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► **To cite this version:**

Alain Blaustein. Large coupling in a FitzHug-Nagumo neural network: quantitative and strong convergence results. 2022. hal-03619446v2

**HAL Id: hal-03619446**

**<https://hal.science/hal-03619446v2>**

Preprint submitted on 7 Apr 2022 (v2), last revised 11 Jun 2023 (v4)

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# LARGE COUPLING IN A FITZHUGH-NAGUMO NEURAL NETWORK: QUANTITATIVE AND STRONG CONVERGENCE RESULTS

ALAIN BLAUSTEIN

Université Paul Sabatier, Institut de Mathématiques de Toulouse,  
118 route de Narbonne - F-31062 TOULOUSE Cedex 9, France;  
*email*: alain.blaustein@math.univ-toulouse.fr

ABSTRACT. We consider a spatially extended mesoscopic FitzHugh-Nagumo model with interactions and prove that in the regime where strong and local interactions dominate, the probability density of the potential throughout the network concentrates into a Dirac distribution whose center of mass solves the classical non-local reaction-diffusion FitzHugh-Nagumo system. In order to refine our comprehension of this regime, we focus on the blow-up profile of this concentration phenomenon. Our main purpose here consists in proving that it is Gaussian by deriving two quantitative and strong convergence estimates: the first one in a  $L^1$  functional framework and the second in a weighted  $L^2$  functional setting. We develop original relative entropy techniques to prove the first result whereas our second result relies on propagation of regularity.

**Keywords:** Diffusive limit, relative entropy, FitzHugh-Nagumo, neural network.

**Mathematics Subject Classification (2010):** 35B40 – 35Q92 – 82C32 – 92B20

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## 1. INTRODUCTION

**1.1. Physical model and motivations.** Neuron models often focus on the regulation of the electrical potential through the membrane of a nerve cell depending on the input it receives. This regulation is the result of ionic exchanges between the neuron and its environment through its cellular membranes. A very precise modeling of these ion exchanges led to the well-known Hodgkin-Huxley model [22]. A simplified version, called the FitzHugh-Nagumo model [18] [27], keeps its most valuable aspects while remaining relatively simple on a mathematical point of view. More precisely, the FitzHugh-Nagumo model accounts for the variations of the membrane potential  $v$  of a neuron coupled to an auxiliary variable  $w$  called the adaptation variable. It is usually written as follows:

$$\begin{cases} dv_t = (N(v_t) - w_t + I_{ext}) dt + \sqrt{2} dB_t, \\ dw_t = A(v_t, w_t) dt, \end{cases}$$

where the drift  $N$  is a confining non-linearity with the typical form

$$N(v) = v - v^3,$$

even though a broader class of drifts  $N$  is considered here, and  $A$  is an affine transformation that has the following form

$$A(v, w) = a v - b w + c,$$

where  $a, c \in \mathbb{R}$  and  $b > 0$ , which means that  $A$  also has linear confining properties. Here, the Brownian motion  $B_t$  has been added in order to take into account random fluctuations in the dynamics of the membrane potential  $v_t$ . Another mathematical reason for looking at this system is that it is a prototypical model of excitable kinetics. Interest in such systems stems from the fact that although the kinetics are relatively simple, couplings between neurons can produce complex dynamics, where well-known examples are the propagation of excitatory pulses, spiral waves in two-dimensions and spatio-temporal chaos. Here, we introduce coupling through the input current  $I_{ext}$ . More specifically, we consider that neurons interact with one another following Ohm's law and that the conductance between two neurons depends on their spatial location  $\mathbf{x} \in K$ , where  $K$  is a compact set of  $\mathbb{R}^d$ . The conductance between two neurons is given by a connectivity kernel  $\Phi : K \times K \rightarrow \mathbb{R}$ . Therefore, in the case of a network composed with  $n$  interacting neurons described by the triplet voltage-adaptation-position  $(v_i, w_i, \mathbf{x}_i)_{1 \leq i \leq n}$ , the current received by neuron  $i$  from the other neurons is given by

$$I_{ext} = -\frac{1}{n} \sum_{j=1}^n \Phi(\mathbf{x}_i, \mathbf{x}_j) (v_t^i - v_t^j),$$

where the scaling parameter  $n$  is introduced here to re-normalize the contribution of each neuron. According to the former discussion, a neural network of size  $n$  is described by the system of equations

$$\begin{cases} dv_t^i = \left( N(v_t^i) - w_t^i - \frac{1}{n} \sum_{j=1}^n \Phi(\mathbf{x}_i, \mathbf{x}_j) (v_t^i - v_t^j) \right) dt + \sqrt{2} dB_t^i, \\ dw_t^i = A(v_t^i, w_t^i) dt, \end{cases}$$

where  $i \in \{1, \dots, n\}$ . In the formal limit  $n \rightarrow +\infty$ , the behavior of the latter system may be described by the evolution of a distribution function  $f := f(t, \mathbf{x}, \mathbf{u})$ , with  $\mathbf{u} = (v, w) \in \mathbb{R}^2$ , representing the density of neurons at time  $t$ , position  $\mathbf{x} \in K$  with a membrane potential  $v$  and adaptation variable  $w \in \mathbb{R}$ . It turns out that the distribution function  $f$  solves the following mean-field equation, see for instance [1, 4, 26, 24] for the FitzHugh-Nagumo system and [3] for a related model in collective dynamics,

$$\partial_t f + \partial_v ((N(v) - w - \mathcal{K}_\Phi[f]) f) + \partial_w (A(v, w) f) - \partial_v^2 f = 0,$$

where the operator  $\mathcal{K}_\Phi[f]$  takes into account spatial interactions and is given by

$$\mathcal{K}_\Phi[f](t, \mathbf{x}, v) = \int_{K \times \mathbb{R}^2} \Phi(\mathbf{x}, \mathbf{x}') (v - v') f(t, \mathbf{x}', \mathbf{u}') d\mathbf{x}' d\mathbf{u}'.$$

Various other types of kinetic models have been derived during the past decades depending on the hypotheses assumed for the dynamics of the emission of an action potential. They include for example integrate-and-fire neural networks [6, 7, 8, 9] and time-elapsd neuronal models [28, 11, 10, 12].

Let us be more specific on the modeling of interactions between neurons. A common assumption consists in considering that there are of two types: strong short range interactions and weak long range interactions (see [5], [29] and [24]). Here we consider a connectivity kernel of the following type

$$\Phi^\varepsilon(\mathbf{x}, \mathbf{x}') = \frac{1}{\varepsilon} \delta_0(\mathbf{x} - \mathbf{x}') + \Psi(\mathbf{x}, \mathbf{x}'),$$

where the Dirac mass  $\delta_0$  accounts for strong short range interactions with strength  $\varepsilon > 0$ , whereas the connectivity kernel  $\Psi : K \times K \rightarrow \mathbb{R}$  is more regular and represents weak long range interactions.

The purpose of this article is to go through the mathematical analysis of the neural network in the regime of strong interactions, that is when  $\varepsilon \ll 1$ . More precisely, we prove that the voltage distribution concentrates to a Dirac mass by providing a comprehensive description of this concentration phenomenon.

**1.2. Formal derivation.** The purpose of this section is to formally derive the result which is at stake in this article and to introduce the quantities we will work with throughout our analysis. First, we define some integrated quantities associated to our system. We consider the spatial distribution of neurons throughout the network

$$\rho_0^\varepsilon(\mathbf{x}) = \int_{\mathbb{R}^2} f^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u}.$$

It is straightforward to check that  $\rho_0^\varepsilon$  is indeed time-homogeneous, integrating the mean field equation with respect to  $\mathbf{u} \in \mathbb{R}^2$ . Second, we introduce the averaged voltage and adaptation variable at a spatial location  $\mathbf{x}$

$$(1.1) \quad \begin{cases} \rho_0^\varepsilon(\mathbf{x}) \mathcal{V}^\varepsilon(t, \mathbf{x}) &= \int_{\mathbb{R}^2} v f^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u}, \\ \rho_0^\varepsilon(\mathbf{x}) \mathcal{W}^\varepsilon(t, \mathbf{x}) &= \int_{\mathbb{R}^2} w f^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u}. \end{cases}$$

In the sequel, we use the vector notation  $\mathcal{U}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$  and we re-normalize  $f^\varepsilon$  as follows

$$\rho_0^\varepsilon \mu^\varepsilon = f^\varepsilon,$$

where, for all positive  $\varepsilon$ ,  $\mu^\varepsilon$  is a non-negative function which lies in  $\mathcal{C}^0(\mathbb{R}^+ \times K, L^1(\mathbb{R}^2))$  and which verifies for all positive  $\varepsilon$

$$\int_{\mathbb{R}^2} \mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u} = 1, \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K.$$

With these notations and our modeling assumptions on the connectivity kernel  $\Phi^\varepsilon$ , the mean field equation rewrites

$$(1.2) \quad \partial_t \mu^\varepsilon + \operatorname{div}_{\mathbf{u}}[\mathbf{b}^\varepsilon \mu^\varepsilon] - \partial_v^2 \mu^\varepsilon = \frac{1}{\varepsilon} \rho_0^\varepsilon \partial_v [(v - \mathcal{V}^\varepsilon) \mu^\varepsilon],$$

where  $\mathbf{b}^\varepsilon$  is defined for all  $(t, \mathbf{x}, \mathbf{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$  as

$$\mathbf{b}^\varepsilon(t, \mathbf{x}, \mathbf{u}) = \begin{pmatrix} N(v) - w - \mathcal{K}_\Psi[\rho_0^\varepsilon \mu^\varepsilon](t, \mathbf{x}, v) \\ A(\mathbf{u}) \end{pmatrix}.$$

Furthermore, one can notice that the non local term  $\mathcal{K}_\Psi[\rho_0^\varepsilon \mu^\varepsilon]$  can be expressed in terms of the macroscopic quantities

$$\mathcal{K}_\Psi[\rho_0^\varepsilon \mu^\varepsilon](t, \mathbf{x}, v) = \Psi *_r \rho_0^\varepsilon(\mathbf{x}) v - \Psi *_r (\rho_0^\varepsilon \mathcal{V}^\varepsilon)(t, \mathbf{x}),$$

where  $*_r$  is a shorthand for the convolution on the right side of any function  $g$  with  $\Psi$

$$\Psi *_r g(\mathbf{x}) = \int_K \Psi(\mathbf{x}, \mathbf{x}') g(\mathbf{x}') d\mathbf{x}'.$$

1.2.1. *Limit of  $\mu^\varepsilon$  as  $\varepsilon$  vanishes.* Coming back to the analysis of the strong interaction regime, we look for the leading order in (1.2). In our case, it is induced by strong short range interactions between neurons, and as  $\varepsilon \rightarrow 0$ , we expect

$$(v - \mathcal{V}^\varepsilon) \mu^\varepsilon = 0,$$

which means that  $\mu^\varepsilon$  concentrates towards a Dirac mass centred in  $\mathcal{V}^\varepsilon$  with respect to the  $v$ -variable. Indeed, multiplying equation (1.2) by  $|v - \mathcal{V}^\varepsilon|^2$  and integrating with respect to  $\mathbf{u} \in \mathbb{R}^2$ , we obtain

$$\mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) \underset{\varepsilon \rightarrow 0}{\sim} \delta_0(v - \mathcal{V}^\varepsilon(t, \mathbf{x})) \otimes \bar{\mu}^\varepsilon(t, \mathbf{x}, w),$$

where  $\mathcal{V}^\varepsilon$  is given by (1.1) and  $\bar{\mu}^\varepsilon$  is defined as the marginal of  $\mu^\varepsilon$  with respect to the voltage variable

$$\bar{\mu}^\varepsilon(t, \mathbf{x}, w) = \int_{\mathbb{R}} \mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) dv.$$

Multiplying equation (1.2) by  $v$  (resp. 1) and then integrating over  $\mathbf{u} \in \mathbb{R}^2$  (resp.  $v \in \mathbb{R}$ ), we obtain that the couple  $(\mathcal{V}^\varepsilon, \bar{\mu}^\varepsilon)$  solves the following system

$$(1.3) \quad \begin{cases} \partial_t \mathcal{V}^\varepsilon = N(\mathcal{V}^\varepsilon) - \mathcal{W}^\varepsilon - \mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon] + \mathcal{E}(\mu^\varepsilon), \\ \partial_t \bar{\mu}^\varepsilon + \partial_w \left( a \int_{\mathbb{R}} v \mu^\varepsilon dv - b w \bar{\mu}^\varepsilon + c \bar{\mu}^\varepsilon \right) = 0. \end{cases}$$

In equation (1.3),  $\mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon]$  is a non local operator given by

$$\mathcal{L}_{\rho_0^\varepsilon}[\mathcal{V}^\varepsilon] = \mathcal{V}^\varepsilon \Psi *_r \rho_0^\varepsilon - \Psi *_r (\rho_0^\varepsilon \mathcal{V}^\varepsilon),$$

and the error term  $\mathcal{E}(\mu^\varepsilon)$  is given by

$$(1.4) \quad \mathcal{E}(\mu^\varepsilon(t, \mathbf{x}, \cdot)) = \int_{\mathbb{R}^2} N(v) \mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u} - N(\mathcal{V}^\varepsilon).$$

Both equations on  $\mathcal{V}^\varepsilon$  and  $\bar{\mu}^\varepsilon$  in (1.3) depend on the distribution function  $\mu^\varepsilon$ . However, our interest lies in the regime of strong interactions, therefore we replace  $\mu^\varepsilon$  in (1.3) by the *ansatz*

$$\delta_{\mathcal{V}^\varepsilon} \otimes \bar{\mu}^\varepsilon,$$

and deduce that as  $\varepsilon$  vanishes,  $(\mathcal{V}^\varepsilon, \bar{\mu}^\varepsilon)$  converges to the couple  $(\mathcal{V}, \bar{\mu})$ , which solves

$$(1.5) \quad \begin{cases} \partial_t \mathcal{V} = N(\mathcal{V}) - \mathcal{W} - \mathcal{L}_{\rho_0}[\mathcal{V}], \\ \partial_t \bar{\mu} + \partial_w (A(\mathcal{V}, w) \bar{\mu}) = 0, \end{cases}$$

with

$$\mathcal{W} = \int_{\mathbb{R}} w \bar{\mu}(t, \mathbf{x}, w) dw.$$

These considerations lead to

$$\mu^\varepsilon \underset{\varepsilon \rightarrow 0}{\longrightarrow} \delta_{\mathcal{V}} \otimes \bar{\mu},$$

which may be interpreted as the expansion at order 0 of  $\mu^\varepsilon$  in the regime of strong interactions.

Before going further into our analysis, we emphasize that the latter result was studied in a deterministic setting in [13] using relative entropy methods. More generally, this result is related to other concentration phenomena occurring in various kinetic models. We mention [23], where the

authors go through the asymptotic analysis of a collisionless and non-diffusive Vlasov-type equation undergoing strong local alignment forces, and also [19] and [16], where the authors study the large time behavior of a Vlasov-Navier-Stokes system respectively in bounded (with periodic boundary conditions) and unbounded domain.

1.2.2. *First correction in the expansion of  $\mu^\varepsilon$ .* In this article, we refine the latter result by investigating the concentration profile of the solution  $\mu^\varepsilon$  in the regime of strong interactions, in order to improve the convergence rates. The strategy consists in considering the following re-scaled version  $\nu^\varepsilon$  of  $\mu^\varepsilon$

$$\mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) = \frac{1}{\theta^\varepsilon} \nu^\varepsilon \left( t, \mathbf{x}, \frac{v - \mathcal{V}^\varepsilon}{\theta^\varepsilon}, w - \mathcal{W}^\varepsilon \right),$$

where  $\theta^\varepsilon$  shall be interpreted as the concentration rate of  $\mu^\varepsilon$  around its mean value  $\mathcal{V}^\varepsilon$ . For a proper choice of  $\theta^\varepsilon$ , we expect  $\nu^\varepsilon$  to converge to some limit as  $\varepsilon$  vanishes.

The first step consists in determining the proper concentration rate  $\theta^\varepsilon$ . To this aim, we derive the equation solved by  $\nu^\varepsilon$ , which is obtained performing the following change of variable

$$(1.6) \quad (t, v, w) \mapsto \left( t, \frac{v - \mathcal{V}^\varepsilon}{\theta^\varepsilon}, w - \mathcal{W}^\varepsilon \right)$$

in equation (1.2) and using the first equation of (1.3) on  $\mathcal{V}^\varepsilon$ . It yields

$$\partial_t \nu^\varepsilon + \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon \nu^\varepsilon] = \frac{1}{(\theta^\varepsilon)^2} \partial_v \left[ \left( (\theta^\varepsilon)' \theta^\varepsilon + \frac{\rho_0^\varepsilon}{\varepsilon} (\theta^\varepsilon)^2 \right) v \nu^\varepsilon + \partial_v \nu^\varepsilon \right],$$

where  $\mathbf{b}_0^\varepsilon$  is a centered version of  $\mathbf{b}^\varepsilon$  and is given by

$$(1.7) \quad \mathbf{b}_0^\varepsilon(t, \mathbf{x}, \mathbf{u}) = \begin{pmatrix} (\theta^\varepsilon)^{-1} B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, w) \\ A_0(\theta^\varepsilon v, w) \end{pmatrix},$$

where  $B_0^\varepsilon$  is defined as

$$B_0^\varepsilon(t, \mathbf{x}, \mathbf{u}) = N(\mathcal{V}^\varepsilon + v) - N(\mathcal{V}^\varepsilon) - w - v \Psi *_r \rho_0^\varepsilon(\mathbf{x}) - \mathcal{E}(\mu^\varepsilon),$$

and where  $A_0$  is the linear version of  $A$

$$A_0(\mathbf{u}) = A(\mathbf{u}) - A(\mathbf{0}).$$

A natural choice for  $\theta^\varepsilon$  would be  $\theta^\varepsilon = \sqrt{\varepsilon}$ , since it leads to the following equation

$$\partial_t \nu^\varepsilon + \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon \nu^\varepsilon] = \frac{1}{\varepsilon} \mathcal{F}_{\rho_0^\varepsilon}[\nu^\varepsilon],$$

where  $\mathbf{b}_0^\varepsilon$  is given by (1.7) with  $\theta^\varepsilon = \sqrt{\varepsilon}$  and where the Fokker-Planck operator is defined as

$$\mathcal{F}_{\rho_0^\varepsilon}[\nu^\varepsilon] = \partial_v [\rho_0^\varepsilon v \nu^\varepsilon + \partial_v \nu^\varepsilon].$$

Keeping only the leading order, it yields

$$\nu^\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon,$$

where the Maxwellian  $\mathcal{M}_{\rho_0^\varepsilon}$  is defined as

$$\mathcal{M}_{\rho_0^\varepsilon(\mathbf{x})}(v) = \sqrt{\frac{\rho_0^\varepsilon(\mathbf{x})}{2\pi}} \exp\left(-\rho_0^\varepsilon(\mathbf{x}) \frac{|v|^2}{2}\right),$$

whereas  $\bar{\nu}^\varepsilon$  is the marginal of  $\nu^\varepsilon$  with respect to the re-scaled adaptation variable

$$\bar{\nu}^\varepsilon(t, \mathbf{x}, w) = \int_{\mathbb{R}} \nu^\varepsilon(t, \mathbf{x}, \mathbf{u}) dv,$$

and solves the following equation, obtained after integrating equation (1.10) with respect to  $v$

$$(1.8) \quad \partial_t \bar{\nu}^\varepsilon - b \partial_w (w \bar{\nu}^\varepsilon) = -a \theta^\varepsilon \partial_w \int_{\mathbb{R}} v \nu^\varepsilon(t, \mathbf{x}, \mathbf{u}) dv.$$

Once again, the dependence with respect to  $\nu^\varepsilon$  is removed from the latter equation when we replace  $\nu^\varepsilon$  with the *ansatz*

$$\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon.$$

Therefore,  $\bar{\nu}^\varepsilon$  converges to  $\bar{\nu}$ , where  $\bar{\nu}$  solves the following linear transport equation

$$(1.9) \quad \partial_t \bar{\nu} - b \partial_w (w \bar{\nu}) = 0,$$

which corresponds to the same equation as (1.5) for  $\bar{\mu}$  after inverting the change of variable (1.6).

The latter approach is the object of [2], where weak convergence results are obtained in probability spaces, as well as [29], which follows a method coming from the analysis of Hamilton-Jacobi equations. In the present article, our purpose consists in deriving quantitative convergence results in strong topology. There are two major difficulties we shall overcome in order to achieve this goal

- (1) considering the concentration rate  $\theta^\varepsilon = \sqrt{\varepsilon}$  in the scaling (1.6) has one major drawback: since the norms associated to strong topology are usually not scaling invariant, this choice comes down to considering well-prepared initial conditions. We seek for a stronger result which also **applies for ill-prepared initial conditions**. Therefore, we need to find a scaling which suits our problem.
- (2) The choice of functional framework is crucial in our analysis: it should be consistent on a modeling point of view and it should also enable to propagate regularity along the trajectories of equation (1.10) in order to recover the optimal convergence rates.

1.2.3. *Derivation of the appropriate concentration rate  $\theta^\varepsilon$ .* We mentioned that a significant difficulty in our analysis stems from the fact that the natural scaling  $\theta^\varepsilon = \sqrt{\varepsilon}$  in (1.6) comes down to considering well-prepared initial conditions. To overcome this difficulty, our strategy consists in adding the following constraint to the concentration rate

$$\theta^\varepsilon(t=0) = 1.$$

The equation obtained on  $\nu^\varepsilon$  after operating the change of variable (1.6) reads as follows

$$\partial_t \nu^\varepsilon + \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon \nu^\varepsilon] = \frac{1}{(\theta^\varepsilon)^2} \partial_v \left[ \left( \frac{1}{2} \frac{d}{dt} |\theta^\varepsilon|^2 + \frac{\rho_0^\varepsilon}{\varepsilon} |\theta^\varepsilon|^2 \right) v \nu^\varepsilon + \partial_v \nu^\varepsilon \right].$$

Since the concentration profile is independent of the initial condition  $\mu_0^\varepsilon$ , we shall still observe concentration with Gaussian profile  $\mathcal{M}_{\rho_0^\varepsilon}$  as  $\varepsilon$  vanishes. Therefore, considering the stiffer term in the latter equation, it turns out that  $\theta^\varepsilon$  should verify

$$\frac{1}{2} \frac{d}{dt} |\theta^\varepsilon|^2 + \frac{\rho_0^\varepsilon}{\varepsilon} |\theta^\varepsilon|^2 = \rho_0^\varepsilon.$$

With this choice, the equation on  $\nu^\varepsilon$  rewrites:

$$(1.10) \quad \partial_t \nu^\varepsilon + \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon \nu^\varepsilon] = \frac{1}{(\theta^\varepsilon)^2} \mathcal{F}_{\rho_0^\varepsilon}[\nu^\varepsilon],$$

where  $\mathbf{b}_0^\varepsilon$  is given by (1.7) and where  $\theta^\varepsilon$  solves the following differential equation

$$\begin{cases} \frac{1}{2} \frac{d}{dt} |\theta^\varepsilon|^2 + \frac{\rho_0^\varepsilon}{\varepsilon} |\theta^\varepsilon|^2 = \rho_0^\varepsilon, \\ \theta^\varepsilon(t=0) = 1, \end{cases}$$

whose solution is given by the following explicit formula

$$(1.11) \quad \theta^\varepsilon(t, \mathbf{x})^2 = \varepsilon (1 - \exp(-(2\rho_0^\varepsilon(\mathbf{x})t)/\varepsilon)) + \exp(-(2\rho_0^\varepsilon(\mathbf{x})t)/\varepsilon).$$

In the end, we deduce the following result

$$(1.12) \quad \nu^\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \nu := \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu},$$

which after inverting the change of variable (1.6) yields

$$(1.13) \quad \mu^\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \mathcal{M}_{\rho_0 |\theta^\varepsilon|^{-2}} (v - \mathcal{V}) \otimes \bar{\mu}.$$

This result may be interpreted as the expansion of  $\mu^\varepsilon$  up to the first corrective term. We can now answer our initial question:  $\mu^\varepsilon$  concentrates around  $\mathcal{V}$  with respect to the  $v$ -variable with Gaussian profile  $\mathcal{M}_{\rho_0}$  and concentration rate  $\theta^\varepsilon$ , which is of order  $\sqrt{\varepsilon}$ , up to an exponentially decaying correction due to ill-preparedness of initial conditions.

