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A short introduction to Carleman estimates¹

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Foreword

These notes correspond to a lecture given by Sylvain Ervedoza within the LIAFSMA International Graduate School in Applied Maths 2021, and are strongly inspired by some lectures taught in 2013 in the master program of Université Paul Sabatier, Toulouse, France.

These lecture notes aim at presenting some insights on the derivation of Carleman estimates and their use in some specific applications, using a self-contained approach requiring only the knowledge of basic theory for elliptic PDE and Fourier transforms.

Of course, we are fully aware that this approach will not allow to reach the many results that have been derived in the extensive literature on Carleman estimates, for which we refer to more advanced textbooks, such as the excellent books [27], [30], [12], [3], among many others.

Still, we believe that our approach offers new perspectives on Carleman estimates and underlines the reasons why they work in a somewhat pedestrian manner. In fact, this methodology is closely related to the ones in [6], [15], among others.

Chapter 0

Prerequisites

This chapter gathers several results which we will assume to be known next.

0.1 Functional spaces

0.1.1 Lebesgue spaces

We assume that the reader is familiar with measurable functions, integrable functions, and more specifically to the Lebesgue space $L^1(\Omega)$ for Ω an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$.

The Lebesgue spaces are then defined as follows:

Definition 0.1 (Lebesgue spaces). *Let Ω an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$ and $p \in [1, \infty)$. We set*

$$\begin{aligned} L^p(\Omega) &= \{f : \Omega \rightarrow \mathbb{C}, \text{ measurable, with } |f|^p \in L^1(\Omega)\}, \\ L^\infty(\Omega) &= \{f : \Omega \rightarrow \mathbb{C}, \text{ measurable,} \\ &\quad \text{such that there exists a constant } C \text{ such that a.e. } |f| \leq C\}, \end{aligned}$$

and

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p dx \right)^{1/p}, \quad \|f\|_{L^\infty(\Omega)} = \inf\{C > 0 \text{ such that } |f| \leq C \text{ a.e.}\}$$

The study of these Lebesgue spaces is classical and can be found for instance in [5, Chapter 4], where the following results are proved:

Proposition 0.2 (Banach spaces). *For all $p \in [1, \infty]$, the Lebesgue space $L^p(\Omega)$ is a Banach space when endowed with the norm $\|\cdot\|_{L^p(\Omega)}$.*

Proposition 0.3 (Duality). *For all $p \in (1, \infty)$, the space $(L^p(\Omega))'$ of all continuous linear forms on $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$, with p' given by*

$$\frac{1}{p} + \frac{1}{p'} = 1$$

We also present some useful estimates:

Proposition 0.4 (Hölder's inequality). *Let $p, q, r \in [1, \infty]^3$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Then, for all $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, fg belongs to $L^r(\Omega)$ and

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Proposition 0.5 (Young's inequality). *Let $p, q, r \in [1, \infty]^3$ with*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

Then for $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, the convolution of f by g , denoted by $f \star g$ and defined by

$$f \star g(x) = \int_{\mathbb{R}^d} f(x-y)g(y) dy, \quad x \in \mathbb{R}^d,$$

is well-defined as a function of $L^r(\mathbb{R}^d)$ and

$$\|f \star g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

0.1.2 A quick reminder on distributions

We start by recalling the definition of distributions in a bounded domain Ω and some elementary properties, see for instance [14].

Definition 0.6 (Test functions). *Let Ω be an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$. Then $\mathcal{D}(\Omega)$ denotes the set of all functions $\varphi : \Omega \rightarrow \mathbb{R}$ which are infinitely differentiable in Ω and with compact support in Ω . The set $\mathcal{D}(\Omega)$ is called the set of test functions.*

Using this set $\mathcal{D}(\Omega)$, we can define the set of distributions on Ω :

Definition 0.7 (Distributions). *Let Ω be an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$. Then a distribution u on Ω is a linear form on $\mathcal{D}(\Omega)$ such that for all compact $K \subset \Omega$, there exist $C_K > 0$ and $p_K \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}(\Omega)$ with $\text{Supp } \varphi \subset K$, then*

$$|u(\varphi)| \leq C_K \sup_{|\alpha| \leq p_K} \|D^\alpha \varphi\|_{L^\infty(K)}.$$

The set of all distributions is denoted by $\mathcal{D}'(\Omega)$.

Notation: In the following, as it is commonly done, for $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we will denote $u(\varphi)$ by $\langle u, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$.

A key remark is that functions can be viewed as distributions through the following identification: if $u \in L^1_{loc}(\Omega)$ ¹, then we define the corresponding distribution by, for all $\varphi \in \mathcal{D}(\Omega)$,

$$\langle u, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \int_{\Omega} u(x)\varphi(x) dx.$$

Note however that they are distributions which cannot be identified with locally integrable functions, such as, for instance, the Dirac delta distribution δ_0 on \mathbb{R} defined by $\forall \varphi \in \mathcal{D}(\mathbb{R})$, $\langle \delta_0, \varphi \rangle_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} = \varphi(0)$.

Another important point is that the classical derivatives can easily be extended on distributions by the following formula:

¹ $L^1_{loc}(\Omega)$ denotes the set of functions whose restriction to any compact K of Ω belongs to $L^1(K)$

Definition 0.8 (Distribution derivative). *Let Ω be an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$, $u \in \mathcal{D}'(\Omega)$, and $i \in \{1, \dots, d\}$. Then $\partial_i u$ is the distribution defined by: for all $\varphi \in \mathcal{D}(\Omega)$,*

$$\langle \partial_i u, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = -\langle u, \partial_i \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)},$$

where in the right hand side of this identity, $\partial_i \varphi$ is the classical derivative of φ with respect to the i -th variable.

One easily checks that Definition 0.8, together with the identification of $L^1_{loc}(\Omega)$ as a subset of $\mathcal{D}'(\Omega)$, extends the classical definition of the derivative, as a straightforward consequence of the following integration by part formula: if $u \in C^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$,

$$-\langle u, \partial_i \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = -\int_{\Omega} u(x) \partial_i \varphi(x) dx = \int_{\Omega} \partial_i u(x) \varphi(x) dx.$$

0.1.3 Sobolev spaces

We shall not give an extensive review of Sobolev spaces, but only some of their properties, which can be found for instance in [10, Chapter 5]. In particular, we will simply focus on the Sobolev spaces which have a Hilbertian structure.

Spaces $H^k(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$

Definition 0.9 (The spaces $H^k(\Omega)$, $k \in \mathbb{N}$). *Let Ω be an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$. For $k \in \mathbb{N}$, we define*

$$H^k(\Omega) = \{f \in L^2(\Omega), \text{ such that for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leq k, \partial^\alpha f \in L^2(\Omega)\},$$

where for $\alpha \in \mathbb{N}^d$, $|\alpha| = \sum_{i=1}^d |\alpha_i|$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ denotes the derivative in the sense of distributions.

The set $H^k(\Omega)$ is endowed with the norm

$$\|f\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^2(\Omega)}^2 \right)^{1/2}. \quad (0.1)$$

We have the following proposition:

Proposition 0.10 (Basic properties of $H^k(\Omega)$). *Let Ω be an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$ and $k \in \mathbb{N}$.*

Then the set $H^k(\Omega)$ is a separable Hilbert space.

If Ω is bounded and its boundary $\partial\Omega$ is of class \mathcal{C}^1 , then the set $\mathcal{C}^\infty(\bar{\Omega})$ is dense in $H^k(\Omega)$.

We shall also often use the space $H_0^1(\Omega)$:

Definition 0.11 (The spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$). *Let Ω be an (non-empty) open set of \mathbb{R}^d , $d \in \mathbb{N} \setminus \{0\}$. Then $H_0^1(\Omega)$ denotes the closure of the set $\mathcal{D}(\Omega)$ for the topology of $H^1(\Omega)$. The set $H^{-1}(\Omega)$ is defined as the topological dual of $H_0^1(\Omega)$.*

Fourier transforms and fractional Sobolev spaces

It turns out that we will precisely deal with boundary data quite often. In order to do that, we will use the Fourier transform and define fractional Sobolev spaces.

Definition 0.12 (The Fourier transform). *Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then the Fourier transform of f is given by*

$$\mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$

When no confusion occurs, we will simply denote it by \widehat{f} .

We have the following classical theorem:

Theorem 0.13 (Parseval's identity). *The Fourier transform \mathcal{F} can be extended as an isometry of $L^2(\mathbb{R}^d)$:*

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 d\xi.$$

Its inverse is given, for $g \in L^2(\mathbb{R}^d)$ by

$$\mathcal{F}^{-1}[g](x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(\xi) e^{ix \cdot \xi} d\xi, \quad x \in \mathbb{R}^d.$$

Definition 0.14 (The fractional Sobolev spaces $H^s(\mathbb{R}^d)$, $s \geq 0$). *For $s \geq 0$, the set $H^s(\mathbb{R}^d)$ is the set of all functions $f \in L^2(\mathbb{R}^d)$ such that $\xi \mapsto (1 + |\xi|^2)^{s/2} \widehat{f}(\xi) \in L^2(\mathbb{R}^d)$. For $f \in H^s(\mathbb{R}^d)$, we set*

$$\|f\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

If $s \in \mathbb{N}$, this norm is equivalent to the norm $H^k(\mathbb{R}^d)$ for $k = s$ given in (0.1).

For all $s \geq 0$, the set $H^s(\mathbb{R}^d)$ is a Hilbert space.

We are then in position to define fractional functional spaces for an open set Ω of \mathbb{R}^d , following for instance [31].

Definition 0.15 (Definition of $H^s(\Omega)$ and $H^s(\partial\Omega)$, $s \geq 0$). *Let Ω be a bounded subset of \mathbb{R}^d whose boundary has \mathcal{C}^∞ regularity. For $s \geq 0$, the set $H^s(\Omega)$ is defined as the restrictions to Ω of functions of $H^s(\mathbb{R}^d)$, and is endowed with the norm*

$$\|u\|_{H^s(\Omega)} = \inf\{\|U\|_{H^s(\mathbb{R}^d)} \text{ for } U \in H^s(\mathbb{R}^d) \text{ with } U|_\Omega = u\}.$$

The definition of $H^s(\partial\Omega)$ for $s \geq 0$ is slightly more involved, and we follow the presentation of [31]. We assume that Ω be a bounded subset of \mathbb{R}^d whose boundary $\partial\Omega$ has \mathcal{C}^∞ regularity. This means that there exists a finite family of open bounded sets \mathcal{O}_j , $j \in \{1, \dots, N\}$ of \mathbb{R}^d which covers $\partial\Omega$ and such that for all $j \in \{1, \dots, N\}$, there exists a \mathcal{C}^∞ mapping φ_j from \mathcal{O}_j to a neighborhood \mathcal{V}_j of 0 in \mathbb{R}^d , such that it is invertible on \mathcal{V}_j as a \mathcal{C}^∞ mapping, and

$$\varphi_j(\mathcal{O}_j \cap \Omega) = \mathcal{V}_j \cap \{y_d > 0\}, \quad \varphi_j(\mathcal{O}_j \cap \partial\Omega) = \mathcal{V}_j \cap \{y_d = 0\}, \quad \varphi_j(\mathcal{O}_j \setminus \overline{\Omega}) = \mathcal{V}_j \cap \{y_d < 0\}.$$

We then require that if $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ then there exists a \mathcal{C}^∞ homeomorphism $J_{i,j}$ from $\varphi_i(\mathcal{O}_i \cap \mathcal{O}_j)$ to $\varphi_j(\mathcal{O}_i \cap \mathcal{O}_j)$ with positive jacobian such that

$$\forall x \in \mathcal{O}_i \cap \mathcal{O}_j, \quad \varphi_j(x) = J_{i,j}(\varphi_i(x)).$$

We then consider a partition of unity associated to the covering $\mathcal{O}_j, j \in \{1, \dots, N\}$ of $\partial\Omega$, by choosing N smooth functions $\alpha_j, j \in \{1, \dots, N\}$, such that for all j , α_j is compactly supported in \mathcal{O}_j and $\sum_{j=1}^N \alpha_j = 1$ on $\partial\Omega$.

We then set

$$H^s(\partial\Omega) = \{u \in L^2(\Gamma), \text{ such that } \forall j \in \{1, \dots, N\}, y' \mapsto (\alpha_j u)(\varphi_j^{-1}(y', 0)) \in H^s(\mathbb{R}^{d-1})\},$$

where the function $y' \mapsto (\alpha_j u)(\varphi_j^{-1}(y', 0))$ is extended by 0 outside $\mathcal{V}_j \cap \{y_d = 0\}$.

We also define the corresponding norm

$$\|u\|_{H^s(\partial\Omega)} = \left(\sum_{j=1}^N \|y' \mapsto (\alpha_j u)(\varphi_j^{-1}(y', 0))\|_{H^s(\mathbb{R}^{d-1})}^2 \right)^{1/2}.$$

Note that the definition of $H^s(\partial\Omega)$ and of $\|\cdot\|_{H^s(\partial\Omega)}$ *a priori* depends on the choice of the covering $\mathcal{O}_j, j \in \{1, \dots, N\}$, the charts $\varphi_j, j \in \{1, \dots, N\}$ and the partition of unity $\alpha_j, j \in \{1, \dots, N\}$.

It turns out that one can prove (see [31]) that the space $H^s(\partial\Omega)$ is in fact independent of all these choices, since two norms corresponding to different choices of covering, charts, and partition of unity, necessarily are equivalent. We refer to [31] for a more detailed discussion.

Trace theorems

Let Ω be a smooth bounded. The question we address here is under which circumstances we can define the value of a function on the boundary of Ω . It turns out that this can be done only under some regularity conditions, that we recall hereafter, see [31]:

Theorem 0.16. *There exists a unique continuous map γ_0 from $H^1(\Omega)$ to $L^2(\partial\Omega)$ such that for all $f \in \mathcal{C}^\infty(\bar{\Omega})$, $\gamma_0(f) = f|_{\partial\Omega}$.*

This map is called the trace map.

The trace map is continuous surjective from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ and its restriction $\gamma_0|_{H^2(\Omega)}$ to $H^2(\Omega)$ is surjective from $H^2(\Omega)$ to $H^{3/2}(\partial\Omega)$.

Besides, $\text{Ker} \gamma_0 = H_0^1(\Omega)$.

Similarly, to define the normal derivative of a function, one needs to add some regularity condition:

Theorem 0.17. *There exists a unique continuous map γ_1 from $H^2(\Omega)$ to $L^2(\partial\Omega)$ such that for all $f \in \mathcal{C}^\infty(\bar{\Omega})$, $\gamma_1(f) = \partial_n f|_{\partial\Omega}$, where $\partial_n f$ denotes the normal derivative of f on $\partial\Omega$.*

The map γ_1 is continuous and surjective from $H^2(\Omega)$ to $H^{1/2}(\partial\Omega)$.

For more properties, we refer to the classical textbook [31].

0.2 Elliptic equations

In this section, we briefly recall the existence theory for the Laplace problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (0.2)$$

where Ω is a smooth (\mathcal{C}^∞) bounded domain of \mathbb{R}^d , and $\Delta = \partial_{11} + \dots + \partial_{dd}$ is the Laplace operator.

Following [5, 10], we get the following result, as a consequence of Lax-Milgram theorem:

Theorem 0.18. *Let $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Then there exists a unique function $u \in H^1(\Omega)$ such that*

- $\gamma_0 u = g$, where γ_0 is the trace map defined in Theorem 0.16,
- for all $v \in H_0^1(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Such a function u is called a weak solution of (0.2).

Besides, there exists a constant C such that

$$\|u\|_{H^1(\Omega)} \leq C \left(\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} \right).$$

One can also prove the following regularity result, see e.g. [10]:

Theorem 0.19. *Let $f \in L^2(\Omega)$ and $g \in H^{3/2}(\partial\Omega)$. Then the function u provided by Theorem 0.18 satisfies $u \in H^2(\Omega)$.*

Besides, there exists a constant C such that

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)} \right).$$

Let us finally remark that these theorems apply as soon as the open set Ω is smooth and bounded in one direction, since then Poincaré's estimate holds: there exists a constant $C > 0$ such that for all $u \in H_0^1(\Omega)$, $\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$.

In our case, we will repeatedly use this property in the case of a vertical strip $\Omega = (0, 1) \times \mathbb{R}^{d-1}$, see the next chapters.

Chapter 1

Introduction: On the Laplace operator in a strip

1.1 The Cauchy problem

Our starting point is the so-called Cauchy problem for the Laplace operator. To be more precise, we consider a solution u of an elliptic equation in a smooth domain Ω of \mathbb{R}^d :

$$\Delta u = f, \quad \text{in } \Omega, \quad (1.1)$$

where $\Delta = \partial_{11} + \dots + \partial_{dd}$ is the Laplace operator and f is assumed to be known.

We also assume that there is a non-empty open part of the boundary $\Gamma \subset \partial\Omega$ on which the Cauchy data are known, that is to say we know

$$\begin{cases} u = g_D, & \text{on } \Gamma, \\ \partial_n u = g_N, & \text{on } \Gamma. \end{cases} \quad (1.2)$$

Here, the indexes D and N respectively stand for the Dirichlet and Neumann data on Γ .

The Cauchy problem is then the following one:

Given the data f in Ω , g_D and g_N on Γ , can we determine the solution u of (1.1)–(1.2)?

Here, recall that if f is given in Ω and g_D is given on the whole boundary $\partial\Omega$, problem (1.1)–(1.2)₁ is well-posed and has a unique solution, see Section 0.2 and e.g. [5] (see also Exercise 1.1), hence the above question is obvious when Γ is the whole boundary, and the difficult case is the one corresponding to $\Gamma \neq \partial\Omega$.

We shall not give any precise answer to that question in this full generality, but we will give some insights on it. In order to do that, we focus on the case

$$\Omega = (0, 1) \times \mathbb{R}^{d-1} \quad (d \in \mathbb{N}), \quad \text{and} \quad \Gamma = \{0\} \times \mathbb{R}^{d-1}, \quad (1.3)$$

i.e. the case of a strip observed from one side.

In that situation, we may rewrite the problem (1.1)–(1.2) as follows:

$$\begin{cases} \partial_{11}u + \Delta' u = f, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ u(0, x') = g_D(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1 u(0, x') = g_N(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (1.4)$$

where $\Delta' = \partial_{22} + \dots + \partial_{dd}$ is the Laplace operator in the variable $x' = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$.

In that setting, a natural approach consists in taking the Fourier transform of u in the x' -variable only. Therefore, for $x_1 \in [0, 1]$ and $\xi' \in \mathbb{R}^{d-1}$, we introduce

$$\begin{aligned}\hat{u}(x_1, \xi') &= \mathcal{F}_{x' \rightarrow \xi'} u(x_1, \cdot), \\ \hat{f}(x_1, \xi') &= \mathcal{F}_{x' \rightarrow \xi'} f(x_1, \cdot), \\ \hat{g}_D(\xi') &= \mathcal{F}_{x' \rightarrow \xi'} g_D(\cdot), \\ \hat{g}_N(\xi') &= \mathcal{F}_{x' \rightarrow \xi'} g_N(\cdot),\end{aligned}$$

where $\mathcal{F}_{x' \rightarrow \xi'}$ is the Fourier transform in the variable x' , defined for function $v \in \mathcal{S}(\mathbb{R}^{d-1})$ of $x' \in \mathbb{R}^{d-1}$ by

$$\forall \xi' \in \mathbb{R}^{d-1}, \quad (\mathcal{F}_{x' \rightarrow \xi'} v)(\xi) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} v(x') e^{-ix' \cdot \xi'} dx'.$$

Taking the partial Fourier transform $\mathcal{F}_{x' \rightarrow \xi'}$ of (1.4), we obtain

$$\begin{cases} \partial_{11} \hat{u} - |\xi'|^2 \hat{u} = \hat{f}, & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{u}(0, \xi') = \hat{g}_D(\xi'), & \text{for } \xi' \in \mathbb{R}^{d-1}, \\ \partial_1 \hat{u}(0, \xi') = \hat{g}_N(\xi'), & \text{for } \xi' \in \mathbb{R}^{d-1}, \end{cases} \quad (1.5)$$

At $\xi' \in \mathbb{R}^{d-1}$ fixed, these equations can be solved explicitly in x_1 and the solution $\hat{u}(\cdot, \xi')$ is given by

$$\begin{aligned}\hat{u}(x_1, \xi') &= \cosh(|\xi'|x_1) \hat{g}_D(\xi') + \frac{\sinh(|\xi'|x_1)}{|\xi'|} \hat{g}_N(\xi') \\ &\quad + \int_0^{x_1} \frac{\sinh(|\xi'|(x_1 - x))}{|\xi'|} \hat{f}(x, \xi') dx. \quad (1.6)\end{aligned}$$

Up to now, our arguments were mainly formal. The above computations require in particular the Fourier transform to be well-defined for $u(x_1, \cdot)$ for almost all $x_1 \in (0, 1)$. It is certainly the case if u is assumed to be for instance in $L^2((0, 1) \times \mathbb{R}^{d-1})$, or even $L^2(0, 1; \mathcal{S}'(\mathbb{R}^{d-1}))$.

An important difficulty arises here. If we do not know *a priori* that u exists and wonder if one can find a solution u to the Cauchy problem (1.4), one needs to guarantee that one can take the inverse Fourier transform of formula (1.6). But, at $x_1 \in (0, 1)$ fixed, $\hat{u}(x_1, \cdot)$ may contain growing exponentials of the form $\xi' \mapsto \exp(\alpha|\xi'|)$ with $\alpha > 0$. Such function does not belong to $\mathcal{S}'(\mathbb{R}^{d-1})$ and its inverse Fourier transform is not defined.

On the opposite, if we know from the beginning that u exists and belongs to some class where the partial Fourier transform is well-defined, then it is uniquely determined by the above formula (1.6). But even in that case, it may be difficult to derive good estimates on u via the use of formula (1.6), as we only have, for all $x_1 \in (0, 1)$ and $\xi' \in \mathbb{R}^{d-1}$,

$$|\hat{u}(x_1, \xi')| \leq C \exp(|\xi'|x_1) \left(|\hat{g}_D(\xi')| + |\hat{g}_N(\xi')| + \int_0^{x_1} |\hat{f}(x, \xi')| dx \right),$$

for some constant C independent of (x_1, ξ') . This allows to derive

$$\begin{aligned} & \int_0^1 \int_{\mathbb{R}^{d-1}} |\hat{u}(x_1, \xi')|^2 \exp(-2|\xi'|x_1) d\xi' dx_1 \\ & \leq C \left(\int_{\mathbb{R}^{d-1}} |\hat{g}_D(\xi')|^2 + \int_{\mathbb{R}^{d-1}} |\hat{g}_N(\xi')|^2 + \int_0^{x_1} \int_{\mathbb{R}^{d-1}} |\hat{f}(x, \xi')|^2 dx d\xi' \right) \\ & \leq C \left(\|g_D\|_{L^2(\mathbb{R}^{d-1})}^2 + \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2 + \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \right). \end{aligned}$$

Of course, the left-hand side defines a norm on u , but it is a very weak one, weaker than any norm of the form $L^2(0, 1; H^{-k}(\mathbb{R}^{d-1}))$, $k > 0$. Actually, the Cauchy problem (1.4) is the prototype of a problem which is not well-posed in the sense of Hadamard, as this norm contains an exponential degeneracy.

1.2 The case of additional information

In the above discussion, we explained that it is difficult to get estimates on u in a reasonable norm in terms of norms of the data f , g_D and g_N . The goal of this paragraph is to explain that it can be much better if we further assume that u is known on the other part of the boundary $\{1\} \times \mathbb{R}^{d-1}$. To simplify the computations, we will assume that the solution u of (1.4) satisfies the homogeneous Dirichlet boundary conditions

$$u(1, x') = 0 \quad \text{for } x' \in \mathbb{R}^{d-1}. \quad (1.7)$$

In that case, equation (1.5) should be completed with the extra boundary conditions

$$\hat{u}(1, \xi') = 0 \quad \text{for } \xi' \in \mathbb{R}^{d-1}. \quad (1.8)$$

Of course, formula (1.6) still holds, but it does not use the full strength of the additional boundary conditions (1.7). We shall rather use the structure of the operator

$$\partial_{11} - |\xi'|^2 = (\partial_1 + |\xi'|)(\partial_1 - |\xi'|).$$

This structure indeed suggests to introduce

$$\hat{v}(x_1, \xi') = (\partial_1 - |\xi'|)\hat{u}(x_1, \xi') \quad \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}. \quad (1.9)$$

Following, the equations (1.5)–(1.7) rewrite:

$$\begin{cases} \partial_1 \hat{u} - |\xi'| \hat{u} = \hat{v} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \partial_1 \hat{v} + |\xi'| \hat{v} = \hat{f} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{u}(1, \xi') = 0, & \text{for } \xi' \in \mathbb{R}^{d-1}, \\ \hat{v}(0, \xi') = \hat{g}_N(\xi') - |\xi'| \hat{g}_D(\xi') & \text{for } \xi' \in \mathbb{R}^{d-1}. \end{cases} \quad (1.10)$$

Here, we check that the equations in \hat{v} do not depend on \hat{u} and can be solved independently. The solution \hat{u} can then be computed in terms of \hat{v} from equations (1.10)_{1,3}. Besides, straightforward computations show

$$\hat{v}(x_1, \xi') = \exp(-|\xi'|x_1) \hat{v}(0, \xi') + \int_0^{x_1} \exp(-|\xi'|(x_1 - x)) \hat{f}(x, \xi') dx, \quad (1.11)$$

$$\hat{u}(x_1, \xi') = - \int_{x_1}^1 \exp(-|\xi'|(x - x_1)) \hat{v}(x, \xi') dx. \quad (1.12)$$

Both formula contains only decaying exponentials (in $|\xi'|$), and we will then be able to derive estimates from them.

