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XIAOFENG XUE AND LINJIE ZHAO

Abstract. We introduce the voter model on the infinite lattice with a slow membrane and investigate its hydrodynamic behavior. The model is defined as follows: a voter adopts one of its neighbors' opinion at rate one except for neighbors crossing the hyperplane $\{x : x_1 = 1/2\}$, where the rate is $\alpha N^{-\beta}$. Above, $\alpha > 0$, $\beta \geq 0$ are two parameters and N is the scaling parameter. The hydrodynamic equation turns out to be heat equation with various boundary conditions depending on the value of β. The proof is based on duality method.

1. Introduction

One of the main issues in statistical physics is to derive partial differential equations from microscopic systems. Roughly speaking, for symmetric systems the macroscopic behaviors are usually governed by diffusive equations, and for asymmetric systems usually by hyperbolic equations. We refer the readers to [9, 13] for a comprehensive understanding of the above subject. Recently, it has been a popular topic to establish PDEs with various boundary conditions from interacting particle systems. For example, Gonçalves, Franco and their collaborators have obtained the heat equation with Dirichlet/Robin/Neumann boundary conditions from the symmetric simple exclusion processes in one dimension [1, 5, 6]. The microscopic models they considered are either defined on a ring with a slow site/slow bond, or defined on a segment with slow boundaries. The results have also been extended to higher dimensions in [7, 14].

The voter model also plays an important role in the theory of interacting particle systems, and its hydrodynamic behavior has been considered by Presutti and Spohn in [12]. For each $x \in \mathbb{Z}^d$, imagine there is a voter at site x. Each voter has one of two possible opinions on an issue, and the possible opinions are denoted by 0 or 1. The dynamics is quite simple: a voter adopts one of its neighbors' opinion at rate one. Although the voter model has the same macroscopic behavior as the symmetric exclusion process, it differs from the later in essential points: the magnetization of the voter model is not locally conserved and the static correlations decay slowly. We refer the readers to [12, 11] for details of the above properties. A natural question is to consider the impact of the slow dynamics introduced in [1, 5, 6] on the voter model, which is the main aim of this article.

In order to obtain boundary conditions, we let the voters at sites x and $x + e_1$ adopt the other's opinion at rate $\alpha N^{-\beta}$ if $x_1 = 0$. Above, x_1 is the first coordinate of site x, and $\{e_i\}_{1 \leq i \leq d}$ is the canonical basis of $\mathbb{Z}^d$. The two parameters $\alpha > 0, \beta \geq 0$ denote the strength of the boundary interaction, and $N \in \mathbb{N}$ is a scaling parameter. We consider the voter model in dimensions $d \geq 4$, and derive Robin boundary conditions if $\beta = 1$ and Neumann boundary conditions if $\beta > 1$. There are no boundary conditions in the case $0 \leq \beta < 1$.

The results are not surprising since such boundary conditions have been well understood for the exclusion process. However, the proof differs a lot from the exclusion process because of the two properties we addressed above. The standard approach to prove a hydrodynamic limit result is as

Key words and phrases. Voter model; heat equation; boundary conditions; duality.

follows: we first write down a martingale formula for the empirical measure of the process by using Dynkin's formula, and then prove the so called replacement lemmas to close the equation. To ensure uniqueness of solution to hydrodynamic equations, an energy estimate is usually also needed. For exclusion process with boundary dynamics, the replacement lemmas and energy estimates are proved by investigating the entropy production of the process. This is not a easy task for the voter model with boundary terms since the invariant measures of the voter model in this case are not explicitly known. Instead, we adopt the duality method and investigate the correlation functions of the model. It is well known that the duality of the voter model is coalescing random walks, *cf.* [11] for example. For the voter model with boundary dynamics we introduced above, the duality turns out to be coalescing random walks with slow bonds. Along the proof, we exploit an invariance principle for the random walk with slow bonds, proved very recently by Erhard *et al.* in [3]. We also remark that no replacement lemmas are needed in Presutti and Spohn's paper [12] since the process is linear.

We believe the results should also hold for dimension $d = 3$. The main issue is to investigate the meeting probabilities of two independent random walks with slow bonds in dimension $d = 3$, which has long been a hard problem and has its own interest. This case is presently out of our reach and we leave it open. For dimensions $d \leq 2$, we refer the readers to the remark at the end of [12, Section 2].

The rest of the paper is organized as follows. In Section 2, we define the model rigorously and state our main results. We outline the standard approach to prove hydrodynamic limit results in Section 3. Replacement lemmas are proved in Section 4, and they are crucial to close the martingale formula as we addressed above. Finally, we prove the tightness and energy estimates for the voter model in Section 5.