1.2.4. *Appropriate functional framework.* In [2], we proved that (1.13) holds up to an error of order  $\varepsilon$  in the sense of weak convergence in some probability space. The analysis is based on an analytic coupling method to estimate Wasserstein distances. To be mentioned that  $O(\varepsilon)$  is, at least formally, the optimal convergence rate in our setting (see [2] for more detail). In the present article, we aim at providing **strong convergence estimates** for (1.13). We present two separate results associated two different functional frameworks.

On the one hand, in section 3, we derive convergence estimates in a  $L^1$  setting, which is the natural space to consider for such type of conservative problem. Our approach follows similar arguments as the ones developed to study various kinetic models. We mention [15] and [20], which focus on the diffusive limit for the Vlasov-Poisson-Fokker-Planck model for (respectively) single species and multi-species. The main difference being that, in our setting, we obtain explicit convergence estimates since we do not make use of any compactness argument.

On the other hand, in section 4, we provide convergence estimates in a weighted  $L^2$  setting which, although less natural on a modeling point of view, allows us to recover the optimal convergence rates obtained in [2] and in which we are able to derive pointwise convergence estimates with respect to time. This analysis is in line with [26], which focuses on the regime of weak interactions between neurons (this corresponds to the asymptotic  $\varepsilon \rightarrow +\infty$  in equation (1.2)), and in which the analysis is performed in a similar setting. We also mention [21], which makes use of a similar functional setting to study the large time behaviour for the Vlasov-Poisson-Fokker-Planck equation in various regimes.

We start this article by Section 2, where we provide the assumptions on the data of the problem:  $N$ ,  $\Psi$ ,  $\mu_0^\varepsilon$  and where we present preliminary results concerning existence and uniqueness for equation (1.2) and convergence of the macroscopic quantities  $\mathcal{V}^\varepsilon$  and  $\mathcal{W}^\varepsilon$  in the regime of strong interactions. We also define a modified relative entropy and justify why our analysis crucially relies on this quantity.

## 2. PRELIMINARIES

2.1. **Mathematical setting.** In this section, we give our set of assumptions on the data of our problem:  $N$ ,  $\Psi$ , and  $\mu_0^\varepsilon$ .

We suppose the drift  $N$  to be of class  $\mathcal{C}^2$  over  $\mathbb{R}$ . Then we set  $\omega(v) = N(v)/v$  and suppose that the following coupled pair of confining assumptions are met

$$(2.1a) \quad \left\{ \begin{array}{l} \limsup_{|v| \rightarrow +\infty} \omega(v) = -\infty, \\ \sup_{|v| \geq 1} \frac{|\omega(v)|}{|v|^{p-1}} < +\infty, \end{array} \right.$$

for some  $p \geq 2$ . A typical choice for  $N$  would be any polynomial of the form

$$P(v) = Q(v) - C v |v|^{p-1},$$

for some positive constant  $C > 0$  and where  $Q$  has degree less than  $p$ .



We turn to the connectivity kernel  $\Psi$ . We suppose  $\Psi \in \mathcal{C}^0(K_{\mathbf{x}}, L^1(K_{\mathbf{x}'}))$  and assume the following bound to hold

$$(2.2) \quad \sup_{\mathbf{x}' \in K} \int_K |\Psi(\mathbf{x}, \mathbf{x}')| \, d\mathbf{x} < +\infty.$$

Moreover, for some  $r > 1$  we define its conjugate  $r' \geq 1$  as  $1/r + 1/r' = 1$  and we suppose

$$(2.3) \quad \sup_{\mathbf{x} \in K} \int_K |\Psi(\mathbf{x}, \mathbf{x}')|^{r'} \, d\mathbf{x}' < +\infty.$$

Our set of assumptions on the connectivity kernel is quite general. On the one hand, we consider non-symmetric interactions between neurons. On the other hand, we authorize the connectivity kernel to follow a power law, a case which is considered in the physical literature (see [24]).

We now state our assumptions on the sequence of initial data  $(\mu_{0,\mathbf{x}}^\varepsilon)_{\varepsilon > 0}$ . We suppose for each  $\varepsilon > 0$

$$(2.4) \quad \mu_0^\varepsilon \in \mathcal{C}^0(K, L^1(\mathbb{R}^2)), \quad \mu^\varepsilon \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \mu_0^\varepsilon(\mathbf{x}, \mathbf{u}) \, d\mathbf{u} = 1, \quad \forall \mathbf{x} \in K.$$

We also suppose the spatial distribution of the network  $\rho_0^\varepsilon$  to be continuous over  $K$  and uniformly bounded from above and below, that is, there exists a positive constant  $m_*$  such that for all  $\varepsilon > 0$

$$(2.5) \quad \rho_0^\varepsilon \in \mathcal{C}^0(K) \quad \text{and} \quad m_* \leq \rho_0^\varepsilon \leq 1/m_*.$$

Furthermore, we assume the following condition: there exists two positive constants  $m_p$  and  $\bar{m}_p$ , independent of  $\varepsilon$ , such that

$$(2.6) \quad \sup_{\mathbf{x} \in K} \int_{\mathbb{R}^2} |\mathbf{u}|^{2p} \mu_0^\varepsilon(\mathbf{x}, \mathbf{u}) \, d\mathbf{u} \leq m_p,$$

and such that

$$(2.7) \quad \int_{K \times \mathbb{R}^2} |\mathbf{u}|^{2pr'} \rho_0^\varepsilon(\mathbf{x}) \mu_0^\varepsilon(\mathbf{x}, \mathbf{u}) \, d\mathbf{u} \, d\mathbf{x} \leq \bar{m}_p,$$

where  $p$  and  $r'$  are given in (2.1b) and (2.3).

**2.2. Modified relative entropy.** Our analysis crucially relies on the following modified relative entropy  $H_\alpha[\mu|\nu]$ , given for any  $\alpha$  in  $(0, 1)$  as

$$(2.8) \quad H_\alpha[\mu|\nu] = \int_{\mathbb{R}^2} \mu \ln \left( \frac{\mu}{\alpha\mu + (1-\alpha)\nu} \right) \, d\mathbf{u},$$

for two non-negative functions  $\mu, \nu \in L^1(\mathbb{R}^2)$  with mass equal to one. In our setting,  $H_\alpha$  has several technical advantages in comparison to the  $L^1$  norm or the usual relative entropy. On the one hand, the latter modified relative entropy, where the denominator is a convex combination of  $\mu$  and  $\nu$  is well defined for all positive functions  $\mu$  and  $\nu$  with mass 1, with no further condition on the domain on which they vanish. Indeed, it holds

$$H_\alpha[\mu|\nu] \leq -\ln(\alpha) < +\infty.$$

We even prove the following stronger result (see [2], Lemma A.1. for details)

**Lemma 2.1.** *For any two non-negative functions  $\mu, \nu \in L^1(\mathbb{R}^2)$  with integral equal to one, the following estimate holds*

$$\frac{(1-\alpha)^2}{2} \|\mu - \nu\|_{L^1(\mathbb{R}^2)}^2 \leq H_\alpha[\mu|\nu] \leq \frac{1-\alpha}{\alpha} \|\mu - \nu\|_{L^1(\mathbb{R}^2)}.$$

On the other hand, relative entropy makes explicit the dissipation due to the Laplace operator, the Fisher information, defined by

$$(2.9) \quad I_\alpha[\mu|\nu] := \int_{\mathbb{R}^2} \left| \partial_v \ln \left( \frac{\mu}{(1-\alpha)\nu + \alpha\mu} \right) \right|^2 \mu \, d\mathbf{u}.$$

Therefore, it enables to derive  $L^1$  estimates in the context of non-linear degenerated parabolic PDEs. We illustrate this idea in the following abstract Lemma, which of course, is stated such that it fits the present problem at best, but which might have extensions in other contexts

**Lemma 2.2.** *Consider  $\delta$  in  $\{0, 1\}$  and smooth solutions  $f$  and  $g$  to the following equations*

$$\begin{cases} \partial_t f + \operatorname{div}_{\mathbf{y}} [\mathbf{a}(t, \mathbf{y}, \xi) f] + \lambda \operatorname{div}_{\xi} [(\mathbf{b}_1 + \mathbf{b}_3)(t, \mathbf{y}, \xi) f] = \lambda^2 \Delta_{\xi} f, \\ \partial_t g + \operatorname{div}_{\mathbf{y}} [\mathbf{a}(t, \mathbf{y}, \xi) g] + \lambda \operatorname{div}_{\xi} [\mathbf{b}_2(t, \mathbf{y}, \xi) g] = \delta \lambda^2 \Delta_{\xi} g. \end{cases}$$

where  $(\mathbf{y}, \xi) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ , with  $d_1 \geq 0$  and  $d_2 \geq 1$ , and where

$$\left( \mathbf{a} : \mathbb{R} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}^{d_1} \right) \text{ and } \left( \mathbf{b}_i : \mathbb{R} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}^{d_2} \right), \quad i \in \{1, 2, 3\},$$

are given vector fields. For all positive  $\lambda$ , it holds

$$(2.10) \quad \frac{d}{dt} H_{1/2}[f(t)|g(t)] \leq \mathcal{R}(t), \quad \forall t \in \mathbb{R}^+,$$

where  $\mathcal{R}$  is defined as

$$\mathcal{R}(t) = \int_{\mathbb{R}^{d_1+d_2}} \left( \frac{1}{4} |\mathbf{b}_1 - \mathbf{b}_2|^2 f + \lambda |\operatorname{div}_{\xi} [\mathbf{b}_3 g + (\delta - 1) \lambda \nabla_{\xi} g]| \right) (t, \mathbf{y}, \xi) d\mathbf{y} d\xi.$$

In particular, for all positive  $\lambda$ , it holds

$$(2.11) \quad \|f(t) - g(t)\|_{L^1(\mathbb{R}^{d_1+d_2})} \leq 2\sqrt{2} \left( \|f_0 - g_0\|_{L^1(\mathbb{R}^{d_1+d_2})}^{\frac{1}{2}} + \left( \int_0^t \mathcal{R}(s) ds \right)^{\frac{1}{2}} \right), \quad \forall t \in \mathbb{R}^+.$$

*Proof.* We consider the quantity  $h := (f + g)/2$ , whose equation is obtained multiplying by  $1/2$  the sum of the equations solved by  $f$  and  $g$ , that is

$$\partial_t h + \operatorname{div}_{\mathbf{y}} [\mathbf{a} h] + \frac{\lambda}{2} \operatorname{div}_{\xi} [(\mathbf{b}_1 + \mathbf{b}_3) f + \mathbf{b}_2 g] - \lambda^2 \Delta_{\xi} h = (\delta - 1) \frac{\lambda^2}{2} \Delta_{\xi} g.$$

Then, we compute the time derivative of the quantity  $H_{1/2}[f|g]$  integrating with respect to both  $\xi$  and  $\mathbf{y}$  the difference between the equation solved by  $f$  multiplied by  $\ln(f/h)$  and the equation solved by  $h$  multiplied by  $f/h$ . After an integration by part, it yields

$$\frac{d}{dt} H_{1/2}[f|g] + \lambda^2 I_{1/2}[f|g] = \mathcal{A} + \mathcal{B},$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are given by

$$\begin{cases} \mathcal{A} = \frac{\lambda}{2} \int_{\mathbb{R}^{d_1+d_2}} (\mathbf{b}_1 - \mathbf{b}_2) \nabla_{\xi} \left( \ln \left( \frac{f}{h} \right) \right) \frac{g}{h} f d\mathbf{y} d\xi, \\ \mathcal{B} = -\frac{\lambda}{2} \int_{\mathbb{R}^{d_1+d_2}} \operatorname{div}_{\xi} [\mathbf{b}_3 g + (\delta - 1) \lambda \nabla_{\xi} g] \frac{f}{h} d\mathbf{y} d\xi. \end{cases}$$

To estimate  $\mathcal{A}$ , we notice that  $|g/h| \leq 2$  and we apply Young's inequality. This yields

$$\mathcal{A} \leq \lambda^2 I_{1/2}[f|g] + \frac{1}{4} \int_{\mathbb{R}^{d_1+d_2}} |\mathbf{b}_1 - \mathbf{b}_2|^2 f d\mathbf{y} d\xi.$$

To estimate  $\mathcal{B}$ , we simply notice that  $|f/h| \leq 2$  and take the absolute value inside the integral. In the end, we obtain

$$\mathcal{A} + \mathcal{B} \leq \mathcal{R}(t) + \lambda^2 I_{1/2}[f|g],$$

where  $\mathcal{R}$  is defined as in Lemma 2.2. Hence it yields the first result (2.10). Then, we deduce (2.11) by integrating (2.10) between 0 and  $t$  and applying Lemma 2.1.  $\square$

**2.3. Existence and uniqueness for equation (1.2).** We first define the notion of solution we consider for equation (1.2).

**Definition 2.3.** For all  $\varepsilon > 0$  we say that  $\mu^\varepsilon$  solves (1.2) with initial condition  $\mu_0^\varepsilon$  if we have

(1)  $\mu^\varepsilon$  lies in

$$\mathcal{C}^0(\mathbb{R}^+ \times K, L^1(\mathbb{R}^2)),$$

(2) for all  $\mathbf{x} \in K, t \geq 0$ , and  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ , it holds

$$\begin{aligned} \int_{\mathbb{R}^2} \varphi(\mathbf{u}) (\mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) - \mu_0^\varepsilon(\mathbf{x}, \mathbf{u})) d\mathbf{u} &= \int_0^t \int_{\mathbb{R}^2} [(\nabla_{\mathbf{u}} \varphi \cdot \mathbf{b}^\varepsilon + \partial_v^2 \varphi) \mu^\varepsilon](s, \mathbf{x}, \mathbf{u}) d\mathbf{u} ds \\ &\quad - \frac{\rho_0^\varepsilon(\mathbf{x})}{\varepsilon} \int_0^t \int_{\mathbb{R}^2} [\partial_v \varphi (v - \mathcal{V}^\varepsilon) \mu^\varepsilon](s, \mathbf{x}, \mathbf{u}) d\mathbf{u} ds, \end{aligned}$$

where  $\mathcal{V}^\varepsilon$  and  $\mathbf{b}^\varepsilon$  are given by (1.1) and (1.2).

With this notion of solution, equation (1.2) is well-posed. Indeed, according to [2] (see Theorem 2.3), it holds

**Theorem 2.4.** For any  $\varepsilon > 0$ , suppose that assumptions (2.1a)-(2.1b) on  $N$ , (2.2)-(2.3) on  $\Psi$  and (2.4)-(2.5) on the initial condition are fulfilled and that  $\mu_0^\varepsilon$  also verifies

$$(2.12) \quad \begin{cases} \sup_{\mathbf{x} \in K} \int_{\mathbb{R}^2} e^{|\mathbf{u}|^2/2} \mu_0^\varepsilon(\mathbf{x}, \mathbf{u}) d\mathbf{u} \leq M^\varepsilon, \\ \sup_{\mathbf{x} \in K} \int_{\mathbb{R}^2} \ln[\mu_0^\varepsilon(\mathbf{x}, \mathbf{u})] \mu_0^\varepsilon(\mathbf{x}, \mathbf{u}) d\mathbf{u} < +\infty, \end{cases}$$

and

$$(2.13) \quad \sup_{\mathbf{x} \in K} \left\| \nabla_{\mathbf{u}} \sqrt{\mu_0^\varepsilon} \right\|_{L^2(\mathbb{R}^2)}^2 \leq m^\varepsilon,$$

where  $M^\varepsilon$  and  $m^\varepsilon$  are two positive constant. Then there exists a unique solution  $\mu^\varepsilon$  to equation (1.2) with initial condition  $\mu_0^\varepsilon$ , in the sense of Definition 2.3 which verifies

$$\sup_{(t, \mathbf{x}) \in [0, T] \times K} \int_{\mathbb{R}^2} e^{|\mathbf{u}|^2/2} \mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u} < +\infty,$$

for all  $T \geq 0$ .

Furthermore, the macroscopic quantities  $\mathcal{V}^\varepsilon$  and  $\mathcal{W}^\varepsilon$  given in (1.1) lie in  $\mathcal{C}^0(\mathbb{R}^+ \times K)$ .

We mention that to prove this result, we extensively make use of the modified relative entropy defined in section 2.2.

We consider the following solutions for the limiting system (1.5)

**Definition 2.5.** We say that  $(\mathcal{V}, \bar{\mu})$  solves (1.5) with initial condition  $(\mathcal{V}_0, \bar{\mu}_0)$  if we have

- (1)  $\mathcal{V} \in \mathcal{C}^0(\mathbb{R}^+ \times K)$  and  $\bar{\mu} \in \mathcal{C}^0(\mathbb{R}^+ \times K, L^1(\mathbb{R}))$ ,
- (2)  $\mathcal{V}$  is a mild solution to (1.5),
- (3) for all  $t \geq 0$ , all  $\mathbf{x} \in K$  and all  $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$  we have

$$\int_{\mathbb{R}} \phi \bar{\mu}(t, \mathbf{x}, w) dw = \int_{\mathbb{R}} \phi \bar{\mu}_0(\mathbf{x}, w) dw + \int_0^t \int_{\mathbb{R}} A(\mathcal{V}, w) \partial_w \phi(w) \bar{\mu}(s, \mathbf{x}, w) dw ds.$$

With this notion of solution, we prove the following well-posedness result (see [2] Theorem 2.6)

**Theorem 2.6.** Under assumptions (2.1a)-(2.1b) on  $N$ , (2.2)-(2.3) on  $\Psi$  and (2.5) on  $\rho_0$ , and for any initial condition

$$(\mathcal{V}_0, \bar{\mu}_0) \in \mathcal{C}^0(K) \times \mathcal{C}^0(K, L^1(\mathbb{R})),$$

there exists a unique solution to (1.5) in the sense of Definition 2.5 with initial condition  $(\mathcal{V}_0, \bar{\mu}_0)$ . Furthermore,  $\mathcal{V}$  and  $\mathcal{W}$  are uniformly bounded over  $\mathbb{R}^+ \times K$ , where  $\mathcal{W}$  is given by

$$\mathcal{W}((t, \mathbf{x})) = \int_{\mathbb{R}} w \bar{\mu}(t, \mathbf{x}, w) dw.$$

**2.4. Moment estimates and convergence of the macroscopic model.** In the next result, we provide convergence estimates for the macroscopic quantities  $(\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$  defined by (1.1) and the error term  $\mathcal{E}(\mu^\varepsilon)$  defined by (1.4). We also provide uniform estimates with respect to  $\varepsilon$  for the moments of  $\mu^\varepsilon$  and for the relative energy given by

$$\begin{cases} M_q[\mu^\varepsilon](t, \mathbf{x}) := \int_{\mathbb{R}^2} |\mathbf{u}|^q \mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u}, \\ D_q[\mu^\varepsilon](t, \mathbf{x}) := \int_{\mathbb{R}^2} |v - \mathcal{V}^\varepsilon(t, \mathbf{x})|^q \mu^\varepsilon(t, \mathbf{x}, \mathbf{u}) d\mathbf{u}, \end{cases}$$

where  $q \geq 2$ . These estimates are important in the forthcoming analysis since we use them to bound the non-linear terms in equation (1.10) on  $\nu^\varepsilon$ . Relying on [2], it holds

**Theorem 2.7.** *Under assumptions (2.1a)-(2.1b) on the drift  $N$ , (2.2)-(2.3) on  $\Psi$ , (2.4)-(2.7) on the initial conditions  $\mu_0^\varepsilon$  and under the additional assumptions of Theorem 2.4, consider the solutions  $\mu^\varepsilon$  and  $(\mathcal{V}, \bar{\mu})$  provided by Theorem 2.4 and 2.6 respectively. Furthermore, define the initial macroscopic error as*

$$\mathcal{E}_{\text{mac}} = \|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_{L^\infty(K)} + \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)}.$$

There exists  $(C, \varepsilon_0) \in (\mathbb{R}_*^+)^2$  such that

(1) for all  $\varepsilon \leq \varepsilon_0$ , it holds

$$\|\mathcal{U}(t) - \mathcal{U}^\varepsilon(t)\|_{L^\infty(K)} \leq C \min(e^{Ct}(\mathcal{E}_{\text{mac}} + \varepsilon), 1), \quad \forall t \in \mathbb{R}^+,$$

where  $\mathcal{U}^\varepsilon$  and  $\mathcal{U}$  are respectively given by (1.1) and (1.5).

(2) For all  $\varepsilon > 0$  and all  $q$  in  $[2, 2p]$  it holds

$$M_q[\mu^\varepsilon](t, \mathbf{x}) \leq C, \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K,$$

where exponent  $p$  is given in assumption (2.1b). In particular,  $\mathcal{U}^\varepsilon$  is uniformly bounded with respect to both  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and  $\varepsilon$ .

(3) For all  $\varepsilon > 0$  and all  $q$  in  $[2, 2p]$  it holds

$$D_q[\mu^\varepsilon](t, \mathbf{x}) \leq C \left[ \exp\left(-q m_* \frac{t}{\varepsilon}\right) + \varepsilon^{\frac{q}{2}} \right], \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K.$$

(4) For all  $\varepsilon > 0$  we have

$$|\mathcal{E}(\mu^\varepsilon(t, \mathbf{x}, \cdot))| \leq C \left[ \exp\left(-2 m_* \frac{t}{\varepsilon}\right) + \varepsilon \right], \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K,$$

where  $\mathcal{E}$  is defined by (1.4).