Indeed, we immediately get from the above formula that

$$\begin{aligned}\hat{u}(x_1, \xi') &= - \int_{x_1}^1 \exp(-|\xi'|(x-x_1)) \exp(-|\xi'|x) \hat{v}(0, \xi') dx \\ &\quad - \int_{x_1}^1 \exp(-|\xi'|(x-x_1)) \left(\int_0^x \exp(-|\xi'|(x-\tilde{x})) \hat{f}(\tilde{x}, \xi') d\tilde{x} \right) dx \\ &= - \hat{v}(0, \xi') \int_{x_1}^1 \exp(-|\xi'|(2x-x_1)) dx\end{aligned}\tag{1.13}$$

$$- \int_0^1 \hat{f}(\tilde{x}, \xi') \left(\int_{\max\{x_1, \tilde{x}\}}^1 \exp(-|\xi'|(2x-x_1-\tilde{x})) dx \right) d\tilde{x}.\tag{1.14}$$

For fixed $\xi' \in \mathbb{R}^{d-1}$, we estimate the $L^2(0, 1)$ -norm of each term of the right hand-side. We get for some constant C independent of ξ' that

$$\left\| \hat{v}(0, \xi') \int_{x_1}^1 \exp(-|\xi'|(2x-x_1)) dx \right\|_{L^2(0,1)} \leq \frac{C}{1+|\xi'|^{3/2}} |\hat{v}(0, \xi')|,$$

and for (1.14), we obtain

$$\begin{aligned}& \left\| \int_0^1 \hat{f}(\tilde{x}, \xi') \left(\int_{\max\{x_1, \tilde{x}\}}^1 \exp(-|\xi'|(2x-x_1-\tilde{x})) dx \right) d\tilde{x} \right\|_{L^2(0,1)} \\ & \leq \left\| \int_0^1 |\hat{f}(\tilde{x}, \xi')| \left(\int_{\max\{x_1, \tilde{x}\}}^1 \exp(-|\xi'|(2x-x_1-\tilde{x})) dx \right) d\tilde{x} \right\|_{L^2(0,1)} \\ & \leq \frac{C}{1+|\xi'|} \left\| \int_0^1 |\hat{f}(\tilde{x}, \xi')| \exp(-|\xi'|\|x_1-\tilde{x}\|) d\tilde{x} \right\|_{L^2(0,1)} \\ & \leq \frac{C}{1+|\xi'|} \left\| \hat{f}(x_1, \xi') \mathbf{1}_{x_1 \in (0,1)} *_{x_1} \exp(-|\xi'|\|x_1\|) \mathbf{1}_{x_1 \in (-2,2)} \right\|_{L^2(0,1)} \\ & \leq \frac{C}{1+|\xi'|} \left\| \hat{f}(\cdot, \xi') \right\|_{L^2(0,1)} \left\| \exp(-|\xi'|\|x_1\|) \mathbf{1}_{x_1 \in (-1,1)} \right\|_{L^1(-1,1)} \\ & \leq \frac{C}{1+|\xi'|^2} \left\| \hat{f}(\cdot, \xi') \right\|_{L^2(0,1)},\end{aligned}$$

for some constant C independent of ξ' . We hence obtain, for C independent of ξ' , that for all $\xi' \in \mathbb{R}^{d-1}$,

$$(1+|\xi'|^4) \|\hat{u}(\cdot, \xi')\|_{L^2(0,1)}^2 \leq C(1+|\xi'|) |\hat{v}(0, \xi')|^2 + C \|\hat{f}(\cdot, \xi')\|_{L^2(0,1)}^2.\tag{1.15}$$

Integrating in $\xi' \in \mathbb{R}^{d-1}$ and using Parseval's identity, we derive

$$\|u\|_{L^2(0,1; H^2(\mathbb{R}^{d-1}))}^2 \leq C \int_{\mathbb{R}^{d-1}} (1+|\xi'|) |\hat{v}(0, \xi')|^2 d\xi' + C \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2.\tag{1.16}$$

Using the equation, $\partial_{11}u = f - \Delta' u$ belongs to $L^2((0, 1) \times \mathbb{R}^{d-1})$ and can be estimated by the right hand side of (1.16). Besides, the explicit form of $\hat{v}(0, \xi')$ in (1.10)₄ shows that

$$\int_{\mathbb{R}^{d-1}} (1+|\xi'|) |\hat{v}(0, \xi')|^2 d\xi' \leq C \|g_D\|_{H^{3/2}(\mathbb{R}^{d-1})}^2 + C \|g_N\|_{H^{1/2}(\mathbb{R}^{d-1})}^2.$$

Similar estimates can also be done on $(1 + |\xi'|)\partial_1 \hat{u}$ in $L^2((0, 1) \times \mathbb{R}^{d-1})$, thus on $\partial_1 u$ in $L^2(0, 1; H^1(\mathbb{R}^{d-1}))$. We thus conclude

$$\|u\|_{H^2((0,1) \times \mathbb{R}^{d-1})} \leq C \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})} + C \|g_D\|_{H^{3/2}(\mathbb{R}^{d-1})} + C \|g_N\|_{H^{1/2}(\mathbb{R}^{d-1})}. \quad (1.17)$$

Note that solutions u of equation (1.4)–(1.7) also satisfy the stronger estimate

$$\|u\|_{H^2((0,1) \times \mathbb{R}^{d-1})} \leq C \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})} + C \|g_D\|_{H^{3/2}(\mathbb{R}^{d-1})}, \quad (1.18)$$

recall Theorem 0.19, which can be proved along the same lines in this geometrical setting, see Exercise 1.1.

Nevertheless, the above proof of (1.17) is important to keep in mind as it is the one that will be used and adapted in more intricate situations in the following chapters.

1.3 Exercises

Exercise 1.1. *Let u be a solution of*

$$\begin{cases} \partial_{11}u + \Delta'u = f, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ u(0, x') = g_0(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \\ u(1, x') = g_1(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (1.19)$$

for some $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$, $g_0, g_1 \in H^{3/2}(\mathbb{R}^{d-1})$.

Our goal is to show the following estimate:

$$\|u\|_{H^2((0,1) \times \mathbb{R}^{d-1})} \leq C \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})} + C \|g_0\|_{H^{3/2}(\mathbb{R}^{d-1})} + C \|g_1\|_{H^{3/2}(\mathbb{R}^{d-1})}. \quad (1.20)$$

1. *Taking the Fourier variable of (1.19)₁ in x' , show that $\hat{u}(x_1, \xi')$ satisfies the identity:*

$$\begin{aligned} \hat{u}(x_1, \xi') &= \hat{g}_0(\xi') \frac{\sinh((1-x_1)|\xi'|)}{\sinh(|\xi'|)} + \hat{g}_1(\xi') \frac{\sinh(x_1|\xi'|)}{\sinh(|\xi'|)} \\ &\quad + \int_{x_1}^1 \frac{\sinh(|\xi|(1-x)) \sinh(x_1|\xi'|)}{\sinh(|\xi'|)} \frac{\hat{f}(x, \xi')}{|\xi'|} dx \\ &\quad + \int_0^{x_1} \left(-\sinh(|\xi'|(x_1-x)) + \frac{\sinh(|\xi'|(1-x)) \sinh(x_1|\xi'|)}{\sinh(|\xi'|)} \right) \frac{\hat{f}(x, \xi')}{|\xi'|} dx. \end{aligned}$$

2. *Using similar estimates as in Section 1.2, show that there exists a constant C independent of ξ' such that*

$$(1 + |\xi|^2) \|\hat{u}(\cdot, \xi')\|_{L^2(0,1)} \leq C \left\| \hat{f}(\cdot, \xi') \right\|_{L^2(0,1)} + C(1 + |\xi|^{3/2}) (|\hat{g}_0(\xi')| + |\hat{g}_1(\xi')|).$$

3. *Deduce estimate (1.20).*

N.B.: The same result can be found in [5] where it is proved using variational formulations in a more elegant way.

Exercise 1.2. 1. *Show that if $u \in H^1((0, 1) \times \mathbb{R}^{d-1})$ solves*

$$\begin{cases} \partial_{11}u + \Delta'u = f, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ u(0, x') = \partial_1 u(0, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ u(1, x') = g_1(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (1.21)$$

for some $g_1 \in H^{3/2}(\mathbb{R}^{d-1})$ and $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$ satisfying

$$f(x_1, x') = 0 \quad \text{for all } x_1 \in (0, 1/2), x' \in \mathbb{R}^{d-1},$$

then $u(x_1, x') = 0$ for all $x_1 \in (0, 1/2), x' \in \mathbb{R}^{d-1}$.

2. Would that property be true if we do not assume $\partial_1 u(0, x') = 0$?

Chapter 2

A Carleman estimate with a linear weight function and application to the Calderón problem

2.1 Introduction

The goal of this chapter is to present the most simple example of a Carleman estimate for an elliptic equation.

Similarly as in the previous chapter, we will again focus on the case of an elliptic equation in a strip:

$$\begin{cases} \partial_{11}u + \Delta' u = f, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ u(0, x') = u(1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1 u(0, x') = g_N(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (2.1)$$

where $\Delta' = \partial_{22} + \dots + \partial_{dd}$ is the Laplace operator in the variable $x' = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$.

Our goal is to explain how the estimate

$$\|u\|_{H^2((0,1) \times \mathbb{R}^{d-1})} \leq C \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})} + C \|g_N\|_{H^{1/2}(\mathbb{R}^{d-1})}, \quad (2.2)$$

proved in (1.17) can be modified by the introduction of a linear weight function¹ of the form e^{-kx_1} , where k is a free parameter assumed to satisfy $k \geq 1$.

To be more precise, we will show that for some constant C independent of k , for all $k \geq 1$, all solutions u of (2.1) with source term $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$ and Neumann data $g_N \in L^2(\mathbb{R}^{d-1})$ satisfy

$$k \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} \leq C \|fe^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} + Ck^{1/2} \|g_N\|_{L^2(\mathbb{R}^{d-1})},$$

see Theorem 2.1 for precise statements.

¹In the context of Carleman estimates, the weight function usually is the function appearing in the exponential. Here, the terminology “linear weight function” comes from the fact that the function $x \mapsto kx_1$ is linear.

The freedom in the parameter $k \geq 1$ and the precise knowledge of the dependence of this estimate in k is what makes Carleman estimates a powerful tool, as we shall see in several situations along this course.

Indeed, it allows in particular to prove similar estimates when the elliptic equation involves lower order terms, potentials for instance. This is precisely one of the properties we shall deeply rely onto in order to show uniqueness in the Calderón problem, see Section 2.3.

2.2 A Carleman estimate

2.2.1 Main result

The goal of this section is to prove the following result:

Theorem 2.1. *There exists a constant $C > 0$ such that for all $k \geq 1$, any solution u of (2.1) with source term $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$ and Neumann data $g_N \in L^2(\mathbb{R}^{d-1})$ satisfies*

$$\begin{aligned} k^2 \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + \|\nabla ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \\ \leq Ck \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2 + C \|fe^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2. \end{aligned} \quad (2.3)$$

We will give three different proofs of Theorem 2.1 in Section 2.2.2, each of them having its own interest, see Section 2.2.3.

But before going into the proof, let us emphasize that the constant C in Theorem 2.1 is independent of $k \geq 1$. For instance, as a straightforward corollary of Theorem 2.1, we get the following:

Corollary 2.2. *Let $q \in L^\infty((0, 1) \times \mathbb{R}^{d-1})$. There exists a constant $C > 0$ such that for all $k \geq 1$, any solution $u \in H^1((0, 1) \times \mathbb{R}^{d-1})$ of*

$$\begin{cases} \partial_{11}u + \Delta' u + qu = f, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ u(0, x') = u(1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1 u(0, x') = g_N(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (2.4)$$

with source term $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$ and Neumann data $g_N \in L^2(\mathbb{R}^{d-1})$ satisfies (2.3).

Proof. If $u \in L^2((0, 1) \times \mathbb{R}^{d-1})$ solves (2.4), it also solves (2.1) with source term $f - qu$. Applying Theorem 2.1, we obtain, for all $k \geq 1$,

$$\begin{aligned} k^2 \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + \|\nabla ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \\ \leq Ck \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2 + 2C \|fe^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \\ + 2C \|q\|_{L^\infty((0,1) \times \mathbb{R}^{d-1})}^2 \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2. \end{aligned}$$

Taking $k^2 \geq 4C \|q\|_{L^\infty((0,1) \times \mathbb{R}^{d-1})}^2$, we can absorb the last term, and we obtain

$$\begin{aligned} k^2 \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + \|\nabla ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \\ \leq 2Ck \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2 + 4C \|fe^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2, \end{aligned}$$

for all $k \geq k_q = 2\sqrt{C} \|q\|_{L^\infty((0,1) \times \mathbb{R}^{d-1})}$. As the weight function is uniformly bounded for $k \in [1, k_q]$, this last estimate is also true for some constant C for $k \in [1, k_q]$. Modifying the constant if needed, we deduce the existence of a constant C such that (2.3) is true for all $k \geq 1$. \square

Let us also remark that the Carleman estimate (2.3) involves an additional information g_N on the hyperplane $x_1 = 0$, corresponding to the set where the weight function $-kx_1$ is the largest on the domain $[0, 1] \times \mathbb{R}^{d-1}$. This is a general phenomenon in Carleman estimates, as the weight function gives more weight where the additional information is available, and reflects the direction of propagation of the information.

In order to illustrate that, let us revisit the result of Exercise 1.2:

Corollary 2.3. *Let $q \in L^\infty((0, 1) \times \mathbb{R}^{d-1})$, and $u \in H_0^1((0, 1) \times \mathbb{R}^{d-1})$ be the solution of (2.4) with Neumann data $g_N = 0$ and source term $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$ satisfying, for some $a \in (0, 1)$,*

$$f(x_1, x') = 0 \quad \text{for } x_1 \in (0, a), x' \in \mathbb{R}^{d-1}.$$

Then u vanishes in $(0, a) \times \mathbb{R}^{d-1}$.

Proof. Apply the Carleman estimate (2.3) to u : for all $k \geq 1$,

$$k \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} \leq C \|fe^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}. \quad (2.5)$$

But on one hand,

$$k \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} \geq k \|ue^{-kx_1}\|_{L^2((0,a) \times \mathbb{R}^{d-1})} \geq ke^{-ka} \|u\|_{L^2((0,a) \times \mathbb{R}^{d-1})}.$$

On the other hand, using the fact that f vanishes in $(0, a) \times \mathbb{R}^{d-1}$,

$$\|fe^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} \leq e^{-ka} \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})}.$$

The estimate (2.5) then yields, for all $k \geq 1$,

$$k \|u\|_{L^2((0,a) \times \mathbb{R}^{d-1})} \leq C \|f\|_{L^2((0,1) \times \mathbb{R}^{d-1})}.$$

As k can be chosen arbitrarily large, u necessarily vanishes in $(0, a) \times \mathbb{R}^{d-1}$. \square

Remark 2.4. *Note that Corollary 2.3 states the same result as in Exercise 1.2 but now allows non-trivial lower order terms in the elliptic equation. Note in particular that the potential q may depend on both x_1 and x' in an intricate way, as we only assumed it to be in L^∞ .*

2.2.2 Proof of Theorem 2.1

Strategy. Since we have to prove estimates on ue^{-kx_1} in terms of fe^{-kx_1} , it will be convenient to set

$$\begin{cases} U(x_1, x') = u(x_1, x')e^{-kx_1}, \\ F(x_1, x') = f(x_1, x')e^{-kx_1}, \end{cases} \quad \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}. \quad (2.6)$$

If u satisfies (2.1), then U satisfies

$$\partial_{11}(e^{kx_1} U) + \Delta'(e^{kx_1} U) = Fe^{kx_1},$$

which yields to the equations

$$\begin{cases} \partial_{11}U + 2k\partial_1U + k^2U + \Delta'U = F, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ U(0, x') = U(1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1U(0, x') = g_N(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (2.7)$$

Starting from there, we will present several proofs of the estimate

$$k^2 \|ue^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \leq Ck \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2 + C \|fe^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2, \quad (2.8)$$

or equivalently

$$k^2 \|U\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \leq Ck \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2 + C \|F\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2. \quad (2.9)$$

Approach 1: Fourier approach and explicit resolution. Similarly as in Section 1.1 and Section 1.2, we can then take the partial Fourier transform in the x' -variable:

$$\begin{aligned} \hat{U}(x_1, \xi') &= \mathcal{F}_{x' \rightarrow \xi'} U(x_1, \cdot), \\ \hat{F}(x_1, \xi') &= \mathcal{F}_{x' \rightarrow \xi'} F(x_1, \cdot), \\ \hat{g}_N(\xi') &= \mathcal{F}_{x' \rightarrow \xi'} g_N(\cdot). \end{aligned}$$

Taking the partial Fourier transform $\mathcal{F}_{x' \rightarrow \xi'}$ of (2.7), we obtain

$$\begin{cases} \partial_{11}\hat{U} + 2k\partial_1\hat{U} + k^2\hat{U} - |\xi'|^2\hat{U} = \hat{F}, & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{U}(0, \xi') = \hat{U}(1, \xi') = 0, & \text{for } \xi' \in \mathbb{R}^{d-1}, \\ \partial_1\hat{U}(0, \xi') = \hat{g}_N(\xi'), & \text{for } \xi' \in \mathbb{R}^{d-1}, \end{cases} \quad (2.10)$$

We now remark the factorization of the operator as follows:

$$\partial_{11} + 2k\partial_1 + k^2 - |\xi'|^2 = (\partial_1 + k)^2 - |\xi'|^2 = (\partial_1 + k + |\xi'|)(\partial_1 + k - |\xi'|).$$

Following the idea of Section 1.2, we then introduce the function

$$\hat{V} = (\partial_1 + k - |\xi'|)\hat{U}.$$

We are then back to the system of equations

$$\begin{cases} \partial_1\hat{U} + (k - |\xi'|)\hat{U} = \hat{V} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \partial_1\hat{V} + (k + |\xi'|)\hat{V} = \hat{F} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{U}(0, \xi') = \hat{U}(1, \xi') = 0 & \text{for } \xi' \in \mathbb{R}^{d-1}, \\ \hat{V}(0, \xi') = \hat{g}_N(\xi') & \text{for } \xi' \in \mathbb{R}^{d-1}. \end{cases} \quad (2.11)$$

For fixed $\xi' \in \mathbb{R}^{d-1}$, we can then solve this system in two steps:

1. Compute \hat{V} in terms of \hat{F} and of \hat{g}_N ;
2. Compute \hat{U} in terms of \hat{V} .

The computation of \hat{V} is straightforward:

$$\hat{V}(x_1, \xi') = \exp(-(k + |\xi'|)x_1)\hat{g}_N(\xi') + \int_0^{x_1} \exp(-(k + |\xi'|)(x_1 - x))\hat{F}(x, \xi') dx. \quad (2.12)$$

The computation of \hat{U} also is straightforward, but now we have two possible formulae, depending whether we use the boundary condition at $x_1 = 0$ or at $x_1 = 1$:

$$\hat{U}(x_1, \xi') = \int_0^{x_1} \exp(-(k - |\xi'|)(x_1 - x)) \hat{V}(x, \xi') dx, \quad (2.13)$$

$$\hat{U}(x_1, \xi') = - \int_{x_1}^1 \exp(-(k - |\xi'|)(x_1 - x)) \hat{V}(x, \xi') dx. \quad (2.14)$$

Here, we see that we have to choose between two formulae, and we may choose the one we prefer to work with. In particular, as we want estimates as small as possible in terms of k , we will always choose formula that do not contain exponentials of positive terms. This means that our choice will depend on the frequency parameter $|\xi'|$: we will take formula (2.13) if $|\xi'| \leq k$, and formula (2.14) if $|\xi'| \geq k$.

In what follows, the constants denoted by C are independent of the parameters ξ' and k and may change from line to line.

The case $|\xi'| \leq k$. In that case, combining (2.12) and (2.13), we get

$$\begin{aligned} \hat{U}(x_1, \xi') &= \hat{g}_N(\xi') \int_0^{x_1} \exp(-(k - |\xi'|)(x_1 - x)) \exp(-(k + |\xi'|)x) dx \\ &\quad + \int_0^{x_1} e^{-(k - |\xi'|)(x_1 - x)} \left(\int_0^x e^{-(k + |\xi'|)(x - \tilde{x})} \hat{F}(\tilde{x}, \xi') d\tilde{x} \right) dx \\ &= \hat{g}_N(\xi') \exp(-(k - |\xi'|)x_1) \int_0^{x_1} \exp(-2|\xi'|x) dx \\ &\quad + \int_0^{x_1} \hat{F}(\tilde{x}, \xi') e^{-k(x_1 - \tilde{x})} \left(\int_{\tilde{x}}^{x_1} e^{|\xi'|(\tilde{x} + x_1 - 2x)} dx \right) d\tilde{x}. \end{aligned}$$

On one hand, direct estimates yield

$$\begin{aligned} &\left\| \hat{g}_N(\xi') \exp(-(k - |\xi'|)x_1) \int_0^{x_1} \exp(-2|\xi'|x) dx \right\|_{L^2(0,1)} \\ &\leq C |\hat{g}_N(\xi')| \left(\frac{1}{1 + |\xi'|} \right) \left(\frac{1}{1 + (k - |\xi'|)^{1/2}} \right) \leq \frac{C}{k^{1/2}} |\hat{g}_N(\xi')|. \end{aligned}$$

On the other hand, writing

$$\int_0^{x_1} \hat{F}(\tilde{x}, \xi') e^{-k(x_1 - \tilde{x})} \left(\int_{\tilde{x}}^{x_1} e^{|\xi'|(\tilde{x} + x_1 - 2x)} dx \right) d\tilde{x} = \int_0^1 \hat{F}(\tilde{x}, \xi') H(x_1 - \tilde{x}, \xi') d\tilde{x}$$

with

$$H(X, \xi') = 1_{X>0} e^{-(k - |\xi'|)X} \int_0^X e^{-2|\xi'|x} dx,$$

we obtain by Young's inequality

$$\begin{aligned}
& \left\| \int_0^{x_1} \hat{F}(\tilde{x}, \xi') e^{-k(x_1 - \tilde{x})} \left(\int_{\tilde{x}}^{x_1} e^{|\xi'|(\tilde{x} + x_1 - 2x)} dx \right) d\tilde{x} \right\|_{L^2(0,1)} \\
& \leq C \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)} \|H(\cdot, \xi')\|_{L^1(0,1)} \\
& \leq C \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)} \left(\frac{1}{1+k-|\xi'|} \right) \left(\frac{1}{1+|\xi'|} \right) \\
& \leq \frac{C}{k} \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)}.
\end{aligned}$$

We then derive, for $\xi' \in \mathbb{R}^{d-1}$ with $|\xi'| \leq k$,

$$\left\| \hat{U}(x_1, \xi') \right\|_{L^2(0,1)} \leq \frac{C}{k^{1/2}} |\hat{g}_N(\xi')| + \frac{C}{k} \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)},$$

which also yields

$$k^2 \left\| \hat{U}(x_1, \xi') \right\|_{L^2(0,1)}^2 \leq Ck |\hat{g}_N(\xi')|^2 + C \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)}^2. \quad (2.15)$$

Note that, in this case, all the informations come from $x_1 = 0$ and our formulas are very close to the ones obtained in Section 1.1.