2. Notation and Results

2.1. **The voter model.** The state space of the voter model is $\Omega_d := \{0,1\}^{\mathbb{Z}^d}$, $d \geq 4$. We use x, y (resp. u, v) to denote points in $\mathbb{Z}^d$ (resp. $\mathbb{R}^d$), and x_i (resp. u_i), $1 \leq i \leq d$, to denote the i-th component of the point x (resp. u). Let $\{e_i\}_{1\leq i\leq d}$ be the canonical basis of $\mathbb{Z}^d$. For a point $x \in \mathbb{Z}^d$, denote $|x| = \sum_{i=1}^d |x_i|$. Fix parameters $\alpha > 0$, $\beta \geq 0$. We use $N \in \mathbb{N}$ to denote the scaling parameter. The generator of the process $\mathcal{L}_N$ acting on local functions $f : \Omega_d \to \mathbb{R}$ is given by

$$\mathcal{L}_N f(\eta) = \sum_{\substack{x,\, y\in\mathbb{Z}^d,\\ |x-y|=1}} \xi^N_{x,y}\big(f(\eta^{x,y}) - f(\eta)\big), \tag{2.1}$$

where $\eta^{x,y}$ is the configuration obtained from η by flipping the value of $\eta(x)$ to $\eta(y)$, i.e., $\eta^{x,y}(x) = \eta(y)$ and $\eta^{x,y}(z) = \eta(z)$ for $z \neq x$, and the flipping rates $\xi^N_{x,y} = \xi^N_{y,x}$, $|x - y| = 1$, are given by

$$\xi^N_{x,y} = \begin{cases} \alpha N^{-\beta} & \text{if } x_1 = 0, y_1 = 1 \\ 1 & \text{otherwise} \end{cases} \tag{2.2}$$

We refer the readers to [11] for a construction of the process.

2.2. **Hydrodynamic limit.** Hereafter, we fix a time horizon $T > 0$. We first introduce weak solutions of heat equations with/without boundary conditions.

Definition 2.1 (Hydrodynamic equation for $0 \leq \beta < 1$)**.** *Let $\rho_0 : \mathbb{R}^d \to [0,1]$ be continuous. A bounded function $\rho : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is said to be a weak solution of the heat equation*

$$\begin{cases} \partial_t \rho(t,u) = \Delta \rho(t,u), & t > 0,\ u \in \mathbb{R}^d \\ \rho(0,u) = \rho_0(u), & u \in \mathbb{R}^d \end{cases} \tag{2.3}$$

if for any $H \in C_c^2(\mathbb{R}^d)$ *and any* $t \in [0,T]$,

$$\int_{\mathbb{R}^d} \rho(t,u)H(u)du = \int_{\mathbb{R}^d} \rho_0(u)H(u)du + \int_0^t \int_{\mathbb{R}^d} \rho(s,u)\Delta H(u)\,du\,ds.$$

For the case $\beta \geq 1$, we first introduce the notion of Sobolev spaces. For an open set $U \in \mathbb{R}^d$, let $\mathcal{H}^1(U)$ be the Sobolev space which consists of locally integrable functions with *weak derivatives* in $L^2(U)$, i.e., for any $\varrho \in \mathcal{H}^1(U)$, there exist d elements in $L^2(U)$, denoted by $\partial_{u_1}\varrho, \ldots, \partial_{u_d}\varrho$, such that

$$\int_U \varrho(u)\partial_{u_i}H(u)du = -\int_U \partial_{u_i}\varrho(u)H(u)du$$

for any test function $H \in C_c^\infty(U)$ and each $1 \leq i \leq d$. Denote by $L^2([0,T],\mathcal{H}^1(U))$ the space of measurable functions $\rho : [0,T] \to \mathcal{H}^1(U)$ such that the norm defined by

$$||\rho||^2_{L^2([0,T],\mathcal{H}^1(U))} := \int_0^T \int_U \sum_{i=1}^d \left(\partial_{u_i}\rho(t,u)\right)^2 du\,dt$$

is finite. We refer the readers to [4, Chapter 5] for properties of Sobolev spaces.

In this paper we are concerned with the case where $U = \mathbb{R}^d\backslash\{u \in \mathbb{R}^d : \ u_1 = 0\}$. For simplicity, from now on we write $\{u \in \mathbb{R}^d : \ u_1 \in A\}$ as $\{u_1 \in A\}$ for any $A \subseteq \mathbb{R}$. For any $f \in \mathcal{H}^1\left(\mathbb{R}^d\backslash\{u_1 = 0\}\right)$, we have that $f\big|_{u_1>0} \in \mathcal{H}^1\left(\{u_1 > 0\}\right)$ and $f\big|_{u_1<0} \in \mathcal{H}^1\left(\{u_1 < 0\}\right)$. Then, we use $f(\cdot^+)$ to denote the trace of $f\big|_{u_1>0}$ on $\{u_1 = 0\}$ and use $f(\cdot^-)$ to denote the trace of $f\big|_{u_1<0}$ on $\{u_1 = 0\}$. The following property of $f(\cdot^+)$ and $f(\cdot^-)$ is crucial for this paper. For any bounded $D \subseteq \{u_1 = 0\}$, according to the trace theorem (*cf.* Theorem 1 in [4, Section 5.5]), it is easy to check that