In this theorem, the constant  $C$  only depends on  $m_p, \bar{m}_p, m_*$  (see (2.5)-(2.7)) and on the data of the problem  $N, A_0$  and  $\Psi$ .

The proof of this result can be found in [2]. More precisely, we refer to [2, Proposition 4.4] for the proof of (1), [2, Proposition 3.1] for the proof of (2), [2, Proposition 3.3] for the proof of (3) and [2, Proposition 3.5] for the proof of (4).

Building on these preliminaries, we may now focus on our main goal which consists in providing quantitative convergence estimates for (1.13) in strong topology. We provide two results: the first

one in a  $L^1$  functional framework (see Section 3) and the second one in a weighted  $L^2$  functional framework (see Section 4).

### 3. CONVERGENCE ANALYSIS IN $L^1$

In this section, we derive convergence estimates for the solution  $\mu^\varepsilon$  to equation (1.2) in the functional space

$$L^\infty(K, L^1(\mathbb{R}^2)).$$

To carry out our analysis, we consider the re-scaled version  $\nu^\varepsilon$  solution to (1.10) and prove that it converges towards  $\nu$  defined by (1.12). Let us outline the main steps of our approach on a simplified example

3.0.1. *An illuminating example: diffusive limit for the kinetic Fokker-Planck equation.* Let us give some insight on our approach in the context of a simplified example: the diffusive limit for the kinetic Fokker-Planck equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon = \frac{1}{\varepsilon^2} \nabla_{\mathbf{v}} \cdot (\mathbf{v} f^\varepsilon + \nabla_{\mathbf{v}} f^\varepsilon),$$

where  $(\mathbf{x}, \mathbf{v})$  lie in the phase space  $\mathbb{R}^d \times \mathbb{R}^d$ . In this context, the challenge consists in proving that as  $\varepsilon$  vanishes, it holds

$$f^\varepsilon(t, \mathbf{x}, \mathbf{v}) \underset{\varepsilon \rightarrow 0}{\sim} \mathcal{M}(\mathbf{v}) \otimes \rho(t, \mathbf{x}),$$

where  $\rho$  is a solution to the heat equation

$$\partial_t \rho = \Delta_{\mathbf{x}} \rho,$$

and where  $\mathcal{M}$  stands for the standard Maxwellian.

Relying on a rather classical relative entropy estimate (see Proposition 3.6) it is possible to prove that  $f^\varepsilon$  converges to the following local equilibrium of the Fokker-Planck operator

$$\mathcal{M} \otimes \rho^\varepsilon,$$

where the spatial density of particles  $\rho^\varepsilon$  is defined by

$$\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon d\mathbf{v}.$$

Then, the difficulty lies in proving that the spatial density of particles  $\rho^\varepsilon$  converges to  $\rho$ . The convergence analysis is made intricate by the transport operator, which keeps us from obtaining a closed equation on  $\rho^\varepsilon$

$$\partial_t \rho^\varepsilon + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \mathbf{v} f^\varepsilon d\mathbf{v} = 0.$$

To overcome this difficulty, our strategy consists in considering the following re-scaled quantity

$$\pi^\varepsilon(t, \mathbf{x}) = \int_{\mathbb{R}^d} f^\varepsilon(t, \mathbf{x} - \varepsilon \mathbf{v}, \mathbf{v}) d\mathbf{v}.$$

On this simplified example, the advantage of considering  $\pi^\varepsilon$  instead of  $\rho^\varepsilon$  is straightforward as it turns out that  $\pi^\varepsilon$  is an exact solution of the limiting equation. Indeed, changing variables in the equation on  $f^\varepsilon$  and integrating with respect to  $\mathbf{v}$ , we obtain

$$\partial_t \pi^\varepsilon = \Delta_{\mathbf{x}} \pi^\varepsilon.$$

Therefore, the convergence analysis comes down to proving that  $\pi^\varepsilon$  is close to  $\rho^\varepsilon$ . It is possible to achieve this final step taking advantage of the following estimate

$$\|\rho^\varepsilon - \pi^\varepsilon\|_{L^1(\mathbb{R}^d)} \leq \mathcal{A} + \mathcal{B},$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are defined as follows

$$\begin{cases} \mathcal{A} = \|f^\varepsilon - \mathcal{M} \otimes \rho^\varepsilon\|_{L^1(\mathbb{R}^{2d})} + \|\mathcal{M} \otimes \tau_{-\varepsilon \mathbf{v}} \rho^\varepsilon - \tau_{-\varepsilon \mathbf{v}} f^\varepsilon\|_{L^1(\mathbb{R}^{2d})}, \\ \mathcal{B} = \int_{\mathbb{R}^d} \mathcal{M}(\tilde{\mathbf{v}}) \|f^\varepsilon - \tau_{-\varepsilon \tilde{\mathbf{v}}} f^\varepsilon\|_{L^1(\mathbb{R}^{2d})} d\tilde{\mathbf{v}}, \end{cases}$$

and where  $\tau_{\mathbf{x}_0}$  stands for the translation of parameter  $\mathbf{x}_0$  with respect to the  $\mathbf{x}$ -variable. To estimate  $\mathcal{A}$ , we use the first step, which ensures that  $f^\varepsilon$  is close to  $\mathcal{M} \otimes \rho^\varepsilon$ . Then, to estimate  $\mathcal{B}$ , it is sufficient to prove equicontinuity estimates for  $f^\varepsilon$ , that is

$$\|f^\varepsilon - \tau_{\mathbf{x}_0} f^\varepsilon\|_{L^1(\mathbb{R}^{2d})} \lesssim |\mathbf{x}_0|.$$

In the forthcoming analysis, we adapt the latter arguments to our context.

**3.0.2. Main steps of the analysis.** We come back to our initial concern, which consists in proving that the solution  $\nu^\varepsilon$  to (1.10) converges towards  $\nu$  defined by (1.12). Our approach relies on a rather classical entropy estimate (see Proposition 3.6) to prove that  $\nu^\varepsilon$  converges to the following local equilibrium of the Fokker-Planck operator

$$\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon.$$

However, the analysis becomes more intricate when it comes to the convergence of the marginal  $\bar{\nu}^\varepsilon$ . Indeed, the proof of convergence is made challenging by the source term in equation (1.8), which still depends on  $\nu^\varepsilon$

$$\partial_w \int_{\mathbb{R}} v \nu^\varepsilon(t, \mathbf{x}, \mathbf{u}) dv.$$

**Remark 3.1.** *As already mentioned, this issue is analogous to the difficulty induced by the free transport operator in the context of kinetic theory, when it comes to closing the equations on the macroscopic quantities.*

To overcome this difficulty, we consider a re-scaled version  $g^\varepsilon$  of  $\nu^\varepsilon$  whose marginal  $\bar{g}^\varepsilon$  solves a similar equation to the one solved by  $\bar{\nu}^\varepsilon$  without the latter source term and with an additional diffusion term which enables to propagate the error between  $\bar{g}^\varepsilon$  and  $\bar{\nu}$ . In Section 3.3.2, we prove that the re-scaled marginal  $\bar{g}^\varepsilon$  converges to the limiting marginal  $\bar{\nu}$ . Then we conclude proving that  $\bar{g}^\varepsilon$  is close to  $\bar{\nu}^\varepsilon$ . This final argument relies on a uniform equicontinuity estimate provided in Proposition 3.4. All along the forthcoming analysis, we extensively make use of the modified relative entropy defined in Section 2.2.

In this section, we denote by  $H$  and  $I$  the Boltzmann relative entropy and Fisher information, defined by (2.8) and (2.9) with  $\alpha = 0$ . Furthermore, we denote by  $H[\cdot] = H[\cdot | 1]$  the Boltzmann entropy. Before stating our main result, we introduce the following notations

$$L_x^\infty L_u^1 := L^\infty(K, L^1(\mathbb{R}^2)).$$

In addition, for any given  $w_0$  lying in  $\mathbb{R}$ , we denote by  $\tau_{w_0}$  the translation by  $w_0$  with respect to the  $w$ -variable

$$\tau_{w_0} \nu(t, \mathbf{x}, v, w) = \nu(t, \mathbf{x}, v, w + w_0).$$

**3.1. Main result.** We proceed in two steps: we first state a convergence result on the abstract quantity  $\nu^\varepsilon$ , which solves the re-scaled equation (1.10). Then we interpret the result on our initial problem, that is on the distribution of the network  $\mu^\varepsilon$ , which solves (1.2).

**3.1.1. Convergence of  $\nu^\varepsilon$ .** In the following result, we provide explicit convergence rate for  $\nu^\varepsilon$  towards the asymptotic concentration profile of the neural network's distribution  $\mu^\varepsilon$  in the regime of strong interactions. We prove that the profile of concentration is Gaussian and we also characterize the limiting distribution with respect to the adaptation variable  $w$ .

**Theorem 3.2.** *Under assumptions (2.1a)-(2.1b) on the drift  $N$  and under assumptions (2.2)-(2.3) on the interaction kernel  $\Psi$ , and under the assumptions of Theorems 2.4 and 2.6, consider the sequence of solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) provided by Theorem 2.4 with initial conditions satisfying assumptions (2.4)-(2.7) and consider the solution  $\bar{\nu}$  to equation (1.9) with an initial condition  $\bar{\nu}_0$  such that*

$$(3.1) \quad \bar{\nu}_0 \in L^\infty(K, W^{2,1}(\mathbb{R})), \quad \text{and} \quad (w \bar{\nu}_0) \in L^\infty(K, L^1(\mathbb{R})).$$

*On top of that, suppose that there exists a positive constant  $m_1$  such that it holds*

$$(3.2) \quad \sup_{\varepsilon>0} \|\nu_0^\varepsilon - \tau_{w_0} \nu_0^\varepsilon\|_{L_x^\infty L_u^1} \leq m_1 |w_0|, \quad \forall w_0 \in \mathbb{R},$$

*and suppose that there exists a positive constant  $m_2$  such that*

$$(3.3) \quad \sup_{\varepsilon>0} \|H[\nu_0^\varepsilon]\|_{L^\infty(K)} \leq m_2^2.$$

*Then, there exists a positive constant  $C$  independent of  $\varepsilon$  such that for all  $\varepsilon$  less than 1, it holds*

$$\int_0^t \|\nu^\varepsilon - \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}\|_{L_x^\infty L_u^1}(s) ds \leq 2\sqrt{2}t \mathcal{E}_{\text{ini}} + \sqrt{\varepsilon} \left(6\sqrt{t}m_2 + C e^{2bt}\right), \quad \forall t \in \mathbb{R}^+,$$

*where the initial error  $\mathcal{E}_{\text{ini}}$  is given by*

$$\mathcal{E}_{\text{ini}} = \|\bar{\nu}_0^\varepsilon - \bar{\nu}_0\|_{L_x^\infty L_w^1}^{\frac{1}{2}}.$$

*In particular, under the compatibility assumption*

$$\mathcal{E}_{\text{ini}} \underset{\varepsilon \rightarrow 0}{=} O(\sqrt{\varepsilon}),$$

*it holds*

$$\sup_{t \in \mathbb{R}^+} \left[ e^{-2bt} \int_0^t \|\nu^\varepsilon(s) - \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}(s)\|_{L_x^\infty L_u^1} ds \right] \underset{\varepsilon \rightarrow 0}{=} O(\sqrt{\varepsilon}).$$

*In this result, the constant  $C$  only depends on  $m_1$ ,  $m_*$ ,  $m_p$  and  $\bar{m}_p$  (see assumptions (2.5)-(2.7)) and the data of the problem  $\bar{\nu}_0$ ,  $N$ ,  $\Psi$  and  $A_0$ .*

The proof of this result is divided into two steps. First we prove that  $\nu^\varepsilon$  converges towards the following local equilibrium of the Fokker-Planck operator

$$\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon.$$

This proof relies on a rather classical relative entropy estimate: see Proposition 3.6. The second step consist in proving that the marginal  $\bar{\nu}^\varepsilon$  converges towards  $\bar{\nu}$  using a re-scaled marginal (see Section 3.3.2).

Let us make a few comments on the result itself before going back to the initial problem on  $\mu^\varepsilon$

- (1) We do not consider "well-prepared" initial condition. Indeed, it is not required for  $\nu_0^\varepsilon$  to be close to the Maxwellian with respect to the voltage variable in order for our result to hold true.
- (2) We point out that our set of assumptions on the initial data  $\nu_0^\varepsilon$  is rather weak. Indeed, we only suppose that the entropy (see (3.3)) and the moments (see (2.6)-(2.7)) of the initial condition stay uniformly bounded as  $\varepsilon$  vanishes. Apart from the equicontinuity assumption (3.2), no additional uniform regularity is required on the initial condition  $\nu_0^\varepsilon$ .

- (3) We obtain  $L^1$  in time convergence estimate. This is a consequence of the relative entropy estimate between  $\nu^\varepsilon$  and  $\mathcal{M}_{\rho_0^\varepsilon}$  (see Proposition 3.6), which in our setting, provides  $L^1$  in time convergence estimate for  $\nu^\varepsilon$  towards  $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon$ . This is somehow similar to what is obtained in various classical kinetic models. Let us mention for instance the diffusive limit for the Vlasov-Poisson-Fokker-Planck equation for single species (see [15], Proposition 5.1) or for multi-species (see [20], Proposition 2.1) or the diffusive limit for the Poisson-Boltzmann model (see [25], Lemma 3.1). We mention that pointwise in time convergence estimate is provided in the Section 4 of the present article (see Theorem 4.1).

3.1.2. *Interpretation on the initial problem: asymptotic expansion for  $\mu^\varepsilon$ .* In the following result, we invert the change of variable (1.6) and deduce from Theorem 3.2 an expansion for the solution  $\mu^\varepsilon$  to equation (1.2) in the regime of strong interactions, that is, as  $\varepsilon$  vanishes.

**Theorem 3.3.** *Under assumptions (2.1a)-(2.1b) on the drift  $N$  and under assumptions (2.2)-(2.3) on the interaction kernel  $\Psi$ , and under the assumptions of Theorems 2.4 and 2.6, consider the sequence of solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) provided by Theorem 2.4 with initial conditions satisfying assumptions (2.4)-(2.7) and (3.2)-(3.3) and consider the solution  $(\mathcal{V}, \bar{\mu})$  to equation (1.5) provided by Theorem 2.6 with an initial condition  $\bar{\mu}_0$  which fulfills assumption (3.1). On top of that, we suppose the following compatibility assumption to be fulfilled*

$$(3.4) \quad \|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_{L^\infty(K)} + \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)} + \|\bar{\mu}_0^\varepsilon - \bar{\mu}_0\|_{L_x^\infty L_w^1} \underset{\varepsilon \rightarrow 0}{=} O(\sqrt{\varepsilon}).$$

There exists  $(C, \varepsilon_0) \in (\mathbb{R}_+^*)^2$  such that for all  $\varepsilon$  less than  $\varepsilon_0$ , it holds

$$\int_0^t \|\mu^\varepsilon - \mu\|_{L_x^\infty L_u^1}(s) ds \leq C e^{Ct} \sqrt{\varepsilon}, \quad \forall t \in \mathbb{R}^+,$$

where the limit  $\mu$  is given by

$$\mu = \mathcal{M}_{\rho_0|\theta^\varepsilon|^{-2}}(v - \mathcal{V}) \otimes \bar{\mu}.$$

In this result, the constant  $C$  and  $\varepsilon_0$  only depend on the implicit constant in assumption (3.4), on the constants  $m_1, m_2, m_*, m_p$  and  $\bar{m}_p$  (see assumptions (3.2)-(3.3) and (2.5)-(2.7)) and on the data of the problem  $\bar{\mu}_0, N, \Psi$  and  $A_0$ .

*Proof.* Since the norm  $\|\cdot\|_{L^1(\mathbb{R}^2)}$  is unchanged by the change of variable (1.6), this Theorem is a straightforward consequence of Theorem 3.2 and Theorem 2.7, which ensures the convergence of the macroscopic quantities  $(\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$ . □

Before detailing the proof of Theorem 3.2, we provide some comments on Theorem 3.3.

- (1) In this theorem, the choice of a non-homogeneous in time concentration rate  $\theta^\varepsilon$  plays a **key role**. Indeed, if  $\theta^\varepsilon = 1/\sqrt{\varepsilon}$ , then the following relation holds true after inverting the change of variable (1.6)

$$H[\mu_0^\varepsilon] = H[\nu_0^\varepsilon] - \frac{1}{2} \ln(\varepsilon) \geq -\frac{1}{2} \ln(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} +\infty,$$

which comes down to considering well-prepared initial conditions. Therefore,  $\theta^\varepsilon(t=0)$  needs to stay lower bounded as  $\varepsilon$  vanishes in order for the result to hold true without considering well-prepared initial data.

- (2) We obtain the convergence rate  $O(\sqrt{\varepsilon})$  instead of the optimal convergence rate, which should be  $O(\varepsilon)$  as rigorously proven for weak convergence metrics (see [2], Theorem 2.7). This is due to the fact that we use the Csizár-Kullback inequality to close our  $L^1$  convergence estimates. Therefore it seems quite unlikely to recover the optimal convergence rate in a  $L^1$  setting.



**3.2. A priori estimates.** The main object of this section consists in deriving equicontinuity estimates for the sequence of solutions  $(\nu^\varepsilon)_{\varepsilon>0}$  to equation (1.10). Namely, we prove

**Proposition 3.4.** *Consider a sequence  $(\nu^\varepsilon)_{\varepsilon>0}$  of smooth solutions to equation (1.10) whose initial conditions meet assumption (3.2). There exists a positive constant  $C$  independent of  $\varepsilon$  such that for all  $\varepsilon > 0$ , it holds*

$$\|\nu^\varepsilon(t, \mathbf{x}) - \tau_{w_0} \nu^\varepsilon(t, \mathbf{x})\|_{L^1(\mathbb{R}^2)} \leq C e^{bt} \left( |e^{bt} w_0| + |e^{bt} w_0|^{\frac{1}{2}} \right), \quad \forall (w_0, \mathbf{x}) \in \mathbb{R} \times K.$$

Constant  $C$  is explicitly given by

$$C = \sqrt{\max(8m_1, 1/b)},$$

where  $m_1$  appears in assumption (3.2).

*Proof.* We fix some  $\mathbf{x}$  in  $K$  and some positive  $\varepsilon$ . Then we consider some  $w_0$  in  $\mathbb{R}$  and the following re-scaled version  $f$  of  $\nu^\varepsilon$

$$f(t, w, v) = e^{-bt} \nu^\varepsilon(t, \mathbf{x}, v, e^{-bt} w).$$

We compute the equation solved by  $f$  performing the change of variable

$$(3.5) \quad w \mapsto e^{-bt} w$$

in equation (1.10).  $f$  solves the following equation

$$\partial_t f + \partial_w \left[ e^{bt} A_0(\theta^\varepsilon v, 0) f \right] + \frac{1}{\theta^\varepsilon} \partial_v \left[ B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, e^{-bt} w) f \right] = \frac{1}{|\theta^\varepsilon|^2} \mathcal{F}_{\rho_0^\varepsilon} [f],$$

where  $A_0$  and  $B_0^\varepsilon$  are given by (1.7). On top of that, we define  $g := \tau_{w_0} f$ , which solves the following equation

$$\partial_t g + \partial_w \left[ e^{bt} A_0(\theta^\varepsilon v, 0) g \right] + \frac{1}{\theta^\varepsilon} \partial_v \left[ B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, e^{-bt}(w + w_0)) g \right] = \frac{1}{|\theta^\varepsilon|^2} \mathcal{F}_{\rho_0^\varepsilon} [g].$$

Thanks to the change of variable (3.5), we can apply Lemma 2.2 to  $f$  and  $g$  with the following parameters

$$\begin{cases} (\delta, \lambda) = (1, 1/\theta^\varepsilon), \\ \mathbf{a}(t, w, v) = e^{bt} A_0(\theta^\varepsilon v, 0), \\ \mathbf{b}_1(t, w, v) = B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, e^{-bt} w) - \frac{1}{\theta^\varepsilon} v, \\ (\mathbf{b}_2, \mathbf{b}_3) = (\tau_{w_0} \mathbf{b}_1, 0). \end{cases}$$

According to (2.11) in Lemma 2.2, it holds

$$\|f(t) - g(t)\|_{L^1(\mathbb{R}^2)} \leq 2\sqrt{2} \|f_0 - g_0\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} + \left( \frac{1 - e^{-2bt}}{b} \right)^{\frac{1}{2}} |w_0|, \quad \forall t \in \mathbb{R}^+.$$

Therefore, according to assumption (3.2), we obtain the result after inverting the change of variable (3.5) and taking the supremum over all  $\mathbf{x}$  in  $K$ .  $\square$

We conclude this section providing regularity estimates for the limiting distribution  $\bar{\nu}$  with respect to the adaptation variable, which solves (1.9). The proof for this result is mainly computational since we have an explicit formula for the solutions to equation (1.9).

**Proposition 3.5.** *Consider some  $\bar{\nu}_0$  satisfying assumption (3.1). The solution  $\bar{\nu}$  to equation (1.9) with initial condition  $\bar{\nu}_0$  verifies*

$$\|\bar{\nu}(t)\|_{L^\infty(K, W^{2,1}(\mathbb{R}))} \leq \exp(2bt) \|\bar{\nu}_0\|_{L^\infty(K, W^{2,1}(\mathbb{R}))}, \quad \forall t \in \mathbb{R}^+,$$

and

$$\|w \partial_w \bar{\nu}(t)\|_{L^\infty(K, L^1(\mathbb{R}))} = \|w \partial_w \bar{\nu}_0\|_{L^\infty(K, L^1(\mathbb{R}))}, \quad \forall t \in \mathbb{R}^+.$$

*Proof.* Since  $\bar{\nu}$  solves (1.9), it is given by the following formula

$$\bar{\nu}(t, \mathbf{x}, w) = e^{bt} \bar{\nu}_0(\mathbf{x}, e^{bt} w), \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K.$$

Consequently, we easily obtain the expected result.  $\square$

**3.3. Proof of Theorem 3.2.** The proof is divided in three steps. First, we prove that the solution  $\nu^\varepsilon$  to (1.10) converges to the local equilibrium

$$\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon,$$

by applying a rather classical entropy estimate. Then we prove that an intermediate quantity  $\bar{g}^\varepsilon$  converges to the solution  $\bar{\nu}$  to equation (1.9) (see Proposition 3.7 and the preceding discussion for more details). We prove this result thanks to a relative entropy argument relying on the modified entropy  $H_{1/2}$ . At last, we prove that  $\bar{g}^\varepsilon$  is close to  $\bar{\nu}$  thanks to the equicontinuity estimate provided by Proposition 3.4 and therefore conclude that the marginal  $\bar{\nu}^\varepsilon$  converges  $\bar{\nu}$  (see Proposition 3.8).