The case $|\xi'| \geq k$. In that case, combining (2.12) and (2.14), we get the following formula:

$$\begin{aligned}
\hat{U}(x_1, \xi') &= -\hat{g}_N(\xi') \int_{x_1}^1 \exp(-(k-|\xi'|)(x_1-x)) \exp(-(k+|\xi'|)x) dx \\
&\quad - \int_{x_1}^1 e^{-(k-|\xi'|)(x_1-x)} \left(\int_0^x e^{-(k+|\xi'|)(x-\tilde{x})} \hat{F}(\tilde{x}, \xi') d\tilde{x} \right) dx \\
&= -\hat{g}_N(\xi') e^{-(k-|\xi'|)x_1} \int_{x_1}^1 e^{-2|\xi'|x} dx \\
&\quad - \int_0^1 \hat{F}(\tilde{x}, \xi') e^{-k(x_1-\tilde{x})} \left(\int_{\max\{\tilde{x}, x_1\}}^1 e^{|\xi'|(x_1+\tilde{x}-2x)} dx \right) d\tilde{x} \\
&= -\hat{g}_N(\xi') e^{-(k-|\xi'|)x_1} \int_{x_1}^1 e^{-2|\xi'|x} dx \\
&\quad - \int_0^1 \hat{F}(\tilde{x}, \xi') e^{-k(x_1-\tilde{x})} e^{-|\xi'|(x_1-\tilde{x})} \left(\int_0^{1-\max\{\tilde{x}, x_1\}} e^{-2|\xi'|x} dx \right) d\tilde{x}.
\end{aligned}$$

The first term above can be bounded directly:

$$\begin{aligned}
& \left\| -\hat{g}_N(\xi') e^{-(k-|\xi'|)x_1} \int_{x_1}^1 e^{-2|\xi'|x} dx \right\|_{L^2(0,1)} \\
& \leq \frac{C}{1+|\xi'|} \left\| -\hat{g}_N(\xi') e^{-(k+|\xi'|)x_1} \right\|_{L^2(0,1)} \\
& \leq \frac{C}{1+|\xi'|} |\hat{g}_N(\xi')| \frac{1}{(k+|\xi'|)^{1/2}} \leq \frac{C}{k^{3/2}} |\hat{g}_N(\xi')|.
\end{aligned}$$

The second term can be bounded as follows:

$$\begin{aligned}
& \left\| - \int_0^1 \hat{F}(\tilde{x}, \xi') e^{-k(x_1 - \tilde{x})} e^{-|\xi'| |x_1 - \tilde{x}|} \left(\int_0^{1 - \max\{\tilde{x}, x_1\}} e^{-2|\xi'|x} dx \right) d\tilde{x} \right\|_{L^2(0,1)} \\
& \leq \left\| \int_0^1 |\hat{F}(\tilde{x}, \xi')| e^{-k(x_1 - \tilde{x})} e^{-|\xi'| |x_1 - \tilde{x}|} \left(\int_0^{1 - \max\{\tilde{x}, x_1\}} e^{-2|\xi'|x} dx \right) d\tilde{x} \right\|_{L^2(0,1)} \\
& \leq \left\| \int_0^1 |\hat{F}(\tilde{x}, \xi')| e^{(k - |\xi'|) |x_1 - \tilde{x}|} \left(\int_0^\infty e^{-2|\xi'|x} dx \right) d\tilde{x} \right\|_{L^2(0,1)} \\
& \leq \frac{C}{|\xi'|} \left\| |\hat{F}(\cdot, \xi')| 1_{x_1 \in (0,1)} *_{x_1} e^{(k - |\xi'|) |x_1|} \right\|_{L^2(0,1)} \\
& \leq \frac{C}{|\xi'|} \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)} \left\| e^{(k - |\xi'|) |x_1|} \right\|_{L^1(-2,2)} \\
& \leq \frac{C}{|\xi'| (1 - k + |\xi'|)} \|F(\cdot, \xi')\|_{L^2(0,1)} \leq \frac{C}{k} \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)}.
\end{aligned}$$

We thus obtain the same estimate as in (2.15) when $|\xi'| \geq k$.

Note that here, similarly as in Section 1.2, we use only one information at $x_1 = 0$, and one more information at $x_1 = 1$.

Conclusion of Approach 1. Since (2.15) holds for all $\xi' \in \mathbb{R}^{d-1}$, we integrate it with respect to $\xi' \in \mathbb{R}^{d-1}$ and we immediately obtain (2.9) by Parseval's identity.

Approach 2: Fourier approach and multiplier type argument. This approach starts similarly as Approach 1, up to the equations (2.11), by taking the Fourier transform in the x' variable and factorizing the operator.

Then, instead of deriving explicitly formula for \hat{U} in terms of \hat{F} and the observation \hat{g}_N , we first estimate \hat{V} in terms of \hat{F} and of \hat{g}_N , then \hat{U} in terms of \hat{V} . These two estimates can be done using multiplier type arguments in a rather straightforward manner.

For convenience, we rewrite system (2.11):

$$\begin{cases} \partial_1 \hat{V} + (k + |\xi'|) \hat{V} = \hat{F} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{V}(0, \xi') = \hat{g}_N(\xi') & \text{for } \xi' \in \mathbb{R}^{d-1}. \end{cases} \quad (2.16)$$

and

$$\begin{cases} \partial_1 \hat{U} + (k - |\xi'|) \hat{U} = \hat{V} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{U}(0, \xi') = \hat{U}(1, \xi') = 0 & \text{for } \xi' \in \mathbb{R}^{d-1}, \end{cases} \quad (2.17)$$

To estimate \hat{V} , we take the square of each side of (2.16) and integrate by parts in x_1 , and we obtain:

$$\begin{aligned}
& \int_0^1 \left(|\partial_1 \hat{V}(x_1, \xi')|^2 + (k + |\xi'|)^2 |\hat{V}(x_1, \xi')|^2 \right) dx_1 + (k + |\xi'|) |\hat{V}(1, \xi')|^2 \\
& = \int_0^1 |\hat{F}(x_1, \xi')|^2 dx_1 + (k + |\xi'|) |\hat{V}(0, \xi')|^2.
\end{aligned}$$

To estimate \hat{U} from \hat{V} , we also take the square of each side of (2.17) and integrate by parts in x_1 :

$$\int_0^1 \left(|\partial_1 \hat{U}(x_1, \xi')|^2 + (k - |\xi'|)^2 |\hat{V}(x_1, \xi')|^2 \right) dx_1 = \int_0^1 |\hat{V}(x_1, \xi')|^2 dx_1.$$

Combining these two estimates, and using Poincaré's estimate in $(0, 1)$, we easily have:

$$\begin{aligned} \int_0^1 |\hat{U}(x_1, \xi')|^2 dx_1 &\leq C \int_0^1 |\partial_1 \hat{U}(x_1, \xi')|^2 dx_1 \\ &\leq \frac{1}{(k + |\xi'|)^2} \left(\int_0^1 |\hat{F}(x_1, \xi')|^2 + (k + |\xi'|) |\hat{V}(0, \xi')|^2 \right), \end{aligned}$$

which of course easily implies (2.9).

Approach 3: A multiplier argument. Here, we directly start from (2.7), by multiplying the equation (2.7) by $\partial_1 U$:

$$\begin{aligned} 2k \|\partial_1 U\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 &= \int_{(0,1) \times \mathbb{R}^{d-1}} F \partial_1 U dx_1 dx' \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^{d-1}} |\partial_1 U(0, x')|^2 dx' - \frac{1}{2} \int_{\mathbb{R}^{d-1}} |\partial_1 U(1, x')|^2 dx', \end{aligned}$$

so that

$$k \|\partial_1 U\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \leq \frac{C}{k} \|F\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + C \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2. \quad (2.18)$$

Third, using the fact that the strip has bounded width in $x_1 \in (0, 1)$ and that U satisfies homogeneous boundary conditions at $x_1 = 0$, we can use Poincaré's identity:

$$\|U\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \leq C \|\partial_1 U\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2.$$

Hence we have from (2.18) that

$$k^2 \|U\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \leq C \|F\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + Ck \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2. \quad (2.19)$$

Conclusion.

Starting from (2.8), equivalently (2.9), we can readily estimate $\nabla u e^{-kx_1}$ using classical elliptic estimates on u . Indeed, In order to estimate $\nabla u e^{-kx_1}$, we multiply the equation (2.10) by $u e^{-2kx_1}$:

$$\begin{aligned} - \int_{(0,1) \times \mathbb{R}^{d-1}} |\nabla u|^2 e^{-2kx_1} dx_1 dx' + k^2 \int_{(0,1) \times \mathbb{R}^{d-1}} |u|^2 e^{-2kx_1} dx_1 dx' \\ = \int_{(0,1) \times \mathbb{R}^{d-1}} f u e^{-2kx_1} dx_1 dx', \end{aligned}$$

so that

$$\begin{aligned} \|\nabla u e^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \\ \leq k^2 \|u e^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + \|f e^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} \|u e^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}. \end{aligned}$$

Using (2.8), we immediately derive

$$\|\nabla u e^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \leq Ck \|g_N\|_{L^2(\mathbb{R}^{d-1})}^2 + C \|f e^{-kx_1}\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2. \quad (2.20)$$

This concludes the proof of Theorem 2.1.

Exercise 2.1. *Let us assume the setting of Theorem 2.1 and use the notations of the above proof.*

1. *Show that there exists $C > 0$ s.t. for all $\xi' \in \mathbb{R}^{d-1}$ and $k \geq 1$,*

$$(1 + |\xi'|^2) \left\| \hat{U}(\cdot, \xi') \right\|_{L^2(0,1)} \leq Ck^{1/2}(1 + |\xi'|^{1/2})|\hat{g}_N(\xi')| + Ck \left\| \hat{F}(\cdot, \xi') \right\|_{L^2(0,1)}.$$

(Following Approach 1 or Approach 2).

2. *Deduce that there exists $C > 0$ such that for all $k \geq 1$,*

$$\|ue^{-kx_1}\|_{L^2(0,1;H^2(\mathbb{R}^{d-1}))}^2 \leq Ck \|g_N\|_{H^{1/2}(\mathbb{R}^{d-1})}^2 + Ck^2 \|F\|_{L^2((0,1)\times\mathbb{R}^{d-1})}^2.$$

3. *Deduce that for all $(i, j) \in \{2, \dots, d\}^2$,*

$$\frac{1}{k^2} \|(\partial_{ij}u)e^{-kx_1}\|_{L^2((0,1)\times\mathbb{R}^{d-1})}^2 \leq Ck \|g_N\|_{H^{1/2}(\mathbb{R}^{d-1})}^2 + C \|F\|_{L^2((0,1)\times\mathbb{R}^{d-1})}^2$$

4. *Estimate $\partial_{11}\hat{U}$ and show the same estimate as above for $\partial_{11}ue^{-kx_1}$.*

5. *Estimating $(1 + |\xi'|)\partial_1\hat{U}$, show that the same estimate also holds for $\partial_{1i}ue^{-kx_1}$ with $i \in \{1, \dots, d\}$.*

At the end, we have proved that

$$\frac{1}{k^2} \|ue^{-kx_1}\|_{H^2((0,1)\times\mathbb{R}^{d-1})}^2 \leq Ck \|g_N\|_{H^{1/2}(\mathbb{R}^{d-1})}^2 + C \|F\|_{L^2((0,1)\times\mathbb{R}^{d-1})}^2 \quad (2.21)$$

2.2.3 Advantages of each approach

In this section, we discuss the advantages of each approach developed in the proof of Theorem 2.1.

When reading the various proofs, it seems that the easiest one is Approach 3. Indeed, this approach, based on pure multiplier arguments, has the advantage of requiring integration by parts only. In particular, one can easily get convinced that it can be adapted quite easily to more general domains (see next section, in which the domain Ω will not need to be a strip orthogonal to the gradient of the weight function), and to general elliptic operator with Lipschitz coefficient (at least for strictly pseudo-convex functions, see next chapter). For this reason, it clearly appears that this is the most robust and efficient approach in general.

However, Approach 3 does not allow to be very precise on the boundary terms. In particular, for a source term in $L^2(\Omega)$ and homogeneous Dirichlet boundary conditions, solutions u of (2.1) belong to $H^2(\Omega)$, and it would be much more natural to consider the observation in the space $H^{1/2}(\partial\Omega)$. This method does not really allow to do this.

Approach 2, which relies on performing a Fourier transform in the transverse variable, factorizing the operator, and doing multiplier type estimates on (2.16)–(2.17), is much more precise when it comes to the boundary conditions, since all the estimates are done frequency by frequency, and thus allows to consider the boundary conditions in appropriate Sobolev spaces. We refer for instance to [16] where such computations are done for strictly pseudo-convex weight functions (see next chapter for the definition) and non-homogeneous Dirichlet boundary conditions, or in the case of elliptic operators with discontinuous conductivities to the recent works [25, 24] (again, for strictly pseudo-convex weight functions).

Still, to use it, one needs to get the possibility of factorizing the operator, which leads to some technicalities in general settings (general bounded domains, or general weight functions), that can be solved using appropriate semi-classical techniques, or, as we will do in

the next chapter, proving it by hand on this particular case by suitable localizing process and change of coordinates.

Approach 1, finally, might seem even more restrictive, since the computations are already quite involved, even in that specific case of a strip with a weight depending linearly on x_1 . However, this is a good point of view when deriving L^p Carleman estimates, since Parseval's identity holds only in Hilbertian setting and has no substitute in the L^p setting. Thus, Approach 1, which provides a "good" (meaning, yielding to nice estimates, at least in L^2) formula for the solution of (2.7) is useful in this setting. Then, to derive good L^p estimates, the strategy consists in going back in the original variables and perform a detailed analysis of the kernel function, using Hardy-Littlewood-Sobolev theory, see for instance the textbook [34] and the articles [19], [8] and [22] and the references therein.

2.2.4 More general setting

The goal of this section is to generalize Theorem 2.1 to more general domains.

Theorem 2.5. *Let Ω be a smooth (C^2) bounded domain of \mathbb{R}^d .*

There exists a constant $C > 0$ such that for all $k \in \mathbb{R}^d$ with $|k| \geq 1$, any solution $u \in H_0^1(\Omega)$ of

$$\begin{cases} \Delta u = f, & \text{for } x \in \Omega \\ u(x) = 0, & \text{for } x \in \partial\Omega, \end{cases} \quad (2.22)$$

with source term $f \in L^2(\Omega)$ satisfies

$$\begin{aligned} |k|^2 \|ue^{-k \cdot x}\|_{L^2(\Omega)}^2 + \|\nabla ue^{-k \cdot x}\|_{L^2(\Omega)}^2 \\ \leq C|k| \|\partial_n ue^{-k \cdot x}\|_{L^2(\Gamma_k)}^2 + C \|fe^{-k \cdot x}\|_{L^2(\Omega)}^2, \end{aligned} \quad (2.23)$$

where $\Gamma_k = \{x \in \partial\Omega, |k \cdot n_x| < 0\}$, where n_x is the outward pointing normal vector at x .

Proof. Setting $U = ue^{-k \cdot x}$ and $F = fe^{-k \cdot x}$, one easily checks that U satisfies

$$\begin{cases} \Delta U + 2k \cdot \nabla U + |k|^2 U = F & \text{for } x \in \Omega, \\ U(x) = 0, & \text{for } x \in \partial\Omega. \end{cases} \quad (2.24)$$

We then follow the strategy developed in Section 2.2.2 based on Approach 3. Multiplying the equation by U , we first derive

$$\|\nabla U\|_{L^2(\Omega)}^2 \leq C|k|^2 \|U\|_{L^2(\Omega)}^2 + C \|F\|_{L^2(\Omega)}^2. \quad (2.25)$$

Multiplying (2.24) by $k \cdot \nabla U$, we derive

$$\begin{aligned} \int_{\Omega} Fk \cdot \nabla U \, dx &= 2 \int_{\Omega} |k \cdot \nabla U|^2 \, dx + \int_{\partial\Omega} \partial_n U k \cdot \nabla U \, d\sigma - \frac{1}{2} \int_{\partial\Omega} k \cdot n_x |\nabla U|^2 \, d\sigma \\ &= 2 \int_{\Omega} |k \cdot \nabla U|^2 \, dx + \frac{1}{2} \int_{\partial\Omega} k \cdot n_x |\partial_n U|^2 \, d\sigma, \end{aligned}$$

where we used that, as $U = 0$ on $\partial\Omega$, $\nabla U = (\partial_n U)n_x$ on $\partial\Omega$.

Similarly as in (2.18), this yields for some constant C independent of $k \in \mathbb{R}^d$ that

$$\int_{\Omega} |k \cdot \nabla U|^2 \, dx \leq \int_{\Omega} |F|^2 \, dx + |k| \int_{\Gamma_k} |\partial_n U|^2 \, d\sigma.$$

Using the fact that Ω is bounded, we can use Poincaré's inequality and we obtain a constant C independent of k such that for all $k \in \mathbb{R}^d$,

$$|k|^2 \|U\|_{L^2(\Omega)}^2 \leq C \|F\|_{L^2(\Omega)}^2 + |k| \|\partial_n U\|_{L^2(\Gamma_k)}^2.$$

Combined with (2.25), we thus derive

$$|k|^2 \|U\|_{L^2(\Omega)}^2 + \|\nabla U\|_{L^2(\Omega)}^2 \leq C \|F\|_{L^2(\Omega)}^2 + |k| \|\partial_n U\|_{L^2(\Gamma_k)}^2. \quad (2.26)$$

Recalling that $ue^{-k \cdot x} = U$ and $\nabla u e^{-k \cdot x} = \nabla U + kU$, we immediately conclude estimate (2.23). \square

Remark 2.6. An H^2 estimate on U similar to (2.21) can also be proved: Indeed, from the equation (2.24),

$$\frac{1}{|k|^2} \|\Delta U\|_{L^2(\Omega)}^2 \leq C |k|^2 \|U\|_{L^2(\Omega)}^2 + C \|\nabla U\|_{L^2(\Omega)}^2 + C \|F\|_{L^2(\Omega)}^2.$$

Since U vanishes on the boundary, we obtain

$$|k|^2 \|U\|_{L^2(\Omega)}^2 + \|\nabla U\|_{L^2(\Omega)}^2 + \frac{1}{|k|^2} \|U\|_{H^2(\Omega)}^2 \leq C \|F\|_{L^2(\Omega)}^2 + |k| \|\partial_n U\|_{L^2(\Gamma_k)}^2. \quad (2.27)$$

2.3 The Calderón problem

2.3.1 Setting and main result

The Calderón problem is also known in the physical literature as the Electrical Impedance Tomography (EIT in short). This corresponds to a medical imaging technique which consists in the recovery of the conductivity of a tissue (or a material) by applying currents on the surface on the body and measuring the electrical potentials on the surface of the body.

To be more precise, let Ω be a bounded domain of \mathbb{R}^d , and consider the elliptic problem

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0, & \text{for } x \in \Omega, \\ u(x) = g_d(x), & \text{for } x \in \partial\Omega. \end{cases} \quad (2.28)$$

Here, $\sigma = \sigma(x)$ is a scalar function modeling the conductivity of the material.

The Dirichlet boundary data g_d is a voltage imposed on the boundary of the object. In the EIT process, it is assumed that for all Voltage g_d , we can measure the current $(\sigma \nabla u) \cdot n_x$ on the whole boundary $\partial\Omega$. This is the so-called Voltage-to-current map.

The question known as the Calderón problem is then the following:

Can we determine the conductivity of the material σ from the knowledge on the Voltage-to-Current map?

Before going further, let us give some more precisions on the mathematical setting under consideration.

First, we assume that the conductivity σ belongs to the class:

$$\sigma \in C^0(\overline{\Omega}), \quad \exists C_* > 0, \text{ s.t. } \forall x \in \Omega, \frac{1}{C_*} \leq \sigma(x) \leq C_*. \quad (2.29)$$

Under these conditions, the elliptic problem (2.28) is well-posed for Dirichlet data $g_d \in H^{1/2}(\partial\Omega)$ (see Theorem 0.18 and [5]), and the solution u of (2.28) belongs to $H^1(\Omega)$.

We are then able to define $(\sigma \nabla u) \cdot n_x$ as a function of $H^{-1/2}(\partial\Omega)$ by the following formula: for all $g \in H^{1/2}(\partial\Omega)$, there exists $v_g \in H^1(\Omega)$ such that $v_g = g$ on $\partial\Omega$, and we define

$$\langle (\sigma \nabla u) \cdot n_x, g \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = \int_{\Omega} \sigma \nabla u \cdot \nabla v_g \, dx. \quad (2.30)$$

Remark 2.7. *More generally, for a function $\vec{v} \in (L^2(\Omega))^d$ such that $\operatorname{div}(\vec{v}) \in L^2(\Omega)$, we can define $\vec{v} \cdot n_x \in H^{-1/2}(\partial\Omega)$ by the formula: for all $g \in H^{1/2}(\partial\Omega)$,*

$$\langle \vec{v} \cdot n_x, g \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = \int_{\Omega} \operatorname{div}(\vec{v}) w_g + \int_{\Omega} \vec{v} \cdot \nabla w_g \, dx,$$

where w_g is any function of $H^1(\Omega)$ such that $w_g = g$ on $\partial\Omega$.

Following, the Voltage-to-Current map is defined as follows:

$$\Lambda_{\sigma} : \begin{array}{l} H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega) \\ g_d \mapsto (\sigma \nabla u) \cdot n_x, \quad \text{where } u \text{ solves (2.28).} \end{array} \quad (2.31)$$

Calderón's problem now consists in studying the map

$$\Lambda : \sigma \mapsto \Lambda_{\sigma}.$$

But let us emphasize that Calderón's problem contains many different subproblems:

- Uniqueness: If $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, can we deduce $\sigma_1 = \sigma_2$? This corresponds to the injectivity of the map Λ .
- Stability: If $\Lambda_{\sigma_1} - \Lambda_{\sigma_2}$ is small (in suitable norms), can we deduce that $\sigma_1 - \sigma_2$ is small (in suitable norms)? This corresponds to the continuity of the inverse of Λ .
- Reconstruction: Given Λ_{σ} , can we compute σ ? This corresponds to the construction of the inverse of Λ . Note that there, this also includes questions like design of numerical schemes to recover σ , being able to propose a good numerical algorithm even in the event of imperfect measurements (noise), ...

In the following, we shall only focus on the uniqueness question, and we shall not address the problem in its full complexity. The interested reader may have a look on [37, 33, 7].

Remark that the map Λ is non-linear. Of course, this is one of the main difficulties of that problem.

Besides that, we shall assume that

- σ is known on the boundary.
- $\sqrt{\sigma}$ belongs to $W^{2,\infty}(\Omega)$.

These conditions allow us to perform the so-called Liouville's transform: Setting $v = \sigma^{1/2}u$, u solves (2.28) if and only if v solves

$$\begin{cases} \Delta v + qv = 0, & \text{for } x \in \Omega, \\ v(x) = h_d(x), & \text{for } x \in \partial\Omega. \end{cases} \quad (2.32)$$

with

$$h_d = \sigma^{1/2}g_d, \quad q = -\frac{\Delta(\sigma^{1/2})}{\sigma^{1/2}}.$$

As σ is known on the boundary, this potential $q \in L^\infty(\Omega)$ determines uniquely the conductivity σ .

It is then natural to introduce the map

$$\Lambda^q : \begin{cases} H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega) \\ h_d \mapsto \partial_n v, \quad \text{where } v \text{ solves (2.32).} \end{cases} \quad (2.33)$$

To simplify notations, we shall further assume that there is no nontrivial solutions of (2.32) when $h_d = 0$. Otherwise, the above map should be defined as a set-valued map (or set of Cauchy data). In the following, we simply omit this issue for sake of simplicity (Also note that, if the potential q corresponds to some conductivity σ satisfying (2.29), there is no non-trivial solutions of (2.32) when $h_d = 0$).

In particular, if we restrict ourselves to classes of conductivity σ for which (2.29) are satisfied, σ is known on the boundary, and $\sqrt{\sigma} \in W^{2,\infty}$, the injectivity of the map $\sigma \mapsto \Lambda_\sigma$ is equivalent to the injectivity of the map

$$\Lambda : q \in L^\infty(\Omega) \mapsto \Lambda^q.$$

Our goal is to prove the following result:

Theorem 2.8. *Assume that the dimension is greater than 3, i.e. $d \geq 3$. If q_1, q_2 belong to $L^\infty(\Omega)$ and $\Lambda^{q_1} = \Lambda^{q_2}$, then $q_1 = q_2$.*

The proof is done in the following sections.

Let us emphasize that our result focuses on the case of a dimension $d \geq 3$. This is due to our proof based on the so-called *Complex Geometric Optics* solutions. The 2d case can also be handled, but using a completely different construction based on complex analysis (see [2]).