$$\text{Aver}_{\varepsilon,+}f\big|_D \to f(\cdot^+)\big|_D \text{ and } \text{Aver}_{\varepsilon,-}f\big|_D \to f(\cdot^-)\big|_D \tag{2.4}$$

in $L^2(D)$ as $\varepsilon \to 0$, where

$$\text{Aver}_{\varepsilon,+}f(u) = \frac{1}{(2\varepsilon)^{d-1}\varepsilon} \int\limits_{\substack{-\varepsilon<v_i<\varepsilon \text{ for } 2\leq i\leq d,\\ 0<v_1<\varepsilon}} f(u+v)dv$$

and

$$\text{Aver}_{\varepsilon,-}f(u) = \frac{1}{(2\varepsilon)^{d-1}\varepsilon} \int\limits_{\substack{-\varepsilon<v_i<\varepsilon \text{ for } 2\leq i\leq d,\\ -\varepsilon<v_1<0}} f(u+v)dv$$

for any $u \in \{u_1 = 0\}$.

To give test functions in definitions of weak solutions to hydrodynamic equations in the cases $\beta \geq 1$, we use $\mathcal{C}$ to denote the set of functions H such that

$$H(u) = H^+(u)1_{\{u_1>0\}} + H^-(u)1_{\{u_1\leq 0\}}$$

for some $H^+, H^- \in C_c^2(\mathbb{R}^d)$. Then for any $H \in \mathcal{C}$ and $u \in \{u_1 = 0\}$, it is reasonable to define

$$H(u^+) = H^+(u),\ H(u^-) = H^-(u),\ \partial^+_{u_1}H(u) = \partial_{u_1}H^+(u) \text{ and } \partial^-_{u_1}H(u) = \partial_{u_1}H^-(u).$$

Note that functions in $\mathcal{C}$ may be discontinuous at the boundary $\{u_1 = 0\}$.

Definition 2.2 (Hydrodynamic equation for $\beta = 1$)**.** *Let* $\rho_0 : \mathbb{R}^d \to [0,1]$ *be continuous. A bounded function* $\rho : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is said to be a weak solution of the heat equation with Robin boundary*

condition

$$\begin{cases} \partial_t \rho(t,u) = \Delta \rho(t,u), & t>0,\ u \in \mathbb{R}^d \backslash \{u_1 = 0\} \\ \partial^+_{u_1} \rho(t,u) = \partial^-_{u_1} \rho(t,u) = \alpha(\rho(t,u^+) - \rho(t,u^-)), & u_1 = 0, \\ \rho(0,u) = \rho_0(u), & u \in \mathbb{R}^d \end{cases} \tag{2.5}$$

if $\rho \in L^2([0,T], \mathcal{H}^1(\mathbb{R}^d \backslash \{u_1 = 0\}))$ *and for every function* $H \in \mathcal{C}$*, for every* $t \in [0,T]$*,*

$$\begin{aligned} \int_{\mathbb{R}^d} \rho(t,u)H(u)du = &\int_{\mathbb{R}^d} \rho_0(u)H(u)du \\ &+ \int_0^t \int_{\mathbb{R}^d} \rho(s,u) \sum_{i=2}^d \partial^2_{u_i} H(u) du\, ds + \int_0^t \int_{u_1 \neq 0} \rho(s,u) \partial^2_{u_1} H(u) du\, ds \\ &+ \int_0^t \int_{u_1 = 0} \rho_s(u^+) \partial^+_{u_1} H(u) - \rho_s(u^-) \partial^-_{u_1} H(u) + \alpha(\rho_s(u^-) - \rho_s(u^+))(H(u^+) - H(u^-))\, dS\, ds. \end{aligned}$$

Definition 2.3 (Hydrodynamic equation for $\beta > 1$)**.** *Let* $\rho_0 : \mathbb{R}^d \to [0,1]$ *be continuous. A bounded function* $\rho : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is said to be a weak solution of the heat equation with Neumann boundary condition*

$$\begin{cases} \partial_t \rho(t,u) = \Delta \rho(t,u), & t>0,\ u \in \mathbb{R}^d \backslash \{u_1 = 0\} \\ \partial^+_{u_1} \rho(t,u) = \partial^-_{u_1} \rho(t,u) = 0, & u_1 = 0, \\ \rho(0,u) = \rho_0(u), & u \in \mathbb{R}^d \end{cases} \tag{2.6}$$

if $\rho \in L^2([0,T], \mathcal{H}^1(\mathbb{R}^d \backslash \{u_1 = 0\}))$ *and for every function* $H \in \mathcal{C}$*, for every* $t \in [0,T]$*,*

$$\begin{aligned} \int_{\mathbb{R}^d} \rho(t,u)H(u)du = &\int_{\mathbb{R}^d} \rho_0(u)H(u)du \\ &+ \int_0^t \int_{\mathbb{R}^d} \rho(s,u) \sum_{i=2}^d \partial^2_{u_i} H(u) du\, ds + \int_0^t \int_{u_1 \neq 0} \rho(s,u) \partial^2_{u_1} H(u) du\, ds \\ &+ \int_0^t \int_{u_1 = 0} \rho_s(u^+) \partial^+_{u_1} H(u) - \rho_s(u^-) \partial^-_{u_1} H(u)\, dS\, ds. \end{aligned}$$