**3.3.1. Convergence of  $\nu^\varepsilon$  towards  $\mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon$  : relative entropy estimate.** This step relies on a rather classical relative entropy estimate for the solution  $\nu^\varepsilon$  to equation (1.10): we investigate the time evolution of the quantity  $H[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}]$  along the trajectories of equation (1.10).

**Proposition 3.6.** *Under assumptions (2.1a)-(2.1b) on the drift  $N$  and (2.2)-(2.3) on the interaction kernel  $\Psi$ , consider a sequence of solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) with initial conditions satisfying assumptions (2.4)-(2.7) and the additional assumption (3.3). Then, there exists a positive constant  $C$  independent of  $\varepsilon$  such that for all for all  $\varepsilon$  less than 1, we have*

(1) *for all  $\mathbf{x}$  lying in  $K$ , it holds*

$$\int_{T^\varepsilon}^t H[\nu^\varepsilon(s, \mathbf{x}) | \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon(s, \mathbf{x})] ds \leq 2\varepsilon H[\nu_0^\varepsilon(\mathbf{x})] + C\varepsilon(t+1), \quad \forall \mathbf{x} \in K,$$

where the time  $T^\varepsilon$  is given by

$$(3.6) \quad T^\varepsilon = \frac{\varepsilon}{2m_*} |\ln(\varepsilon)|.$$

(2) *In particular, it holds*

$$\int_0^t \|\nu^\varepsilon(s) - \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon\|_{L_x^\infty L_u^1} ds \leq 2\sqrt{\varepsilon} t m_2 + C\sqrt{\varepsilon}(t+1), \quad \forall t \geq 0,$$

where  $m_2$  is given in assumption (3.3).

In this result, the constant  $C$  only depends on  $m_*$ ,  $m_p$  and  $\bar{m}_p$  (see assumptions (2.5)-(2.7)) and the data of the problem  $N$ ,  $\Psi$  and  $A_0$ .

*Proof.* All along this proof, we choose some  $\mathbf{x}$  lying in  $K$  and we omit the dependence with respect to  $(t, \mathbf{x})$  when the context is clear. We compute the time derivative of  $H[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}]$  multiplying equation (1.10) by  $\ln(\nu^\varepsilon / \mathcal{M}_{\rho_0^\varepsilon})$ . After integrating by part the stiffer term, it yields

$$\frac{d}{dt} H[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] + \frac{1}{|\theta^\varepsilon|^2} I[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] = \mathcal{A},$$

where  $\mathcal{A}$  is given by

$$\mathcal{A} = - \int_{\mathbb{R}^2} \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon \nu^\varepsilon] \ln\left(\frac{\nu^\varepsilon}{\mathcal{M}_{\rho_0^\varepsilon}}\right) d\mathbf{u}.$$

After an integration by part,  $\mathcal{A}$  rewrites as follows

$$\mathcal{A} = \frac{1}{\theta^\varepsilon} \int_{\mathbb{R}^2} B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, w) \partial_v \left[ \ln\left(\frac{\nu^\varepsilon}{\mathcal{M}_{\rho_0^\varepsilon}}\right) \right] \nu^\varepsilon d\mathbf{u} + b,$$

where  $B_0^\varepsilon$  is given by (1.7). According to items (2) and (4) in Theorem 2.7,  $\nu^\varepsilon$  and  $\mathcal{E}(\mu^\varepsilon)$  are uniformly bounded with respect to both  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and  $\varepsilon > 0$ . On top of that, according to assumptions (2.3) and (2.5) on  $\Psi$  and  $\rho_0^\varepsilon$ ,  $\Psi *_{r} \rho_0^\varepsilon$  is uniformly bounded with respect to both  $\mathbf{x} \in K$  and  $\varepsilon > 0$ . Consequently, applying Young's inequality, assumption (2.1b) and since  $N$  is locally Lipschitz, we obtain

$$\mathcal{A} \leq \frac{1}{2|\theta^\varepsilon|^2} I[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] + C \left( 1 + \int_{\mathbb{R}^2} (|\theta^\varepsilon v|^{2p} + w^2) \nu^\varepsilon d\mathbf{u} \right),$$

for some positive constant  $C$  only depending on  $m_*$ ,  $m_p$ ,  $\bar{m}_p$  and the data of the problem:  $N$ ,  $A$  and  $\Psi$ . Then we invert the change of variable (1.6) in the integral in the right-hand side of the latter inequality and apply item (2) in Theorem 2.7. In the end, it yields

$$\mathcal{A} \leq \frac{1}{2|\theta^\varepsilon|^2} I[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] + C.$$

Consequently, we end up with following differential inequality

$$\frac{d}{dt} H[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] + \frac{1}{2|\theta^\varepsilon|^2} I[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] \leq C,$$

which we integrate between 0 and  $t$  to get

$$H[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] + \int_0^t \frac{1}{2|\theta^\varepsilon(s)|^2} I[\nu^\varepsilon(s) | \mathcal{M}_{\rho_0^\varepsilon}] ds \leq H[\nu_0^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}] + Ct.$$

Then we point out that the logarithmic Sobolev inequality for Gaussian measures reads as follows in our context (see [17])

$$2H[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon] \leq I[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon}].$$

We inject this inequality in the former estimate, and multiply it by  $\varepsilon$ . Then, making use of the explicit expression of  $\theta^\varepsilon$  (see (1.11)), we deduce that

$$\frac{1}{2} \leq \frac{\varepsilon}{|\theta^\varepsilon(s)|^2},$$

as long as  $\varepsilon$  is less than 1 and  $s$  is greater than  $T^\varepsilon$ , where  $T^\varepsilon$  is given by (3.6). Consequently, we obtain item (1) in Proposition 3.6. Item (2), is obtained after applying the Csiszar-Kullback inequality

$$\|\nu^\varepsilon - \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon\|_{L^1(\mathbb{R}^2)}^2 \leq 2H[\nu^\varepsilon | \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon],$$

taking the supremum over all  $\mathbf{x}$  in  $K$ , and noticing that since equation (1.10) is conservative, it holds

$$\int_0^{T^\varepsilon} \|\nu^\varepsilon(s) - \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon\|_{L^\infty L_u^1} ds \leq 2T^\varepsilon \leq C\sqrt{\varepsilon}.$$

□

3.3.2. *Convergence of  $\bar{g}^\varepsilon$  towards  $\bar{\nu}$ .* We mentioned that proving that  $\bar{\nu}^\varepsilon$  converges is intricate because of the source term in equation (1.8)

$$\partial_w \int_{\mathbb{R}} v \nu^\varepsilon(t, \mathbf{x}, \mathbf{u}) dv.$$

To overcome this difficulty, we remove this term by considering the following re-scaled version  $g^\varepsilon$  of  $\nu^\varepsilon$

$$\nu^\varepsilon(t, \mathbf{x}, v, w) = g^\varepsilon(t, \mathbf{x}, v, w + \gamma^\varepsilon(t, \mathbf{x})v),$$

where  $\gamma^\varepsilon$  is given by

$$\gamma^\varepsilon(t, \mathbf{x}) = \frac{a\varepsilon}{\rho_0^\varepsilon(\mathbf{x})} \theta^\varepsilon(t, \mathbf{x}).$$

Indeed, operating the following change of variable in equation (1.10)

$$(3.7) \quad (t, v, w) \mapsto (t, v, w + \gamma^\varepsilon v),$$

and integrating the equation with respect to  $v$ , the equation on the marginal  $\bar{g}^\varepsilon$  of  $g^\varepsilon$  defined as

$$\bar{g}^\varepsilon(t, \mathbf{x}, w) = \int_{\mathbb{R}} g^\varepsilon(t, \mathbf{x}, v, w) dv,$$

reads as follows

$$(3.8) \quad \partial_t \bar{g}^\varepsilon + \frac{a\varepsilon}{\rho_0^\varepsilon} \partial_w \left[ \int_{\mathbb{R}} (B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, w - \gamma^\varepsilon v) + b\theta^\varepsilon v) g^\varepsilon dv \right] - \left( \frac{a\varepsilon}{\rho_0^\varepsilon} \right)^2 \partial_w^2 \bar{g}^\varepsilon = \partial_w [bw \bar{g}^\varepsilon],$$

where  $B_0^\varepsilon$  is defined by (1.7). In our analysis, the main advantage of considering equation (3.8) on  $\bar{g}^\varepsilon$  rather than equation (1.8) on  $\bar{v}^\varepsilon$  stems from the fact that equation (3.8) displays a vanishing diffusion with respect to the  $w$ -variable. This diffusion should be interpreted as the result of regularizing effects with respect to the  $v$ -variable due to the Fokker-Planck operator, which are transferred to the  $w$ -variable thanks to the source term in equation (1.8). Taking advantage of the regularizing properties of the diffusion, we derive convergence estimates for  $\bar{g}^\varepsilon$ : it is the object of the following result

**Proposition 3.7.** *Under assumptions (2.1a)-(2.1b) on the drift  $N$  and (2.2)-(2.3) on the interaction kernel  $\Psi$ , consider a sequence of solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) with initial conditions satisfying assumption (2.4)-(2.7) and the solution  $\bar{v}$  to equation (1.9) with initial condition  $\bar{v}_0$  satisfying assumption (3.1). There exists a positive constant  $C$  independent of  $\varepsilon$  such that for all  $\varepsilon$  less than 1, it holds*

$$\|\bar{g}^\varepsilon(t) - \bar{v}(t)\|_{L^\infty_x L^1_w} \leq 2\sqrt{2} \|\bar{g}_0^\varepsilon - \bar{v}_0\|_{L^\infty_x L^1_w} + C e^{bt} \sqrt{\varepsilon}, \quad \forall t \in \mathbb{R}^+.$$

In this result, the constant  $C$  only depends on  $m_*$ ,  $m_p$  and  $\bar{m}_p$  (see assumptions (2.5)-(2.7)) and the data of the problem  $\bar{v}_0$ ,  $N$ ,  $\Psi$  and  $A_0$ .

*Proof.* All along this proof, we choose some  $\mathbf{x}$  lying in  $K$  and some positive  $\varepsilon$ ; we omit the dependence with respect to  $(t, \mathbf{x})$  when the context is clear. Since  $\bar{v}$  and  $\bar{g}^\varepsilon$  solve respectively equations (1.10) and (3.8), Lemma 2.2 applies with the following parameters

$$\begin{cases} (d_1, d_2, \delta, \lambda) = (0, 1, 0, a\varepsilon/\rho_0^\varepsilon), \\ \mathbf{b}_2(t, w) = -\frac{\rho_0^\varepsilon}{a\varepsilon} bw, \\ \mathbf{b}_1(t, w) = \mathbf{b}_2(t, w) + \int_{\mathbb{R}} (B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, -\gamma^\varepsilon v) + b\theta^\varepsilon v) \frac{g^\varepsilon}{\bar{g}^\varepsilon} dv, \\ \mathbf{b}_3(t, w) = -w, \end{cases}$$

where  $B_0^\varepsilon$  is given by (1.7). According to (2.11) in Lemma 2.2, it holds

$$\|\bar{g}^\varepsilon(t) - \bar{v}(t)\|_{L^1(\mathbb{R})} \leq 2\sqrt{2} \left( \|\bar{g}_0^\varepsilon - \bar{v}_0\|_{L^1(\mathbb{R})} + \left( \int_0^t \mathcal{R}_1(s) + \mathcal{R}_2(s) ds \right)^{\frac{1}{2}} \right), \quad \forall t \in \mathbb{R}^+,$$

where  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are given by

$$\begin{cases} \mathcal{R}_1(t) = \frac{1}{4} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, -\gamma^\varepsilon v) + b\theta^\varepsilon v) \frac{g^\varepsilon}{\bar{g}^\varepsilon} dv \right|^2 \bar{g}^\varepsilon dw, \\ \mathcal{R}_2(t) = \frac{a\varepsilon}{\rho_0^\varepsilon} \int_{\mathbb{R}} |\partial_w [w \bar{v}]| + \frac{a\varepsilon}{\rho_0^\varepsilon} |\partial_w^2 \bar{v}| dw. \end{cases}$$

Let us estimate  $\mathcal{R}_1$ . According item (2) in Theorem 2.7,  $\mathcal{V}^\varepsilon$  is uniformly bounded with respect to both  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and  $\varepsilon > 0$ . On top of that, according to assumptions (2.3) and (2.5) on  $\Psi$  and  $\rho_0^\varepsilon$ ,  $\Psi *_r \rho_0^\varepsilon$  is uniformly bounded with respect to both  $\mathbf{x} \in K$  and  $\varepsilon > 0$ . Consequently, applying Young's inequality, assumption (2.1b) and since  $N$  is locally Lipschitz, we obtain

$$\mathcal{R}_1(t) \leq C \int_{\mathbb{R}^2} \left( |\theta^\varepsilon v|^{2p} + |\theta^\varepsilon v|^2 + |\mathcal{E}(\mu^\varepsilon)|^2 \right) g^\varepsilon d\mathbf{u},$$

as long as  $\varepsilon$  is less than 1 to ensure that  $\gamma^\varepsilon$  given by (3.7) is less than  $a\theta^\varepsilon/m_*$ . We invert the changes of variable (3.7) and (1.6) and apply items (3) and (4) in Theorem 2.7

$$\mathcal{R}_1(t) \leq C \left( e^{-2\rho_0^\varepsilon(\mathbf{x})t/\varepsilon} + \varepsilon \right).$$

Then to estimate  $\mathcal{R}_2$ , we apply Proposition 3.5. According to assumption (2.5) it holds

$$\mathcal{R}_2(t) \leq C\varepsilon e^{2bt}.$$

We gather the former computations and take the supremum over all  $\mathbf{x}$  in  $K$ : it yields the expected result. □

**3.3.3. Convergence of  $\bar{v}^\varepsilon$  towards  $\bar{v}$ .** In this section, we gather the result from the last steps to deduce that  $\bar{v}^\varepsilon$  converges towards  $\bar{v}$ .

**Proposition 3.8.** *Under the assumptions of Theorem 3.2, there exists a positive constant  $C$  independent of  $\varepsilon$  such that for all  $\varepsilon$  less than 1, it holds*

$$\int_0^t \|\bar{v}^\varepsilon - \bar{v}\|_{L_x^\infty L_w^1} ds \leq 2\sqrt{2}t \|\bar{v}_0^\varepsilon - \bar{v}_0\|_{L_x^\infty L_w^1}^{\frac{1}{2}} + 4\sqrt{\varepsilon}t m_2 + C e^{2bt} \sqrt{\varepsilon}, \quad \forall t \in \mathbb{R}^+.$$

In this result, the constant  $C$  only depends on  $m_1$ ,  $m_*$ ,  $m_p$  and  $\bar{m}_p$  (see assumptions (2.5)-(2.7)) and the data of the problem  $\bar{v}_0$ ,  $N$ ,  $\Psi$  and  $A_0$ .

*Proof.* All along this proof, we omit the dependence with respect to  $(t, \mathbf{x})$  when the context is clear. We consider the following triangular inequality

$$\|\bar{v}^\varepsilon - \bar{v}\|_{L_x^\infty L_w^1} \leq \mathcal{A} + \mathcal{B},$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are given by

$$\begin{cases} \mathcal{A} = \|\bar{v}^\varepsilon - \bar{g}^\varepsilon\|_{L_x^\infty L_w^1}, \\ \mathcal{B} = \|\bar{g}^\varepsilon - \bar{v}\|_{L_x^\infty L_w^1}. \end{cases}$$

We estimate  $\mathcal{B}$  applying Proposition 3.7, which ensures

$$\mathcal{B} \leq 2\sqrt{2} \|\bar{g}_0^\varepsilon - \bar{v}_0\|_{L_x^\infty L_w^1}^{\frac{1}{2}} + C e^{bt} \sqrt{\varepsilon},$$

at all times  $t$  in  $\mathbb{R}^+$ . Then we notice that inverting the change of variable (3.7), it holds

$$\|\bar{g}_0^\varepsilon - \bar{v}_0\|_{L_x^\infty L_w^1} \leq \|\bar{v}_0^\varepsilon - \bar{v}_0\|_{L_x^\infty L_w^1} + \int_{\mathbb{R}} \mathcal{M}(v) \left\| \bar{v}_0 - \tau_{(\gamma_0^\varepsilon v)} \bar{v}_0 \right\|_{L_x^\infty L_w^1} dv.$$

Therefore, according to the regularity assumption (3.1) on  $\bar{v}_0$ , we obtain

$$\mathcal{B} \leq 2\sqrt{2} \|\bar{v}_0^\varepsilon - \bar{v}_0\|_{L_x^\infty L_w^1}^{\frac{1}{2}} + C e^{bt} \sqrt{\varepsilon},$$

for all  $\varepsilon$  less than 1.

To estimate  $\mathcal{A}$ , we invert the change of variable (3.7) and apply the triangular inequality. It yields

$$\mathcal{A} \leq \mathcal{A}_1 + \mathcal{A}_2,$$

where  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are defined as follows

$$\begin{cases} \mathcal{A}_1 = \|\nu^\varepsilon - \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{v}^\varepsilon\|_{L_x^\infty L_w^1} + \|\mathcal{M}_{\rho_0^\varepsilon} \otimes \tau_{-(\gamma^\varepsilon v)} \bar{v}^\varepsilon - \tau_{-(\gamma^\varepsilon v)} \nu^\varepsilon\|_{L_x^\infty L_w^1}, \\ \mathcal{A}_2 = \sup_{\mathbf{x} \in K} \int_{\mathbb{R}} \mathcal{M}_{\rho_0^\varepsilon}(v) \|\nu^\varepsilon - \tau_{-(\gamma^\varepsilon v)} \nu^\varepsilon\|_{L^1(\mathbb{R}^2)} dv. \end{cases}$$

According to Proposition 3.6, it holds

$$\int_0^t \mathcal{A}_1(s) ds \leq 4\sqrt{\varepsilon} t m_2 + C\sqrt{\varepsilon}(t+1), \quad \forall t \geq 0,$$

where  $m_2$  is given in assumption (3.3). In addition, we apply Proposition 3.4 and derive the following estimate for  $\mathcal{A}_2$

$$\mathcal{A}_2 = C\sqrt{\varepsilon} e^{2bt}.$$

We obtain the result gathering the former estimates.  $\square$

Theorem 3.2 is obtained gathering the results from the first and the last step.

#### 4. CONVERGENCE ANALYSIS IN WEIGHTED $L^2$ SPACES

In this section, we derive convergence estimates for  $\mu^\varepsilon$  in a weighted  $L^2$  functional framework. We take advantage of the variational structure of  $L^2$  spaces in order to derive uniform regularity estimates for  $\mu^\varepsilon$ . Thanks to these regularity estimates, we prove a stronger convergence result than the one presented in [2] while keeping the same rates (which are, at least formally, the optimal convergence rates in our setting).

**4.1. Functional framework.** We consider the following weighted  $L^2$  space

$$L^2(m_{\mathbf{x}}^\varepsilon) = \left\{ \nu : \mathbb{R}^2 \rightarrow \mathbb{R} \mid \int_{\mathbb{R}^2} |\nu(\mathbf{u})|^2 m_{\mathbf{x}}^\varepsilon(\mathbf{u}) d\mathbf{u} < +\infty \right\},$$

where the weight  $m_{\mathbf{x}}^\varepsilon$  is given by

$$(4.1) \quad m_{\mathbf{x}}^\varepsilon(\mathbf{u}) = \frac{2\pi}{\sqrt{\rho_0^\varepsilon} \kappa} \exp\left(\frac{1}{2}(\rho_0^\varepsilon(\mathbf{x})|v|^2 + \kappa|w|^2)\right),$$

for some exponent  $\kappa > 0$  which will be prescribed later. We equip  $L^2(m_{\mathbf{x}}^\varepsilon)$  with its natural Hilbertian structure

$$\langle \mu, \nu \rangle_{L^2(m_{\mathbf{x}}^\varepsilon)} = \int_{\mathbb{R}^2} \mu \nu m_{\mathbf{x}}^\varepsilon(\mathbf{u}) d\mathbf{u},$$

to which is associated the norm:  $\|\nu\|_{L^2(m_{\mathbf{x}}^\varepsilon)}^2 = \langle \nu, \nu \rangle_{L^2(m_{\mathbf{x}}^\varepsilon)}$ . We also introduce the associated weight with respect to the adaptation variable

$$\bar{m}(w) = \sqrt{\frac{2\pi}{\kappa}} \exp\left(\frac{\kappa}{2}|w|^2\right).$$

We carry out our analysis in the functional space  $H_w^k(m_{\mathbf{x}}^\varepsilon)$ , given by

$$H_w^k(m_{\mathbf{x}}^\varepsilon) = \left\{ \nu : \mathbb{R}^2 \rightarrow \mathbb{R} \mid \forall l \in \mathbb{N}, l \leq k : \partial_w^l \nu \in L^2(m_{\mathbf{x}}^\varepsilon) \right\},$$

for  $k$  in  $\{0, 1, 2\}$ . We equip the latter functional space with the norm

$$\|\nu\|_{H_w^k(m_{\mathbf{x}}^\varepsilon)}^2 = \sum_{l \leq k} \left\| \partial_w^l \nu \right\|_{L^2(m_{\mathbf{x}}^\varepsilon)}^2.$$

On top of that, we define the spaces  $\mathcal{H}^k(m^\varepsilon)$  for  $k$  in  $\{0, 1, 2\}$  as follows

$$\mathcal{H}^k(m^\varepsilon) = \left\{ \nu : K \times \mathbb{R}^2 \rightarrow \mathbb{R} \mid \forall \mathbf{x} \in K : \nu(\mathbf{x}, \cdot) \in H_w^k(m_{\mathbf{x}}^\varepsilon) \right\},$$

which are equipped with the norm

$$\|\nu\|_{\mathcal{H}^k(m^\varepsilon)} = \sup_{\mathbf{x} \in K} \left\{ \left\| \nu(\mathbf{x}, \cdot) \right\|_{H_w^k(m_{\mathbf{x}}^\varepsilon)} \right\}.$$

We denote by  $\mathcal{H}^k(\bar{m})$  the associated functional space without the voltage variable

$$\mathcal{H}^k(\bar{m}) = \left\{ \bar{\nu} : K \times \mathbb{R} \rightarrow \mathbb{R} \mid \forall \mathbf{x} \in K : \bar{\nu}(\mathbf{x}, \cdot) \in H^k(\bar{m}) \right\},$$

which are equipped with the norm

$$\|\nu\|_{\mathcal{H}^k(\bar{m})} = \sup_{\mathbf{x} \in K} \left\{ \|\nu(\mathbf{x}, \cdot)\|_{H^k(\bar{m})} \right\}.$$

**4.2. Main results.** We proceed in two steps: we first state a convergence result on the abstract quantity  $\nu^\varepsilon$ , which solves the re-scaled equation (1.10). Then we interpret the result on our initial problem, that is on the distribution of the network  $\mu^\varepsilon$ , which solves (1.2).