2.3.2 Preliminaries

Let us begin with the following remark:

Proposition 2.9. *Let $q \in L^\infty(\Omega)$. Then the map Λ^q is self-adjoint.*

Proof. Let $q \in L^\infty(\Omega)$ and, for h_1 and h_2 in $H^{1/2}(\partial\Omega)$, set v_1 and v_2 as the respective solutions of

$$\begin{cases} \Delta v_1 + qv_1 = 0, & \text{for } x \in \Omega, \\ v_1(x) = h_1(x), & \text{for } x \in \partial\Omega, \end{cases} \quad \begin{cases} \Delta v_2 + qv_2 = 0, & \text{for } x \in \Omega, \\ v_2(x) = h_2(x), & \text{for } x \in \partial\Omega. \end{cases}$$

According to the definition of the normal derivative on the boundary,

$$\begin{aligned} \langle \Lambda^q h_1, h_2 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} &= \langle \partial_n v_1, v_2 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \\ &= \int_{\Omega} \Delta v_1 v_2 \, dx + \int_{\Omega} \nabla v_1 \cdot \nabla v_2 \, dx \\ &= - \int_{\Omega} qv_1 v_2 \, dx + \int_{\Omega} \nabla v_1 \cdot \nabla v_2 \, dx \\ &= \langle h_1, \Lambda^q h_2 \rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)}, \end{aligned}$$

and this concludes the proof of Proposition 2.9. □

We then show the following polarization formula:

Proposition 2.10. *Let q_1 and q_2 in $L^\infty(\Omega)$. Then for all h_1, h_2 in $H^{1/2}(\partial\Omega)$, the solutions v_1, v_2 of*

$$\begin{cases} \Delta v_1 + q_1 v_1 = 0, & \text{for } x \in \Omega, \\ v_1(x) = h_1(x), & \text{for } x \in \partial\Omega, \end{cases} \quad \begin{cases} \Delta v_2 + q_2 v_2 = 0, & \text{for } x \in \Omega, \\ v_2(x) = h_2(x), & \text{for } x \in \partial\Omega, \end{cases} \quad (2.34)$$

satisfy

$$\langle (\Lambda^{q_1} - \Lambda^{q_2})h_1, h_2 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = \int_{\Omega} (q_2 - q_1)v_1 v_2 dx. \quad (2.35)$$

Proof. We compute

$$\begin{aligned} \langle \Lambda^{q_1} h_1, h_2 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} &= \langle \partial_n v_1, v_2 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \\ &= \int_{\Omega} \Delta v_1 v_2 dx + \int_{\Omega} \nabla v_1 \cdot \nabla v_2 dx \\ &= - \int_{\Omega} q_1 v_1 v_2 dx + \int_{\Omega} \nabla v_1 \cdot \nabla v_2 dx. \end{aligned}$$

Similar computations yield

$$\langle h_1, \Lambda^{q_2} h_2 \rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} = - \int_{\Omega} q_2 v_1 v_2 dx + \int_{\Omega} \nabla v_1 \cdot \nabla v_2 dx.$$

As Λ^{q_2} is self-adjoint from Proposition 2.9,

$$\langle h_1, \Lambda^{q_2} h_2 \rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} = \langle \Lambda^{q_2} h_1, h_2 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)},$$

and the subtraction of the two above identities immediately yield (2.35). \square

The polarization formula is very important as it gives a simple way to describe the condition $\Lambda^{q_1} = \Lambda^{q_2}$. In particular, we see that, if we are able to show that the set described by the products $v_1 v_2$ is dense, then $q_2 = q_1$. This is where the Complex Geometric Optics solutions come into play, see afterwards.

2.3.3 Complex Geometric Optics solutions

Let us begin by a brief description of the main idea. The first basic remark is that, if ρ denotes a constant vector,

$$\Delta e^{\rho \cdot x} = \rho \cdot \rho e^{\rho \cdot x}.$$

In particular, we see that, if $\rho \cdot \rho = 0$, $e^{\rho \cdot x}$ is an harmonic solution. This suggests to take complex vectors $\rho \in \mathbb{C}^d$ with

$$\rho = a + \mathbf{i}b, \quad a, b \in \mathbb{R}^d, \quad a \cdot b = 0, \quad |a| = |b|, \quad (2.36)$$

which implies that $\rho \cdot \rho = |a|^2 - |b|^2 + 2\mathbf{i}a \cdot b = 0$. As $e^{\rho \cdot x}$ is an harmonic function and a potential function is a low-order perturbation of the Laplace operator, we expect it to “almost” solve

$$(\Delta + q)e^{\rho \cdot x} = 0$$

at high-frequency.

To give a precise meaning to these insights, we show the following result:

Theorem 2.11. *Let $q \in L^\infty(\Omega)$. There exists $C > 0$, such that for all $\rho = a + ib \in \mathbb{C}^d$ as in (2.36) with $|a| \geq 1$, there exists a solution $v_\rho \in L^2(\Omega)$ of*

$$\Delta v_\rho + qv_\rho = 0 \text{ in } \Omega, \quad (2.37)$$

that can be written as

$$v_\rho(x) = e^{\rho \cdot x} + e^{a \cdot x} r_\rho(x), \quad (2.38)$$

with r_ρ satisfying the estimate

$$|a| \|r_\rho\|_{L^2(\Omega)} \leq C. \quad (2.39)$$

Proof. First remark that v_ρ as in (2.38) solves (2.37) if and only if r_ρ solves

$$e^{-a \cdot x}(\Delta + q)(e^{a \cdot x} r_\rho) = -qe^{ib \cdot x} \text{ in } \Omega.$$

We set $\tilde{q} = -qe^{ib \cdot x}$. The question is then reduced to show that there exists $r_\rho \in L^2(\Omega)$ solution of

$$e^{-a \cdot x}(\Delta + q)(e^{a \cdot x} r_\rho) = \tilde{q} \text{ in } \Omega, \quad (2.40)$$

and satisfying estimate (2.39). Though it is completely straightforward that one can choose $r_\rho \in L^2(\Omega)$ solving (2.40) (for instance by setting $r_\rho(x) = e^{-a \cdot x} R$, where R solves $(\Delta + q)R = \tilde{q}e^{a \cdot x}$ in Ω with $R = 0$ on the boundary $\partial\Omega$), the difficult part is to show that one can choose r_ρ solution of (2.40) such that it moreover satisfies the estimate (2.39).

The function r_ρ solves (2.40) if and only if for all $w \in \mathcal{D}(\Omega)$,

$$\int_{\Omega} r_\rho (e^{a \cdot x}(\Delta + q)(e^{-a \cdot x} w)) \, dx = \int_{\Omega} \tilde{q} w \, dx, \quad (2.41)$$

or, by density, for all $w \in H_0^2(\Omega)$.

It follows that the set of r_ρ solving (2.40) is an affine space with vector space

$$\{e^{a \cdot x}(\Delta + q)(e^{-a \cdot x} w), w \in H_0^2(\Omega)\}^{\perp L^2(\Omega)}.$$

Therefore, the solution r_ρ of (2.40) of smallest norm should belong to the set

$$\overline{\{e^{a \cdot x}(\Delta + q)(e^{-a \cdot x} w), w \in H_0^2(\Omega)\}}^{L^2(\Omega)}.$$

Note that

$$e^{a \cdot x}(\Delta + q)(e^{-a \cdot x} w) = \Delta w - 2a \cdot \nabla w + a^2 w + qw.$$

Hence, applying Carleman estimate (2.27) with $k = -a$ and $U = w$, we get

$$|a|^2 \|w\|_{L^2(\Omega)}^2 + \frac{1}{|a|^2} \|w\|_{H^2(\Omega)}^2 \leq C \|e^{a \cdot x}(\Delta + q)(e^{-a \cdot x} w)\|_{L^2(\Omega)}^2, \quad (2.42)$$

for some constant C independent of a (Cf Corollary 2.2 and its proof: this is the place where we use the fact that $q \in L^\infty(\Omega)$). Following, the set

$$\{e^{a \cdot x}(\Delta + q)(e^{-a \cdot x} w), w \in H_0^2(\Omega)\}$$

is closed in $L^2(\Omega)$. Indeed, if w_n is a sequence of $H_0^2(\Omega)$ for which $(e^{a \cdot x}(\Delta + q)e^{-a \cdot x} w_n)$ is a Cauchy sequence in $L^2(\Omega)$, then (2.42) implies that w_n is a Cauchy sequence in $H^2(\Omega)$.

Thus there exists $W \in H_0^2(\Omega)$ such that

$$r_\rho = e^{a \cdot x}(\Delta + q)(e^{-a \cdot x} W) \quad (2.43)$$

satisfies (2.41), i.e. r_ρ solves (2.40), and is the solution of (2.40) of minimal $L^2(\Omega)$ -norm.

Now, to derive estimates on r_ρ , we apply (2.41) to W : that way, we obtain

$$\int_{\Omega} |e^{a \cdot x}(\Delta + q)(e^{-a \cdot x}W)|^2 dx = \int_{\Omega} \tilde{q}W dx \leq \|\tilde{q}\|_{L^2} \|W\|_{L^2},$$

Using (2.42), we check that

$$\left(\int_{\Omega} |e^{a \cdot x}(\Delta + q)(e^{-a \cdot x}W)|^2 dx \right)^{1/2} \leq \frac{C \|\tilde{q}\|_{L^\infty}}{|a|} \leq \frac{C \|q\|_{L^\infty}}{|a|}.$$

Using the definition of r_ρ in (2.43), we conclude (2.39). \square

2.3.4 Proof of Theorem 2.8

Let q_1 and q_2 in $L^\infty(\Omega)$ be such that $\Lambda^{q_1} = \Lambda^{q_2}$. According to Proposition 2.10, this implies that

$$\int_{\Omega} (q_2 - q_1)v_1v_2 = 0 \quad (2.44)$$

for all v_1, v_2 solutions of (2.34).

The idea is to use the CGO solutions v_ρ given by Theorem 2.11 in order to construct, for all $\xi \in \mathbb{R}^d$, solutions v_1 and v_2 corresponding respectively to potentials q_1 and q_2 according to (2.34), so that

$$v_1v_2 \simeq e^{-i\xi x}.$$

We will not be able to do that exactly, but we will manage to get sequence of solutions $v_{1,n}, v_{2,n}$ such that the product $v_{1,n}v_{2,n}$ converges to $e^{-i\xi x}$.

Indeed, let $\xi \in \mathbb{R}^d$. Consider α, β in \mathbb{R}^d of unit norm such that

$$\alpha \cdot \beta = \beta \cdot \xi = \alpha \cdot \xi = 0. \quad (2.45)$$

Note that this requires $d \geq 3$.

For each $n \in \mathbb{N}$ satisfying $n \geq |\xi|$, there exists $\gamma_n > 0$ such that

$$\rho_{n,1} = n\alpha + \mathbf{i} \left(\gamma_n \beta - \frac{\xi}{2} \right), \quad \rho_{n,2} = -n\alpha - \mathbf{i} \left(\gamma_n \beta + \frac{\xi}{2} \right)$$

both satisfy (2.36) (take $\gamma_n = \sqrt{n^2 - |\xi|^2/4}$). Note that this step requires the orthogonality conditions (2.45), and that by construction,

$$\rho_{n,1} + \rho_{n,2} = -\mathbf{i}\xi.$$

According to Theorem 2.11, one can construct

$$v_1 = e^{\rho_{n,1} \cdot x} + e^{n\alpha} r_{n,1}(x), \quad v_2 = e^{\rho_{n,2} \cdot x} + e^{-n\alpha} r_{n,2}(x),$$

solutions of (2.34) corresponding respectively to potentials q_1 and q_2 , with

$$\|r_{n,1}\|_{L^2(\Omega)} + \|r_{n,2}\|_{L^2(\Omega)} \leq \frac{C}{n}. \quad (2.46)$$

Using the polarization formula (2.44), we obtain

$$\begin{aligned} \int_{\Omega} (q_2 - q_1) e^{-i\xi x} dx \\ = - \int_{\Omega} (q_2 - q_1) \left(e^{-i(\gamma_n \beta - \xi/2)x} r_{n,1}(x) + e^{i(\gamma_n \beta + \xi/2)x} r_{n,2}(x) + r_{n,1} r_{n,2} \right) dx. \end{aligned}$$

and thus we have

$$\begin{aligned} |\mathcal{F}((q_2 - q_1)1_{\Omega})(\xi)| \\ \leq C \|q_2 - q_1\|_{L^{\infty}(\Omega)} \left(\|r_{n,1}\|_{L^2(\Omega)} + \|r_{n,2}\|_{L^2(\Omega)} + \|r_{n,1}\|_{L^2(\Omega)} \|r_{n,2}\|_{L^2(\Omega)} \right). \end{aligned}$$

Using (2.46) and letting $n \rightarrow \infty$, we deduce that the Fourier transform of $(q_2 - q_1)1_{\Omega}$ vanishes identically, hence $q_2 = q_1$ on Ω . This concludes the proof of Theorem 2.8.

Exercise 2.2. *The goal of this exercise is to study the stability of the map $q \mapsto \Lambda^q$ when $d \geq 3$.*

Let $m > 0$ and let $L_{\leq m}^{\infty}(\Omega) = \{q \in L^{\infty}(\Omega) \mid \|q\|_{L^{\infty}(\Omega)} \leq m\}$.

1. Show that there exist C and $a_0 \geq 1$ depending only on m such that for all $q \in L_{\leq m}^{\infty}(\Omega)$, for all $\rho = a + i\mathbf{b} \in \mathbb{C}^d$ as in (2.36) with $|a| \geq a_0$, there exists a solution $v_{\rho} \in H^1(\Omega)$ of (2.37) that can be written as (2.38) with r_{ρ} satisfying the estimate (2.39).
2. Show that for all $\eta \in C_c^{\infty}(\Omega)$, the above constructed r_{ρ} satisfy, for some constant C depending on m and independent of ρ and $q \in L_{\leq m}^{\infty}(\Omega)$, that

$$\|\eta \nabla r_{\rho}\|_{L^2(\Omega)} \leq C.$$

Hint: Use (2.40).

3. Let \mathcal{O} be a bounded open set such that $\bar{\Omega} \subset \mathcal{O}$. Extending the potentials by 0 outside Ω and applying the above estimates to derive solutions v_{ρ} on \mathcal{O} , show the following result: there exist C and $a_0 \geq 1$ depending only on m such that for all $q \in L_{\leq m}^{\infty}(\Omega)$, for all $\rho = a + i\mathbf{b} \in \mathbb{C}^d$ as in (2.36) with $|a| \geq a_0$, there exists a solution $v_{\rho} \in H^1(\Omega)$ of (2.37) that can be written as (2.38) with r_{ρ} satisfying the estimate (2.39) and

$$\|\nabla r_{\rho}\|_{L^2(\Omega)} \leq C.$$

4. Deduce from these estimates that for some constant C independent of ρ and $q \in L_{\leq m}^{\infty}(\Omega)$,

$$\|v_{\rho}\|_{H^1(\Omega)} \leq C e^{C|a|} (1 + |a|).$$

5. We recall that for $v \in H^1(\Omega)$ satisfying $\Delta v \in L^2(\Omega)$, we have $v|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ and $\partial_n v|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ with

$$\|v|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)} + \|\partial_n v|_{\partial\Omega}\|_{H^{-1/2}(\partial\Omega)} \leq C \left(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{L^2(\Omega)} \right).$$

Using the polarization formula (2.35) and following the above proof, show that there exists $C > 0$ and n_0 such that for all $\xi \in \mathbb{R}^d$, and $n \in \mathbb{N}$ larger than $\max\{n_0, |\xi|\}$,

$$|\mathcal{F}((q_1 - q_2)1_{\Omega})(\xi)| \leq C \|\Lambda^{q_1} - \Lambda^{q_2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} e^{Cn} + \frac{C}{n} \|q_1 - q_2\|_{L^{\infty}(\Omega)}.$$

6. If $\|\Lambda^{q_1} - \Lambda^{q_2}\|_{\mathfrak{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}$ is small enough, deduce that for all $\xi \in \mathbb{R}^d$ with $|\xi| < \log(1/\|\Lambda^{q_1} - \Lambda^{q_2}\|_{\mathfrak{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))})$,

$$|\mathcal{F}((q_1 - q_2)1_\Omega)(\xi)| \leq \frac{C}{\log\left(\frac{1}{\|\Lambda^{q_1} - \Lambda^{q_2}\|_{\mathfrak{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}}\right)}.$$

7. Recalling that $q_1 - q_2 \in L^2(\Omega)$, and extending $q_1 - q_2$ by 0 outside Ω , for $k > 0$, deduce an estimate on $\|q_1 - q_2\|_{H^{-k}(\mathbb{R}^d)}$.

Hint: Estimate

$$\int_{\mathbb{R}^d} |\mathcal{F}((q_1 - q_2)1_\Omega)(\xi)|^2 (1 + |\xi|^2)^{-k} d\xi$$

and divide the integral in the parts $|\xi| \leq \rho$ and $|\xi| > \rho$ and optimize in ρ .

2.3.5 Further comments

In this chapter, we focused on the Calderón problem for a scalar conductivity σ . But this question also makes sense for conductivity σ taking value in the set of positive definite matrices: physically, this corresponds to anisotropic materials.

In the case of anisotropic conductivities, L. Tartar proposed a simple construction to show that uniqueness cannot hold. Indeed, given any C^2 diffeomorphism Ψ on Ω with $\Psi = Id$ on the boundary $\partial\Omega$, then

$$\Lambda_{\tilde{\sigma}} = \Lambda_\sigma \quad \text{where } \tilde{\sigma}(y) = \left(\frac{D\Psi^T(\Psi^{-1}(y)) \times \sigma(\Psi^{-1}(y)) \times D\Psi(\Psi^{-1}(y))}{|\det(D\Psi(\Psi^{-1}(y)))|} \right).$$

This shows in particular that one cannot distinguish between σ and $\tilde{\sigma}$ when allowing anisotropic conductivities.

Actually, this counterexample is the basis of several recent works on invisibility, see for instance the article [38], where one possible idea is to construct diffeomorphisms that approximate the singular transformation between $B(0, 2) \setminus \{0\}$ and $B(0, 2) \setminus B(0, 1)$ given by

$$\Psi(x) = \frac{x}{|x|}(1 + |x|).$$

In such case, the material inside the ball $B(0, 1)$ will be invisible from the boundary.

Chapter 3

Carleman estimates with general weights in a strip

3.1 Introduction

In Corollary 2.3, we saw that if $u \in H^1((0,1) \times \mathbb{R}^{d-1})$ solves

$$\Delta u + qu = 0 \text{ in } (0,1) \times \mathbb{R}^{d-1}, \quad (3.1)$$

for some $q \in L^\infty(\Omega)$, satisfy

$$u(0, x') = \partial_1 u(0, x') = 0, \text{ for all } x' \in \mathbb{R}^{d-1}, \quad (3.2)$$

and $u(1, x') = 0$ for all $x' \in \mathbb{R}^{d-1}$, then u vanishes everywhere.

In this chapter, our goal is to generalize this result to more intricate situations. In order to do that, we will produce new Carleman estimates corresponding to more general weight functions, still considering the case of a strip $\Omega = (0,1) \times \mathbb{R}^{d-1}$ with an additional observation on $\Gamma = \{0\} \times \mathbb{R}^{d-1}$.

3.2 General Carleman weights in a strip for a weight function depending on x_1 .

3.2.1 Goal

The goal of this section is to understand if the Carleman estimate obtained in Theorem 2.1 in the case of a strip can be generalized to more general functions, but depending on x_1 only.

To be more precise, we want to know under which conditions on $\varphi = \varphi(x_1)$ we can guarantee an estimate of the form

$$s^{\theta_1} \|ue^{s\varphi}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} \leq C \|\Delta ue^{s\varphi}\|_{L^2((0,1) \times \mathbb{R}^{d-1})} + Cs^{\theta_2} \left\| \partial_n u(0, \cdot) e^{s\varphi(0)} \right\|_{L^2(\mathbb{R}^{d-1})}, \quad (3.3)$$

for all solution u of (2.1), with some constants C and θ_1, θ_2 independent of u and s , and valid for all $s \geq 1$.

3.2.2 Analysis

In this section, we explain what would be the conditions on φ to develop the same type of proof as in Section 2.2.

In order to do that, for u solution of

$$\begin{cases} \partial_{11}u + \Delta' u = f, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ u(0, x') = u(1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1 u(0, x') = g_N(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (3.4)$$

we set

$$U(x_1, x') = u(x_1, x')e^{s\varphi(x_1)}, \quad F(x_1, x') = f(x_1, x')e^{s\varphi(x_1)}, \quad G_N(x') = g_N(x')e^{s\varphi(0)}. \quad (3.5)$$

Easy computations show that U satisfies the equation

$$\begin{cases} \partial_{11}U - 2s\partial_1\varphi\partial_1U + (s^2|\partial_1\varphi|^2 - s\partial_{11}\varphi)U + \Delta'U = F, & \text{for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1}, \\ U(0, x') = U(1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1U(0, x') = G_N(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (3.6)$$

Similarly as before, we can then take the partial Fourier transform in the x' -variable:

$$\begin{aligned} \hat{U}(x_1, \xi') &= \mathcal{F}_{x' \rightarrow \xi'} U(x_1, \cdot), \\ \hat{F}(x_1, \xi') &= \mathcal{F}_{x' \rightarrow \xi'} F(x_1, \cdot), \\ \hat{G}_N(\xi') &= \mathcal{F}_{x' \rightarrow \xi'} G_N(\cdot). \end{aligned}$$

Using these notations, $\hat{U}(x_1, \xi')$ solves

$$\begin{cases} \partial_{11}\hat{U} - 2s\partial_1\varphi\partial_1\hat{U} + (s^2|\partial_1\varphi|^2 - s\partial_{11}\varphi)\hat{U} - |\xi'|^2\hat{U} = \hat{F}, & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{U}(0, \xi') = \hat{U}(1, \xi') = 0, & \text{for } \xi' \in \mathbb{R}^{d-1}, \\ \partial_1\hat{U}(0, \xi') = \hat{G}_N(\xi'), & \text{for } \xi' \in \mathbb{R}^{d-1}, \end{cases} \quad (3.7)$$

Here again, the key is that this operator can be factorized into

$$(\partial_1 - s\partial_1\varphi + |\xi'|)(\partial_1 - s\partial_1\varphi - |\xi'|),$$

or

$$(\partial_1 - s\partial_1\varphi - |\xi'|)(\partial_1 - s\partial_1\varphi + |\xi'|).$$

But the information on the first derivative can only come from the left. It follows that one should have for all x_1 and ξ' ,

$$-s\partial_1\varphi - |\xi'| \geq 0 \quad \text{or} \quad -s\partial_1\varphi + |\xi'| \geq 0.$$

Taking $\xi' = 0$, we will thus require the condition

$$\partial_1\varphi(x_1) \leq 0, \quad \text{for all } x_1 \in (0, 1). \quad (3.8)$$

Using this condition, it is natural to set

$$\hat{V}(x_1, \xi') = (\partial_1 - s\partial_1\varphi - |\xi'|)\hat{U}(x_1, \xi'),$$

so that we have

$$\begin{cases} \partial_1 \hat{U} + (-s\partial_1 \varphi - |\xi'|)\hat{U} = \hat{V} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \partial_1 \hat{V} + (-s\partial_1 \varphi + |\xi'|)\hat{V} = \hat{F} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{U}(0, \xi') = \hat{U}(1, \xi') = 0 & \text{for } \xi' \in \mathbb{R}^{d-1}, \\ \hat{V}(0, \xi') = \hat{G}_N(\xi') & \text{for } \xi' \in \mathbb{R}^{d-1}. \end{cases} \quad (3.9)$$

Similarly as in Section 2.2.2, we first solve the equation in \hat{V} , which is well-posed:

$$\hat{V}(x_1, \xi') = e^{s\varphi(x_1) - |\xi'|x_1} \hat{G}_n(\xi') + \int_0^{x_1} e^{-s\varphi(x) + |\xi'|x} e^{s\varphi(x_1) - |\xi'|x_1} \hat{F}(x, \xi') dx.$$

One should then solve the equation on \hat{U} . We see there that the solution \hat{U} can be expressed as an integral of exponentials involving only negative coefficients if for all $\xi' \in \mathbb{R}^{d-1}$, there exists $x_\xi \in [0, 1]$ such that

$$-s\partial_1 \varphi(x_\xi) - |\xi'| = 0,$$

and

$$\begin{cases} \forall x_1 \in [0, x_\xi], & -s\partial_1 \varphi(x_1) - |\xi'| \geq 0, \\ \forall x_1 \in [x_\xi, 1], & -s\partial_1 \varphi(x_1) - |\xi'| \leq 0, \end{cases}$$

In that case indeed, for $x_1 \leq x_\xi$, we solve the equation from the left, whereas for $x_1 \geq x_\xi$, we solve the equation from the right.