Remark 2.4 (Uniqueness of weak solutions)**.** *It is well known that the weak solution to the heat equation* (2.3) *is unique. Following the ideas in* [5, Section 7.2]*, it is easy to prove the weak solutions to the above PDEs* (2.5) *and* (2.6) *are unique. We also refer the readers to* [7, Section 7] *for the uniqueness of weak solutions to the above PDEs if the underlying space is the torus* $\mathbb{T}^d$*.*

Throughout the paper, we assume the initial distribution, denoted by μ_N, of the process satisfies the following condition.

Assumption (A). *Fix an initial density profile* $\rho_0 : \mathbb{R}^d \to [0,1]$ *such that* ρ_0 *has continuous and bounded partial derivative with respect to the first coordinate* u_1*. Under* μ_N*,* $\{\eta(x)\}_{x \in \mathbb{Z}^d}$ *are independent and*

$$\mu_N(\eta(x) = 1) = \rho_0\left(\tfrac{x}{N}\right)$$

for every $x \in \mathbb{Z}^d$*.*

Let η_t be the accelerated process with generator $N^2 \mathcal{L}_N$ and with initial condition μ_N. To make notations short, we omit the dependence of the process η_t on N. Denote by $\mathbb{P}$ the probability measure

on the path space $D([0,T],\Omega_d)$ associated with the process η_t and the initial distribution μ_N, and by $\mathbb{E}$ the corresponding expectation.

Now we are ready to state the main result of the article.

Theorem 2.5. *Assume $d \geq 4$. For every $t \in [0,T]$, every $\varepsilon > 0$ and every $H \in C_c(\mathbb{R}^d)$,*

$$\lim_{N\to\infty} \mathbb{P}\Big(\Big| \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \eta_t(x) H(x/N) - \int_{\mathbb{R}^d} \rho(t,u) H(u) du \Big| > \varepsilon \Big) = 0,$$

where $\rho(t,u)$ is the unique weak solution

(i) to the heat equation (2.3) if $0 \leq \beta < 1$;
(ii) to the heat equation with Robin boundary condition (2.5) if $\beta = 1$;
(iii) to the heat equation with Neumann boundary condition (2.6) if $\beta > 1$.

3. Proof Outline

In this section we outline the proof of Theorem 2.5. The procedures are quite standard, *cf.* [9, Chapter 4], and the main ingredients are the replacement lemmas proved in Section 4.

Denote by $\mathcal{M}_+(\mathbb{R}^d)$ the space of Radon measures on $\mathbb{R}^d$ endowed with the vague topology, that is, for a sequence $\{\nu^N\}_{N\geq 1}$, $\nu \in \mathcal{M}_+(\mathbb{R}^d)$, $\nu^N \to \nu$ as $N \to \infty$ if for every $f \in C_c(\mathbb{R}^d)$,

$$\lim_{N\to\infty} \langle \nu^N, f \rangle = \langle \nu, f \rangle.$$

For a configuration $\eta \in \Omega_d$, define the empirical measure $\pi^N(\eta) \in \mathcal{M}_+(\mathbb{R}^d)$ as

$$\pi^N(\eta; du) = \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \eta(x) \delta_{x/N}(du).$$

Put $\pi^N_t(du) = \pi^N(\eta_t; du)$. Let Q^N be the distribution on the path space $D([0,T], \mathcal{M}_+(\mathbb{R}^d))$ associated to the process π^N_t and the initial measure μ_N.

By Lemma 5.1, the sequence $\{Q^N\}_{N\geq 1}$ is tight. Whence, any subsequence of Q^N further has a subsequence that converges as $N \to \infty$, whose limit is denoted by Q^*. Moreover, Q^* is concentrated on trajectories which are absolutely continuous with respect to the Lebesgue measure. Denote by $\rho(t,u)$ the corresponding density.

Now we characterize the limit. Fix a test function $H : \mathbb{R}^d \to \mathbb{R}$ which will be specified later depending on whether $\beta < 1$ or $\beta \geq 1$. By Dynkin's martingale formula,

$$\pi^N_t(H) = \pi^N_0(H) + \int_0^t N^2 \mathcal{L}_N \langle \pi^N_s, H \rangle ds + M^N_t(H), \tag{3.1}$$

where $M^N_t(H)$ is a martingale, whose quadratic variation is given by

$$\int_0^t \big\{ N^2 \mathcal{L}_N \langle \pi^N_s, H \rangle^2 - 2N^2 \langle \pi^N_s, H \rangle \mathcal{L}_N \langle \pi^N_s, H \rangle \big\} ds.$$