**4.2.1. The re-scaled problem.** In the following result, we provide explicit convergence rate for  $\nu^\varepsilon$  towards the asymptotic concentration profile of the neural network's distribution  $\mu^\varepsilon$  in the regime of strong interactions. We prove that the profile of concentration is Gaussian and we also characterize the limiting distribution with respect to the adaptation variable  $w$ .

**Theorem 4.1.** *Under assumptions (2.1a)-(2.1b) on the drift  $N$  and the additional assumption*

$$(4.2) \quad \sup_{|v| \geq 1} (v^2 \omega(v) - C_0 N'(v)) < +\infty,$$

for all positive constant  $C_0 > 0$ , supposing assumptions (2.2)-(2.3) on the interaction kernel  $\Psi$ , and under the assumptions of Theorem 2.4 and 2.6, consider the sequence of solutions  $(\mu^\varepsilon)_{\varepsilon > 0}$  to (1.2) provided by Theorem 2.4 with initial conditions satisfying assumptions (2.4)-(2.7) and the solution  $\bar{\nu}$  to equation (1.9) with an initial condition  $\bar{\nu}_0$ . On top of that, consider an exponent  $\kappa$  which verifies the condition

$$(4.3) \quad \kappa \in \left( \frac{1}{2b}, +\infty \right),$$

and consider a rate  $\alpha_*$  lying in  $(0, 1 - (2b\kappa)^{-1})$ . There exists a positive constant  $C$  independent of  $\varepsilon$  such that for all  $\varepsilon$  between 0 and 1 the following results hold true

- (1) consider  $k$  less than 2 and suppose that the initial condition  $\mu_0^\varepsilon$  is such that the quantity  $\nu_0^\varepsilon$ , obtained performing the change of variable (1.6), lies in  $\mathcal{H}^k(m^\varepsilon)$ , where  $m^\varepsilon$  is defined by (4.1) with exponent  $\kappa$ . Then the unique solution  $\mu^\varepsilon$  to equation (1.2) provided by Theorem 2.4 is such that for all time  $t$ , the quantity  $\nu^\varepsilon(t)$  lies in  $\mathcal{H}^k(m^\varepsilon)$ . On top of that, it holds

$$\int_0^t \sup_{\mathbf{x} \in K} \int_{\mathbb{R}^2} \left[ \left| \partial_v \left( \partial_w^k \nu^\varepsilon m^\varepsilon \right) \right|^2 (m^\varepsilon)^{-1} \right] (s, \mathbf{x}, \mathbf{u}) \, d\mathbf{u} \, ds < +\infty, \quad \forall t \in \mathbb{R}^+.$$

- (2) Consider  $k$  in  $\{0, 1\}$  and suppose that the sequence of initial profiles  $(\nu_0^\varepsilon)_{\varepsilon > 0}$  stays uniformly bounded in  $\mathcal{H}^{k+1}$  for weight  $m^\varepsilon$  defined by (4.1) with exponent  $\kappa$ , that is

$$(4.4) \quad \sup_{\varepsilon > 0} \|\nu_0^\varepsilon\|_{\mathcal{H}^{k+1}(m^\varepsilon)} < +\infty,$$

and that  $\bar{\nu}_0$  verifies

$$(4.5) \quad \bar{\nu}_0 \in \mathcal{H}^k(\bar{m}).$$

Then for all time  $t$  in  $\mathbb{R}^+$  it holds

$$\|\nu^\varepsilon(t) - \nu(t)\|_{\mathcal{H}^k(m^\varepsilon)} \leq e^{Ct} \left( \mathcal{E}_{\text{ini}}^k + C \|\nu_0^\varepsilon\|_{\mathcal{H}^{k+1}(m^\varepsilon)} \left( \sqrt{\varepsilon} + \min \left\{ 1, e^{-\alpha_* \frac{t}{\varepsilon}} - \frac{\alpha_*}{2m_*} \right\} \right) \right),$$

where the asymptotic profile  $\nu$  is given by

$$\nu = \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu},$$

and the initial error  $\mathcal{E}_{\text{ini}}^k$  is given by

$$\mathcal{E}_{\text{ini}}^k = \|\bar{\nu}_0^\varepsilon - \bar{\nu}_0\|_{\mathcal{H}^k(\bar{m})}.$$

(3) Supposing assumption (4.4) with index  $k = 1$  and assumption (4.5) with index  $k = 0$ , it holds

$$\|\bar{\nu}^\varepsilon(t) - \bar{\nu}(t)\|_{\mathcal{H}^0(\bar{m})} \leq e^{Ct} \left( \mathcal{E}_{\text{ini}}^0 + C \|\nu_0^\varepsilon\|_{\mathcal{H}^2(m^\varepsilon)} \varepsilon \sqrt{|\ln \varepsilon| + 1} \right), \quad \forall t \in \mathbb{R}_+.$$

In this theorem, the positive constant  $C$  only depends on  $\kappa$ ,  $\alpha_*$ ,  $m_*$ ,  $m_p$ ,  $\bar{m}_p$  (see assumptions (2.5), (2.6) and (2.7)) and on the data of the problem:  $N$ ,  $A_0$  and  $\Psi$ .

We prove items (2)-(3) in section 4.4: it is the main object of the forthcoming analysis. The proof of these results relies on regularity estimates for the solution  $\nu^\varepsilon$  to equation (1.10) (see Proposition 4.7). These regularity estimates allow us to bound the source term which appears in the right hand side of equation (1.8), and therefore to deduce the convergence of the marginal  $\bar{\nu}^\varepsilon$  of  $\nu^\varepsilon$  towards  $\bar{\nu}$ .

Before coming back to the initial problem on  $\mu^\varepsilon$ , let us comment on this result

- (1) We achieve pointwise in time convergence estimates. This is an improvement in comparison to our result in the  $L^1$  setting, where we only proved  $L^1$  in time convergence estimates (see Theorem 3.2). This is made possible thanks to the regularity results obtained for  $\nu^\varepsilon$  (see Proposition 4.7), which we were not able to obtain in the  $L^1$  setting.
- (2) We make the additional assumption (4.2) on  $N$  due to the structure of  $L^2$  spaces with inverse Gaussian weights. This condition arises naturally in the analysis. Indeed, if  $\nu$  solves

$$\partial_t \nu + \partial_v [N \nu] = 0,$$

then multiplying the latter equation by  $\nu m^\varepsilon$  and integrating by part with respect to  $v$ , we obtain

$$\frac{d}{dt} \|\nu\|_{L^2(m^\varepsilon)}^2 = \int_{\mathbb{R}^2} (\rho_0^\varepsilon v N(v) - N'(v)) |\nu|^2 m^\varepsilon d\mathbf{u}.$$

Therefore, the  $L^2(m^\varepsilon)$  - norm is propagated if the quantity inside the integral is upper bounded, which leads to assumption (4.2) (see Lemma 4.4 for more details). At the end of the day, this assumption is not constraining in our setting since it is verified by polynomial of the form

$$P(v) = Q(v) - C v |v|^{p-1},$$

for any  $p \geq 2$  and any positive constant  $C > 0$ , where  $Q$  has degree less than  $p$ .

4.2.2. *Interpretation on the initial problem.* In the following result, we invert the change of variable (1.6) and deduce from Theorem 3.2 an expansion for the solution  $\mu^\varepsilon$  to equation (1.2) in the regime of strong interactions, that is, as  $\varepsilon$  vanishes. Since the  $m^\varepsilon$  defined by (4.1) depends on  $\varepsilon$  through the spatial distribution  $\rho_0^\varepsilon$ , we introduce weights which do not depend on  $\varepsilon$  anymore and which are meant to upper and lower bound  $m^\varepsilon$ . We consider  $(\mathbf{x}, \mathbf{u})$  lying in  $K \times \mathbb{R}^2$  and define

$$\left\{ \begin{array}{l} m_{\mathbf{x}}^-(\mathbf{u}) = (\rho_0(\mathbf{x}) \kappa)^{-\frac{1}{2}} \exp\left(\frac{1}{8} (\rho_0(\mathbf{x}) |v|^2 + \kappa |w|^2)\right), \\ \bar{m}^-(w) = \kappa^{-\frac{1}{2}} \exp\left(\frac{\kappa}{8} |w|^2\right), \\ m_{\mathbf{x}}^+(\mathbf{u}) = (\rho_0(\mathbf{x}) \kappa)^{-\frac{1}{2}} \exp\left(2 (\rho_0(\mathbf{x}) |v|^2 + \kappa |w|^2)\right), \\ \bar{m}^+(w) = \kappa^{-\frac{1}{2}} \exp\left(2 \kappa |w|^2\right). \end{array} \right.$$

With these notations, our result reads as follows

**Theorem 4.2.** *Under assumptions (2.1a)-(2.1b) and (4.2) on the drift  $N$ , (2.2)-(2.3) on the interaction kernel  $\Psi$  and supposing the assumptions of Theorems 2.4 and 2.6, consider the sequence of solutions  $(\mu^\varepsilon)_{\varepsilon > 0}$  to (1.2) provided by Theorem 2.4 with initial conditions satisfying assumptions*



(2.4)-(2.7) and the solution  $(\mathcal{V}, \bar{\mu})$  to (1.5) provided by Theorem 2.6 with initial condition  $\bar{\mu}_0$ . Consider an exponent  $\kappa$  which verifies the condition

$$\kappa \in \left( \frac{1}{2b}, +\infty \right),$$

and a rate  $\alpha_*$  lying in  $(0, \min\{1 - (2b\kappa)^{-1}, m_*/2\})$ , where  $m_*$  the uniform lower bound of  $\rho_0^\varepsilon$  (see (2.5)). The following results hold true

(1) Consider  $k$  in  $\{0, 1\}$  and suppose

$$(4.6) \quad \sup_{\varepsilon > 0} \|\mu_0^\varepsilon\|_{\mathcal{H}^{k+1}(m^+)} < +\infty,$$

as well as the following compatibility assumption

$$(4.7) \quad \|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_{L^\infty(K)} + \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)} + \|\bar{\mu}_0^\varepsilon - \bar{\mu}_0\|_{\mathcal{H}^k(\bar{m}^+)} \underset{\varepsilon \rightarrow 0}{=} O(\varepsilon).$$

On top of that, suppose that there exists a constant  $C$  such that

$$(4.8) \quad \sup_{\varepsilon > 0} \|\bar{\mu}_0 - \tau_{w_0} \bar{\mu}_0\|_{\mathcal{H}^k(\bar{m}^+)} \leq C |w_0|, \quad \forall w_0 \in \mathbb{R}.$$

Then for all integer  $i$ , there exists  $(C_i, \varepsilon_0) \in (\mathbb{R}_+^*)^2$  such that for all  $\varepsilon$  less than  $\varepsilon_0$ , it holds

$$\|(v - \mathcal{V})^i (\mu^\varepsilon - \mu)(t)\|_{\mathcal{H}^k(m^-)} \leq C_i e^{C_i(t + \varepsilon e^{C_i t})} \left( \varepsilon^{\frac{i}{2} + \frac{1}{4}} + e^{-\alpha_* \frac{t}{\varepsilon}} \varepsilon^{-\frac{1}{2}} \right), \quad \forall t \in \mathbb{R}_+,$$

where the limit  $\mu$  is given by

$$\mu = \mathcal{M}_{\rho_0 |\theta^\varepsilon|^{-2}} (v - \mathcal{V}) \otimes \bar{\mu}.$$

(2) Suppose assumption (4.6) with  $k = 1$ , assumption (4.8) with  $k = 0$  and

$$\|\mathcal{U}_0 - \mathcal{U}_0^\varepsilon\|_{L^\infty(K)} + \|\rho_0 - \rho_0^\varepsilon\|_{L^\infty(K)} + \|\bar{\mu}_0^\varepsilon - \bar{\mu}_0\|_{\mathcal{H}^0(\bar{m}^+)} \underset{\varepsilon \rightarrow 0}{=} O\left(\varepsilon \sqrt{|\ln \varepsilon|}\right).$$

There exists  $(C, \varepsilon_0) \in (\mathbb{R}_+^*)^2$  such that for all  $\varepsilon$  less than  $\varepsilon_0$ , it holds

$$\|\bar{\mu}^\varepsilon(t) - \bar{\mu}(t)\|_{\mathcal{H}^0(\bar{m}^-)} \leq C e^{Ct} \varepsilon \sqrt{|\ln \varepsilon|}, \quad \forall t \in \mathbb{R}_+.$$

This result is a straightforward consequence of Theorem 4.1 and the convergence estimates for the macroscopic quantities given by item (1) in Theorem 2.7. We postpone the proof to Section 4.5. Let us first comment on these results.

(1) Similarly to Theorem 3.2, the choice of a non-homogeneous in time concentration rate  $\theta^\varepsilon$  plays a **key role**. Indeed, we have the following relation after inverting the change of variable (1.6)

$$\|\mu_0^\varepsilon\|_{\mathcal{H}^0(m^+)} = \sup_{\mathbf{x} \in K} \frac{1}{\theta_0^\varepsilon} \left( \int_{\mathbb{R}^2} |\nu_0^\varepsilon|^2 m^+(\mathbf{x}, \mathcal{V}_0^\varepsilon + \theta_0^\varepsilon v, \mathcal{W}_0^\varepsilon + w) d\mathbf{u} \right)^{\frac{1}{2}}.$$

Therefore,  $\theta^\varepsilon(t = 0)$  needs to stay lower bounded as  $\varepsilon$  vanishes in order for the result to hold true without considering well-prepared initial data.

(2) This result justifies our approach, which consists in working with the re-scaled version  $\nu^\varepsilon$  of  $\mu^\varepsilon$ . Indeed, it would not have been possible to derive  $L^2$  convergence estimates to quantify the distance between  $\mu^\varepsilon$  and  $\delta_{\mathcal{V}} \otimes \bar{\mu}$  simply because the limit displays a Dirac mass and therefore does not lie in  $L^2$  spaces.

(3) We recover the optimal convergence rate for the marginal  $\bar{\mu}^\varepsilon$  of  $\mu^\varepsilon$  towards the limit  $\bar{\mu}$ , up to a logarithmic correction. The logarithmic correction arises due to the fact that we do not consider well prepared initial data (see Proposition 4.12 for more details). In the statement (1), we prove convergence with rate  $O(\varepsilon^i)$  for all  $i$ . This is specific to the structure of the weighted  $L^2$  at play in this result (see Section 4.5 for more details).

**4.3. A priori estimates.** The main purpose of this section is to propagate the  $\mathcal{H}^k$ -norms along the trajectories of equation (1.10) uniformly with respect to  $\varepsilon$ . We outline the strategy in the case of the  $\mathcal{H}^0$ -norm. Its time derivative along the trajectories of equation (1.10) are obtained multiplying (1.10) by  $\nu^\varepsilon m^\varepsilon$  and integrating with respect to  $\mathbf{u}$

$$\frac{1}{2} \frac{d}{dt} \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2 = \frac{1}{(\theta^\varepsilon)^2} \langle \mathcal{F}_{\rho_0^\varepsilon}[\nu^\varepsilon], \nu^\varepsilon \rangle_{L^2(m^\varepsilon)} - \langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu^\varepsilon], \nu^\varepsilon \rangle_{L^2(m^\varepsilon)}.$$

We first point out that according to the following Lemma, the term associated to the Fokker-Planck operator is dissipative and is consequently a helping term in the upcoming analysis

**Lemma 4.3.** *For all  $\mathbf{x}$  in  $K$ , it holds*

$$\left\langle \mathcal{F}_{\rho_0^\varepsilon(\mathbf{x})}[\nu], \nu \right\rangle_{L^2(m_{\mathbf{x}}^\varepsilon)} = -\mathcal{D}_{\rho_0^\varepsilon(\mathbf{x})}[\nu] \leq 0,$$

for all  $\nu \in H^1(m_{\mathbf{x}}^\varepsilon)$ , where the dissipation  $\mathcal{D}_{\rho_0^\varepsilon}$  is given by

$$\mathcal{D}_{\rho_0^\varepsilon(\mathbf{x})}[\nu] = \int_{\mathbb{R}^2} |\partial_v(\nu m_{\mathbf{x}}^\varepsilon)|^2 (m_{\mathbf{x}}^\varepsilon)^{-1} d\mathbf{u} \geq 0.$$

*Proof.* The Fokker-Planck operator rewrites as follows

$$\mathcal{F}_{\rho_0^\varepsilon(\mathbf{x})}[\nu] = \partial_v \left[ (m_{\mathbf{x}}^\varepsilon)^{-1} \partial_v(\nu m_{\mathbf{x}}^\varepsilon) \right].$$

Consequently, the result is obtained integrating by part  $\mathcal{F}_{\rho_0^\varepsilon(\mathbf{x})}[\nu]$  against  $\nu m^\varepsilon$  with respect to  $\mathbf{u}$ .  $\square$

Therefore, the main challenge is to control the contribution of the transport operator  $\operatorname{div}_{\mathbf{u}} \mathbf{b}_0^\varepsilon$  with the dissipation  $\mathcal{D}_{\rho_0^\varepsilon}$  brought by the Fokker-Planck operator. This is done in the following key Lemma, on which rely all our results. The main difficulty to prove this result is to take advantage of the confining properties of  $N$  and  $A$ .

**Lemma 4.4.** *Under assumptions (2.1a)-(2.1b) and (4.2) on the drift  $N$ , (2.2)-(2.3) on the interaction kernel  $\Psi$ , consider a sequence of solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) with initial conditions satisfying assumptions (2.4)-(2.7). Then, for all positive  $\varepsilon$  and any  $\alpha$  greater than  $1/(2b\kappa)$ , holds the following estimate*

$$-\langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu], \nu \rangle_{L^2(m_{\mathbf{x}}^\varepsilon)} \leq \frac{\alpha}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon(\mathbf{x})}[\nu] + C \|\nu\|_{L^2(m_{\mathbf{x}}^\varepsilon)}^2,$$

for all  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and all  $\nu \in H^1(m_{\mathbf{x}}^\varepsilon)$ ; where  $C$  is a positive constant only depending on  $\alpha, \kappa, m_*, m_p, \bar{m}_p$  (see assumptions (2.5), (2.6) and (2.7)) and the data of the problem:  $N, A_0$  and  $\Psi$ .

Before getting into the heart of the proof, we point out that as long as the latter Lemmas hold with some  $\alpha$  less than 1, we have

$$\frac{1}{2} \frac{d}{dt} \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2 \leq C \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2,$$

which ensures that the  $\mathcal{H}^0$ -norm is propagated along the curves of (1.10) uniformly with respect to  $\varepsilon$ . We follow the exact same strategy in order to propagate the  $\mathcal{H}^k$ -norms when  $k$  is not 0: see Proposition 4.7 for more details. On top of that, we emphasize that the constraint (4.3) on  $\kappa$  in Theorem 4.1 arises from the lower bound on  $\alpha$  in Lemma 4.4.

**Remark 4.5.** *Due to the structure of the space  $L^2(m^\varepsilon)$ , we added the confining assumption (4.2) on the drift  $N$  to Theorem 4.1. Our proof of Lemma 4.4 crucially relies on this assumption; it is the only time that we use it as well.*

*Proof.* All along this proof, we consider some  $\varepsilon > 0$  and some  $(t, \mathbf{x})$  in  $\mathbb{R}_+ \times K$ ; we omit the dependence with respect to  $\mathbf{x}$  when the context is clear. Furthermore, we choose some  $\nu$  in  $H^1(m_{\mathbf{x}}^\varepsilon)$ . Since  $\mathbf{b}_0^\varepsilon$  is given by (1.7), we have

$$-\langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu], \nu \rangle_{L^2(m^\varepsilon)} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3,$$

where

$$\begin{cases} \mathcal{A}_1 = \frac{1}{\theta^\varepsilon} \int_{\mathbb{R}^2} \partial_v [w \nu] \nu m^\varepsilon \, d\mathbf{u} - \int_{\mathbb{R}^2} \partial_w [A_0(\theta^\varepsilon v, w) \nu] \nu m^\varepsilon \, d\mathbf{u}, \\ \mathcal{A}_2 = -\frac{1}{\theta^\varepsilon} \int_{\mathbb{R}^2} \partial_v [(N(\mathcal{V}^\varepsilon + \theta^\varepsilon v) - N(\mathcal{V}^\varepsilon)) \nu] \nu m^\varepsilon \, d\mathbf{u}, \\ \mathcal{A}_3 = \int_{\mathbb{R}^2} \partial_v \left[ \left( v \Psi *_r \rho_0^\varepsilon + \mathcal{E}(\mu^\varepsilon) (\theta^\varepsilon)^{-1} \right) \nu \right] \nu m^\varepsilon \, d\mathbf{u}. \end{cases}$$

To estimate  $\mathcal{A}_1$ , we take advantage of the confining properties of  $A$ . When it comes to  $\mathcal{A}_2$ , the estimate relies on the confining properties of the non-linear drift  $N$ .  $\mathcal{A}_3$  gathers the lower order terms and adds no difficulty.

Now let us detail the computation and start with  $\mathcal{A}_{21}$ , which rewrites as follows after exact computations and an integration by part,

$$\mathcal{A}_1 = \frac{1}{2} \int_{\mathbb{R}^2} (\kappa w A_0(\theta^\varepsilon v, w) - \partial_w A_0) |\nu|^2 m^\varepsilon \, d\mathbf{u} - \int_{\mathbb{R}^2} w \nu (m^\varepsilon)^{1/2} \frac{1}{\theta^\varepsilon} \partial_v [\nu m^\varepsilon] (m^\varepsilon)^{-1/2} \, d\mathbf{u}.$$

According to the definition of  $A_0$  and applying Young's inequality, we obtain

$$\mathcal{A}_1 \leq \frac{1}{2\eta_2 (\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon} [\nu] + \int_{\mathbb{R}^2} \left( \frac{C}{\eta_1} |\theta^\varepsilon v|^2 + \left( C\eta_1 + \frac{\eta_2}{2} - \frac{b\kappa}{2} \right) w^2 \right) |\nu|^2 m^\varepsilon \, d\mathbf{u} + C \|\nu\|_{L^2(m^\varepsilon)}^2,$$

for all positive  $\eta_1$  and  $\eta_2$  and for some positive constant  $C$ . We set  $\alpha_- = (\alpha + 1/(2b\kappa))/2$ ,  $\eta_2 = 1/(2\alpha_-)$  and  $\eta_1 = (b\kappa - \eta_2)/(2C)$ . According to the condition on  $\alpha$  in Lemma 4.4, we have  $\eta_1 > 0$ . Consequently, we obtain

$$(4.9) \quad \mathcal{A}_1 \leq \frac{\alpha_-}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon} [\nu] + C \int_{\mathbb{R}^2} |\theta^\varepsilon v|^2 |\nu|^2 m^\varepsilon \, d\mathbf{u} + C \|\nu\|_{L^2(m^\varepsilon)}^2,$$

for some positive constant  $C$  only depending on  $A_0$ ,  $\kappa$  and  $\alpha$ .