As for all X , there exists $\xi' \in \mathbb{R}^d$ such that $-s\partial_1 \varphi(X) - |\xi'| = 0$, this later condition is satisfied if

$$\forall X \in (0, 1), \quad \begin{cases} \forall x_1 < X, & -s\partial_1 \varphi(x_1) \geq -s\partial_1 \varphi(X), \\ \forall x_1 > X, & -s\partial_1 \varphi(x_1) \leq -s\partial_1 \varphi(X), \end{cases}$$

i.e. if $\partial_1 \varphi$ is non-decreasing, i.e.

$$\partial_{11} \varphi \geq 0. \quad (3.10)$$

Under this convexity assumption, for all $\xi' \in \mathbb{R}^{d-1}$, we have the following cases:

- Low-frequency case: for all $x_1 \in [0, 1]$, $-s\partial_1 \varphi(x_1) - |\xi'| > 0$. In that case, we have

$$\hat{U}(x_1, \xi') = \int_0^{x_1} e^{s\varphi(x_1) + |\xi'|x_1} e^{-s\varphi(x) - |\xi'|x} \hat{V}(x, \xi') dx. \quad (3.11)$$

- High-frequency case: for all $x_1 \in [0, 1]$, $-s\partial_1 \varphi(x_1) - |\xi'| < 0$. In that case, we have

$$\hat{U}(x_1, \xi') = - \int_{x_1}^1 e^{s\varphi(x_1) + |\xi'|x_1} e^{-s\varphi(x) - |\xi'|x} \hat{V}(x, \xi') dx. \quad (3.12)$$

- Intermediate range: there exists $x_\xi \in [0, 1]$ such that $-s\partial_1 \varphi(x_\xi) - |\xi'| = 0$. Then, for $x_1 < x_\xi$, we use formula (3.11), whereas for $x_1 > x_\xi$, we use formula (3.12).

One may then estimate the solutions $\hat{U}(\cdot, \xi')$ in $L^2(0, 1)$ for fixed $\xi' \in \mathbb{R}^{d-1}$ as in Section 2.2, distinguishing between the above cases.

Roughly speaking, the low frequency and high frequency cases will follow the same path as in Section 2.2. In order to deal with these cases, we will use that φ is strictly decreasing, i.e. the assumption

$$\sup_{x_1 \in (0, 1)} \partial_1 \varphi(x_1) < -\alpha,$$

for some $\alpha > 0$.

The new case corresponding to the intermediate range of frequency will require a deeper analysis. First, to make it simpler, it is usually assumed that φ is strictly convex instead of (3.10), i.e. we will assume

$$\inf_{x_1 \in (0,1)} \partial_{11}\varphi(x_1) \geq \beta,$$

for some $\beta > 0$. In particular, this condition implies that in the intermediate range of frequency, there exists a *unique* x_ξ such that $-s\partial_1\varphi(x_\xi) - |\xi'| = 0$. One then needs to get estimates on the weight function $s\varphi(x_1) + |\xi'|x_1$, which can be deduced by the facts that $\Psi(x_1, \xi') = s\varphi(x_1) + |\xi'|x_1$ satisfies $\Psi(0, \xi') = 0$, $\partial_1\Psi(x_\xi, \xi') = 0$, and $\partial_{11}\Psi \geq s\beta$. We will not give more details about that explicit approach as the computations rapidly become heavy.

Still, Approach 2 in the proof of Theorem 2.1 can be performed here assuming that

$$\exists \alpha > 0 \text{ and } \beta > 0, \text{ such that } \sup_{x_1 \in (0,1)} \partial_1\varphi(x_1) \leq -\alpha, \text{ and } \inf_{x_1 \in (0,1)} \partial_{11}\varphi(x_1) \geq \beta. \quad (3.13)$$

Indeed, one can perform multiplier type arguments on the system (3.9), by estimating \hat{V} in terms of \hat{F} and \hat{G}_N , and on \hat{U} in terms of \hat{V} .

Indeed, since \hat{V} satisfies

$$\begin{cases} \partial_1 \hat{V} + (-s\partial_1\varphi + |\xi'|)\hat{V} = \hat{F} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{V}(0, \xi') = \hat{G}_N(\xi') & \text{for } \xi' \in \mathbb{R}^{d-1}, \end{cases} \quad (3.14)$$

taking the square of the equation and integrating in x_1 , we obtain:

$$\begin{aligned} & \int_0^1 \left(|\partial_1 \hat{V}(x_1, \xi')|^2 + ((-s\partial_1\varphi(x_1) + |\xi'|)^2 + s\partial_{11}\varphi) |\hat{V}(x_1, \xi')|^2 \right) dx_1 \\ & + (-s\partial_1\varphi + |\xi'|) |\hat{V}(1, \xi')|^2 = \int_0^1 |\hat{F}(x_1, \xi')|^2 dx_1 + (-s\partial_1\varphi + |\xi'|) |\hat{G}_N(\xi')|^2, \end{aligned}$$

hence, due to (3.13), for some constant $C > 0$ independent of s and ξ' ,

$$\begin{aligned} & \int_0^1 \left(|\partial_1 \hat{V}(x_1, \xi')|^2 + (s + |\xi'|)^2 |\hat{V}(x_1, \xi')|^2 \right) dx_1 \\ & \leq C \int_0^1 |\hat{F}(x_1, \xi')|^2 dx_1 + C(s + |\xi'|) |\hat{G}_N(\xi')|^2. \end{aligned} \quad (3.15)$$

To estimate \hat{U} , we also take the square of both sides of

$$\begin{cases} \partial_1 \hat{U} + (-s\partial_1\varphi - |\xi'|)\hat{U} = \hat{V} & \text{for } (x_1, \xi') \in (0, 1) \times \mathbb{R}^{d-1}, \\ \hat{U}(0, \xi') = \hat{U}(1, \xi') = 0 & \text{for } \xi' \in \mathbb{R}^{d-1}, \end{cases} \quad (3.16)$$

and integrate in the x_1 variable:

$$\begin{aligned} & \int_0^1 \left(|\partial_1 \hat{U}(x_1, \xi')|^2 + ((-s\partial_1\varphi(x_1) - |\xi'|)^2 + s\partial_{11}\varphi) |\hat{U}(x_1, \xi')|^2 \right) dx_1 \\ & = \int_0^1 |\hat{V}(x_1, \xi')|^2 dx_1. \end{aligned} \quad (3.17)$$

Here, we emphasize that the term $(-s\partial_1\varphi(x_1) - |\xi'|)^2$ may vanish, but this does not matter as the other term $s\partial_{11}\varphi$ is always strictly positive according to (3.13). Therefore, we get that for some constant C independent of s and ξ' ,

$$\int_0^1 \left(|\partial_1 \hat{U}(x_1, \xi')|^2 + s|\hat{U}(x_1, \xi')|^2 \right) dx_1 \leq C \int_0^1 |\hat{V}(x_1, \xi')|^2 dx_1. \quad (3.18)$$

Combining the estimates on \hat{U} and on \hat{V} , we easily deduce that there exists a constant C independent of s and ξ' , such that

$$s(s + |\xi'|)^2 \int_0^1 |\hat{U}(x_1, \xi')|^2 dx_1 \leq C \int_0^1 |\hat{F}(x_1, \xi')|^2 dx_1 + C(s + |\xi'|) |\hat{G}_N(\xi')|^2.$$

Accordingly,

$$\begin{aligned} s^3 \int_0^1 |\hat{U}(x_1, \xi')|^2 dx_1 &\leq C \frac{s^2}{(s + |\xi'|)^2} \int_0^1 |\hat{F}(x_1, \xi')|^2 dx_1 + C \frac{s^2}{(s + |\xi'|)} |\hat{G}_N(\xi')|^2 \\ &\leq C \int_0^1 |\hat{F}(x_1, \xi')|^2 dx_1 + Cs |\hat{G}_N(\xi')|^2. \end{aligned}$$

Integrating in ξ' and using Parseval's identity, we obtain:

$$s^3 \int_{\Omega} |U(x)|^2 dx \leq C \int_{\Omega} |F(x)|^2 dx + Cs \|G_N\|_{L^2(\Gamma)}^2. \quad (3.19)$$

3.2.3 A Carleman estimate for strictly convex weight functions

Based on the analysis done in the previous section, we derive the following result:

Theorem 3.1. *Let $\varphi = \varphi(x_1)$ be a $C^4([0, 1])$ function such that there exist $\alpha > 0$ and $\beta > 0$ for which*

$$\sup_{x_1 \in (0,1)} \partial_1 \varphi(x_1) \leq -\alpha, \quad \text{and} \quad \inf_{x_1 \in (0,1)} \partial_{11} \varphi(x_1) \geq \beta. \quad (3.20)$$

Then there exists a constant $C > 0$ such that for all $u \in L^2((0, 1) \times \mathbb{R}^{d-1})$ solution of (3.4) with source term $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$, and for all $s \geq 1$,

$$\begin{aligned} s^3 \|e^{s\varphi} u\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + s \|e^{s\varphi} \nabla u\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 \\ \leq C \left(\|e^{s\varphi} f\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + s \left\| e^{s\varphi(0)} \partial_1 u(0, \cdot) \right\|_{L^2(\mathbb{R}^{d-1})}^2 \right). \end{aligned} \quad (3.21)$$

Remark 3.2. *Note that the fact that the powers of the parameter s in (3.21) are strictly greater than the powers of the parameter $|k|$ in Theorem 2.5. This allows to handle first-order potentials W and zero-order potentials q , while Corollary 2.3 based on Theorem 2.5 only yields unique continuation for zero-order potentials q .*

In fact, strictly speaking, Carleman estimates usually refer to cases in which the strict convexity condition in (3.20) is satisfied, so that the linear weight does not truly correspond to a Carleman weight function. In fact, using the wording of [20], it is usually referred to as a limiting Carleman weight.

Proof. We present below a proof of Theorem 3.1 which relies on similar arguments as the Approach 3 presented for the proof of Theorem 2.1.

Let $u \in L^2((0, 1) \times \mathbb{R}^{d-1})$ be a solution of (3.4) with source term $f \in L^2((0, 1) \times \mathbb{R}^{d-1})$. Let U and F be as in (3.5). Then U solves (3.6).

Multiply the equation (3.6) by $s\partial_1\varphi\partial_1U$:

$$\begin{aligned}
& \iint_{(0,1) \times \mathbb{R}^{d-1}} F s \partial_1 \varphi \partial_1 U \\
&= \iint_{(0,1) \times \mathbb{R}^{d-1}} (\partial_{11} U - 2s \partial_1 \varphi \partial_1 U + (s^2 |\partial_1 \varphi|^2 - s \partial_{11} \varphi) U + \Delta' U) s \partial_1 \varphi \partial_1 U \\
&= \frac{s}{2} \int_{\mathbb{R}^{d-1}} \partial_1 \varphi |\nabla U|^2 \Big|_{x_1=0}^{x_1=1} + \frac{s}{2} \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_{11} \varphi |\nabla' U|^2 \\
&\quad - \iint_{(0,1) \times \mathbb{R}^{d-1}} \left(\frac{s}{2} \partial_{11} \varphi |\partial_1 U|^2 + 2s^2 (\partial_1 \varphi)^2 |\partial_1 U|^2 + \frac{3s^3}{2} \partial_{11} \varphi (\partial_1 \varphi)^2 |U|^2 \right) \\
&\quad + \iint_{(0,1) \times \mathbb{R}^{d-1}} \frac{s^2}{2} ((\partial_{11} \varphi)^2 + \partial_1 \varphi \partial_{111} \varphi) |U|^2
\end{aligned}$$

Using the conditions (3.20) and using the bound

$$\left| \iint_{(0,1) \times \mathbb{R}^{d-1}} F s \partial_1 \varphi \partial_1 U \right| \leq \frac{1}{4} \iint_{(0,1) \times \mathbb{R}^{d-1}} F^2 + s^2 \iint_{(0,1) \times \mathbb{R}^{d-1}} (\partial_1 \varphi)^2 (\partial_1 U)^2,$$

we have proved:

$$\begin{aligned}
& \iint_{(0,1) \times \mathbb{R}^{d-1}} \left(\frac{s}{2} \partial_{11} \varphi |\partial_1 U|^2 + s^2 (\partial_1 \varphi)^2 |\partial_1 U|^2 + \frac{3s^3}{2} \partial_{11} \varphi (\partial_1 \varphi)^2 |U|^2 \right) \\
&\quad \leq \frac{s}{2} \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_{11} \varphi |\nabla' U|^2 + C s \int_{\mathbb{R}^{d-1}} |\nabla U(0, x')|^2 dx' \\
&\quad \quad + C \iint_{(0,1) \times \mathbb{R}^{d-1}} |F|^2 + C s^2 \iint_{(0,1) \times \mathbb{R}^{d-1}} |U|^2. \quad (3.22)
\end{aligned}$$

In order to estimate

$$\frac{s}{2} \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_{11} \varphi |\nabla' U|^2, \quad (3.23)$$

we multiply the equation (3.6) by $s\partial_{11}\varphi U$:

$$\begin{aligned}
& \iint_{(0,1) \times \mathbb{R}^{d-1}} F s \partial_{11} \varphi U \\
&= \iint_{(0,1) \times \mathbb{R}^{d-1}} (\partial_{11} U - 2s \partial_1 \varphi \partial_1 U + (s^2 |\partial_1 \varphi|^2 - s \partial_{11} \varphi) U + \Delta' U) s \partial_{11} \varphi U \\
&= -s \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_{11} \varphi |\nabla U|^2 + \frac{s}{2} \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_1^{(4)} \varphi |U|^2 + s^2 \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_1 (\partial_1 \varphi \partial_{11} \varphi) |U|^2 \\
&\quad + s^3 \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_{11} \varphi (\partial_1 \varphi)^2 |U|^2 - s^2 \iint_{(0,1) \times \mathbb{R}^{d-1}} (\partial_{11} \varphi)^2 |U|^2.
\end{aligned}$$

Accordingly,

$$s \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_{11} \varphi |\nabla U|^2 \leq s^3 \iint_{(0,1) \times \mathbb{R}^{d-1}} \partial_{11} \varphi (\partial_1 \varphi)^2 |U|^2 + C \iint_{(0,1) \times \mathbb{R}^{d-1}} |F|^2 + Cs^2 \iint_{(0,1) \times \mathbb{R}^{d-1}} |U|^2 \quad (3.24)$$

Combined with (3.22), we easily deduce the existence of a constant $C > 0$ such that

$$\begin{aligned} & \iint_{(0,1) \times \mathbb{R}^{d-1}} \left(\frac{s}{2} \partial_{11} \varphi |\partial_1 U|^2 + s^2 (\partial_1 \varphi)^2 |\partial_1 U|^2 + \frac{s^3}{2} \partial_{11} \varphi (\partial_1 \varphi)^2 |U|^2 + \frac{s}{2} \partial_{11} \varphi |\nabla U|^2 \right) \\ & \leq C \|F\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + Cs \|\partial_1 U(0, \cdot)\|_{L^2(\mathbb{R}^{d-1})}^2 + Cs^2 \|U\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2. \end{aligned}$$

Using the assumptions (3.20) on φ , we deduce the existence of a constant $C > 0$ and of $s_0 \geq 1$ large enough such that for all $s \geq s_0$,

$$\begin{aligned} & \iint_{(0,1) \times \mathbb{R}^{d-1}} (s |\nabla U|^2 + s^2 |\partial_1 U|^2 + s^3 |U|^2) \\ & \leq C \|F\|_{L^2((0,1) \times \mathbb{R}^{d-1})}^2 + Cs \|\partial_1 U(0, \cdot)\|_{L^2(\mathbb{R}^{d-1})}^2. \end{aligned}$$

We then write $ue^{s\varphi} = U$ and $\nabla ue^{s\varphi} = \nabla(Ue^{-s\varphi})e^{s\varphi} = \nabla U - s\nabla\varphi U$. Straightforward estimates then yield (3.21) for $s \geq s_0$. As the weight function is bounded, modifying the constant if needed, the Carleman estimate (3.21) holds for all $s \geq 1$. \square

Remark 3.3. Note that a direct approach consists in multiplying the equation of U by

$$2s\partial_1\varphi\partial_1U + s\partial_{11}\varphi U.$$

But the two above terms do not play the same role. Here $s\partial_1\varphi\partial_1U$ corresponds to a multiplier chosen for analyzing the propagation in the x_1 variable. The other term, $s\partial_{11}\varphi U$, rather corresponds to an energy method to compensate the “bad” term (3.23) coming from the multiplier $s\partial_1\varphi\partial_1U$. Note in particular that $s\partial_{11}\varphi U$ is a weaker order term in U than $s\partial_1\varphi\partial_1U$.

Remark 3.4. Another approach would consist in writing (3.6) under the form

$$AU + SU = F \text{ for } (x_1, x') \in (0, 1) \times \mathbb{R}^{d-1},$$

with

$$\begin{aligned} AU &= -2s\partial_1\varphi\partial_1U - s\partial_{11}\varphi U, \\ SU &= \Delta U + s^2|\partial_1\varphi|^2U, \end{aligned}$$

corresponding respectively to the skew-adjoint and symmetric parts of the conjugated operator. Then we can write:

$$\int_{\Omega} |F|^2 dx = \int_{\Omega} (|AU|^2 + |SU|^2) dx + 2 \int_{\Omega} AU SU dx.$$

and computations will yield that

$$\int_{\Omega} AU SU dx = 2s^3 \int_{\Omega} \partial_{11} \varphi (\partial_1 \varphi)^2 |U|^2 dx - \frac{s}{2} \int_{\Omega} \partial_{1111} \varphi |U|^2 dx - s \int_{\partial\Omega} \partial_1 \varphi \vec{n} \cdot \vec{e}_1 |\partial_1 U|^2 d\sigma.$$

According to Assumptions (3.20), we get some positive constants c_* and C such that for all s large enough,

$$\int_{\Omega} AU SU dx \geq c_* s^3 \int_{\Omega} |U|^2 dx - Cs \int_{\mathbb{R}^{d-1}} |\partial_1 U(0, x')|^2 dx'.$$

This gives another proof of Theorem 3.1 using multiplier type arguments, also in the spirit of the third approach developed for the proof of Theorem 2.1.

Exercise 3.1. (Difficult) Prove the same estimate as in (3.21) using Approach 1 in Section 3.2.2.

3.2.4 A Carleman estimate for coefficients depending on x_1

One may wonder under which general settings the above strategy can be applied. It is quite clear that it can be extended when the coefficients of the tangential derivatives possibly depend on x_1 , while the coefficients in the derivatives with respect to x_1 should not depend on x_1 .

We present below a class of coefficients for which we can perform the above strategy. As it turns out, we will be able to use these computations in the next Chapter in a much more general setting.

We introduce constants $\lambda_{j,k} = \lambda_{k,j}$ defined for $j, k \in \{2, \dots, d\}$, and we let $X_1 > 0$ be such that there exists $c > 0$ such that for all $\xi' \in \mathbb{R}^{d-1}$ and $x_1 \in [-X_1, X_1]$,

$$|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi'_j \xi'_k \geq c |\xi'|^2. \quad (3.25)$$

We also assume the following condition

$$\exists c_* > 0, \forall \xi' \in \mathbb{R}^{d-1}, \quad \sum_{j,k=2}^d \lambda_{j,k} \xi'_j \xi'_k \geq c_* |\xi'|^2, \quad (3.26)$$

We then let

$$X_0 \in [-X_1, 0], \quad \Omega_x = (X_0, X_1) \times \mathbb{R}^{d-1} \quad (3.27)$$

and, for $s > 0$, and $G \in L^2(\Omega_x)$, we consider the equation:

$$\begin{cases} \Delta w - x_1 \sum_{j,k=2}^d \lambda_{j,k} \partial_j \partial_k w - 2s \partial_1 w + s^2 w = G & \text{in } \Omega_x, \\ w(X_0, x') = w(X_1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1 w(X_1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}. \end{cases} \quad (3.28)$$

Remark 3.5. Setting $\tilde{w}(x) = w(x)e^{-sx_1}$, one easily checks that w solves (3.28) with $s > 0$ and $G \in L^2(\Omega_x)$ if and only if \tilde{w} solves

$$\begin{cases} \Delta \tilde{w} - x_1 \sum_{j,k=2}^d \lambda_{j,k} \partial_j \partial_k \tilde{w} = \tilde{G} & \text{in } \Omega_x, \\ \tilde{w}(X_0, x') = \tilde{w}(X_1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \\ \partial_1 \tilde{w}(X_1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \quad (3.29)$$

with $\tilde{G}(x) = G(x)e^{-sx_1}$. Accordingly, equation (3.28) should be seen as the conjugated operator of (3.29).

We then claim the following result:

Lemma 3.6. *Let $(\lambda_{j,k})$, X_1 satisfying assumption (3.25) with c and (3.26) with c_* , and consider the geometry given by (3.27).*

Then there exist positive constants C and $s_0 \geq 1$, depending only on c and c_ , such that for all $s \geq s_0$ and w satisfying (3.28),*

$$s^3 \|w\|_{L^2(\Omega_x)}^2 + s \|\nabla w\|_{L^2(\Omega_x)}^2 + \frac{1}{s} \|D^2 w\|_{L^2(\Omega_x)}^2 \leq C \|G\|_{L^2(\Omega_x)}^2. \quad (3.30)$$

Proof. Taking the Fourier transform in the transverse variable, we obtain

$$L_- L_+ \widehat{w} = \widehat{H},$$

where

$$\begin{aligned} L_- &= \partial_1 - s - \sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k}, & x_1 \in [X_0, X_1], \xi' \in \mathbb{R}^{d-1}, \\ L_+ &= \partial_1 - s + \sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k}, & x_1 \in [X_0, X_1], \xi' \in \mathbb{R}^{d-1}, \\ \widehat{H} &= \widehat{G} - \partial_1 \left(\sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k} \right) \widehat{w}, & x_1 \in [X_0, X_1], \xi' \in \mathbb{R}^{d-1}. \end{aligned}$$

It is easy to prove that

$$\widehat{z} = L_+ \widehat{w}, \quad (3.31)$$

satisfies, for some constant C independent of s and $\xi' \in \mathbb{R}^{d-1}$,

$$(s + |\xi'|) \|\widehat{z}(\cdot, \xi')\|_{L^2(X_0, X_1)} + \|\partial_1 \widehat{z}(\cdot, \xi')\|_{L^2(X_0, X_1)} \leq \left\| \widehat{H}(\cdot, \xi') \right\|_{L^2(X_0, X_1)}. \quad (3.32)$$

Indeed, for $\xi' \in \mathbb{R}^{d-1}$,

$$L_- \widehat{z} = \widehat{H} \text{ in } (X_0, X_1), \quad \widehat{z}(X_0, \xi') = 0. \quad (3.33)$$

Taking then the L^2 norm of both-sides of this identity, we easily deduce the estimate (3.32) from the assumptions (3.25) with c and (3.26).