A simple computation shows that the last line is bounded by $C_H N^{2-d}$ for some finite constant C_H, hence, by Doob's inequality,

$$\lim_{N\to\infty} \mathbb{E}\Big[\sup_{0\leq t\leq T} \big(M^N_t(H) \big)^2 \Big] = 0. \tag{3.2}$$

A tedious but elementary computation shows that the integrand in Eq. (3.1) is equal to

$$N^{-d} \sum_{x\in\mathbb{Z}^d} \sum_{i=2}^{d} \eta_s(x) \partial^2_{u_i} H\big(\tfrac{x}{N}\big) + N^{-d} \sum_{x_1 \neq 0,1} \eta_s(x) \partial^2_{u_1} H\big(\tfrac{x}{N}\big) + o_N(1) \tag{3.3}$$

$$+ N^{1-d} \sum_{x_1=1} \eta_s(x) \partial^+_{u_1} H\big(\tfrac{x}{N}\big) - N^{1-d} \sum_{x_1=0} \eta_s(x) \partial^-_{u_1} H\big(\tfrac{x}{N}\big) \tag{3.4}$$

$$+ \alpha N^{2-d-\beta} \sum_{x_1=0} \eta_s(x) \Big(H\big(\tfrac{x+e_1}{N}\big) - H\big(\tfrac{x}{N}\big)\Big) + \alpha N^{2-d-\beta} \sum_{x_1=1} \eta_s(x) \Big(H\big(\tfrac{x-e_1}{N}\big) - H\big(\tfrac{x}{N}\big)\Big). \tag{3.5}$$

For a configuration η and a positive integer $k \in \mathbb{N}$, define the space average of η over a box of size k centered at x as

$$\bar\eta^k(x) = \frac{1}{(2k+1)^d} \sum_{|y-x|\le k} \eta(y).$$

Similarly, the space averages over the right/left boxes are defined as

$$\bar\eta^{k,+}(x) = \frac{1}{|\Lambda^+_{x,k}|} \sum_{y\in\Lambda^+_{x,k}} \eta(y), \quad \bar\eta^{k,-}(x) = \frac{1}{|\Lambda^-_{x,k}|} \sum_{y\in\Lambda^-_{x,k}} \eta(y),$$

where

$$\Lambda^+_{x,k} = \{y : |y-x| \le k, y_1 \ge x_1\}, \quad \Lambda^-_{x,k} = \{y : |y-x| \le k, y_1 \le x_1\}.$$

Next we discuss the three cases respectively.

The case $0 \le \beta < 1$. Take $H \in C^2_c(\mathbb{R}^d)$. Whence

$$\partial^+_{u_1} H\big(\tfrac{x}{N}\big) = \partial^-_{u_1} H\big(\tfrac{x}{N}\big).$$

By Lemma 4.1 (i), we could replace $\eta_s(x)$ by $\bar\eta^{\varepsilon N}_s(x)$ in (3.4). Note also that if $|x-y|=1$, then

$$|\bar\eta^{\varepsilon N}(x) - \bar\eta^{\varepsilon N}(y)| \le \frac{C}{\varepsilon N}$$

for some finite constant C. Whence, the time integral of the term (3.4) converges in probability to zero as $N\to\infty$. The term (3.5) vanishes in the limit in the same way. Therefore, the limit Q^* concentrates on trajectories whose densities $\rho(t,u)$ satisfy

$$\int_{\mathbb{R}^d} \rho(t,u)H(u)du = \int_{\mathbb{R}^d} \rho_0(u)H(u)du + \int_0^t \int_{\mathbb{R}^d} \rho(s,u)\Delta H(u)du\,ds$$

The case $\beta = 1$. Take $H \in \mathcal{C}$. By Lemma 4.1 (ii), we could replace the time integral of the term (3.4) with the time integral of

$$N^{1-d} \sum_{x_1=1} \bar\eta^{\varepsilon N,+}_s(x) \partial^+_{u_1} H\big(\tfrac{x}{N}\big) - N^{1-d} \sum_{x_1=0} \bar\eta^{\varepsilon N,-}_s(x) \partial^-_{u_1} H\big(\tfrac{x}{N}\big).$$

The term (3.5) is handled in the same way. Observe that

$$\bar\eta^{\varepsilon N,+}_s(x) = \big\langle \pi^N_s, \iota^+_{\varepsilon,x/N}\big\rangle + o_N(1), \quad \bar\eta^{\varepsilon N,-}_s(x) = \big\langle \pi^N_s, \iota^-_{\varepsilon,x/N}\big\rangle + o_N(1),$$

where

$$\iota^+_{\varepsilon,u}(v) = 2^{-d+1}\varepsilon^{-d}\mathbb{1}\{0 \le v_1 - u_1 \le \varepsilon, |v_i - u_i| \le \varepsilon, 2 \le i \le d\},$$
$$\iota^-_{\varepsilon,u}(v) = 2^{-d+1}\varepsilon^{-d}\mathbb{1}\{-\varepsilon \le v_1 - u_1 \le 0, |v_i - u_i| \le \varepsilon, 2 \le i \le d\}.$$