To estimate  $\mathcal{A}_2$ , we take advantage of the super-linear decay of  $N$  (see assumption (2.1a)) in order to control the terms growing at most linearly. We emphasize that the decaying property of  $N$  is prescribed at infinity. Consequently, it may not have confining property on bounded sets. Hence, the main point here consists in isolating the domain where  $N$  decays super linearly.

After some exact computations and an integration by part,  $\mathcal{A}_2$  rewrites

$$\mathcal{A}_2 = \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{\rho_0^\varepsilon}{\theta^\varepsilon} v (N(\mathcal{V}^\varepsilon + \theta^\varepsilon v) - N(\mathcal{V}^\varepsilon)) - N'(\mathcal{V}^\varepsilon + \theta^\varepsilon v) \right] |\nu|^2 m^\varepsilon \, d\mathbf{u}.$$

We consider some  $R > 0$  and split the former expression in three different parts

$$\mathcal{A}_2 = \mathcal{A}_{21} + \mathcal{A}_{22} + \mathcal{A}_{23},$$

where

$$\begin{cases} \mathcal{A}_{21} = \frac{\rho_0^\varepsilon}{2\theta^\varepsilon} \int_{\mathbb{R}^2} \mathbb{1}_{|\theta^\varepsilon v| > R} v (N(\mathcal{V}^\varepsilon + \theta^\varepsilon v) - N(\mathcal{V}^\varepsilon)) |\nu|^2 m^\varepsilon \, d\mathbf{u}, \\ \mathcal{A}_{22} = \frac{1}{2} \int_{\mathbb{R}^2} \mathbb{1}_{|\theta^\varepsilon v| \leq R} v \left[ \frac{\rho_0^\varepsilon}{\theta^\varepsilon} (N(\mathcal{V}^\varepsilon + \theta^\varepsilon v) - N(\mathcal{V}^\varepsilon)) - N'(\mathcal{V}^\varepsilon + \theta^\varepsilon v) \right] |\nu|^2 m^\varepsilon \, d\mathbf{u}, \\ \mathcal{A}_{23} = -\frac{1}{2} \int_{\mathbb{R}^2} \mathbb{1}_{|\theta^\varepsilon v| > R} N'(\mathcal{V}^\varepsilon + \theta^\varepsilon v) |\nu|^2 m^\varepsilon \, d\mathbf{u}. \end{cases}$$

$\mathcal{A}_{21}$  corresponds to the the contribution of  $N$  on the domain where it decays super-linearly. Consequently,  $\mathcal{A}_{21}$  is non positive for  $R$  great enough. We take advantage of the helping term  $\mathcal{A}_{21}$  to control  $\mathcal{A}_{22}$ , which corresponds to the contribution of  $N$  on bounded sets. We estimate  $\mathcal{A}_3$  taking

advantage of the confining term  $\mathcal{A}_{21}$  coupled with the confining assumption (4.2) on  $N$ .

Let us estimate  $\mathcal{A}_{21}$ . According to item (2) in Theorem 2.7,  $\mathcal{V}^\varepsilon$  is uniformly bounded with respect to both  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and  $\varepsilon$ . Furthermore, since  $N$  is continuous, we obtain

$$\mathbb{1}_{|\theta^\varepsilon v| > R} \theta^\varepsilon v (N(\mathcal{V}^\varepsilon + \theta^\varepsilon v) - N(\mathcal{V}^\varepsilon)) \leq \mathbb{1}_{|\theta^\varepsilon v| > R} \left( |\mathcal{V}^\varepsilon + \theta^\varepsilon v|^2 \omega(\mathcal{V}^\varepsilon + \theta^\varepsilon v) \frac{\theta^\varepsilon v}{\mathcal{V}^\varepsilon + \theta^\varepsilon v} + C |\theta^\varepsilon v| \right),$$

where  $\omega$  is given by assumption (2.1a) and where the constant  $C$  depends on both  $N$  and the uniform upper bound on  $\mathcal{V}^\varepsilon$ . Since  $\mathcal{V}^\varepsilon$  is uniformly bounded and applying assumption (2.1a), the right hand side in the latter inequality is non positive for  $R$  sufficiently large. Hence, since  $\theta^\varepsilon \leq 1$ , we obtain a radius  $R$  only depending on  $N$  and the uniform bound on  $|\mathcal{V}^\varepsilon|$  such that

$$\mathcal{A}_{21} \leq \int_{\mathbb{R}^2} \left( \frac{m_*}{4} \mathbb{1}_{|\theta^\varepsilon v| > R} |\mathcal{V}^\varepsilon + \theta^\varepsilon v|^2 \omega(\mathcal{V}^\varepsilon + \theta^\varepsilon v) + C |\theta^\varepsilon v| \right) |\nu|^2 m^\varepsilon d\mathbf{u},$$

where  $m_*$  is the lower bound of  $\rho_0^\varepsilon$  given by assumption (2.5). From now on, **we fix**  $R$  such that the latter estimate holds. Furthermore, we introduce the following notation

$$\mathcal{N} = \int_{\mathbb{R}^2} \mathbb{1}_{|\theta^\varepsilon v| > R} |\mathcal{V}^\varepsilon + \theta^\varepsilon v|^2 \omega(\mathcal{V}^\varepsilon + \theta^\varepsilon v) |\nu|^2 m^\varepsilon d\mathbf{u} \leq 0 \quad \text{when } R \gg 1.$$

The term  $\mathcal{N}$  corresponds to the contribution of  $N$  on the domain where it has super-linear decaying properties. In the sequel, we use  $\mathcal{N}$  to control the contribution of the other terms. With this notation, the estimate on  $\mathcal{A}_{21}$  rewrites

$$\mathcal{A}_{21} \leq C \int_{\mathbb{R}^2} |\theta^\varepsilon v| |\nu|^2 m^\varepsilon d\mathbf{u} + \frac{m_*}{4} \mathcal{N},$$

where  $C$  and  $R$  only depend on  $N$  and the uniform bound on  $|\mathcal{V}^\varepsilon|$ .

We turn to  $\mathcal{A}_{22}$ . Since  $N$  has  $\mathcal{C}^1$  regularity and relying item (2) in Theorem 2.7, which ensures that  $\mathcal{V}^\varepsilon$  stays uniformly bounded, we obtain

$$\mathcal{A}_{22} \leq C \frac{\rho_0^\varepsilon}{2} \int_{\mathbb{R}^2} |v|^2 |\nu|^2 m^\varepsilon d\mathbf{u} + C \|\nu\|_{L^2(m^\varepsilon)}^2,$$

where  $C$  is a positive constant which may depend on  $m_*$ ,  $N$  and the uniform bound on  $|\mathcal{V}^\varepsilon|$ . We estimate the quadratic term in the latter inequality with the following relation

$$\frac{1}{2} \int_{\mathbb{R}^2} (\rho_0^\varepsilon |v|^2 - 1) |\nu|^2 m^\varepsilon d\mathbf{u} = \int_{\mathbb{R}^2} v \nu \partial_v (\nu m^\varepsilon) d\mathbf{u}.$$

It is obtained after exact computations and an integration by part in the right hand side of the latter equality. We apply Young's inequality to the former relation and in the end it yields

$$\mathcal{A}_{22} \leq \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu] + C \left( \frac{1}{\eta} \int_{\mathbb{R}^2} |\theta^\varepsilon v|^2 |\nu|^2 m^\varepsilon d\mathbf{u} + \|\nu\|_{L^2(m^\varepsilon)}^2 \right),$$

for all positive  $\eta$  and for some positive constant  $C$  depending on  $m_*$ ,  $N$  and the uniform bound on  $|\mathcal{V}^\varepsilon|$ .

We estimate the last term  $\mathcal{A}_{23}$  taking advantage of the confining properties corresponding to  $\mathcal{N}$ . Indeed we have

$$\mathcal{A}_{23} + \frac{m_*}{8} \mathcal{N} = \int_{\mathbb{R}^2} \mathbb{1}_{|\theta^\varepsilon v| > R} \left( \frac{m_*}{8} |v'|^2 \omega(v') - \frac{1}{2} N'(v') \right) |\nu|^2 m^\varepsilon d\mathbf{u},$$

where we used the shorthand notation  $v' = \mathcal{V}^\varepsilon + \theta^\varepsilon v$ . Hence, according to assumption (4.2), we deduce

$$\mathcal{A}_{23} + \frac{m_*}{8} \mathcal{N} \leq C \|\nu\|_{L^2(m^\varepsilon)}^2,$$

for some positive constant  $C$  depending on  $m_*$  and  $N$ . Gathering these computations, we obtain

$$(4.10) \quad \mathcal{A}_2 \leq \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu] + C \left( \frac{1}{\eta} + 1 \right) \int_{\mathbb{R}^2} |\theta^\varepsilon v|^2 |\nu|^2 m^\varepsilon \, d\mathbf{u} + C \|\nu\|_{L^2(m^\varepsilon)}^2 + \frac{m_*}{8} \mathcal{N},$$

for all  $\eta > 0$  and where  $C$  is a positive constant which may depend on  $m_*$ ,  $R$ ,  $N$  and the uniform bound on  $|\mathcal{V}^\varepsilon|$ .

We turn to  $\mathcal{A}_3$ , which gathers terms of lower-order. We integrate by part and apply Young's inequality. It yields

$$\mathcal{A}_3 \leq \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu] + \frac{1}{2\eta} \left( |\Psi *_{r} \rho_0^\varepsilon|^2 \int_{\mathbb{R}^2} |\theta^\varepsilon v|^2 |\nu|^2 m^\varepsilon \, d\mathbf{u} + |\mathcal{E}(\mu^\varepsilon)|^2 \|\nu\|_{L^2(m^\varepsilon)}^2 \right),$$

for all positive  $\eta$ . According to item (4) in Theorem 2.7,  $\mathcal{E}(\mu^\varepsilon)$  is uniformly bounded with respect to both  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and  $\varepsilon > 0$ . On top of that, according to assumptions (2.3) and (2.5) on  $\Psi$  and  $\rho_0^\varepsilon$ ,  $\Psi *_{r} \rho_0^\varepsilon$  is uniformly bounded with respect to both  $\mathbf{x} \in K$  and  $\varepsilon > 0$ . Consequently, we obtain

$$(4.11) \quad \mathcal{A}_3 \leq \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu] + \frac{C}{2\eta} \left( \int_{\mathbb{R}^2} |\theta^\varepsilon v|^2 |\nu|^2 m^\varepsilon \, d\mathbf{u} + \|\nu\|_{L^2(m^\varepsilon)}^2 \right),$$

for some positive constant  $C$  which may depend on  $m_*$  (see assumption (2.5)),  $m_p$  and  $\bar{m}_p$  (see assumptions (2.6) and (2.7)) and the data of our problem  $N$ ,  $\Psi$  and  $A_0$ .

Gathering estimates (4.9), (4.10) and (4.11), it yields

$$-\langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu], \nu \rangle \leq \frac{\alpha_- + 2\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu] + C \left( 1 + \frac{1}{\eta} \right) \int_{\mathbb{R}^2} (|\theta^\varepsilon v|^2 + 1) |\nu|^2 m^\varepsilon \, d\mathbf{u} + \frac{m_*}{8} \mathcal{N},$$

for all positive  $\eta$ . Hence, we choose  $2\eta = \alpha - \alpha_-$ , for which condition on  $\alpha$  in Lemma 4.4 ensures that  $\eta$  is positive. Therefore, replacing  $\mathcal{N}$  by its definition, the former estimate rewrites

$$-\langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu], \nu \rangle \leq \frac{\alpha}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu] + \int_{\mathbb{R}^2} \left( C (|\theta^\varepsilon v|^2 + 1) + \mathbb{1}_{|\theta^\varepsilon v| > R} \frac{m_*}{8} |v'|^2 \omega(v') \right) |\nu|^2 m^\varepsilon \, d\mathbf{u},$$

where we used the shorthand notation  $v' = \mathcal{V}^\varepsilon + \theta^\varepsilon v$ . To conclude, we estimate the right-hand side in the latter inequality applying assumption (2.1a) on  $N$ . Since  $\mathcal{V}^\varepsilon$  is uniformly bounded, we obtain

$$-\langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu], \nu \rangle_{L^2(m^\varepsilon)} \leq \frac{\alpha}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu] + C \|\nu\|_{L^2(m^\varepsilon)}^2,$$

for some constant  $C$  only depending on  $\alpha$ ,  $\kappa$ ,  $m_*$ ,  $m_p$ ,  $\bar{m}_p$  and the data of the problem:  $N$ ,  $A_0$  and  $\Psi$ .  $\square$

We also mention the following general result, which may be interpreted as a Poincaré inequality in the functional space  $L^2(m^\varepsilon)$

**Lemma 4.6.** *For all  $\mathbf{x} \in K$  and all function  $\nu$  in  $H_w^1(m_{\mathbf{x}}^\varepsilon)$ , holds the following estimates*

$$\|\nu\|_{L^2(m^\varepsilon)} \leq \frac{1}{\sqrt{\kappa}} \|\partial_w \nu\|_{L^2(m^\varepsilon)} \quad \text{and} \quad \|w \nu\|_{L^2(m^\varepsilon)} \leq \frac{2}{\kappa} \|\partial_w \nu\|_{L^2(m^\varepsilon)}.$$

*Proof.* The proof relies on the following relation

$$\frac{1}{2} \int_{\mathbb{R}^2} (1 + \kappa w^2) |\nu|^2 m_{\mathbf{x}}^\varepsilon(\mathbf{u}) \, d\mathbf{u} = - \int_{\mathbb{R}^2} w \nu (\partial_w \nu) m_{\mathbf{x}}^\varepsilon(\mathbf{u}) \, d\mathbf{u},$$

which is obtained by an integration by part in the left-hand side of the equality. From the latter relation we obtain the result applying Young's inequality in the right-hand side for the first estimate and Cauchy-Schwarz inequality for the second one.  $\square$

From Lemma 4.4, we deduce regularity estimates for the solution  $\nu^\varepsilon$  to equation (1.10). The main challenge consists in propagating the  $\mathcal{H}^0$ -norm. Then we easily adapt our analysis to the case of the  $\mathcal{H}^k$ -norms, when  $k$  is greater than 0. Indeed, the  $w$ -derivatives of  $\nu^\varepsilon$  solve equation (1.10) with additional source terms which we are able to control with the dissipation brought by the Fokker-Planck operator. More precisely, equation (1.10) on  $\nu^\varepsilon$  reads as follows

$$\partial_t \nu^\varepsilon + \mathcal{A}^\varepsilon [\nu^\varepsilon] = 0,$$

where the operator  $\mathcal{A}^\varepsilon$  is given by

$$(4.12) \quad \mathcal{A}^\varepsilon [\nu^\varepsilon] = \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon \nu^\varepsilon] - \frac{1}{(\theta^\varepsilon)^2} \mathcal{F}_{\rho_0^\varepsilon} [\nu^\varepsilon].$$

With this notation, the equations on the  $w$ -derivatives read as follows

$$(4.13) \quad \partial_t h^\varepsilon + \mathcal{A}^\varepsilon [h^\varepsilon] = \frac{1}{\theta^\varepsilon} \partial_w \nu^\varepsilon + b h^\varepsilon,$$

where  $h^\varepsilon = \partial_w \nu^\varepsilon$ , and

$$(4.14) \quad \partial_t g^\varepsilon + \mathcal{A}^\varepsilon [g^\varepsilon] = \frac{2}{\theta^\varepsilon} \partial_w h^\varepsilon + 2b g^\varepsilon,$$

where  $g^\varepsilon$  is given by  $\partial_w^2 \nu^\varepsilon$ .

**Proposition 4.7.** *Under assumptions (2.1a)-(2.1b) and (4.2) on the drift  $N$ , (2.2)-(2.3) on the interaction kernel  $\Psi$ , consider a sequence of smooth solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) with initial conditions satisfying assumptions (2.4)-(2.7) and (4.4) with an exponent  $\kappa$  greater than  $1/(2b)$ . Then, for all  $\varepsilon > 0$ , holds the following estimate*

$$\left\| \partial_w^k \nu^\varepsilon(t, \mathbf{x}) \right\|_{L^2(m_{\mathbf{x}}^\varepsilon)} \leq e^{Ct} \left\| \partial_w^k \nu_0^\varepsilon(\mathbf{x}) \right\|_{L^2(m_{\mathbf{x}}^\varepsilon)}, \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K,$$

for all integer  $k$  less than or equal to 2 and some positive constant  $C$  only depending on  $\kappa$ ,  $m_*$ ,  $m_p$ ,  $\bar{m}_p$  (see assumptions (2.5), (2.6) and (2.7)) and the data of the problem:  $N$ ,  $A_0$  and  $\Psi$ .

*Proof.* We start with  $k = 0$ . We compute the time derivative of  $\|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2$  multiplying equation (1.10) by  $\nu^\varepsilon m$  and integrating with respect to  $\mathbf{u}$ . After integrating by part the stiffer term, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \frac{1}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon} [\nu^\varepsilon] = - \langle \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon \nu^\varepsilon], \nu^\varepsilon \rangle_{L^2(m^\varepsilon)},$$

for all  $\varepsilon > 0$  and all  $(t, \mathbf{x}) \in \mathbb{R}_+ \times K$ . Since  $\kappa$  is greater than  $1/(2b)$ , we apply Lemma 4.4 with  $\alpha = 1$ . This leads to the following inequality

$$\frac{d}{dt} \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2 \leq C \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2,$$

for some constant  $C$  only depending on  $\kappa$ ,  $m_*$ ,  $m_p$ ,  $\bar{m}_p$  and on the data of the problem:  $N$ ,  $A_0$  and  $\Psi$ . According to Gronwall's lemma, it yields

$$\|\nu^\varepsilon(t, \mathbf{x})\|_{L^2(m_{\mathbf{x}}^\varepsilon)} \leq e^{Ct} \|\nu^\varepsilon(0, \mathbf{x})\|_{L^2(m_{\mathbf{x}}^\varepsilon)}, \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K.$$

In the remaining of this proof, we shall adapt the former computations to the cases  $\partial_w^k \nu^\varepsilon$  when  $k$  less than or equal to 2. Due to the linear properties of equation (1.10) with respect to  $w$ , these terms solve the same equation as  $\nu^\varepsilon$  with additional source terms which add no further difficulty.

Let us now treat the case  $k = 1$ . We write  $h^\varepsilon = \partial_w \nu^\varepsilon$ . We compute the derivative of  $\|h^\varepsilon\|_{L^2(m^\varepsilon)}^2$  multiplying equation (4.13) by  $h^\varepsilon m^\varepsilon$  and integrating with respect to  $\mathbf{u}$ . After integrating by part the stiffer term, we obtain

$$\frac{1}{2} \frac{d}{dt} \|h^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \frac{1}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon} [h^\varepsilon] = - \langle \operatorname{div}_{\mathbf{u}} [\mathbf{b}_0^\varepsilon h^\varepsilon], h^\varepsilon \rangle_{L^2(m^\varepsilon)} + b \|h^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \mathcal{B},$$

for all  $\varepsilon > 0$  and all  $(t, \mathbf{x}) \in \mathbb{R}_+ \times K$ , where  $\mathcal{B}$  is given by

$$\mathcal{B} = \frac{1}{\theta^\varepsilon} \int_{\mathbb{R}^2} \partial_v \nu^\varepsilon h^\varepsilon m^\varepsilon d\mathbf{u}.$$

We estimate  $\mathcal{B}$  integrating by part and applying Young's inequality. It yields

$$\mathcal{B} \leq \frac{C}{\eta} \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon} [h^\varepsilon].$$

for some positive constant  $C$  and for all positive  $\eta$ . Then we apply Lemma 4.6, which yields

$$\mathcal{B} \leq \frac{C}{\eta} \|h^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon} [h^\varepsilon],$$

and conclude this step following the same method as in the former step of the proof.

The last case  $k = 2$  relies on the same arguments as the former step. Indeed, equation (4.14) on  $\partial_w^2 \nu^\varepsilon$  is the same as equation (4.13) on  $\partial_w \nu^\varepsilon$  up to a constant. Consequently, we skip the details and conclude this proof.  $\square$

Due to the cross terms between the  $v$  and  $w$  variables in equation (1.2), we are led to estimate mixed quantities of the form  $w^{k_1} \partial_w^{k_2} \nu^\varepsilon$ . These estimates are easily obtained from Proposition 4.7 and Lemma 4.6

**Corollary 4.8.** *Under the assumptions of Proposition 4.7, we consider  $(k, k_1, k_2)$  in  $\mathbb{N}^3$  such that  $k_1 + k_2 = k$  and  $k \leq 2$ . Then, for all  $\varepsilon > 0$ , it holds*

$$\left\| \left( w^{k_1} \partial_w^{k_2} \right) \nu^\varepsilon(t, \mathbf{x}) \right\|_{L^2(m_{\bar{\mathbf{x}}}^\varepsilon)} \leq \left( \frac{2}{\kappa} \right)^{k_1} e^{Ct} \left\| \partial_w^k \nu_0^\varepsilon(\mathbf{x}) \right\|_{L^2(m_{\bar{\mathbf{x}}}^\varepsilon)}, \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K,$$

for some positive constant  $C$  only depending on  $\kappa, m_*, m_p, \bar{m}_p$  (see assumptions (2.5), (2.6) and (2.7)) and the data of the problem:  $N, A_0$  and  $\Psi$ .

*Proof.* We consider  $(k, k_1, k_2)$  in  $\mathbb{N}^3$  such that  $k_1 + k_2 = k$  and  $k \leq 2$  and point out that according to Lemma 4.6, we have

$$\left\| \left( w^{k_1} \partial_w^{k_2} \right) \nu^\varepsilon(t, \mathbf{x}) \right\|_{L^2(m_{\bar{\mathbf{x}}}^\varepsilon)} \leq \left( \frac{2}{\kappa} \right)^{k_1} \left\| \partial_w^k \nu^\varepsilon(t, \mathbf{x}) \right\|_{L^2(m_{\bar{\mathbf{x}}}^\varepsilon)}.$$

Consequently, we obtain the result applying Proposition 4.7.  $\square$

We conclude this section with providing regularity estimates for the limiting distribution  $\bar{\nu}$  with respect to the adaptation variable, which solves (1.9). The proof for this result is mainly computational since we have an explicit formula for the solutions to equation (1.9).