The estimate of \widehat{w} is more involved. Taking the $L^2(X_0, X_1)$ norm of both sides of (3.31)

at ξ' fixed, we obtain

$$\begin{aligned}
& \|\widehat{z}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 \\
&= \|\partial_1 \widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 + \left\| \left(s - \sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k} \right) \widehat{w}(\cdot, \xi') \right\|_{L^2(X_0, X_1)}^2 \\
&\quad - \int_{X_0}^{X_1} \partial_1 \left(\sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k} \right) |\widehat{w}(x_1, \xi')|^2 dx_1 \\
&= \|\partial_1 \widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 + \left\| \left(s - \sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k} \right) \widehat{w}(\cdot, \xi') \right\|_{L^2(X_0, X_1)}^2 \\
&\quad + \int_{X_0}^{X_1} \frac{\sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k}{\sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k}} |\widehat{w}(x_1, \xi')|^2 dx_1
\end{aligned}$$

In particular, in view of (3.25) and (3.26), the last identity provides, for some $C > 0$ depending only on c and c_* ,

$$\begin{aligned}
& \|\widehat{z}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 \\
&\geq \|\partial_1 \widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 + \left\| \left(s - \sqrt{|\xi'|^2 - x_1 \sum_{j,k=2}^d \lambda_{j,k} \xi_j \xi_k} \right) \widehat{w}(\cdot, \xi') \right\|_{L^2(X_0, X_1)}^2 \\
&\quad + C \int_{X_0}^{X_1} |\xi'| |w(x_1, \xi')|^2 dx_1.
\end{aligned}$$

It follows that we always have, for some C strictly positive independent of s and ξ' ,

$$\|\widehat{z}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 \geq \|\partial_1 \widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 + C(s + |\xi'|) \|\widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2.$$

We then easily conclude that

$$\begin{aligned}
& (s + |\xi'|)^2 \|\partial_1 \widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 + (s + |\xi'|)^3 \|\widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 \\
&\leq C \|\widehat{H}\|_{L^2(X_0, X_1)}^2 \leq C \|\widehat{G}\|_{L^2(X_0, X_1)}^2 + C|\xi'|^2 \|\widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2.
\end{aligned}$$

In particular, for s large enough, we get

$$(s + |\xi'|)^2 \|\partial_1 \widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 + (s + |\xi'|)^3 \|\widehat{w}(\cdot, \xi')\|_{L^2(X_0, X_1)}^2 \leq C \|\widehat{G}\|_{L^2(X_0, X_1)}^2.$$

We can then integrate in $\xi' \in \mathbb{R}^{d-1}$, use Parseval's identity and obtain

$$s^3 \|w\|_{L^2(\Omega_x)}^2 + s^2 \|\partial_1 w\|_{L^2(\Omega_x)}^2 + s \|\nabla' w\|_{L^2(\Omega_x)}^2 + \|\partial_1 \nabla' w\|_{L^2(\Omega_x)}^2 \leq C \|G\|_{L^2(\Omega_x)}^2.$$

From the equation satisfied by w , we easily check that

$$\left\| \Delta w - x_1 \sum_{j,k=2}^d \lambda_{j,k} \partial_j \partial_k w \right\|_{L^2(\Omega_x)} \leq C s^{1/2} \|G\|_{L^2(\Omega_x)}.$$

Besides, the operator $\Delta - x_1 \sum_{j,k=2}^d \lambda_{j,k} \partial_j \partial_k$ is elliptic due to the assumption (3.25), so the standard elliptic regularity results give

$$\frac{1}{s^{1/2}} \|w\|_{H^2(\Omega_x)} \leq C \|G\|_{L^2(\Omega_x)}.$$

This concludes the proof of Lemma 3.6. \square

Remark 3.7. *In fact, the estimates on w in the direction of e_1 are better than in the other directions. We have indeed proved that there exists $C > 0$ such that for all $s \geq s_0$,*

$$s^2 \|\partial_1 w\|_{L^2(\Omega_y)}^2 + \|\partial_1 \nabla' w\|_{L^2(\Omega_y)}^2 \leq C \|G\|_{L^2(\Omega_y)}^2.$$

As we also have the estimate

$$\|\partial_1 \widehat{z}\|_{L^2(\Omega_y)}^2 \leq C \|\widehat{G}\|_{L^2(\Omega_y)}^2,$$

recalling the definition of \widehat{z} , we easily check that we have in fact

$$s^2 \|\partial_1 w\|_{L^2(\Omega_y)}^2 + \|\partial_1 \nabla w\|_{L^2(\Omega_y)}^2 \leq C \|G\|_{L^2(\Omega_y)}^2. \quad (3.34)$$

Chapter 4

Global Carleman estimates

4.1 Elliptic Carleman estimate under Hörmander's strict pseudo-convexity condition: the distributed case

The goal of this chapter is to prove a Carleman estimate for solutions of Laplace equation with homogeneous Dirichlet boundary conditions. More precisely, we consider a smooth bounded domain Ω of \mathbb{R}^d , and u such that

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

Then, we have the following result:

Theorem 4.1. *Let Ω be a smooth bounded domain. Let ω be an open subset of Ω with $\bar{\omega} \subset \Omega$. Let $\varphi \in C^4(\bar{\Omega})$ be such that there exists $\alpha, \beta > 0$ such that*

$$\inf_{x \in \bar{\Omega} \setminus \omega} |\nabla\varphi(x)| > \alpha \quad (4.2)$$

and $\forall x \in \bar{\Omega} \setminus \omega, \forall \xi \in \mathbb{R}^d$,

$$|\nabla\varphi(x)| = |\xi| \text{ and } \nabla\varphi(x) \cdot \xi = 0 \Rightarrow D^2\varphi_x(\nabla\varphi(x), \nabla\varphi(x)) + D^2\varphi_x(\xi, \xi) \geq \beta |\nabla\varphi(x)|^2. \quad (4.3)$$

and

$$\forall x \in \partial\Omega, \quad \varphi(x) = 0, \quad \text{and} \quad \partial_n\varphi(x) < 0. \quad (4.4)$$

Then for all $\varepsilon > 0$, there exist $C, s_0 > 0$ such that for all $s \geq s_0$, for all $u \in L^2(\Omega)$ solution of (4.1) with source term $f \in L^2(\Omega)$,

$$s^3 \|e^{s\varphi} u\|_{L^2(\Omega)}^2 + s \|e^{s\varphi} \nabla u\|_{L^2(\Omega)}^2 \leq C (\|e^{s\varphi} f\|_{L^2(\Omega)}^2 + s^3 \|e^{s\varphi} u\|_{L^2(\omega_\varepsilon)}^2), \quad (4.5)$$

where $\omega_\varepsilon = \{x \in \Omega, d(x, \omega) \leq \varepsilon\}$.

Remark 4.2. Assumption (4.3) is often referred as Hörmander's strict pseudo-convexity condition.

Proof. Here, we provide a proof of Theorem 4.1 using Fourier techniques as we did earlier. As before, we start by setting

$$w = e^{s\varphi} u, \quad F = e^{s\varphi} f, \quad (4.6)$$

which verifies

$$\begin{cases} \Delta w - 2s\nabla\varphi \cdot \nabla w + s^2|\nabla\varphi|^2 w - s\Delta\varphi w = F, & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.7)$$

In order to prove Theorem 4.1, we will prove it locally. Namely, for $x_0 \in \overline{\Omega} \setminus \omega$, we introduce $\eta_{x_0}(x)$ a cut-off function and

$$w_{x_0}(x) = \eta_{x_0}(x)w(x), \quad (4.8)$$

which solves

$$\begin{cases} \Delta w_{x_0} - 2s\nabla\varphi \cdot \nabla w_{x_0} + s^2|\nabla\varphi|^2 w_{x_0} = F_{x_0}, & \text{in } \Omega, \\ w_{x_0} = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.9)$$

where F_{x_0} is defined by

$$F_{x_0} = \eta_{x_0}F + s\Delta\varphi w_{x_0} + [\Delta, \eta_{x_0}]w - 2s[\nabla\varphi \cdot \nabla, \eta_{x_0}]w, \quad (4.10)$$

where for operators A and B , $[A, B]$ denotes the operator $AB - BA$, i.e. the commutator of A and B : in the above cases, $[\Delta, \eta_{x_0}]w = 2\nabla\eta_{x_0} \cdot \nabla w + \Delta\eta_{x_0}w$ and $[\nabla\varphi \cdot \nabla, \eta_{x_0}]w = \nabla\varphi \cdot \nabla\eta_{x_0}w$.

In particular, all our proof is based on the following Lemma, whose proof will be postponed to Section 4.2:

Lemma 4.3. *There exist constants $C > 0$ and $s_0 > 0$ such that for all $x_0 \in \overline{\Omega} \setminus \omega$, and for all $F_{x_0} \in L^2(\Omega)$ and w_{x_0} satisfying (4.9), we have*

$$s^3\|w_{x_0}\|_{L^2(\Omega)}^2 + s\|\nabla w_{x_0}\|_{L^2(\Omega)}^2 + \frac{1}{s}\|w_{x_0}\|_{H^2(\Omega)}^2 \leq C\|F_{x_0}\|_{L^2(\Omega)}^2. \quad (4.11)$$

for all w_{x_0} supported in $B(x_0, s^{-1/3}) \cap \overline{\Omega}$.

We will thus choose

$$\eta_{x_0}(x) = \eta^0(s^{1/3}|x - x_0|)$$

with $\text{Supp } \eta^0 \subset [0, 1)$ and $\eta^0(\rho) = 1$ for $\rho \in [0, 1/2)$. Therefore, we obtain, for C and s independent of x_0 ,

$$\begin{aligned} s^3\|w_{x_0}\|_{L^2(\Omega)}^2 + s\|\nabla w_{x_0}\|_{L^2(\Omega)}^2 &\leq C\|\eta_{x_0}F\|_{L^2(\Omega)}^2 \\ &\quad + C\|[\Delta, \eta_{x_0}]w\|_{L^2(\Omega)}^2 + C\|[\nabla\varphi \cdot \nabla, \eta_{x_0}]w\|_{L^2(\Omega)}^2 + Cs^2\|\nabla\eta_{x_0}|w|\|_{L^2(\Omega)}^2, \end{aligned}$$

and then,

$$\begin{aligned} s^3\|\eta_{x_0}w\|_{L^2(\Omega)}^2 + s\|\eta_{x_0}\nabla w\|_{L^2(\Omega)}^2 &\leq C\|\eta_{x_0}F\|_{L^2(\Omega)}^2 \\ &\quad + C\|[\Delta, \eta_{x_0}]w\|_{L^2(\Omega)}^2 + C\|[\nabla\varphi \cdot \nabla, \eta_{x_0}]w\|_{L^2(\Omega)}^2 + Cs^2\|\nabla\eta_{x_0}|w|\|_{L^2(\Omega)}^2, \end{aligned}$$

Now, integrating in $x_0 \in \overline{\Omega} \setminus \omega$ and using Fubini's identity, we get

$$\begin{aligned} s^3 \int_{\Omega} \rho_0(x)^2 |w(x)|^2 dx + s \int_{\Omega} \rho_0(x) |\nabla w(x)|^2 dx &\leq C \int_{\Omega} \rho_0(x) |F(x)|^2 dx \\ &\quad + C \int_{\Omega} (\rho_{r,1}(x) + s^2 \rho_{r,2}(x)) |w(x)|^2 dx + C \int_{\Omega} \rho_{r,2}(x) |\nabla w(x)|^2 dx, \end{aligned}$$

where the weights $\rho_0, \rho_{r,i}$ are defined as follows:

$$\begin{aligned}\rho_0(x) &= \int_{\Omega \setminus \omega} |\eta_{x_0}(x)|^2 dx_0 \\ \rho_{r,1}(x) &= \int_{\Omega \setminus \omega} |\Delta \eta_{x_0}(x)|^2 dx_0, \\ \rho_{r,2}(x) &= \int_{\Omega \setminus \omega} |\nabla \eta_{x_0}(x)|^2 dx_0,\end{aligned}$$

Now let $\varepsilon > 0$. We check that:

$$\begin{aligned}\forall x \in \Omega, \quad |\rho_0(x)| &\leq C s^{-d/3}, \\ \forall x \in \Omega, \quad |\rho_{r,1}(x)| &\leq C s^{4/3-d/3}, \\ \forall x \in \Omega, \quad |\rho_{r,2}(x)| &\leq C s^{2/3-d/3}, \\ \text{If } s \geq \frac{1}{\min\{\varepsilon^3, d(\omega, \Omega)^3\}}, \forall x \in \Omega \setminus \omega_\varepsilon, \quad |\rho_0(x)| &\geq \frac{s^{-d/3}}{3} \|\eta^0\|_{L^2}^2.\end{aligned}$$

Thus, for s large enough,

$$\begin{aligned}s^3 \int_{\Omega \setminus \omega_\varepsilon} |w(x)|^2 dx + s \int_{\Omega \setminus \omega_\varepsilon} |\nabla w(x)|^2 dx &\leq C \int_{\Omega} |F(x)|^2 dx \\ &\quad + C s^{8/3} \int_{\omega_\varepsilon} |w(x)|^2 dx + C s^{2/3} \int_{\omega_\varepsilon} |\nabla w(x)|^2 dx.\end{aligned}\quad (4.12)$$

Adding then

$$s^3 \int_{\omega_\varepsilon} |w(x)|^2 dx + s \int_{\omega_\varepsilon} |\nabla w(x)|^2 dx$$

to both sides of (4.12), we get:

$$\begin{aligned}s^3 \int_{\Omega} |w(x)|^2 dx + s \int_{\Omega} |\nabla w(x)|^2 dx \\ \leq C \int_{\Omega} |F(x)|^2 dx + C s^3 \int_{\omega_\varepsilon} |w(x)|^2 dx + C s \int_{\omega_\varepsilon} |\nabla w(x)|^2 dx\end{aligned}$$

To remove the term in ∇w in the right hand-side we take a smooth cut-off function η_ω taking

value 1 in ω_ε and vanishing in $\bar{\Omega} \setminus \omega_{2\varepsilon}$ and write¹

$$\begin{aligned}
s \int_{\omega_\varepsilon} |\nabla w(x)|^2 dx &\leq s \int_{\omega_{2\varepsilon}} \eta_\omega(x) |\nabla w(x)|^2 dx \\
&\leq -s \int_{\omega_{2\varepsilon}} w \nabla \eta_\omega \cdot \nabla w - s \int_{\omega_{2\varepsilon}} \eta_\omega w \Delta w \\
&\leq \frac{s}{2} \int_{\omega_{2\varepsilon}} \Delta \eta_\omega |w|^2 - s \int_{\omega_{2\varepsilon}} \eta_\omega w (F + 2s \nabla \varphi \cdot \nabla w - s^2 |\nabla \varphi|^2 w + s \Delta \varphi w) \\
&\leq \frac{s}{2} \int_{\omega_{2\varepsilon}} \Delta \eta_\omega |w|^2 - s \int_{\omega_{2\varepsilon}} \eta_\omega w F + s^2 \int_{\omega_{2\varepsilon}} \operatorname{div}(\eta_\omega \nabla \varphi) |w|^2 \\
&\quad + s^3 \int_{\omega_{2\varepsilon}} \eta_\omega |\nabla \varphi|^2 |w|^2 - s^2 \int_{\omega_{2\varepsilon}} \eta_\omega \Delta \varphi |w|^2 \\
&\leq C s^3 \int_{\omega_{2\varepsilon}} |w|^2 + C \int_{\Omega} |F|^2.
\end{aligned}$$

This concludes the proof of Theorem 4.1 up to the proof of Lemma 4.3 since $\varepsilon > 0$ is arbitrary. \square

Remark 4.4. *The above argument relies on the local character of Carleman estimates: A global Carleman estimate, such as (4.5), can be deduced by gluing local Carleman estimates such as (4.11).*

4.2 Proof of Lemma 4.3

In order to prove the local Carleman estimate stated in Lemma 4.3, we first rewrite the problem (4.9) in suitable coordinates, which will turn out to be adapted to prove the local Carleman estimate (4.11).

4.2.1 A suitable change of variable

We fix $x_0 \in \bar{\Omega} \setminus \omega$ and introduce $L_1 \in \mathbb{R}^d$ and $A_1 \in \mathbb{R}^{d \times d}$ as follows:

$$L_1 = \nabla \varphi(x_0) \in \mathbb{R}^d, \quad A_1 = D^2 \varphi(x_0) \in \mathbb{R}^{d \times d}. \quad (4.13)$$

We then introduce $d-1$ vectors $(L_i)_{i \in \{2, \dots, d\}}$ such that the family $(L_1, (L_i)_{i \in \{2, \dots, d\}})$ form an orthogonal basis of \mathbb{R}^d , and such that for all $i \in \{2, \dots, d\}$,

$$|L_i| = |L_1|.$$

For $i \in \{2, \dots, d\}$, we then introduce self-adjoint matrices A_i such that

1. $A_i L_1 = -A_1 L_i$,
2. for $j \in \{2, \dots, d\}$, $A_i L_i \cdot L_j = A_1 L_1 \cdot L_j$.
3. for $(j, k) \in \{2, \dots, d\}^2$ with i, j, k two by two distinct, $A_i L_j \cdot L_k = 0$,
4. for $j \in \{2, \dots, d\}$ with $j \neq i$, $A_i L_j \cdot L_j = -A_1 L_1 \cdot L_i$.

¹This estimate is also known as Cacciopoli's inequality.

It is then easy to check that each matrix A_i is fully determined. Indeed, writing the matrix A_i in the base of $(L_1, (L_i)_{i \in \{2, \dots, d\}})$, the first condition imposes the first column of A_i , and thus its first line as A_i is symmetric. The second condition imposes the lines 2 to d of the i -th column of A_i , and thus also its lines. The two last conditions then imposes the lines 2 to d of the j -th line of A_i for $j \notin \{1, i\}$. It remains to check that A_i is symmetric. This is a consequence of the third condition.

We shall then introduce the following coordinates for x in a neighborhood of x_0 :

$$y_1(x) = \varphi(x) - \varphi(x_0), \quad (4.14)$$

$$\text{for } j \in \{2, \dots, d\}, y_j(x) = L_j \cdot (x - x_0) + \frac{1}{2} A_j (x - x_0) \cdot (x - x_0), \quad (4.15)$$

By construction, there exists a neighborhood, whose size depends on the C^2 norm of φ only, such that $x \mapsto y(x)$ is a local diffeomorphism between a neighborhood of x_0 and a neighborhood of 0. In particular, for s large enough, we can ensure that the ball of center x_0 and radius $s^{-1/3}$ is included in a set on which $x \mapsto y(x)$ is a diffeomorphism, and its image is included in a ball $B(0, Cs^{-1/3})$. Therefore, for w_{x_0} solving (4.9), we set

$$\tilde{w}(y) = w_{x_0}(x) \quad \text{for } y = y(x), \quad \tilde{F}(y) = F_{x_0}(x) \quad \text{for } y = y(x). \quad (4.16)$$

Explicit computations then give that \tilde{w} satisfies

$$\begin{aligned} \sum_{j,k=1}^d b_{j,k}(x) \partial_{y_j} \partial_{y_k} \tilde{w}(y(x)) + \nabla_y \tilde{w}(y(x)) \cdot \Delta_x y(x) \\ - 2s \sum_{j=1}^d c_j \partial_{y_j} \tilde{w}(y(x)) + s^2 |\nabla \varphi(x)|^2 \tilde{w}(y(x)) = \tilde{F}(y(x)), \quad \text{in } \Omega, \end{aligned} \quad (4.17)$$

where

$$b_{j,k}(x) = \nabla y_j(x) \cdot \nabla y_k(x), \quad \text{and} \quad c_j(x) = \nabla \varphi(x) \cdot \nabla y_j(x),$$

We then remark that $c_j(x) = b_{j,1}(x)$ and that $b_{j,k}(x) = b_{k,j}(x)$ for all x . We now briefly analyze the coefficients $b_{j,k}$. By construction of the coordinates y_j , we easily check that

$$\begin{aligned} b_{j,k}(x_0) &= |L_1|^2 \delta_{j,k}, \\ \partial_\ell b_{j,k}(x_0) &= \sum_{i=1}^d (\partial_\ell \partial_i y_j(x_0) \partial_i y_k(x_0) + \partial_i y_j(x_0) \partial_\ell \partial_i y_k(x_0)) \\ &= \sum_{i=1}^d ((A_j e_i \cdot e_\ell)(L_k \cdot e_i) + (L_j \cdot e_i)(A_k e_i \cdot e_\ell)) \\ &= A_j e_\ell \cdot L_k + A_k e_\ell \cdot L_j = (A_j L_k + A_k L_j, e_\ell), \end{aligned}$$

so that we have in particular that

$$L_\ell \cdot \nabla b_{j,k}(x_0) = (A_j L_k + A_k L_j) \cdot L_\ell,$$

and, corresponding to $j = k = 1$,

$$|\nabla \varphi(x_0)|^2 = |L_1|^2, \quad \text{and} \quad \partial_\ell (|\nabla \varphi(x_0)|^2) = 2 \sum_i \partial_\ell \partial_i \varphi(x_0) \partial_i \varphi(x_0) = 2A_1 L_1 \cdot e_\ell.$$

We can thus analyze $b_{j,k}/|\nabla\varphi|$ close to $x = x_0$:

$$\begin{aligned} \frac{b_{j,k}}{|\nabla\varphi(x)|^2}(x_0) &= \delta_{j,k}, \\ L_\ell \cdot \nabla \left(\frac{b_{j,k}}{|\nabla\varphi(x)|^2} \right) (x_0) &= \frac{1}{|L_1|^2} ((A_j L_k + A_k L_j) \cdot L_\ell - 2\delta_{j,k} A_1 L_1 \cdot L_\ell). \end{aligned}$$

In particular, for all $j, k \in \{1, \dots, d\}^2$,

$$L_\ell \cdot \nabla \left(\frac{b_{j,k}}{|\nabla\varphi(x)|^2} \right) (x_0) = 0 \text{ when } \ell \in \{2, \dots, d\}.$$

When $\ell = 1$, the behavior of this quantity is more interesting:

- If $j = k \geq 2$,

$$L_1 \cdot \nabla \left(\frac{b_{j,j}}{|\nabla\varphi(x)|^2} \right) (x_0) = -\frac{2}{|L_1|^2} (A_1 L_j \cdot L_j + A_1 L_1 \cdot L_1),$$

- If $j = k = 1$,

$$L_1 \cdot \nabla \left(\frac{b_{1,1}}{|\nabla\varphi(x)|^2} \right) (x_0) = 0,$$

- If $j \neq k$ and $j, k \geq 2$,

$$L_1 \cdot \nabla \left(\frac{b_{j,k}}{|\nabla\varphi(x)|^2} \right) (x_0) = -\frac{2}{|L_1|^2} A_1 L_j \cdot L_k,$$

- If j or k equals 1, and $j \neq k$,

$$L_1 \cdot \nabla \left(\frac{b_{j,k}}{|\nabla\varphi(x)|^2} \right) (x_0) = 0.$$

Consequently, as a consequence of Taylor expansion of $b_{j,k}/|\nabla\varphi(x)|^2$ close to $x = x_0$,

$$\begin{aligned} & \left| \sum_{j,k=1}^d b_{j,k}(x) \partial_{y_j} \partial_{y_k} \tilde{w}(y(x)) \right. \\ & - \left(\partial_{11} \tilde{w}(y(x)) + \sum_{j=2}^d \left(1 - \frac{2}{|L_1|^2} (A_1 L_j \cdot L_j + A_1 L_1 \cdot L_1) \right) y_1(x) \partial_{jj} \tilde{w}(y(x)) \right. \\ & \quad \left. \left. - \frac{2}{|L_1|^2} \sum_{j,k=2, j \neq k}^d y_1(x) A_1 L_j \cdot L_k \partial_j \partial_k \tilde{w}(y(x)) \right) \right| \\ & \leq C|x - x_0|^2 \|w\|_{H^2}^2 \leq C|y|^2 \|w\|_{H^2}^2. \quad (4.18) \end{aligned}$$

We thus obtain that

$$\begin{aligned} & \partial_{11} \tilde{w} + \sum_{j=2}^d \left(1 - \frac{2}{|L_1|^2} (A_1 L_j \cdot L_j + A_1 L_1 \cdot L_1) \right) y_1 \partial_{jj} \tilde{w} \\ & - y_1 \sum_{j,k=2, j \neq k}^d \frac{2}{|L_1|^2} A_1 L_j \cdot L_k \partial_j \partial_k \tilde{w} - 2s \partial_1 \tilde{w} + s^2 \tilde{w} = \tilde{G}, \quad \text{in } B(0, Cs^{-1/3}), \quad (4.19) \end{aligned}$$

where

$$\left| \tilde{G}(y) - \frac{1}{|\nabla\varphi(x(y))|^2} (\tilde{F}(y) - \nabla_y \tilde{w}(y) \cdot \Delta_x y(x(y))) \right| \leq C \| |y|^2 \tilde{w} \|_{H^2}. \quad (4.20)$$

If the ball $B(0, Cs^{-1/3})$ intersects the boundary of Ω , then we simply recall that the weight function φ has been chosen such that $\varphi = 0$ on the boundary. In particular, the boundary is locally parametrized by $y_1(x) = Y_0$, where $Y_0 = -\varphi(x_0) \leq 0$, and Ω can be locally defined by $y_1 > Y_0$. Thus, in this case, the equation (4.19) of \tilde{w} should be completed with

$$\tilde{w}(Y_0, y') = 0, \quad \text{for } y' \in \mathbb{R}^{d-1}. \quad (4.21)$$

Remark 4.5. *To better understand this change of variable, it is interesting to focus on the case of a weight function depending only on the x_1 variable. Then, after some computations left as an exercise, the above strategy amounts to transform equation (3.6) into equation (3.28), and the strict convexity of the weight function φ in x_1 is then equivalent to (3.26).*

4.2.2 Proof of Lemma 4.3

For $(j, k) \in \{2, \dots, d\}^2$, we set

$$\lambda_{j,j} = \frac{2}{|L_1|^2} (A_1 L_j \cdot L_j + A_1 L_1 \cdot L_1) \quad \text{and} \quad \lambda_{j,k} = \frac{2}{|L_1|^2} (A_1 L_j \cdot L_k).$$

In particular, for all $\xi' \in \mathbb{R}^{d-1}$,

$$\begin{aligned} \sum_{j,k=2}^d \lambda_{j,k} \xi'_j \xi'_k &= \frac{2}{|L_1|^2} \left(A_1 L_1 \cdot L_1 |\xi'|^2 + A_1 \left(\sum_{j=2}^d \xi_j L_j \right) \cdot \left(\sum_{j=2}^d \xi_j L_j \right) \right) \\ &= \frac{2|\xi'|^2}{|\nabla\varphi(x_0)|^2} (D^2\varphi_{x_0}(\nabla\varphi(x_0), \nabla\varphi(x_0)) + D^2\varphi_{x_0}(\zeta', \zeta')), \end{aligned}$$

where $\zeta' = \frac{1}{|\xi'|} \sum_{j=2}^d \xi_j L_j$ is of modulus $|\nabla\varphi(x_0)|$ and satisfies $\zeta' \cdot \nabla\varphi(x_0) = 0$. Thus, condition (4.3) at a point x_0 is in fact equivalent to the condition (3.25). We can thus apply Lemma 3.6 to each \tilde{w}_{x_0} : There exist $C > 0$ and $s_0 > 0$ independent of x_0 such that for all $s \geq s_0$ and all w_{x_0} supported in $B(x_0, s^{-1/3})$,

$$s^3 \|\tilde{w}_{x_0}\|_{L^2(\Omega_y)}^2 + s \|\nabla \tilde{w}_{x_0}\|_{L^2(\Omega_y)}^2 + \frac{1}{s} \|w_{x_0}\|_{H^2(\Omega_y)}^2 \leq C \|\tilde{G}_{x_0}\|_{L^2(\Omega_y)}^2.$$

Using the estimate (4.20) and the fact that w_{x_0} is supported in $B(0, s^{-1/3})$, we get for s large enough,

$$s^3 \|\tilde{w}_{x_0}\|_{L^2(\Omega_y)}^2 + s \|\nabla \tilde{w}_{x_0}\|_{L^2(\Omega_y)}^2 + \frac{1}{s} \|w_{x_0}\|_{H^2(\Omega_y)}^2 \leq C \|\tilde{F}_{x_0}\|_{L^2(\Omega_y)}^2 + C \|\nabla \tilde{w}_{x_0}\|_{L^2(\Omega_y)}^2.$$

Taking s large enough, the last term can be absorbed and we obtain

$$s^3 \|\tilde{w}_{x_0}\|_{L^2(\Omega_y)}^2 + s \|\nabla \tilde{w}_{x_0}\|_{L^2(\Omega_y)}^2 + \frac{1}{s} \|w_{x_0}\|_{H^2(\Omega_y)}^2 \leq C \|\tilde{F}_{x_0}\|_{L^2(\Omega_y)}^2.$$

We then undo the change of variable, and conclude immediately the proof of Lemma 4.3.

