^{k\ell} \partial_\ell V + \xi^k \right) \xi_i \quad \text{due to (15)} \\
&= \left( \partial_\ell \Gamma_{jk}^i + \Gamma_{jk}^s \Gamma_{sl}^i - \Gamma_{kl}^s \Gamma_{sj}^i \right) \dot{q}^k \dot{q}^\ell \xi_i - \left( \frac{d^2 \xi}{dt^2} \right)_j - \Gamma_{kj}^i M^{k\ell} \partial_\ell V \xi_i + \Gamma_{kj}^i \xi^k \xi_i
\end{aligned}$$

and

$$(26) \quad \dot{p}_j = \left( R_{kjl}^i + \partial_j \Gamma_{kl}^i \right) \dot{q}^k \dot{q}^\ell \xi_i - \left( \frac{d^2 \xi}{dt^2} \right)_j - \Gamma_{kj}^i M^{k\ell} \partial_\ell V \xi_i + \Gamma_{kj}^i \xi^k \xi_i$$

taking into account the expression (12) of the Riemann tensor. We confront the relations (26) and (22). We deduce

$$(27) \quad \begin{cases} R_{kjl}^i \dot{q}^k \dot{q}^\ell \xi_i - \left( \frac{d^2 \xi}{dt^2} \right)_j - \Gamma_{kj}^i M^{k\ell} \partial_\ell V \xi_i + \Gamma_{kj}^i \xi^k \xi_i \\ \quad = \partial_j (M^{i\ell} \partial_\ell V) \xi_i - \frac{1}{2} (\partial_j M^{k\ell}) \xi_k \xi_\ell. \end{cases}$$

We take into account the second Ricci identity (10). It comes

$$(\partial_j M^{k\ell}) \xi_k \xi_\ell = -\Gamma_{js}^k \xi_k \xi^s - \Gamma_{js}^\ell \xi_\ell \xi^s = -2\Gamma_{j\ell}^k \xi_k \xi^\ell = -2\Gamma_{jk}^i \xi^k \xi_i.$$

Then we can write the relation (27) in a simpler way:

$$\begin{aligned}
R_{kjl}^i \dot{q}^k \dot{q}^\ell \xi_i - \left( \frac{d^2 \xi}{dt^2} \right)_j &= \left[ \Gamma_{kj}^i M^{k\ell} \partial_\ell V + \partial_j (M^{i\ell} \partial_\ell V) \right] \xi_i \\
&= \left[ \Gamma_{kj}^i M^{k\ell} \partial_\ell V - \Gamma_{js}^i M^{s\ell} \partial_\ell V - \Gamma_{js}^\ell M^{is} \partial_\ell V + M^{i\ell} \partial_\ell \partial_j V \right] \xi_i \\
&= \left( \partial_\ell \partial_j V - \Gamma_{j\ell}^s \partial_s V \right) M^{i\ell} \xi_i = (\nabla_{j\ell}^2 V) \xi^\ell
\end{aligned}$$

due to the expression (11) of the second gradient of a scalar field. We have established the following evolution equation

$$\left( \frac{d^2 \xi}{dt^2} \right)_j + (\nabla_{jk}^2 V) \xi^k = R_{kjl}^i \dot{q}^k \dot{q}^\ell \xi_i$$

and the relation (24) is a simple consequence of the anti-symmetry of the Riemann tensor and of the identity (20).  $\square$

## Conclusion

We have established that the methods of Euler-Lagrange and Pontryagin conduct to two second order differential systems that couples state and control variables. The choice of a Riemannian metric allows the two systems to be in a well-defined tensorial nature: contravariant for the equation of motion and covariant for the equation of the control variables. The study of a robotic system, of which we try to optimize the control, shows how important is the introduction of an appropriate geometric structure. Riemannian geometry selected on the configuration parameter space favors the metric directly related to the mass tensor as suggested by the expression of the kinetic energy. An undeniable impact is the choice of an invariant cost function with respect to the choice of parameters, this

is a stabilizing factor for numerical developments. Pontryagin's principle applied to contravariant equation of motion associated with the cost function conducts to a mechanical interpretation of adjoint states.

The adjoint control equation is established in a condensed form by the introduction of second order covariant derivatives and shows the Riemann curvature tensor. Moreover, this framework exhibits a numerically stable method when discretization is considered. The resolution of the coupled system gives a direct access to control variables without any additional calculation. Thus, future numerical developments will have to juggle between two coupled systems of second-order ordinary differential equations: the equation of motion and the equation for the control.

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