**Lemma 4.9.** *Consider some index  $k$  lying in  $\{0, 1\}$  and some  $\bar{\nu}_0$  lying in  $\mathcal{H}^k(\bar{m})$ . The solution  $\bar{\nu}$  to equation (1.9) with initial condition  $\bar{\nu}_0$  verifies*

$$\|\bar{\nu}(t)\|_{\mathcal{H}^k(\bar{m})} \leq \exp\left(\left(k + \frac{1}{2}\right)bt\right) \|\bar{\nu}_0\|_{\mathcal{H}^k(\bar{m})}, \quad \forall t \in \mathbb{R}^+.$$

*Proof.* Since  $\bar{\nu}$  solves (1.9), it is given by the following formula

$$\bar{\nu}_{t, \mathbf{x}}(w) = e^{bt} \bar{\nu}_{0, \mathbf{x}}\left(e^{bt} w\right), \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K.$$

Consequently, we easily obtain the expected result.  $\square$

**4.4. Proof of Theorem 4.1.** In the forthcoming analysis we quantify the convergence of  $\nu^\varepsilon$  towards the asymptotic profile  $\nu$  given by

$$\nu = \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu},$$

in the functional spaces  $\mathcal{H}^k(m^\varepsilon)$ . We introduce the orthogonal projection of  $\nu^\varepsilon$  onto the space of function with marginal  $\mathcal{M}_{\rho_0^\varepsilon}$  with respect to the voltage variable

$$\Pi \nu^\varepsilon = \mathcal{M}_{\rho_0^\varepsilon} \otimes \bar{\nu}^\varepsilon.$$

Furthermore, we consider the orthogonal component  $\nu_\perp^\varepsilon$  of  $\nu^\varepsilon$  with respect to the latter projection

$$\nu_\perp^\varepsilon = \nu^\varepsilon - \Pi \nu^\varepsilon.$$

With these notations we have

$$\|\nu^\varepsilon - \nu\|_{\mathcal{H}^k(m^\varepsilon)}^2 = \|\nu_\perp^\varepsilon\|_{\mathcal{H}^k(m^\varepsilon)}^2 + \|\bar{\nu}^\varepsilon - \bar{\nu}\|_{\mathcal{H}^k(\bar{m})}^2,$$

for  $k$  in  $\{0, 1\}$ . Therefore, we prove that  $\nu_\perp^\varepsilon$  and  $\bar{\nu}^\varepsilon - \bar{\nu}$  vanish as  $\varepsilon$  goes to zero in both  $\mathcal{H}^0$  and  $\mathcal{H}^1$ .

**4.4.1. Estimates for  $\nu_\perp^\varepsilon$ .** Our strategy relies on the same arguments as the ones we developed in the former section to prove Proposition 4.7. Indeed, the equation satisfied by  $\nu_\perp^\varepsilon$  is the same equation as equation (1.10) solved by  $\nu^\varepsilon$  with additional source terms. It reads as follows

$$(4.15) \quad \partial_t \nu_\perp^\varepsilon + \mathcal{A}^\varepsilon[\nu_\perp^\varepsilon] = \mathcal{S}[\nu^\varepsilon, \Pi \nu^\varepsilon],$$

where the operator  $\mathcal{A}^\varepsilon$  is given by (4.12) and the source terms are given by

$$\mathcal{S}[\nu^\varepsilon, \Pi \nu^\varepsilon] = \partial_w \left[ a \theta^\varepsilon \int v \nu^\varepsilon dv \mathcal{M}_{\rho_0^\varepsilon} - b w \Pi \nu^\varepsilon \right] - \operatorname{div}_u [\mathbf{b}_0^\varepsilon \Pi \nu^\varepsilon].$$

Consequently, our strategy consists in estimating the source terms using the regularity estimates provided by Proposition 4.7. Then, we adapt our analysis to the case of  $\partial_w \nu_\perp^\varepsilon$ , which we write  $h_\perp^\varepsilon = \partial_w \nu_\perp^\varepsilon$  and which solves again the same equation up to extra terms that add no difficulty

$$(4.16) \quad \partial_t h_\perp^\varepsilon + \mathcal{A}^\varepsilon[h_\perp^\varepsilon] = \mathcal{S}[h^\varepsilon, \Pi h^\varepsilon] + b h_\perp^\varepsilon + \frac{1}{\theta^\varepsilon} \partial_v \nu^\varepsilon,$$

where we used the notation  $h^\varepsilon = \partial_w \nu^\varepsilon$ .

**Proposition 4.10.** *Under assumptions (2.1a)-(2.1b) and (4.2) on the drift  $N$  and (2.2)-(2.3) on the interaction kernel  $\Psi$ , consider a sequence of solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) with initial conditions satisfying assumptions (2.4)-(2.7) and (4.4) with an index  $k$  in  $\{0, 1\}$  and an exponent  $\kappa$  greater than  $1/(2b)$ . Then, for all  $\varepsilon$  between 0 and 1 and any  $\alpha_*$  lying in  $(0, 1 - (2b\kappa)^{-1})$ , there exists a positive constant  $C$  independent of  $\varepsilon$  such that*

$$\|\nu_\perp^\varepsilon(t)\|_{\mathcal{H}^k(m^\varepsilon)} \leq e^{Ct} \|\nu_0^\varepsilon\|_{\mathcal{H}^{k+1}(m^\varepsilon)} \left( C \sqrt{\varepsilon} + \min \left\{ 1, e^{-\alpha_* t / \varepsilon} \varepsilon^{-\alpha_* / (2m_*)} \right\} \right), \quad \forall t \in \mathbb{R}^+.$$

*In this proposition, the constant  $C$  only depends on  $\alpha_*$ ,  $\kappa$ ,  $m_*$ ,  $m_p$ ,  $\bar{m}_p$  (see assumptions (2.5), (2.6) and (2.7)) and on the data of the problem:  $N$ ,  $A_0$  and  $\Psi$ .*

*Proof.* We first treat the case  $k = 0$ . All along this step of the proof, we consider some  $\varepsilon > 0$  and some  $(t, \mathbf{x})$  in  $\mathbb{R}_+ \times K$ ; we omit the dependence with respect to  $(t, \mathbf{x})$  when the context is clear. We compute the time derivative of  $\|\nu_\perp^\varepsilon\|_{L^2(m^\varepsilon)}^2$  multiplying equation (4.15) by  $\nu_\perp^\varepsilon m^\varepsilon$  and integrating with respect to  $\mathbf{u}$

$$\frac{1}{2} \frac{d}{dt} \|\nu_\perp^\varepsilon\|_{L^2(m^\varepsilon)}^2 = \langle \mathcal{S}[\nu^\varepsilon, \Pi \nu^\varepsilon], \nu_\perp^\varepsilon \rangle_{L^2(m^\varepsilon)} - \langle \mathcal{A}^\varepsilon[\nu_\perp^\varepsilon], \nu_\perp^\varepsilon \rangle_{L^2(m^\varepsilon)},$$

and we split the contribution of the source terms as follows

$$\langle \mathcal{S}[\nu^\varepsilon, \Pi \nu^\varepsilon], \nu_\perp^\varepsilon \rangle_{L^2(m^\varepsilon)} = \mathcal{A}_1 + \mathcal{A}_2,$$



where the terms  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are given by

$$\begin{cases} \mathcal{A}_1 = -\frac{1}{\theta^\varepsilon} \int_{\mathbb{R}^2} \partial_v [B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, w) \Pi \nu^\varepsilon] \nu_\perp^\varepsilon m^\varepsilon \, d\mathbf{u}, \\ \mathcal{A}_2 = -a \theta^\varepsilon \int_{\mathbb{R}^2} v \partial_w [\Pi \nu^\varepsilon] \nu_\perp^\varepsilon m^\varepsilon \, d\mathbf{u}, \end{cases}$$

where  $B_0^\varepsilon$  is given by (1.7).

Let us estimate  $\mathcal{A}_1$ . After an integration by part, this term rewrites as follows

$$\mathcal{A}_1 = \int_{\mathbb{R}^2} B_0^\varepsilon(t, \mathbf{x}, \theta^\varepsilon v, w) \Pi \nu^\varepsilon (m^\varepsilon)^{\frac{1}{2}} \frac{1}{\theta^\varepsilon} \partial_v [\nu_\perp^\varepsilon m^\varepsilon] (m^\varepsilon)^{-\frac{1}{2}} \, d\mathbf{u}.$$

According to items (2) and (4) in Theorem 2.7,  $\mathcal{V}^\varepsilon$  and  $\mathcal{E}(\mu^\varepsilon)$  are uniformly bounded with respect to both  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and  $\varepsilon$ . On top of that, according to assumptions (2.3) and (2.5),  $\Psi_{*r} \rho_0^\varepsilon$  stays uniformly bounded with respect to both  $\mathbf{x} \in K$  and  $\varepsilon$  as well. Consequently, applying Young's inequality to the former relation and using assumption (2.1b), which ensures that  $N$  has polynomial growth, we obtain

$$\mathcal{A}_1 \leq \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu_\perp^\varepsilon] + \frac{C}{\eta} \int_{\mathbb{R}^2} \left( |\theta^\varepsilon v|^{2p} + |w|^2 + 1 \right) |\bar{\nu}^\varepsilon|^2 \mathcal{M}_{\rho_0^\varepsilon} \bar{m} \, d\mathbf{u},$$

for all positive  $\eta$  and for some positive constant  $C$  which only depends on  $m_*$ ,  $m_p$  and  $\bar{m}_p$  and the data of the problem  $N$ ,  $\Psi$  and  $A_0$ . Taking advantage of the properties of the Maxwellian  $\mathcal{M}_{\rho_0^\varepsilon}$  and since  $\rho_0^\varepsilon$  meets assumption (2.5), the latter estimate simplifies into

$$\mathcal{A}_1 \leq \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu_\perp^\varepsilon] + \frac{C}{\eta} \left( \|\bar{\nu}^\varepsilon\|_{L^2(\bar{m})}^2 + \|w \bar{\nu}^\varepsilon\|_{L^2(\bar{m})}^2 \right).$$

Furthermore, according to Jensen's inequality, it holds

$$\|\bar{\nu}^\varepsilon\|_{L^2(\bar{m})}^2 + \|w \bar{\nu}^\varepsilon\|_{L^2(\bar{m})}^2 \leq \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \|w \nu^\varepsilon\|_{L^2(m^\varepsilon)}^2.$$

Therefore, we apply Corollary 4.8 and obtain the following estimate for  $\mathcal{A}_1$

$$\mathcal{A}_1 \leq \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu_\perp^\varepsilon] + \frac{C}{\eta} e^{Ct} \|\nu_0^\varepsilon\|_{H_w^1(m^\varepsilon)}^2.$$

To estimate  $\mathcal{A}_2$ , we apply the Cauchy-Schwarz inequality, use the properties of the Maxwellian  $\mathcal{M}_{\rho_0^\varepsilon}$  and assumption (2.5). It yields

$$\mathcal{A}_2 \leq C \left( \|\partial_w \bar{\nu}^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \|\nu_\perp^\varepsilon\|_{L^2(m^\varepsilon)}^2 \right).$$

According to the same remark as in the former step, it holds

$$\|\partial_w \bar{\nu}^\varepsilon\|_{L^2(\bar{m})}^2 + \|\nu_\perp^\varepsilon\|_{L^2(m^\varepsilon)}^2 \leq \|\nu^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \|\partial_w \nu^\varepsilon\|_{L^2(m^\varepsilon)}^2,$$

hence, applying Proposition 4.7, we obtain

$$\mathcal{A}_2 \leq C e^{Ct} \|\nu_0^\varepsilon\|_{H_w^1(m^\varepsilon)}^2.$$

To evaluate the contribution of  $\mathcal{A}^\varepsilon$  we replace it by its definition (4.12) and integrate by part the stiffer term. It yields

$$-\langle \mathcal{A}^\varepsilon[\nu_\perp^\varepsilon], \nu_\perp^\varepsilon \rangle_{L^2(m^\varepsilon)} = -\frac{1}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu_\perp^\varepsilon] - \langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu_\perp^\varepsilon], \nu_\perp^\varepsilon \rangle_{L^2(m^\varepsilon)}.$$

In order to close the estimate, we apply Lemma 4.4 and Proposition 4.7 to control the term associated to linear transport. It yields

$$-\langle \operatorname{div}_{\mathbf{u}}[\mathbf{b}_0^\varepsilon \nu_\perp^\varepsilon], \nu_\perp^\varepsilon \rangle_{L^2(m^\varepsilon)} \leq \frac{\alpha}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon}[\nu_\perp^\varepsilon] + C e^{Ct} \|\nu_0^\varepsilon\|_{L^2(m^\varepsilon)}^2,$$

for all positive constant  $\alpha$  greater than  $1/(2b\kappa)$ . Consequently, the former computations lead to the following differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\nu_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 + \frac{1 - \alpha - \eta}{(\theta^{\varepsilon})^2} \mathcal{D}_{\rho_0^{\varepsilon}}[\nu_{\perp}^{\varepsilon}] \leq \frac{C}{\eta} e^{Ct} \|\nu_0^{\varepsilon}\|_{H_w^1(m^{\varepsilon})}^2.$$

Based on our assumptions, exponent  $\kappa$  is greater than  $1/(2b)$  and it is therefore possible to choose the constants  $\alpha$  and  $\eta$  such that  $\alpha_* = 1 - \alpha - \eta$  lies in  $]0, 1 - (2b\kappa)^{-1}[$ . Furthermore, in our context, the Gaussian-Poincaré inequality rewrites

$$\|\nu_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 \leq \mathcal{D}_{\rho_0^{\varepsilon}}[\nu_{\perp}^{\varepsilon}],$$

see [14] for a comprehensive analysis of this inequality and its extensions. According to the latter remarks, the former inequality rewrites

$$\frac{d}{dt} \|\nu_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 + \frac{2\alpha_*}{(\theta^{\varepsilon})^2} \|\nu_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 \leq C e^{Ct} \|\nu_0^{\varepsilon}\|_{H_w^1(m^{\varepsilon})}^2.$$

We multiply this estimate by

$$\exp\left(2\alpha_* \int_0^t \frac{1}{(\theta^{\varepsilon})^2} ds\right),$$

and integrate between 0 and  $t$ . In the end, we deduce the following inequality

$$\|\nu_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 \leq \|\nu_0^{\varepsilon}\|_{H_w^1(m^{\varepsilon})}^2 e^{-2\alpha_* I(t)} \left(1 + C \int_0^t e^{Cs} e^{2\alpha_* I(s)} ds\right),$$

where  $I$  is given by

$$I(t) = \int_0^t \frac{1}{(\theta^{\varepsilon})^2} ds.$$

Taking advantage of the ODE solved by  $\theta^{\varepsilon}$  (see (1.11)), we compute explicitly  $I$

$$I(t) = \frac{t}{\varepsilon} + \frac{1}{2\rho_0^{\varepsilon}} \ln(\theta^{\varepsilon}(t)^2).$$

Consequently, the latter estimate rewrites

$$\|\nu_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 \leq \|\nu_0^{\varepsilon}\|_{H_w^1(m^{\varepsilon})}^2 e^{-2\alpha_* \frac{t}{\varepsilon}} (\theta^{\varepsilon}(t))^{-2\frac{\alpha_*}{\rho_0^{\varepsilon}}} \left(1 + C \int_0^t e^{Cs} e^{2\alpha_* \frac{s}{\varepsilon}} (\theta^{\varepsilon}(s))^{2\frac{\alpha_*}{\rho_0^{\varepsilon}}} ds\right).$$

Then we notice that according to the explicit formula (1.11) for  $\theta^{\varepsilon}$ , given that  $\varepsilon$  lies in  $(0, 1)$ , we have on the one hand

$$e^{-2\alpha_* \frac{t}{\varepsilon}} (\theta^{\varepsilon}(t))^{-2\frac{\alpha_*}{\rho_0^{\varepsilon}}} \leq \min\left(1, e^{-2\alpha_* \frac{t}{\varepsilon}} \varepsilon^{-\frac{\alpha_*}{\rho_0^{\varepsilon}}}\right),$$

and on the other hand

$$\left(\frac{\theta^{\varepsilon}(s)}{\theta^{\varepsilon}(t)}\right)^{2\frac{\alpha_*}{\rho_0^{\varepsilon}}} \leq \min\left(2e^{2\alpha_* \frac{t-s}{\varepsilon}}, C\left(1 + e^{-2\alpha_* \frac{t}{\varepsilon}} \varepsilon^{-\frac{\alpha_*}{\rho_0^{\varepsilon}}}\right)\right).$$

We inject these bounds and take the supremum over all  $\mathbf{x}$  in  $K$  in the latter estimate. In the end, we obtain the estimate for the case where  $k = 0$  in Proposition 4.10.

We turn to the case  $k = 1$  in Proposition 4.10. We make use of the shorthand notation  $h_{\perp}^{\varepsilon} = \partial_w \nu_{\perp}^{\varepsilon}$ . We compute the time derivative of  $\|h_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2$  multiplying equation (4.16) by  $h_{\perp}^{\varepsilon} m^{\varepsilon}$  and integrating with respect to  $\mathbf{u}$

$$\frac{1}{2} \frac{d}{dt} \|h_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 = \langle \mathcal{S}^{\varepsilon}[h^{\varepsilon}, \Pi h^{\varepsilon}], h_{\perp}^{\varepsilon} \rangle_{L^2(m^{\varepsilon})} - \langle \mathcal{A}^{\varepsilon}[h_{\perp}^{\varepsilon}], h_{\perp}^{\varepsilon} \rangle_{L^2(m^{\varepsilon})} + b \|h_{\perp}^{\varepsilon}\|_{L^2(m^{\varepsilon})}^2 + \mathcal{A},$$

where  $\mathcal{A}$  is given by

$$\mathcal{A} = \frac{1}{\theta^\varepsilon} \int_{\mathbb{R}^2} \partial_v \nu^\varepsilon h_\perp^\varepsilon m^\varepsilon \, d\mathbf{u}.$$

We estimate  $\mathcal{A}$  integrating by part with respect to  $v$  and applying Young's inequality. After applying Proposition 4.7, it yields

$$\mathcal{A} \leq \frac{1}{4\eta} e^{Ct} \|\nu_0^\varepsilon\|_{L^2(m^\varepsilon)}^2 + \frac{\eta}{(\theta^\varepsilon)^2} \mathcal{D}_{\rho_0^\varepsilon} [h_\perp^\varepsilon],$$

for some positive constant  $C$  and all positive  $\eta$ . Then we follow the same argument as in the last step and obtain the expected result.  $\square$

4.4.2. *Estimate for  $(\bar{\nu}^\varepsilon - \bar{\nu})$ .* It solves the following equation

$$(4.17) \quad \partial_t (\bar{\nu}^\varepsilon - \bar{\nu}) - b \partial_w [w (\bar{\nu}^\varepsilon - \bar{\nu})] = -a \theta^\varepsilon \partial_w \left[ \int_{\mathbb{R}} v \nu^\varepsilon \, dv \right],$$

obtained taking the difference between equations (1.9) and (1.8). It is the same equation as (1.9) solved by  $\bar{\nu}$  with the additional source term on the right-hand side of the latter equation. Consequently, our strategy consists in estimating the source term. We point out that since the source term is weighted by  $\theta^\varepsilon$ , it is sufficient to prove that it is bounded in order to obtain convergence. However, it is not hard to check that the source term cancels if we replace  $\nu^\varepsilon$  by its projection  $\Pi \nu^\varepsilon$

$$\partial_w \left[ \int_{\mathbb{R}} v \Pi \nu^\varepsilon \, dv \right] = 0.$$

Consequently, based on the estimates obtained in the first step on  $\nu_\perp^\varepsilon$  (see Proposition 4.10), we expect

$$\theta^\varepsilon \partial_w \left[ \int_{\mathbb{R}} v \nu^\varepsilon \, dv \right] \underset{\varepsilon \rightarrow 0}{=} O(\varepsilon).$$

This formal approach was already rigorously justified in a weak convergence setting in [2]. In our setting and due to the structure of the source term, we use regularity estimates to achieve the latter convergence rate.

**Lemma 4.11.** *Consider a sequence of solutions  $(\mu^\varepsilon)_{\varepsilon > 0}$  to (1.2) with initial conditions satisfying assumption (4.4) with an index  $k$  in  $\{0, 1\}$  as well as the solution  $\bar{\nu}$  to equation (1.9) with some initial condition  $\bar{\nu}_0$  lying in  $\mathcal{H}^k(\bar{m})$ . The following estimate holds for all positive  $\varepsilon$*

$$\|\bar{\nu}^\varepsilon - \bar{\nu}\|_{H^k(\bar{m})} \leq e^{Ct} \left( \|\bar{\nu}_0^\varepsilon - \bar{\nu}_0\|_{H^k(\bar{m})} + \int_0^t e^{-Cs} \theta^\varepsilon \|\nu_\perp^\varepsilon\|_{H_w^{k+1}(m^\varepsilon)} \, ds \right), \quad \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times K,$$

where  $k$  lies in  $\{0, 1\}$  and for some positive constant  $C$  only depending on  $\kappa$ ,  $m_*$  and  $A$ .