Remark 4.6. *We can add a non-homogeneous boundary condition in $H^{3/2}(\partial\Omega)$ and do the same thing.*

We can also get a similar estimate with a source term F lying in $H^{-1}(\Omega)$ and a non-homogeneous boundary condition in $H^{1/2}(\partial\Omega)$, see [15].

4.3 Elliptic Carleman estimate under Hörmander's strict pseudo-convexity condition: the boundary case

We next derive a similar estimate when the observation is on the boundary.

Again, we consider a smooth bounded domain Ω of \mathbb{R}^d , and u a solution of (4.1).

Then, we have the following result:

Theorem 4.7. *Let $\varphi \in C^4(\overline{\Omega})$ be such that there exists $\alpha, \beta > 0$ such that*

$$\inf_{x \in \overline{\Omega}} |\nabla \varphi(x)| \geq \alpha > 0 \quad (4.22)$$

and $\forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^d$,

$$|\nabla \varphi(x)| = |\xi| \text{ and } \nabla \varphi(x) \cdot \xi = 0 \Rightarrow D^2 \varphi(\nabla \varphi(x), \nabla \varphi(x)) + D^2 \varphi(\xi, \xi) \geq \beta |\nabla \varphi(x)|^2. \quad (4.23)$$

Then there exists $C, s_0 > 0$ such that for all $s \geq s_0$, for all $u \in L^2(\Omega)$ solution of (4.1) with source term $f \in L^2(\Omega)$,

$$s^3 \|e^{s\varphi} u\|_{L^2(\Omega)}^2 + s \|e^{s\varphi} \nabla u\|_{L^2(\Omega)}^2 \leq C (\|e^{s\varphi} f\|_{L^2(\Omega)} + s \|e^{s\varphi} \partial_\nu u\|_{L^2(\Gamma_\varphi)}^2), \quad (4.24)$$

where

$$\Gamma_\varphi = \{x \in \partial\Omega, \partial_n \varphi(x) > 0\}.$$

Remark 4.8. *Note that by standard elliptic regularity result, if $u \in L^2(\Omega)$ is solution of (4.1) with source term $f \in L^2(\Omega)$, then u belongs to $H^2(\Omega)$, and therefore expression (4.24) makes sense.*

Remark 4.9. *Assumption (4.23) is often referred as Hörmander's strict pseudo-convexity condition.*

Note that the linear weight function $x \mapsto \vec{e} \cdot x$ for a unitary vector \vec{e} does not satisfy the convexity condition (4.23). This explains why the powers of the parameter $|k|$ in Theorem 2.5 are not the same as the powers of the parameter s in Theorem 4.7.

Actually, it is easy to check that it satisfies the degenerate convexity condition (4.23) with $\beta = 0$.

When condition (4.3) is saturated, that is when

$$\begin{aligned} \forall x \in \Omega, \forall \xi \in \mathbb{R}^d \text{ with } |\xi| = |\nabla \varphi(x)| \text{ and } \xi \cdot \nabla \varphi(x) = 0, \\ D^2 \varphi_x(\nabla \varphi(x), \nabla \varphi(x)) + D^2 \varphi_x(\xi, \xi) = 0, \end{aligned}$$

the weight function belongs to the so-called Limiting Carleman Weights, see e.g. [9, 20], where it is used to solve the Calderón problem in dimension $d \geq 3$ with partial knowledge of the Dirichlet-to-Neumann map.

Remark 4.10. *Actually, Ω does not necessarily need to be bounded. It is sufficient to assume φ and all its derivative to be bounded on $\overline{\Omega}$ to obtain the same result.*

As usual, to prove Theorem 4.7, we work on the conjugated variable and source term

$$w = e^{s\varphi} u, \quad F = e^{s\varphi} f$$

which verifies

$$\begin{cases} \Delta w + s^2 |\nabla \varphi|^2 u - 2s \nabla \varphi \cdot \nabla u - s \Delta \varphi u = F \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases} \quad (4.25)$$

Then Theorem 4.7 is implied by the following Carleman estimate on w :

Theorem 4.11. *Let $\varphi \in C^4(\bar{\Omega})$ verify (4.22) and (4.23). Then there exists $s_0, C > 0$ such that for all $s \geq s_0$ and all $w \in L^2(\Omega)$ satisfying (4.25) with source term $F \in L^2(\Omega)$, we have*

$$s^3 \|w\|_{L^2(\Omega)}^2 + s^2 \|\xi_2\|_{L^2(\Omega)}^2 + s \|\nabla w\|_{L^2(\Omega)}^2 \leq C (\|F\|_{L^2(\Omega)}^2 + s \|\partial_n w\|_{L^2(\Gamma_\varphi)}^2), \quad (4.26)$$

where

$$\xi_2 = (\nabla w \cdot \eta) \eta,$$

with for all $x \in \Omega$

$$\eta(x) = \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|}.$$

In Theorem 4.11, the term $s^2 \|\xi_2\|_{L^2(\Omega)}^2$ emphasizes that we obtain a better estimate on the component of the gradient *in the direction of $\nabla \varphi$* than on the total gradient.

We now focus on the proof of Theorem 4.11. From now on, we consider $\varphi \in C^4(\bar{\Omega})$ verifying (4.22) and (4.23).

4.3.1 Some basic estimates

For any $w \in H_0^1(\Omega)$, we define

$$P_S w = \Delta w + s^2 |\nabla \varphi|^2 w$$

and

$$P_A w = -2s \nabla \varphi \cdot \nabla w - s \Delta \varphi w.$$

The operators P_S and P_A are respectively symmetric and skew-adjoint, as for any $w, \tilde{w} \in H_0^1(\Omega)$,

$$\langle P_S w, \tilde{w} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle w, P_S \tilde{w} \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}$$

and

$$\langle P_A w, \tilde{w} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = -\langle w, P_A \tilde{w} \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}.$$

We start with some basic estimates that shall be useful in the following.

Proposition 4.12. *There exists a constant $C > 0$ such that for any $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and $s \geq 1$,*

$$s^2 \|\xi_2\|_{L^2(\Omega)}^2 \leq C (\|P_A w\|_{L^2(\Omega)}^2 + s^2 \|w\|_{L^2(\Omega)}^2), \quad (4.27)$$

$$\|\xi_1\|_{L^2(\Omega)}^2 \leq C (s^2 \|w\|_{L^2(\Omega)}^2 + \frac{1}{s^2} (\|P_S w\|_{L^2(\Omega)}^2 + \|P_A w\|_{L^2(\Omega)}^2)), \quad (4.28)$$

and

$$s \int_{\Omega} |D^2 \varphi(\xi_1, \xi_2)| dx \leq C \left(s^{5/2} \|w\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{s}} (\|P_S w\|_{L^2(\Omega)}^2 + \|P_A w\|_{L^2(\Omega)}^2) \right), \quad (4.29)$$

where

$$\nabla w = \xi_1 + \xi_2,$$

with

$$\xi_2 = (\nabla w \cdot \nu) \nu, \quad \nu(x) = \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|}, \quad \xi_1 \cdot \xi_2 = 0.$$

Proof. Estimate (4.27) is direct as

$$|\xi_2| = \frac{1}{|\nabla\varphi|} |\nabla\varphi \cdot \nabla w|$$

and

$$2s \nabla\varphi \cdot \nabla w = -P_A w - s \nabla\varphi w.$$

To prove estimate (4.28), we note that

$$\int_{\Omega} P_S w w \, dx = - \int_{\Omega} |\nabla w|^2 \, dx + s^2 \int_{\Omega} |\nabla\varphi|^2 |w|^2 \, dx.$$

As $|\nabla w|^2 = |\xi_1|^2 + |\xi_2|^2$, we obtain for some constant $C > 0$

$$\int_{\Omega} |\xi_1|^2 \, dx \leq C \left(\int_{\Omega} |\xi_2|^2 \, dx + s^2 \int_{\Omega} |w|^2 \, dx + \left| \int_{\Omega} P_S w w \, dx \right| \right).$$

Estimate (4.28) then follows from

$$\left| \int_{\Omega} P_S w w \, dx \right| \leq \frac{1}{2s^2} \int_{\Omega} |P_S w|^2 \, dx + \frac{s^2}{2} \int_{\Omega} |w|^2 \, dx,$$

and estimate (4.27). Finally, estimate (4.29) simply comes from the inequality

$$s \int_{\Omega} |D^2\varphi(\xi_1, \xi_2)| \, dx \leq C\sqrt{s} \int_{\Omega} |\xi_1|^2 \, dx + C s^{3/2} \int_{\Omega} |\xi_2|^2 \, dx,$$

and estimates (4.28) and (4.27). □

Lemma 4.13. *For any $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and any $s \geq 1$,*

$$\begin{aligned} \int_{\Omega} P_S w P_A w \, dx &= 2s^3 \int_{\Omega} D^2\varphi(\nabla\varphi, \nabla\varphi) |w|^2 \, dx - \frac{s}{2} \int_{\Omega} \Delta^2\varphi |w|^2 \, dx \\ &\quad + 2s \int_{\Omega} D^2\varphi(\nabla w, \nabla w) \, dx - s \int_{\Omega} \partial_n\varphi |\partial_n w|^2 \, ds(x). \end{aligned}$$

Proof. By definition, for any smooth function w ,

$$\int_{\Omega} P_S w P_A w \, dx = \int_{\Omega} (\Delta w + s^2 |\nabla\varphi|^2 w) (-2s \nabla\varphi \nabla w - s \nabla\varphi w) \, dx,$$

so we have to compute each term appearing in the product.

First term: we have

$$\int_{\Omega} \Delta w (-2s \nabla\varphi \nabla w) \, dx = -2s \int_{\partial\Omega} (\partial_n w) (\nabla\varphi \cdot \nabla w) \, ds(x) + 2s \int_{\Omega} \nabla w \cdot \nabla (\nabla\varphi \cdot \nabla w) \, dx.$$

On the boundary of Ω , as $w = 0$, we have $\nabla w = (\nabla w \cdot n)n$, hence

$$-2s \int_{\partial\Omega} (\partial_n w) (\nabla\varphi \cdot \nabla w) \, ds(x) = -2s \int_{\partial\Omega} \partial_n\varphi |\partial_n w|^2 \, ds(x).$$

On the other hand, an easy computation shows that

$$\nabla w \cdot \nabla (\nabla\varphi \cdot \nabla w) = D^2\varphi(\nabla w, \nabla w) + \frac{1}{2} \nabla\varphi \cdot \nabla (|w|^2),$$

which leads to

$$\begin{aligned} 2s \nabla w \cdot \nabla(\nabla\varphi \cdot \nabla w) dx &= 2s \int_{\Omega} D^2\varphi(\nabla w, \nabla w) dx + s \int_{\Omega} \nabla\varphi \cdot \nabla(|w|^2) dx \\ &= 2s \int_{\Omega} D^2\varphi(\nabla w, \nabla w) dx + s \int_{\partial\Omega} \partial_n\varphi |\partial_n w|^2 ds(x) - s \int_{\Omega} \Delta\varphi |w|^2 dx, \end{aligned}$$

where we have again used that on $\partial\Omega$, $\nabla w = (\nabla w \cdot n)n$. So we finally obtain

$$\begin{aligned} \int_{\Omega} \Delta w (-2s \nabla\varphi \nabla w) dx &= 2s \int_{\Omega} D^2\varphi(\nabla w, \nabla w) dx \\ &\quad - s \int_{\Omega} \Delta\varphi |w|^2 dx - s \int_{\partial\Omega} \partial_n\varphi |\partial_n w|^2 ds(x). \end{aligned} \quad (4.30)$$

Second term: as $w = 0$ on $\partial\Omega$, we have

$$\begin{aligned} -s \int_{\Omega} \Delta w \Delta\varphi w dx &= s \int_{\Omega} \Delta\varphi |\nabla w|^2 dx + s \int_{\Omega} \nabla w \cdot \nabla(\Delta\varphi) w dx \\ &= s \int_{\Omega} \Delta\varphi |\nabla w|^2 dx + \frac{s}{2} \int_{\Omega} \nabla(|w|^2) \cdot \nabla(\Delta\varphi) dx \\ &= s \int_{\Omega} \Delta\varphi |\nabla w|^2 dx - \frac{s}{2} \int_{\Omega} \Delta^2\varphi |w|^2 dx. \end{aligned} \quad (4.31)$$

Third term: first we note that

$$\nabla \cdot (\nabla\varphi |\nabla\varphi|^2) = 2D^2\varphi(\nabla\varphi, \nabla\varphi) + |\nabla\varphi|^2 \Delta\varphi.$$

Therefore, we obtain

$$\begin{aligned} -2s^3 \int_{\Omega} |\nabla\varphi|^2 w \nabla\varphi \cdot \nabla w ds &= -s^3 \int_{\Omega} |\nabla\varphi|^2 \nabla\varphi \cdot \nabla(|w|^2) dx \\ &= s^3 \int_{\Omega} \nabla \cdot (\nabla\varphi |\nabla\varphi|^2) |w|^2 dx \\ &= 2s^3 \int_{\Omega} D^2\varphi(\nabla\varphi, \nabla\varphi) |w|^2 dx + s^3 \int_{\Omega} |\nabla\varphi|^2 \Delta\varphi |w|^2 dx. \end{aligned} \quad (4.32)$$

Fourth term: nothing to do here, the fourth term simply reads

$$-s^3 \int_{\Omega} |\nabla\varphi|^2 \Delta\varphi |w|^2 dx. \quad (4.33)$$

Adding equations (4.30), (4.31), (4.32) and (4.33), we obtain the desired result. \square

Lemma 4.14. *For any $w \in H^2(\Omega)$, we have*

$$D^2\varphi(\nabla w, \nabla w) \geq |\xi_2|^2 \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} + \beta |\xi_1|^2 - |\xi_1|^2 \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} + 2D^2\varphi(\xi_1, \xi_2), \quad (4.34)$$

with

$$\nabla w = \xi_1 + \xi_2$$

and

$$\xi_2 = (\nabla w \cdot \nu)\nu, \quad \nu(x) = \frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}, \quad \xi_1 \cdot \xi_2 = 0.$$

Proof. It is clear that

$$D^2\varphi(\nabla w, \nabla w) = D^2\varphi(\xi_1, \xi_1) + 2D^2\varphi(\xi_1, \xi_2) + D^2\varphi(\xi_2, \xi_2).$$

By construction,

$$\tilde{\xi}_1 = |\nabla\varphi| \frac{\xi_1}{|\xi_1|}$$

verifies

$$|\tilde{\xi}_1| = |\nabla\varphi| \text{ and } \tilde{\xi}_1 \cdot \nabla\varphi = 0.$$

Therefore, Hörmander's pseudo-convexity condition (4.23) implies that

$$D^2\varphi(\tilde{\xi}_1, \tilde{\xi}_1) \geq \beta |\nabla\varphi|^2 - D^2\varphi(\nabla\varphi, \nabla\varphi).$$

An easy computation shows that

$$D^2\varphi(\xi_2, \xi_2) = |\xi_2|^2 \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2},$$

which ends the proof. \square

Lemma 4.15. *For any $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\begin{aligned} & s^3 \int_{\Omega} D^2\varphi(\nabla\varphi, \nabla\varphi) |w|^2 dx - s \int_{\Omega} \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} |\nabla w|^2 dx \\ &= s \int_{\Omega} P_S w w \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} dx - \frac{s}{2} \int_{\Omega} \Delta \left(\frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} \right) |w|^2 dx. \end{aligned} \quad (4.35)$$

Proof. We have

$$\begin{aligned} & s \int_{\Omega} P_S w w \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} dx \\ &= s^3 \int_{\Omega} D^2\varphi(\nabla\varphi, \nabla\varphi) |w|^2 dx + s \int_{\Omega} \Delta w \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} w dx \\ &= s^3 \int_{\Omega} D^2\varphi(\nabla\varphi, \nabla\varphi) |w|^2 dx - s \int_{\Omega} \frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} |\nabla w|^2 dx \\ &\quad - s \int_{\Omega} \nabla \left(\frac{D^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^2} \right) \cdot \nabla w w dx. \end{aligned}$$

As $\nabla w w = \frac{1}{2} \nabla (|w|^2)$, another integration by parts of the last term gives the result. \square

4.3.2 Proof of theorem 4.11

To prove Theorem 4.11, we first note that w solution of (4.25) verifies

$$P_S w + P_A w = F \in L^2(\Omega),$$

with implies in particular $w \in H^2(\Omega)$ (see remark 4.8), hence $P_S w \in L^2(\Omega)$ and $P_A w \in L^2(\Omega)$. Therefore, it is readily seen that

$$\|P_S w\|_{L^2(\Omega)}^2 + 2(P_S w, P_A w)_{L^2(\Omega)} + \|P_A w\|_{L^2(\Omega)}^2 \leq \|F\|_{L^2(\Omega)}^2. \quad (4.36)$$

Then, Theorem 4.11 is a direct consequence of the following result, whose proof is postponed afterwards:

Proposition 4.16. *There exists $s_0, C > 0$ such that for any w solution of (4.25) and $s \geq s_0$,*

$$s^3 \|w\|_{L^2(\Omega)}^2 \leq C \left(\|P_S w\|_{L^2(\Omega)}^2 + (P_S w, P_A w)_{L^2(\Omega)} + \|P_A w\|_{L^2(\Omega)}^2 + s \|\partial_n w\|_{\Gamma_\varphi}^2 \right). \quad (4.37)$$

Proof of Theorem 4.11. Proposition 4.16 and equation (4.36) implies that for any $s \geq s_0$,

$$s^3 \|w\|_{L^2(\Omega)}^2 \leq C \left(\|F\|_{L^2(\Omega)}^2 + s \|\partial_n w\|_{\Gamma_\varphi}^2 \right).$$

Then, if we write

$$\nabla w = \xi_1 + \xi_2$$

with

$$\xi_2 = (\nabla w \cdot \nu) \nu, \quad \nu(x) = \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|}, \quad \xi_1 \cdot \xi_2 = 0,$$

Proposition 4.12 implies that for any $s \geq \max(s_0, 1)$,

$$s^2 \|\xi_2\|_{L^2(\Omega)}^2 \leq C \left(\|P_A w\|_{L^2(\Omega)}^2 + s^3 \|w\|_{L^2(\Omega)}^2 \right),$$

and

$$s \|\xi_1\|_{L^2(\Omega)}^2 \leq C \left(s^3 \|w\|_{L^2(\Omega)}^2 + \|P_S w\|_{L^2(\Omega)}^2 + \|P_A w\|_{L^2(\Omega)}^2 \right),$$

which, combined again with Proposition 4.16 and equation (4.36), gives (4.26). \square

Proof of Proposition 4.16. From Lemma 4.13, we know that

$$\begin{aligned} \int_{\Omega} P_S w P_A w \, dx &= 2 s^3 \int_{\Omega} D^2 \varphi(\nabla \varphi, \nabla \varphi) |w|^2 \, dx - \frac{s}{2} \int_{\Omega} \Delta^2 \varphi |w|^2 \, dx \\ &\quad + 2 s \int_{\Omega} D^2 \varphi(\nabla w, \nabla w) \, dx - s \int_{\partial \Omega} \partial_n \varphi |\partial_n w|^2 \, ds(x), \end{aligned}$$

which immediately implies

$$\begin{aligned} (P_S w, P_A w)_{L^2(\Omega)} + s \int_{\Gamma_\varphi} \partial_n \varphi |\partial_n w|^2 \, ds(x) \\ \geq 2 s^3 \int_{\Omega} D^2 \varphi(\nabla \varphi, \nabla \varphi) |w|^2 \, dx - \frac{s}{2} \int_{\Omega} \Delta^2 \varphi |w|^2 \, dx + 2 s \int_{\Omega} D^2 \varphi(\nabla w, \nabla w) \, dx. \end{aligned} \quad (4.38)$$

Now, from Lemma 4.14 we get

$$\begin{aligned} 2 s \int_{\Omega} D^2 \varphi(\nabla w, \nabla w) \, dx &\geq 2 s \int_{\Omega} G(\varphi) |\xi_2|^2 \, dx + 2 s \beta \int_{\Omega} |\xi_1|^2 \, dx \\ &\quad - 2 s \int_{\Omega} G(\varphi) |\xi_1|^2 \, dx + 2 s \int_{\Omega} D^2 \varphi(\xi_1, \xi_2), \end{aligned} \quad (4.39)$$

where we have introduced $G(\varphi) = \frac{D^2 \varphi(\nabla \varphi, \nabla \varphi)}{|\nabla \varphi|^2}$ to shorten notations. Using that $|\nabla w|^2 = |\xi_1|^2 + |\xi_2|^2$, we see that (4.39) is equivalent to

$$\begin{aligned} 2 s \int_{\Omega} D^2 \varphi(\nabla w, \nabla w) \, dx &\geq 4 s \int_{\Omega} G(\varphi) |\xi_2|^2 \, dx + 2 s \beta \int_{\Omega} |\xi_1|^2 \, dx \\ &\quad - 2 s \int_{\Omega} G(\varphi) |\nabla w|^2 \, dx + 4 s \int_{\Omega} D^2 \varphi(\xi_1, \xi_2). \end{aligned} \quad (4.40)$$

Hence, equation (4.38) implies

$$\begin{aligned}
& (P_S w, P_A w)_{L^2(\Omega)} + s \int_{\Gamma_\varphi} \partial_n \varphi |\partial_n w|^2 ds(x) \\
& \geq 2s^3 \int_{\Omega} D^2 \varphi (\nabla \varphi, \nabla \varphi) |w|^2 dx - \frac{s}{2} \int_{\Omega} \Delta^2 \varphi |w|^2 dx + 4s \int_{\Omega} G(\varphi) |\xi_2|^2 dx \\
& \quad + 2s\beta \int_{\Omega} |\xi_1|^2 dx - 2s \int_{\Omega} G(\varphi) |\nabla w|^2 dx + 4s \int_{\Omega} D^2 \varphi (\xi_1, \xi_2),
\end{aligned}$$

from which we obtain using Lemma 4.15

$$\begin{aligned}
& (P_S w, P_A w)_{L^2(\Omega)} + s \int_{\Gamma_\varphi} \partial_n \varphi |\partial_n w|^2 ds(x) \\
& \geq 2s \int_{\Omega} P_S w w G(\varphi) dx - s \int_{\Omega} \Delta G(\varphi) |w|^2 dx - \frac{s}{2} \int_{\Omega} \Delta^2 \varphi |w|^2 dx \\
& \quad + 4s \int_{\Omega} G(\varphi) |\xi_2|^2 dx + 2s\beta \int_{\Omega} |\xi_1|^2 dx + 4s \int_{\Omega} D^2 \varphi (\xi_1, \xi_2). \quad (4.41)
\end{aligned}$$

We have already seen that in the proof of Proposition 4.12 that

$$\int_{\Omega} |\xi_1|^2 dx = s^2 \int_{\Omega} |\nabla \varphi|^2 |w|^2 dx - \int_{\Omega} |\xi_2|^2 dx - \int_{\Omega} P_S w w dx$$

hence (4.41) and $|\nabla \varphi| \geq \alpha$ gives

$$\begin{aligned}
& (P_S w, P_A w)_{L^2(\Omega)} + s \int_{\Gamma_\varphi} \partial_n \varphi |\partial_n w|^2 ds(x) \\
& \geq 2s \int_{\Omega} P_S w w G(\varphi) dx - s \int_{\Omega} \Delta G(\varphi) |w|^2 dx - \frac{s}{2} \int_{\Omega} \Delta^2 \varphi |w|^2 dx \\
& \quad + 2s^3 \beta \alpha^2 \int_{\Omega} |w|^2 dx - 2s \int_{\Omega} |\xi_2|^2 dx - 2s \int_{\Omega} P_S w w dx \\
& \quad + 4s \int_{\Omega} G(\varphi) |\xi_2|^2 dx + 4s \int_{\Omega} D^2 \varphi (\xi_1, \xi_2) dx. \quad (4.42)
\end{aligned}$$