Therefore, by Eq. (2.4), as $N\to\infty$ and $\varepsilon\to 0$, the limit density $\rho(t,u)$ satisfies

$$\int_{\mathbb{R}^d} \rho(t,u)H(u)du = \int_{\mathbb{R}^d} \rho_0(u)H(u)du$$

$$+\int_0^t\int_{\mathbb{R}^d}\rho(s,u)\sum_{i=2}^d\partial^2_{u_i}H(u)du\,ds+\int_0^t\int_{u_1\neq 0}\rho(s,u)\partial^2_{u_1}H(u)du\,ds$$
$$+\int_0^t\int_{u_1=0}\rho_s(u^+)\partial^+_{u_1}H(u)-\rho_s(u^-)\partial^-_{u_1}H(u)+\alpha(\rho_s(u^-)-\rho_s(u^+))(H(u^+)-H(u^-))\,dS\,ds.$$

The case $\beta>1$. Take $H\in\mathcal{C}$. This case is similar to the case $\beta=1$. The difference is that the term (3.5) converges in probability to zero as $N\to\infty$ since $\beta>1$. Therefore, the limit density $\rho(t,u)$ satisfies

$$\int_{\mathbb{R}^d}\rho(t,u)H(u)du=\int_{\mathbb{R}^d}\rho_0(u)H(u)du$$
$$+\int_0^t\int_{\mathbb{R}^d}\rho(s,u)\sum_{i=2}^d\partial^2_{u_i}H(u)du\,ds+\int_0^t\int_{u_1\neq 0}\rho(s,u)\partial^2_{u_1}H(u)du\,ds$$
$$+\int_0^t\int_{u_1=0}\rho_s(u^+)\partial^+_{u_1}H(u)-\rho_s(u^-)\partial^-_{u_1}H(u)\,dS\,ds.$$

In Lemma 5.2, we show that any limit Q^* of the sequence $\{Q^N\}_{N\geq 1}$ is concentrated on trajectories whose densities belong to the space $L^2([0,T],\mathcal{H}^1(\mathbb{R}^d\backslash\{u_1=0\}))$ if $\beta\geq 1$. Together with the above observations, Q^* is concentrated on trajectories whose densities are weak solutions to the corresponding hydrodynamic equations. Since the weak solution is unique, the limit Q^* is uniquely determined. This concludes the proof.

4. Replacement Lemma

The aim of this section is to prove the following replacement lemma.

Lemma 4.1 (Replacement Lemma)**.** *For every* $\delta>0$,

(*i*) *if* $0\leq\beta<1$, *then for every* $H\in C^2_c(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{P}\Big(\Big|\int_0^t N^{1-d}\sum_{x_1=1}\big(\eta_s(x)-\bar\eta^{\varepsilon N}_s(x)\big)H\left(\tfrac{x}{N}\right)ds\Big|>\delta\Big)=0.$$

The same result holds with the summation over $\{x_1=1\}$ *replaced by over* $\{x_1=0\}$.

(*ii*) *if* $\beta\geq 1$, *then for every* $H\in\mathcal{C}$

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{P}\Big(\Big|\int_0^t N^{1-d}\sum_{x_1=1}\big(\eta_s(x)-\bar\eta^{\varepsilon N,+}_s(x)\big)H\left(\tfrac{x}{N}\right)ds\Big|>\delta\Big)=0.$$

The same result holds with the summation over $\{x_1=1\}$ *replaced by over* $\{x_1=0\}$, *and with* $\bar\eta^{\varepsilon N,+}_s(x)$ *replaced by* $\bar\eta^{\varepsilon N,-}_s(x)$.

Before proving the above Lemma, we first recall in Subsection 4.1 the duality relationship between the voter model and the coalescing random walk introduced in [8], and in Subsection 4.2 an invariance principle for one dimensional symmetric random walks with slow bond proved recently in [3]. The proof of the above Lemma is presented in Subsection 4.3.

4.1. **Duality.** In this subsection we recall the duality relationship introduced in [8]. Let $\{Y^N_{t,\beta}\}_{t\geq 0}$ be the random walk on the one dimensional integer lattice $\mathbb{Z}$ with generator $\Omega_{N,\beta}$ given by

$$\Omega_{N,\beta}f(i)=\begin{cases} f(i+1)-f(i)+f(i-1)-f(i) & \text{if } i\neq 0,1,\\ \alpha N^{-\beta}\left[f(1)-f(0)\right]+f(-1)-f(0) & \text{if } i=0,\\ \alpha N^{-\beta}\left[f(0)-f(1)\right]+f(2)-f(1) & \text{if } i=1\end{cases}$$