*Proof.* We start with the case  $k = 0$ . We consider some  $\varepsilon > 0$  and some  $(t, \mathbf{x})$  in  $\mathbb{R}_+ \times K$ ; we omit the dependence with respect to  $(t, \mathbf{x})$  when the context is clear. We compute the time derivative of  $\|\bar{\nu}^\varepsilon - \bar{\nu}\|_{L^2(\bar{m})}^2$  multiplying equation (4.17) by  $(\bar{\nu}^\varepsilon - \bar{\nu}) \bar{m}$  and integrating with respect to  $w$ . We integrate by part the term associated to linear transport and end up with the following relation

$$\frac{1}{2} \frac{d}{dt} \|\bar{\nu}^\varepsilon - \bar{\nu}\|_{L^2(\bar{m})}^2 = \frac{b}{2} \int_{\mathbb{R}} (1 - \kappa w^2) |\bar{\nu}^\varepsilon - \bar{\nu}|^2 \bar{m} \, dw - a \theta^\varepsilon \int_{\mathbb{R}^2} v \partial_w \nu_\perp^\varepsilon (\bar{\nu}^\varepsilon - \bar{\nu}) \bar{m} \, d\mathbf{u}.$$

According to Cauchy-Schwarz inequality and applying assumption (2.5), the source term admits the bound

$$-a \theta^\varepsilon \int_{\mathbb{R}^2} v \partial_w \nu_\perp^\varepsilon (\bar{\nu}^\varepsilon - \bar{\nu}) \bar{m} \, d\mathbf{u} \leq C \theta^\varepsilon \|h_\perp^\varepsilon\|_{L^2(m^\varepsilon)} \|\bar{\nu}^\varepsilon - \bar{\nu}\|_{L^2(\bar{m})},$$

for some positive constant  $C$  only depending on  $A$  and  $m_*$ . On top of that we bound the term associated to linear transport using that the polynomial  $1 - \kappa w^2$  is upper-bounded over  $\mathbb{R}$ . Gathering the former considerations we end up with the following differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\bar{v}^\varepsilon - \bar{v}\|_{L^2(\bar{m})}^2 \leq C \left( \|\bar{v}^\varepsilon - \bar{v}\|_{L^2(\bar{m})}^2 + C \theta^\varepsilon \|h_\perp^\varepsilon\|_{L^2(m^\varepsilon)} \|\bar{v}^\varepsilon - \bar{v}\|_{L^2(\bar{m})} \right),$$

for some positive constant  $C$  only depending on  $\kappa$ ,  $m_*$  and  $A$ . we divide the latter inequality by  $\|\bar{v}^\varepsilon - \bar{v}\|_{L^2(\bar{m})}$  and obtain

$$\frac{d}{dt} \|\bar{v}^\varepsilon - \bar{v}\|_{L^2(\bar{m})} \leq C \left( \|\bar{v}^\varepsilon - \bar{v}\|_{L^2(\bar{m})} + C \theta^\varepsilon \|h_\perp^\varepsilon\|_{L^2(m^\varepsilon)} \right),$$

We conclude this step applying Gronwall's Lemma.

We treat the case  $k = 1$  applying the same method. Indeed,  $\bar{v}^\varepsilon - \bar{v}$  and  $\partial_w(\bar{v}^\varepsilon - \bar{v})$  solve the same equation up to an additional source term which adds no difficulty.  $\square$

**Proposition 4.12.** *Under assumptions (2.1a)-(2.1b) and (4.2) on the drift  $N$  and (2.2)-(2.3) on the interaction kernel  $\Psi$ , consider a sequence of solutions  $(\mu^\varepsilon)_{\varepsilon>0}$  to (1.2) with initial conditions satisfying assumptions (2.4)-(2.7) as well as the solution  $\bar{v}$  to equation (1.9) with some initial condition  $\bar{v}_0$  and an exponent  $\kappa$  greater than  $1/(2b)$ . Then there exists a positive constant  $C$  independent of  $\varepsilon$  such that for all  $\varepsilon$  between 0 and 1 hold the following statements*

(1) *suppose that assumptions (4.4)-(4.5) are fulfilled with an index  $k$  in  $\{0, 1\}$ , then*

$$\|\bar{v}^\varepsilon(t) - \bar{v}(t)\|_{\mathcal{H}^k(\bar{m})} \leq e^{Ct} \left( \|\bar{v}_0^\varepsilon - \bar{v}_0\|_{\mathcal{H}^k(\bar{m})} + C \|\nu_0^\varepsilon\|_{\mathcal{H}^{k+1}(m^\varepsilon)} \sqrt{\varepsilon} \right), \quad \forall t \in \mathbb{R}^+.$$

(2) *Supposing assumption (4.4) with index  $k = 1$  and assumption (4.5) with index  $k = 0$ , it holds*

$$\|\bar{v}^\varepsilon(t) - \bar{v}(t)\|_{\mathcal{H}^0(\bar{m})} \leq e^{Ct} \left( \|\bar{v}_0^\varepsilon - \bar{v}_0\|_{\mathcal{H}^0(\bar{m})} + C \|\nu_0^\varepsilon\|_{\mathcal{H}^2(m^\varepsilon)} \varepsilon \sqrt{|\ln \varepsilon| + 1} \right), \quad \forall t \in \mathbb{R}^+.$$

*In this proposition the positive constant  $C$  only depends on  $\kappa$ ,  $m_*$ ,  $m_p$ ,  $\bar{m}_p$  (see assumptions (2.5), (2.6) and (2.7)) and the data of the problem:  $N$ ,  $A_0$  and  $\Psi$ .*

*Proof.* We prove item (2) in the latter proposition. According to Lemma 4.4, we have

$$\|\bar{v}^\varepsilon - \bar{v}\|_{L^2(\bar{m})} \leq e^{Ct} \left( \|\bar{v}_0^\varepsilon - \bar{v}_0\|_{L^2(\bar{m})} + \int_0^t e^{-Cs} \theta^\varepsilon \|\nu_\perp^\varepsilon\|_{H_w^1(m^\varepsilon)} ds \right).$$

Therefore, the proof comes down to estimating the integral in the right-hand side of the latter inequality

$$\mathcal{A} := \int_0^t e^{-Cs} \theta^\varepsilon \|\nu_\perp^\varepsilon\|_{H_w^1(m^\varepsilon)} ds.$$

We apply the second estimate in Proposition 4.10 and Cauchy-Schwarz inequality. This yields

$$\mathcal{A} \leq C \|\nu_0^\varepsilon\|_{H_w^2(m^\varepsilon)} \left( \int_0^t |\theta^\varepsilon|^2 ds \right)^{1/2} \left( \int_0^t \varepsilon + \min \left\{ 1, e^{-2\alpha_* \frac{s}{\varepsilon}} \varepsilon^{-\frac{\alpha_*}{m_*}} \right\} ds \right)^{1/2}.$$

Then we inject the following estimate in the latter inequality

$$\min \left\{ 1, e^{-2\alpha_* s / \varepsilon} \varepsilon^{-\alpha_* / m_*} \right\} \leq \mathbf{1}_{\left\{ s \leq -\frac{1}{2m_*} \varepsilon \ln \varepsilon \right\}} + \mathbf{1}_{\left\{ s > -\frac{1}{2m_*} \varepsilon \ln \varepsilon \right\}} e^{-2\alpha_* \frac{s}{\varepsilon}} \varepsilon^{-\frac{\alpha_*}{m_*}}.$$

On top of that, we use the explicit formula for  $\theta^\varepsilon$  given by (1.11). In the end, we obtain

$$\mathcal{A} \leq C \|\nu_0^\varepsilon\|_{H_w^2(m^\varepsilon)} \varepsilon \sqrt{t+1} \sqrt{|\ln \varepsilon| + 1}.$$

Hence, we obtain the expected result taking the supremum over all  $\mathbf{x}$  in  $K$  in the latter estimate.

Item (1) in Proposition 4.12 is obtained following the same method excepted that we estimate  $\mathcal{A}$  using Proposition 4.7 with index  $k$  instead of Proposition 4.10.  $\square$

Let us now conclude the proof of Theorem 4.1. Item (1) is a consequence of Theorem 2.4 coupled with the regularity estimates provided in Proposition 4.7. Item (3) corresponds to item (2) in Proposition 4.12. Finally, we treat item (2) gathering the estimate in Proposition 4.10 and item (1) in Proposition 4.12.

**4.5. Proof of Theorem 4.2.** All along this proof, we consider some  $\varepsilon_0$  small enough so that the following condition is fulfilled

$$\|\rho_0^\varepsilon - \rho_0\|_{L^\infty(K)} < m_*/2,$$

for all  $\varepsilon$  less than  $\varepsilon_0$ . We omit the dependence with respect to  $(t, \mathbf{x}, \mathbf{u}) \in \mathbb{R}^+ \times K \times \mathbb{R}^2$  when the context is clear. We start by proving item (1) in Theorem 4.2. Since the cases  $k = 0$  and  $k = 1$  are treated the same way, we only detail the case  $k = 0$ . We consider some integer  $i$  and take some  $\varepsilon$  less than  $\varepsilon_0$ . Then we decompose the error as follows

$$\|(v - \mathcal{V})^i (\mu^\varepsilon - \mu)(t)\|_{\mathcal{H}^0(m^-)} \leq \mathcal{A} + \mathcal{B},$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are given by

$$\begin{cases} \mathcal{A} = \|(v - \mathcal{V})^i (\mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon}(\nu))\|_{\mathcal{H}^0(m^-)}, \\ \mathcal{B} = \|(v - \mathcal{V})^i (\tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon}(\nu) - \mu)\|_{\mathcal{H}^0(m^-)}, \end{cases}$$

where  $\nu$  is the limit of  $\nu^\varepsilon$  in Theorem 4.1 and is defined by (1.12) and where the operators  $\tau_{-\mathcal{U}^\varepsilon}$  and  $D_{\theta^\varepsilon}$  respectively stand for the translation of vector  $-\mathcal{U}^\varepsilon$  with respect to the  $\mathbf{u}$ -variable and the dilatation with parameter  $(\theta^\varepsilon)^{-1}$  with respect to the  $v$ -variable, that is

$$\tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon}(\nu)(t, \mathbf{x}, \mathbf{u}) = \frac{1}{\theta^\varepsilon} \nu\left(t, \mathbf{x}, \frac{v - \mathcal{V}^\varepsilon}{\theta^\varepsilon}, w - \mathcal{W}^\varepsilon\right).$$

The first term  $\mathcal{A}$  corresponds to the convergence of the re-scaled version  $\nu^\varepsilon$  of  $\mu^\varepsilon$  towards  $\nu$  whereas  $\mathcal{B}$  corresponds to the convergence of the macroscopic quantities.

We estimate  $\mathcal{A}$  as follows

$$\mathcal{A} \leq C(\mathcal{A}_1 + \mathcal{A}_2),$$

where  $C$  is a positive constant which only depends on  $i$  and where  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are given by

$$\begin{cases} \mathcal{A}_1 = \left\| (v - \mathcal{V}^\varepsilon)^i (\mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon}(\nu)) \right\|_{\mathcal{H}^0(m^-)}, \\ \mathcal{A}_2 = \|\mathcal{V} - \mathcal{V}^\varepsilon\|_{L^\infty(K)}^i \|\mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon}(\nu)\|_{\mathcal{H}^0(m^-)}. \end{cases}$$

According to item (2) in Theorem 2.7,  $\mathcal{V}^\varepsilon$  and  $\mathcal{W}^\varepsilon$  are uniformly bounded with respect to both  $(t, \mathbf{x}) \in \mathbb{R}^+ \times K$  and  $\varepsilon > 0$ , and  $0 \leq \theta^\varepsilon \leq 1$ , hence

$$m^-(\mathbf{x}, v, w) \leq C m^\varepsilon\left(\mathbf{x}, \frac{v - \mathcal{V}^\varepsilon}{\sqrt{2}\theta^\varepsilon}, w - \mathcal{W}^\varepsilon\right),$$

which yields

$$\mathcal{A}_1 \leq C \sup_{\mathbf{x} \in K} \left( \int_{\mathbb{R}^2} (v - \mathcal{V}^\varepsilon)^{2i} |\mu^\varepsilon - \tau_{-\mathcal{U}^\varepsilon} \circ D_{\theta^\varepsilon}(\nu)|^2 m^\varepsilon\left(\mathbf{x}, \frac{v - \mathcal{V}^\varepsilon}{\sqrt{2}\theta^\varepsilon}, w - \mathcal{W}^\varepsilon\right) d\mathbf{u} \right)^{\frac{1}{2}},$$

for another constant  $C > 0$  depending only on  $\kappa$ ,  $m_*$ ,  $m_p$  and  $\bar{m}_p$  (see assumptions (2.5)-(2.7)) and on the data of the problem  $N$ ,  $\Psi$  and  $A_0$ . Then we invert the change of variable (1.6) and notice that

$$v^{2i} m^\varepsilon \left( \mathbf{x}, \frac{v}{\sqrt{2}}, w \right) \leq C m^\varepsilon (\mathbf{x}, \mathbf{u}),$$

for some constant  $C > 0$  only depending on  $i$  and  $m_*$ . Consequently, we deduce

$$\mathcal{A}_1 \leq C \left\| (\theta^\varepsilon)^{i-\frac{1}{2}} \right\|_{L^\infty(K)} \|\nu^\varepsilon - \nu\|_{\mathcal{H}^0(m^\varepsilon)}.$$

Therefore, applying Theorem 4.1, using the compatibility assumption (4.7), and thanks to the constraint  $\theta^\varepsilon(t=0) = 1$ , which ensures

$$\|\nu_0^\varepsilon\|_{\mathcal{H}^1(m^\varepsilon)} \leq C \|\mu_0^\varepsilon\|_{\mathcal{H}^1(m^+)},$$

for some constant  $C$  depending only on  $m_p$ ,  $\kappa$  and  $m_*$ , we finally get

$$\mathcal{A}_1 \leq C e^{Ct} \varepsilon^{-\frac{1}{4}} \left( \varepsilon^{\frac{i}{2}} + e^{-i m_* \frac{t}{\varepsilon}} \right) \left( \varepsilon^{\frac{1}{2}} + e^{-\alpha_* \frac{t}{\varepsilon}} \varepsilon^{-\frac{\alpha_*}{2m_*}} \right).$$

Moreover, since  $\alpha_* < m_*/2$ , we deduce

$$\mathcal{A}_1 \leq C e^{Ct} \left( \varepsilon^{\frac{i}{2} + \frac{1}{4}} + e^{-\alpha_* \frac{t}{\varepsilon}} \varepsilon^{-\frac{1}{2}} \right).$$

To estimate  $\mathcal{A}_2$ , we apply item (1) in Theorem 2.7 and the compatibility assumption (4.7), which ensure

$$\|\mathcal{V} - \mathcal{V}^\varepsilon\|_{L^\infty(K)}^i \leq C e^{Ct} \varepsilon^i.$$

Then we follow the same method as before. In the end, we end up with the following bound for  $\mathcal{A}_2$

$$\mathcal{A}_2 \leq C e^{Ct} \left( \varepsilon^{i+\frac{1}{4}} + e^{-\alpha_* \frac{t}{\varepsilon}} \varepsilon^{i-\frac{1}{2}} \right).$$

Gathering these results, we obtain the following estimate for  $\mathcal{A}$

$$\mathcal{A} \leq C e^{Ct} \left( \varepsilon^{\frac{i}{2} + \frac{1}{4}} + e^{-\alpha_* \frac{t}{\varepsilon}} \varepsilon^{-\frac{1}{2}} \right).$$

We turn to  $\mathcal{B}$ . Similarly as before, we apply the triangular inequality and invert the change of variable (1.6). Then we apply Theorem 2.7, which yields

$$\mathcal{B} \leq C e^{Ct} \varepsilon^{-\frac{1}{4}} \left( \varepsilon^{\frac{i}{2}} + e^{-i m_* \frac{t}{\varepsilon}} \right) \left\| \nu - \tau_{\left(\frac{\nu^\varepsilon - \nu}{\theta^\varepsilon}, \mathcal{W}^\varepsilon - \mathcal{W}\right)} (\mathcal{M}_{\rho_0} \otimes \bar{\nu}) \right\|_{\mathcal{H}^0(D_{\sqrt{2}}(m^\varepsilon))},$$

where  $D_{\sqrt{2}}(m^\varepsilon)$  is a short-hand notation for

$$D_{\sqrt{2}}(m^\varepsilon)(\mathbf{x}, \mathbf{u}) = m^\varepsilon \left( \mathbf{x}, \frac{v}{\sqrt{2}}, w \right).$$

Then we decompose the right-hand side of the latter inequality as follows

$$\left\| \nu - \tau_{\left(\frac{\nu^\varepsilon - \nu}{\theta^\varepsilon}, \mathcal{W}^\varepsilon - \mathcal{W}\right)} (\mathcal{M}_{\rho_0} \otimes \bar{\nu}) \right\|_{\mathcal{H}^0(D_{\sqrt{2}}(m^\varepsilon))} \leq \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3,$$

where  $\mathcal{B}_1$ ,  $\mathcal{B}_2$  and  $\mathcal{B}_3$  are given by

$$\begin{cases} \mathcal{B}_1 = \|\bar{\nu} - \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})} \bar{\nu}\|_{\mathcal{H}^0(\bar{m})}, \\ \mathcal{B}_2 = \|\tau_{(\mathcal{W}^\varepsilon - \mathcal{W})} \bar{\nu}\|_{\mathcal{H}^0(\bar{m})} \left\| \mathcal{M}_{\rho_0^\varepsilon} - \tau_{\left(\frac{\nu^\varepsilon - \nu}{\theta^\varepsilon}\right)} \mathcal{M}_{\rho_0^\varepsilon} \right\|_{\mathcal{H}^0(\mathcal{M}_{\rho_0^\varepsilon}^{-1})}, \\ \mathcal{B}_3 = \|\tau_{(\mathcal{W}^\varepsilon - \mathcal{W})} \bar{\nu}\|_{\mathcal{H}^0(\bar{m})} \left\| \mathcal{M}_{\rho_0^\varepsilon} - \mathcal{M}_{\rho_0} \right\|_{\mathcal{H}^0(\mathcal{M}_{\rho_0^\varepsilon}^{-1})} e^{\|\mathcal{V}^\varepsilon - \mathcal{V}\|_{L^\infty(K)}^2 / (2m_* \varepsilon)}, \end{cases}$$

where we used that

$$D_{\sqrt{2}}(m^\varepsilon) \leq m^\varepsilon, \quad \text{and} \quad m^\varepsilon = \mathcal{M}_{\rho_0^\varepsilon}^{-1} \bar{m}.$$

Since equation (1.9) is linear,  $\bar{\nu} - \tau_{w_0} \bar{\nu}$  also solves the equation, therefore, applying Lemma 4.9 with  $w_0 = \mathcal{W}^\varepsilon - \mathcal{W}$ , it yields

$$\mathcal{B}_1 \leq C e^{\frac{b}{2}t} \left\| \bar{\nu}_0 - \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})} \bar{\nu}_0 \right\|_{\mathcal{H}^0(\bar{m})}.$$

Furthermore, since  $\bar{m} \leq \bar{m}^+$  and relying on assumption (4.8), we deduce

$$\mathcal{B}_1 \leq C e^{\frac{b}{2}t} \|\mathcal{W}^\varepsilon - \mathcal{W}\|_{L^\infty(K)}.$$

Therefore, according to item (1) in Theorem 2.7 we conclude

$$\mathcal{B}_1 \leq C e^{Ct} \varepsilon.$$

To estimate  $\mathcal{B}_2$  and  $\mathcal{B}_3$ , we follow the same method as for  $\mathcal{B}_1$ : we first apply the following relation

$$\left\| \mathcal{M}_{\rho_0^\varepsilon} - \tau_{\left(\frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon}\right)} \mathcal{M}_{\rho_0^\varepsilon} \right\|_{\mathcal{H}^0\left(\mathcal{M}_{\rho_0^\varepsilon}^{-1}\right)} = \left\| e^{\rho_0^\varepsilon \left| \frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon} \right|^2} - 1 \right\|_{L^\infty(K)}^{\frac{1}{2}},$$

which ensures

$$\left\| \mathcal{M}_{\rho_0^\varepsilon} - \tau_{\left(\frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon}\right)} \mathcal{M}_{\rho_0^\varepsilon} \right\|_{\mathcal{H}^0\left(\mathcal{M}_{\rho_0^\varepsilon}^{-1}\right)} \leq \left\| e^{\rho_0^\varepsilon \left| \frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon} \right|^2} \rho_0^\varepsilon \left| \frac{\mathcal{V}^\varepsilon - \mathcal{V}}{\theta^\varepsilon} \right|^2 \right\|_{L^\infty(K)}^{\frac{1}{2}}.$$

Furthermore, we apply Lemma 4.9, which ensures that

$$\left\| \tau_{(\mathcal{W}^\varepsilon - \mathcal{W})} \bar{\nu} \right\|_{\mathcal{H}^0(\bar{m})} \leq C e^{Ct}.$$

Therefore, applying item (1) in Theorem 2.7, we obtain

$$\mathcal{B}_2 \leq C e^{C(t + \varepsilon e^{Ct})} \varepsilon^{\frac{1}{2}}.$$

Then to estimate  $\mathcal{B}_3$ , an exact computation yields

$$\left\| \mathcal{M}_{\rho_0^\varepsilon} - \mathcal{M}_{\rho_0} \right\|_{\mathcal{H}^0\left(\mathcal{M}_{\rho_0^\varepsilon}^{-1}\right)} = \sup_{\mathbf{x} \in K} \left( \frac{|\rho_0 - \rho_0^\varepsilon|^2}{\sqrt{\rho_0^2 - (\rho_0 - \rho_0^\varepsilon)^2} \left( \rho_0 + \sqrt{\rho_0^2 - (\rho_0 - \rho_0^\varepsilon)^2} \right)} \right)^{\frac{1}{2}}.$$

Therefore, according to assumption (4.7) and item (1) in Theorem 2.7, which ensures that

$$e^{\|\mathcal{V}^\varepsilon - \mathcal{V}\|_{L^\infty(K)}^2 / (2m_* \varepsilon)} \leq e^{C e^{Ct} \varepsilon},$$

this yields

$$\mathcal{B}_3 \leq C e^{C(t + e^{Ct} \varepsilon)} \varepsilon.$$

In the end, we deduce the following estimate for  $\mathcal{B}$

$$\mathcal{B} \leq C e^{C(t + \varepsilon e^{Ct})} \left( \varepsilon^{\frac{i}{2} + \frac{1}{4}} + e^{-im_* \frac{t}{\varepsilon}} \varepsilon^{\frac{1}{4}} \right).$$

The proof for the statement (2) in Theorem 4.2 follows the same lines as the former one excepted that we apply item (3) in Theorem 4.1 instead of item (2) to quantify the convergence of  $\bar{\nu}^\varepsilon$  towards  $\bar{\nu}$ . Therefore, we do not detail the proof.

## 5. CONCLUSION

In this paper, we have characterized the blow-up profile of the voltage distribution in the regime of strong and short-range coupling between neurons and have computed the limiting distribution for the adaptation variable as well. Our result should be interpreted as the expansion up to the first term of the network's distribution as  $\varepsilon$  vanishes. A significant aspect of our analysis lies in the fact that we obtained quantitative convergence estimates in strong topology. More specifically, we provide two results. On the one hand, we prove convergence in a  $L^1$  functional framework. Our analysis relies on a modified Boltzmann entropy (see section 2.2) which is original to our knowledge. On the other hand, we prove convergence in a weighted  $L^2$  setting, in which we take advantage of the variational structure in order to obtain regularity estimates. This enables to recover the optimal convergence rates obtained in a weak convergence setting in probability spaces (see [2]). In many ways, the problem at hands in this article shows similarities with other problems coming from kinetic theory. Therefore, we hope for the relative entropy method developed in Section 3 that it might have applications in a broader context.

## ACKNOWLEDGMENT

The author thanks warmly Francis Filbet for all the discussions, advises and proofreading without which it would not have been possible to achieve this work.

The author gratefully acknowledges the support of ANITI (Artificial and Natural Intelligence Toulouse Institute). This project has received support from ANR ChaMaNe No: ANR-19-CE40-0024.

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