As $\varphi \in C^4(\bar{\Omega})$, there exists a constant C such that $|G(\varphi)| \leq C$ on $\bar{\Omega}$. Combined with estimate (4.27), we obtain that there exists a constant $C > 0$ such that

$$2s \left| \int_{\Omega} (2G(\varphi) - 1) |\xi_2|^2 dx \right| \leq \frac{C}{s} \|P_A w\|_{L^2(\Omega)}^2 + C s \int_{\Omega} |w|^2 dx, \quad (4.43)$$

whereas a simple computation shows that

$$\begin{aligned}
2s \left| \int_{\Omega} (G(\varphi) - 1) P_S w w dx \right| & \leq C s \int_{\Omega} |P_S w w| dx \\
& \leq \frac{C}{2\sqrt{s}} \|P_S w\|_{L^2(\Omega)}^2 + \frac{C s^{5/2}}{2} \int_{\Omega} |w|^2 dx. \quad (4.44)
\end{aligned}$$

Additionally, estimate (4.29) gives a constant $C > 0$ such that

$$4s \left| \int_{\Omega} D^2 \varphi (\xi_1, \xi_2) dx \right| \leq C s^{5/2} \int_{\Omega} |w|^2 + \frac{C}{\sqrt{s}} \left(\|P_S w\|_{L^2(\Omega)}^2 + \|P_A w\|_{L^2(\Omega)}^2 \right). \quad (4.45)$$

Inserting estimates (4.43), (4.44) and (4.45) in (4.42), and furthermore using that there exists a constant C such that

$$s \left| \int_{\Omega} (\Delta G(\varphi) + \frac{1}{2} \Delta^4 \varphi) |w|^2 dx \right| \leq C s \int_{\Omega} |w|^2 dx,$$

we obtain that there exist constants $c_* > 0$ and $C > 0$ such that

$$\begin{aligned} & (P_S w, P_A w)_{L^2(\Omega)} + s \int_{\Gamma_\varphi} \partial_n \varphi |\partial_n w|^2 ds(x) \\ & \geq (c_* s^3 - C s^{5/2} - C s) \int_{\Omega} |w|^2 dx - \frac{C}{\sqrt{s}} \left(\|P_S w\|_{L^2(\Omega)}^2 + \|P_A w\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (4.46)$$

We obtain the result using the existence of $C > 0$ such that $|\partial_n \varphi| \leq C$ on $\partial\Omega$ and choosing $s \geq s_0$, with $s_0 > 0$ such that $c_* s^3/2 - C s^{5/2} - C s \geq 0$ for all $s \geq s_0$. \square

4.4 Some comments on the Hörmander's strict pseudo-convexity condition

In this section, we give some comments on the Hörmander pseudo-convexity condition:

4.4.1 Case of a weight function depending on $x \cdot e$ only

We suppose in this section that $\varphi \in C^4(\overline{\Omega})$ depends only on the coordinate $x \cdot e$, where e is a fixed direction, that is $e \in \mathbb{S}^{d-1}$. Let us denote

$$m = \min_{x \in \overline{\Omega}} x \cdot e, \quad M = \max_{x \in \overline{\Omega}} x \cdot e,$$

and suppose $\varphi(x) = f(x \cdot e)$, for some $f \in C^4([m, M])$. Then $\nabla \varphi = f'(x \cdot e) e$, hence $|\nabla \varphi| > \alpha > 0$ directly reads $|f'| > \alpha$ on $[m, M]$, that is a strict monotonicity condition for f on $[m, M]$. Furthermore, for all $a, b \in \mathbb{R}^d$, it is straightforward to see that

$$D^2 \varphi(a, b) = (a \cdot e)(b \cdot e) f''(x \cdot e).$$

As furthermore all $\xi \in \mathbb{R}^d$ verifying $\xi \cdot \nabla \varphi = 0$ necessarily verifies $\xi \cdot e = 0$, the condition that there exists $\beta > 0$ such that for all $x \in \overline{\Omega}$, all $\xi \in \mathbb{R}^d$,

$$|\nabla \varphi(x)| = |\xi| \text{ and } \nabla \varphi(x) \cdot \xi = 0 \Rightarrow D^2 \varphi(\nabla \varphi(x), \nabla \varphi(x)) + D^2 \varphi(\xi, \xi) \geq \beta |\nabla \varphi(x)|^2,$$

simply rewrites

$$f''(x \cdot e) > \beta \text{ on } \overline{\Omega},$$

which reads as a strict convexity condition for f on $[m, M]$.

These are precisely the two conditions verified by the weight function we used in our study of the previous chapter, in the case of a strip.

4.4.2 Construction of a weight function

Let us now consider Ω an arbitrary smooth bounded domain of \mathbb{R}^d , and $\Gamma \subset \Omega$ nonempty and open. Suppose that we want to use Theorem 4.7 with an observation localized on Γ , that is such that

$$\Gamma_\varphi = \{x \in \partial\Omega, \partial_n \varphi(x) > 0\} \subset \Gamma,$$

where φ is the weight function in the Carleman estimate (4.24). The existence of such a weight function satisfying furthermore $|\nabla\varphi| \geq \alpha > 0$ on $\bar{\Omega}$ and Hörmander's strict pseudoconvexity condition (4.23) is not obvious. But it turns out it is always possible to construct a function φ verifying all the require properties. The key result to do so is the following proposition.

Proposition 4.17. *There exists a function ψ belonging to $C^4(\bar{\Omega})$ such that*

$$\inf_{x \in \bar{\Omega}} \{|\nabla\psi(x)|\} > 0, \quad \text{and} \quad \forall x \in \partial\Omega \setminus \Gamma, \partial_n \psi(x) < 0. \quad (4.47)$$

Proof. See for instance [36, Appendix 3]. □

The process to construct φ from ψ is often called convexification. It consists in defining φ as $e^{\lambda\psi}$, for a parameter $\lambda > 0$ chosen sufficiently large.

Proposition 4.18. *Let ψ be given by Proposition 4.17, and $\varphi = e^{\lambda\psi}$ with $\lambda > 0$. There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, φ satisfies $\Gamma_\varphi \subset \Gamma$,*

$$\inf_{x \in \bar{\Omega}} |\nabla\varphi| > 0,$$

and Hörmander's strict pseudoconvexity condition (4.23).

Proof. First of all, as $\nabla\varphi = \lambda\varphi\nabla\psi$, $\lambda\varphi > 0$ on $\bar{\Omega}$ and $\Gamma_\psi \subset \Gamma$, we clearly have

$$\inf_{x \in \bar{\Omega}} |\nabla\varphi| = \alpha > 0 \text{ and } \Gamma_\varphi \subset \Gamma.$$

It then remains to consider Hörmander's pseudoconvexity condition.

First of all, we denote

$$\sigma_\psi = \inf_{x \in \bar{\Omega}} \psi(x), \quad \alpha_\psi = \inf_{x \in \bar{\Omega}} |\nabla\psi| > 0, \quad \gamma_\psi = \sup_{e \in \mathbb{S}^{d-1}} |D^2\psi(e, e)|.$$

A simple computation shows that for all $a, b \in \mathbb{R}^d$, one has

$$D^2\varphi(a, b) = \lambda^2 \varphi (a \cdot \nabla\psi) (b \cdot \nabla\psi) + \lambda\varphi D^2\psi(a, b).$$

Therefore, for all $x \in \bar{\Omega}$ and all $\xi \in \mathbb{R}^d$ such that

$$|\xi| = |\nabla\varphi(x)| = \lambda\varphi(x) |\nabla\psi(x)|,$$

and $\xi \cdot \nabla\varphi = 0$, or equivalently $\xi \cdot \nabla\psi = 0$, we have

$$\begin{aligned} & D^2\varphi(\nabla\varphi(x), \nabla\varphi(x)) + D^2\varphi(\xi, \xi) \\ &= |\nabla\varphi(x)|^2 \varphi(x) \left[\lambda^2 |\nabla\psi|^2 + \lambda D^2\psi \left(\frac{\nabla\psi}{|\nabla\psi|}, \frac{\nabla\psi}{|\nabla\psi|} \right) + \lambda D^2\psi \left(\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \right) \right], \end{aligned}$$

hence

$$D^2\varphi(\nabla\varphi(x), \nabla\varphi(x)) + D^2\varphi(\xi, \xi) \geq |\nabla\varphi(x)|^2 e^{\lambda\sigma_\psi} [\lambda^2 \alpha_\psi - 2\lambda\gamma_\psi].$$

Choosing $\lambda_0 > 2 \frac{\gamma_\psi}{\alpha_\psi}$ gives the result. □

Remark 4.19. Interestingly, choosing a weight function φ of the form $e^{\lambda\psi}$ gives a new parameter λ that can be chosen arbitrarily large. This extra parameter is of paramount importance in some applications, see e.g. [23].

Remark 4.20. In fact, once a function ψ is constructed satisfying the conditions of Proposition 4.17, it is interesting to look for φ under the form $\varphi = f(\psi)$ for some strictly monotonic function f . In particular, such process will not modify the level sets of the function ψ , so that the quantity $D^2\varphi_x(\xi, \xi)$ for ξ' satisfying $\xi \cdot \nabla\varphi(x) = 0$, which corresponds to curvatures of the level set of the function, is barely modified, while the convexity in the direction of $\nabla\varphi$ will be strongly modified. In fact, condition (4.24) then reads: for all $x \in \overline{\Omega}$, $\forall \xi \in \mathbb{R}^d$ with $|\xi| = 1$ and $\xi \cdot \nabla\psi(x) = 0$

$$f''(\psi(x)) + f'(\psi(x)) \left(D^2\psi_x \left(\frac{\nabla_x\psi(x)}{|\nabla_x\psi(x)|}, \frac{\nabla_x\psi(x)}{|\nabla_x\psi(x)|} \right) + D^2\psi_x(\xi, \xi) \right) \geq \beta.$$

It is then clear that this could be achieved by taking f'' large enough compared to f' , which is precisely what the choice $f(y) = \exp(\lambda y)$ does when λ is large enough.

This choice “convexifies” the function ψ . In particular, one needs f to be sufficiently convex in the direction of $\nabla\psi$ compared to the curvature of the level sets $\{\psi = c\}$. This process is called a “convexification” process.

4.4.3 Easier set of assumptions

Here, we consider again the setting of Theorem 4.7, and we mention that if we impose some additional conditions, then we can derive an easier proof than the one of Theorem 4.7.

Theorem 4.21. Let $\varphi \in C^4(\overline{\Omega})$ and assume that there exist $\alpha > 0$ and $\beta > 0$ for which we have

$$\inf_{x \in \overline{\Omega}} |\nabla\varphi(x)| \geq \alpha, \quad (4.48)$$

and

$$\begin{cases} -|\nabla\varphi|^2 \Delta\varphi + 2D^2\varphi(\nabla\varphi, \nabla\varphi) \geq \beta|\nabla\varphi|^2. \\ \inf_{\substack{x \in \overline{\Omega} \\ \xi \in \mathbb{R}^d, |\xi|=1}} \{ \Delta\varphi|\xi|^2 + 2D^2\varphi(\xi, \xi) \} \geq \beta. \end{cases} \quad (4.49)$$

Then there exists a constant $C > 0$ such that for all $u \in L^2(\Omega)$ solution of (4.1) with source term $f \in L^2(\Omega)$, for all $s \geq 1$,

$$s^3 \|e^{s\varphi} u\|_{L^2(\Omega)}^2 + s \|e^{s\varphi} u\|_{L^2(\Omega)}^2 \leq C \left(\|e^{s\varphi} f\|_{L^2(\Omega)}^2 + s \|e^{s\varphi} \partial_n u\|_{L^2(\Gamma_\varphi)}^2 \right), \quad (4.50)$$

where

$$\Gamma_\varphi = \{x \in \partial\Omega, \partial_n \varphi(x) > 0\}.$$

Conditions (4.48)–(4.49) have to be compared with (4.22)–(4.23), which are more general.

Before going into the proof of Theorem 4.21, let us mention the following result, which indicates that the additional observation may be done on any arbitrary subset of the boundary:

Lemma 4.22. Let Γ be a non-empty open subset of the boundary $\partial\Omega$. Then there exists a smooth function φ satisfying condition (4.48)–(4.49) and $\Gamma_\varphi \subset \Gamma$.

Proof of Lemma 4.22. Similarly as for the proof of Proposition 4.18, we start by taking ψ as in Proposition 4.17, then look for φ of the form $\varphi(x) = e^{\lambda\psi(x)}$ for some parameter $\lambda \geq 1$ large enough. Explicit computations yield

$$\begin{aligned}\nabla\varphi &= \lambda\nabla\psi\varphi \\ \Delta\varphi &= \lambda^2|\nabla\psi|^2\varphi + \lambda\Delta\psi\varphi \\ D^2\varphi &= \lambda^2\nabla\psi^T\nabla\psi\varphi + \lambda D^2\psi\varphi.\end{aligned}$$

First, according to condition (4.47), we easily check that $\Gamma_\varphi \subset \Gamma$.

We then compute

$$\begin{aligned}& -|\nabla\varphi|^2\Delta\varphi + 2D^2\varphi(\nabla\varphi, \nabla\varphi) \\ &= -\lambda^4|\nabla\psi|^4\varphi^3 - \lambda^3D^2\psi(\nabla, \nabla\psi)\varphi^3 + 2\lambda^4|\nabla\psi|^4\varphi^3 + \lambda^3\Delta\psi|\nabla\psi|^2\varphi^3 \\ &= \lambda^4|\nabla\psi|^4\varphi^3 - \lambda^3D^2\psi(\nabla, \nabla\psi)\varphi^3 + \lambda^3\Delta\psi|\nabla\psi|^2\varphi^3.\end{aligned}$$

According to conditions (4.47), condition (4.49)₁ is satisfied for $\lambda \geq \lambda_1$.

Similarly, explicit computations yield

$$\begin{aligned}\Delta\varphi|\xi|^2 + 2D^2\varphi(\xi, \xi) \\ = \lambda^2|\nabla\psi|^2|\xi|^2\varphi + \lambda\Delta\psi|\xi|^2\varphi + 2\lambda^2(\nabla\psi \cdot \xi)^2\varphi + \lambda D^2\psi(\xi, \xi)\varphi.\end{aligned}$$

Again, due to condition (4.47), we can make this term positive on $\overline{\Omega}$ by taking $\lambda \geq \lambda_1$ large enough.

Therefore, the choice $\varphi = e^{\lambda\psi}$ for λ sufficiently large satisfy all the assumptions of Theorem 4.21 with $\Gamma_\varphi \subset \Gamma$. \square

Proof of Theorem 4.21. We set $U = u \exp(s\varphi)$ and $F = fe^{s\varphi}$. Then

$$F = e^{s\varphi}\Delta u = e^{s\varphi}\Delta(e^{-s\varphi}U) = \Delta U - 2s\nabla\varphi \cdot \nabla U + s^2|\nabla\varphi|^2U - s\Delta\varphi U,$$

so that U solves

$$\begin{cases} \Delta U - 2s\nabla\varphi \cdot \nabla U + s^2|\nabla\varphi|^2U - s\Delta\varphi U = F & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.51)$$

We thus write

$$P_1U = \Delta U + s^2|\nabla\varphi|^2U, \quad P_2U = -2s\Delta\varphi U - 2s\nabla\varphi \cdot \nabla U, \quad RU = -s\Delta\varphi U.$$

so that

$$P_1U + P_2U = F + RU,$$

and P_1, P_2 roughly correspond to the self-adjoint and skew-adjoint parts of the operator in (4.51).

It follows

$$2 \int_{\Omega} P_1U P_2U \leq 2 \int_{\Omega} |F|^2 + 2 \int_{\Omega} |RU|^2. \quad (4.52)$$

We thus compute the product of P_1U by P_2U . We will denote by $I_{i,j}$ the cross product between the i -th term of P_1 and the j -th term of P_2 . Of course, our computations will

strongly use the fact that u and thus U vanish on the boundary.

$$\begin{aligned}
I_{11} &= -2s \int_{\Omega} \Delta \varphi U \Delta U = 2s \int_{\Omega} \Delta \varphi |\nabla U|^2 - s \int_{\Omega} \Delta^2 \varphi |U|^2 \\
I_{12} &= -2s \int_{\Omega} \Delta U \nabla \varphi \cdot \nabla U \\
&= 2s \int_{\Omega} \nabla U \cdot \nabla (\nabla \varphi \cdot \nabla U) - 2s \int_{\partial \Omega} \partial_n \varphi |\partial_n U|^2 \\
&= 2s \int_{\Omega} D^2 \varphi (\nabla U, \nabla U) - s \int_{\Omega} \Delta \varphi |\nabla U|^2 - s \int_{\partial \Omega} \partial_n \varphi |\partial_n U|^2 \\
I_{21} &= -2s^3 \int_{\Omega} |\nabla \varphi|^2 \Delta \varphi |U|^2 \\
I_{22} &= -2s^3 \int_{\Omega} |\nabla \varphi|^2 \nabla \varphi \cdot \nabla U U = -s^3 \int_{\Omega} |\nabla \varphi|^2 \nabla \varphi \cdot \nabla (|U|^2) \\
&= s^3 \int_{\Omega} (|\nabla \varphi|^2 \Delta \varphi + 2D^2 \varphi (\nabla \varphi, \nabla \varphi)) |U|^2.
\end{aligned}$$

Combining these estimates,

$$\begin{aligned}
\int_{\Omega} P_1 U P_2 U &= s^3 \int_{\Omega} (-|\nabla \varphi|^2 \Delta \varphi + 2D^2 \varphi (\nabla \varphi, \nabla \varphi)) |U|^2 - s \int_{\Omega} \Delta^2 \varphi |U|^2 \\
&\quad + s \int_{\Omega} \Delta \varphi |\nabla U|^2 + 2s \int_{\Omega} D^2 \varphi (\nabla U, \nabla U) - s \int_{\partial \Omega} \partial_n \varphi |\partial_n U|^2. \quad (4.53)
\end{aligned}$$

Using Assumptions (4.48)–(4.49), we obtain, for $s \geq s_0$ large enough, that

$$s^3 \int_{\Omega} |U|^2 + s \int_{\Omega} |\nabla U|^2 \leq C \int_{\Omega} P_1 U P_2 U + Cs \int_{\Gamma_{\varphi}} |\partial_n U|^2.$$

Using (4.52), we obtain

$$s^3 \int_{\Omega} |U|^2 + s \int_{\Omega} |\nabla U|^2 \leq C \int_{\Omega} |F|^2 + Cs^2 \int_{\Omega} |U|^2 + Cs \int_{\Gamma_{\varphi}} |\partial_n U|^2.$$

Hence taking s_0 larger if necessary, for all $s \geq s_0$,

$$s^3 \int_{\Omega} |U|^2 + s \int_{\Omega} |\nabla U|^2 \leq C \int_{\Omega} |F|^2 + Cs \int_{\Gamma_{\varphi}} |\partial_n U|^2.$$

We now remark that $U = ue^{s\varphi}$ hence

$$|u|^2 e^{2s\varphi} \leq |U|^2 \quad \text{and} \quad |\nabla u|^2 e^{2s\varphi} \leq 2|\nabla U|^2 + 2s^2 |U|^2,$$

that yields the claimed result. \square

4.5 Application to unique continuation

The main result of this section is the following one:

Theorem 4.23. Let $q \in L^\infty(\Omega)$ and $W \in (L^\infty(\Omega))^d$, and let Γ be a non-empty open subset of the boundary $\partial\Omega$. Then any solution $u \in H_0^1(\Omega)$ of

$$\begin{cases} \Delta u + qu + W \cdot \nabla u = 0 & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

which further satisfies $\partial_n u = 0$ on Γ vanishes identically on Ω .

This property is the so-called unique continuation property through Γ .

Proof. Choose a weight function φ so that $\Gamma_\varphi \subset \Gamma$ according to Lemma 4.22, and apply the Carleman estimate of Theorem 4.21 with $f = -qu - W \cdot \nabla u$: For all $s \geq 1$,

$$s^3 \|e^{s\varphi} u\|_{L^2(\Omega)}^2 + s \|e^{s\varphi} \nabla u\|_{L^2(\Omega)}^2 \leq C \left(\|q\|_{L^\infty}^2 \|e^{s\varphi} u\|_{L^2(\Omega)}^2 + \|W\|_{L^\infty}^2 \|e^{s\varphi} \nabla u\|_{L^2(\Omega)}^2 \right),$$

Taking s large enough, we easily obtain $u = 0$ in Ω . □

Exercise 4.1. Arguing as in Corollary 2.3, check that if $u \in H_0^1(\Omega)$ solves

$$\begin{cases} \Delta u + qu + W \cdot \nabla u = f & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

with a source term that vanishes on a set of the form $\{\varphi > a\}$ for some φ satisfying (3.8)–(4.49) and $\partial_n u$ vanishing on Γ_φ and $q \in L^\infty(\Omega)$, $W \in (L^\infty(\Omega))^d$, then u vanishes on the whole set $\{\varphi > a\}$.

Chapter 5

Conclusion

In this lecture, we have explained how Carleman estimates can be derived for the Laplace operator, using several different techniques and approaches.

We emphasize that Carleman estimates can be derived in much more general settings, regarding the PDE under consideration or the question at stake. The approach we presented here are quite far from being general, and somehow shortcuts semi-classical arguments by a careful study of what happens in the case of a vertical strip and coefficients depending only on x_1 .

For more about Carleman estimates, in particular regarding more PDE and presenting several applications, we refer to the textbooks [27], [7], [12], [30], [21], [17], and to the survey articles [26], [11], among many others.

This lecture was focused on Carleman estimates considered as a tool, for which we give only few applications, namely to the Calderón problem and to the unique continuation for elliptic equations. We should nevertheless close this lecture by emphasizing that Carleman estimates appeared in many fields and have much more applications.

As we discussed, Carleman estimates can be applied to derive *unique continuation properties* in many settings, and yield several related properties. For instance, they can be used to derive *propagation of smallness* of solutions of PDE, in which the question is the following: if the solution u of some PDE is small in some set ω , can we prove that the solution is small in the whole domain? See an instance of such result in Theorem 4.1. As a matter of fact, this question is also related to the Cauchy problem for the Laplace equation that we saw in the beginning, in which this question can be recast into a stability problem. Carleman estimates can also be used to prove the so-called three sphere inequality, which is a classical property for harmonic functions, and can be general to solutions of elliptic PDE, see e.g. the survey article [1]: there exist $C > 0$ and $\theta \in (0, 1)$, such that for $u \in L^2(B(4))$ solution of $\Delta u = 0$ in the ball $B(4)$ of radius 4,

$$\|u\|_{L^2(B(2))} \leq C \|u\|_{L^2(B(1))}^\theta \|u\|_{L^2(B(4))}^{1-\theta}.$$

Related to these unique continuation properties, we may quote the *Landis conjecture*: is it true that, if $u \in L^1_{loc}(\mathbb{R}^d; \mathbb{R})$ satisfies $|\Delta u| \leq |u|$ in \mathbb{R}^d and for some $C > 0$ sufficiently large, $|u(x)| \exp(C|x|) \in L^\infty(\mathbb{R}^d)$, then u vanishes? This example should be thought of as a unique continuation problem at ∞ , meaning that the solution is supposed to be appropriately small in a neighborhood of the infinity. It should be emphasized that the Landis conjecture is still a conjecture, except in the 1d case, but decisive results have been made using Carleman estimates, see e.g. [4] for results in general dimensions, used in the proof of Anderson

localization for the Bernoulli models, and the recent work [32] for an almost proof in the 2-d case.

As we have explained for the Calderón problem, Carleman estimates can also be used to derive stability results for many *inverse problems*, and we refer to the textbooks [17], [21], [7], [3] for many applications in the context of inverse problems. Of course, this is a very rich field and there is no time to present it in more details.

Another broad application of Carleman estimates is to derive *observability estimates* for PDE, which basically corresponds to quantification of the unique continuation for PDE. Such observability properties are often used to derive *controllability results*, which are equivalent one to another through classical duality arguments (see for instance [36]). Therefore, Carleman estimates are of the main tools used to derive controllability results for PDE, in particular since the pioneering works [13, 28] which proved null-controllability of the heat equation in arbitrary geometric settings.

We will not list more the various applications which use Carleman estimates, but we would like to point out that this list is not exhaustive (by far!), and they can be encountered as well to establish the logarithmic decay rate of waves when no geometric control condition is satisfied ([29]), or to prove that a weak solution of the Navier-Stokes equations which belongs to $L^\infty(0, T; L^3(\mathbb{R}^d))$ is smooth up to the time T ([18, 35]).

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