for every $f:\mathbb{Z}\to\mathbb{R}$. Roughly speaking, the above random walk jumps to its neighbor at rate one everywhere except across the bond $(0,1)$, where the rate is $\alpha N^{-\beta}$. We further denote by $\{U_t\}_{t\geq 0}$ the usual simple symmetric random walk on $\mathbb{Z}$, i.e., the generator of U_t is given by $\Omega_{N,\beta}$ with (α,β) replaced by $(1,0)$. Let $\{X^N_{t,\beta}\}_{t\geq 0}$ be the random walk on $\mathbb{Z}^d$ that $\{\{X^N_{t,\beta}(i)\}_{t\geq 0}\}_{1\leq i\leq d}$ are independent, $\{X^N_{t,\beta}(1)\}_{t\geq 0}$ is a copy of $\{Y^N_{t,\beta}\}_{t\geq 0}$ and $\{X^N_{t,\beta}(i)\}_{t\geq 0}$ is a copy of $\{U_t\}_{t\geq 0}$ for $i=2,3,\dots,d$, where $X^N_{t,\beta}(i)$ is the i-th coordinate of $X^N_{t,\beta}$. We write $X^N_{t,\beta}$ as $X^{N,x}_{t,\beta}$ when $X^N_{0,\beta}=x$.

For given $s>0$ and $x\in\mathbb{Z}^d$, we denote by $\{\hat{X}^{N,x,s}_{t,\beta}\}_{t\geq 0}$ the *frozen* random walk on $\mathbb{Z}^d$ that $\hat{X}^{N,x,s}_{u,\beta}=x$ for $u\leq s$ and $\{\hat{X}^{N,x,s}_{s+t,\beta}\}_{t\geq 0}$ is an independent copy of $\{X^{N,x}_{t,\beta}\}_{t\geq 0}$. Intuitively, the random walk $\{\hat{X}^{N,x,s}_{t,\beta}\}_{t\geq 0}$ stays frozen at site x until time s, and then performs the random walk as $\{X^{N,x}_{t,\beta}\}_{t\geq 0}$ after time s. For given $x,y\in\mathbb{Z}^d, s>0$ and $\{X^{N,x}_{t,\beta}\}_{t\geq 0},\{\hat{X}^{N,y,s}_{t,\beta}\}_{t\geq 0}$ which are independent, we define

$$\tau^{N,\beta,s}_{x,y}=\inf\{u\geq s:\ X^{N,x}_{u,\beta}=\hat{X}^{N,y,s}_{u,\beta}\}-s,$$

i.e., $\tau^{N,\beta,s}_{x,y}$ is the time it takes for $X^{N,x}_{\cdot,\beta}$ and $\hat{X}^{N,y,s}_{\cdot,\beta}$ to meet after $\hat{X}^{N,y,s}_{\cdot,\beta}$ is unfrozen from y.

For given $x,y\in\mathbb{Z}^d$ and $s>0$, we denote by $\{\tilde{X}^{N,y,s}_{t,\beta,x}\}_{t\geq 0}$ the random walk on $\mathbb{Z}^d$ that

$$\tilde{X}^{N,y,s}_{t,\beta,x}=\begin{cases}\hat{X}^{N,y,s}_{t,\beta} & \text{if } t\leq s+\tau^{N,\beta,s}_{x,y},\\ X^{N,x}_{t,\beta} & \text{if } t>s+\tau^{N,\beta,s}_{x,y}.\end{cases}$$

Note that $\{\tilde{X}^{N,y,s}_{t,\beta,x}\}_{t\geq 0}$ and $\{\hat{X}^{N,y,s}_{t,\beta}\}_{t\geq 0}$ have the same distribution but $\{\tilde{X}^{N,y,s}_{t,\beta,x}\}_{t\geq 0}$ is not independent of $\{X^{N,x}_{t,\beta}\}_{t\geq 0}$. The process $\left\{\left(X^{N,x}_{t,\beta},\tilde{X}^{N,y,s}_{t,\beta,x}\right)\right\}_{t\geq 0}$ is the so-called *coalescing* random walk.

To distinguish from the accelerated voter model, let $\{\tilde{\eta}_t\}_{t\geq 0}$ be the process with generator given by $\mathcal{L}_N$. According to the duality-relationship between the voter model and the coalescing random walk given in [8], for given $x,y\in\mathbb{Z}^d$ and $t>s$,

$$\mathbb{P}\left(\tilde{\eta}_t(x)=1\right)=\sum_{u\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=u\right)\mathbb{P}(\tilde{\eta}_0(u)=1),\tag{4.1}$$

and

$$\begin{aligned}\mathbb{P}\left(\tilde{\eta}_t(x)=\tilde{\eta}_s(y)=1\right)&=\sum_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=u,\tilde{X}^{N,y,t-s}_{t,\beta,x}=v\right)\mathbb{P}(\tilde{\eta}_0(u)=\tilde{\eta}_0(v)=1)\\&=\sum_{u\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=u,\tau^{N,\beta,t-s}_{x,y}\leq s\right)P(\tilde{\eta}_0(u)=1)\\&\quad+\sum_{u\in\mathbb{Z}^d}\sum_{v\neq u}\mathbb{P}\left(X^{N,x}_{t,\beta}=u,\hat{X}^{N,y,t-s}_{t,\beta}=v,\tau^{N,\beta,t-s}_{x,y}>s\right)\mathbb{P}(\tilde{\eta}_0(u)=\tilde{\eta}_0(v)=1).\end{aligned}\tag{4.2}$$

Above, we also use $\mathbb{P}$ to denote the law of the process $\{\tilde{\eta}_t\}_{t\geq 0}$ and the random walks since this will not cause confusion. We refer the readers to [8] or [11, Section 3.4] for proofs of the above duality relations.

4.2. **Invariance principle.** In this subsection we recall the invariance principle given in [3] of the random walk $\{Y^N_{t,\beta}\}_{t\geq 0}$ on $\mathbb{Z}$ with slow bond $(0,1)$. We denote by $\{B_t\}_{t\geq 0}$ the 1-dimensional standard Brownian motion and write B_t as B^u_t when $B_0 = u$. For any $\beta \geq 0$, we define $\{B_{t,\beta}\}_{t\geq 0}$ as follows. If $0 \leq \beta < 1$, then $B_{t,\beta} = B_t$, i.e., the standard Brownian motion. When $\beta > 1$, then $B_{t,\beta} = |B^u_t|$ when $B_{0,\beta} = u \in [0^+, \infty)$ and $B_{t,\beta} = -|B^u_t|$ when $B_{0,\beta} = u \in (-\infty, 0^-]$, i.e., $B_{t,\beta}$ is the reflected Brownian motion. When $\beta = 1$, then $B_{t,\beta}$ is the *snapping out* Brownian motion with parameter 2α introduced in [10]. More precisely, for every $u \in (-\infty, 0^-] \cup [0^+, +\infty)$ and $f \in C_b\left((-\infty, 0^-] \cup [0^+, +\infty)\right)$,

$$\mathbb{E}_u[f(B_{t,\beta})] = \mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_t}}{2}\right) f\left(\mathrm{sgn}(u)|B^u_t|\right) + \left(\frac{1-e^{-2\alpha L_t}}{2}\right) f\left(-\mathrm{sgn}(u)|B^u_t|\right)\right],$$

where L_t is the local time of $\{B_t\}_{t\geq 0}$ at point 0. We write $B_{t,\beta}$ (resp. $Y^N_{t,\beta}$) as $B^u_{t,\beta}$ (resp. $Y^{N,u}_{t,\beta}$) when $B_{0,\beta} = u$ (resp. $Y^N_{0,\beta} = u$). The following invariance principle is proved by Erhard *et al.* in [3].

Theorem 4.2 ([3, Theorem 2.2])**.** *The following invariance principle holds for the one dimensional symmetric random walk $\{Y^N_{t,\beta}\}_{t\geq 0}$ with a slow bond $(0,1)$,*

(i) for every $u \neq 0$ and $t \geq 0$, $\frac{Y^{N,\lfloor uN\rfloor}_{tN^2,\beta}}{N}$ converges weakly to $B^u_{2t,\beta}$ as $N \to +\infty$;

(ii) for every $x \in \{1, 2, \ldots\}$ and $t \geq 0$, $\frac{Y^{N,x}_{tN^2,\beta}}{N}$ converges weakly to $B^{0^+}_{2t,\beta}$ as $N \to +\infty$;

(iii) for every $x \in \{-1, -2, \ldots\}$ and $t \geq 0$, $\frac{Y^{N,x}_{tN^2,\beta}}{N}$ converges weakly to $B^{0^-}_{2t,\beta}$ as $N \to +\infty$.

Note that although statements (ii) and (iii) in the above theorem are not listed in the main theorem of [3], they are direct corollaries of Lemma 5.2 and Eq. (5.11) of [3].

4.3. **Proof of Lemma 4.1.** Now we are ready to prove Lemma 4.1. We only prove statement (ii) since (i) could be handled with in the same way.

Proof of Lemma 4.1, (ii). By Chebyshev's inequality, we only need to show that

$$\lim_{\varepsilon\to 0}\limsup_{N\to+\infty} \mathbb{E}\left[\left(N^{1-d}\int_0^t \sum_{x_1=1} \left(\eta_s(x) - \bar{\eta}^{\varepsilon N,+}_s(x)\right) H\left(\tfrac{x}{N}\right) ds\right)^2\right] = 0. \tag{4.3}$$

We start with writing the expectation above as $\mathrm{I}^2 + \mathrm{II}$, where

$$\mathrm{I} = \mathbb{E}\left[N^{1-d}\int_0^t \sum_{x_1=1} \left(\eta_s(x) - \bar{\eta}^{\varepsilon N,+}_s(x)\right) H\left(\tfrac{x}{N}\right) ds\right]$$

and

$$\mathrm{II} = \mathrm{Var}\left(N^{1-d}\int_0^t \sum_{x_1=1} \left(\eta_s(x) - \bar{\eta}^{\varepsilon N,+}_s(x)\right) H\left(\tfrac{x}{N}\right) ds\right).$$

Then, Eq. (4.3) holds if we can check that

$$\lim_{\varepsilon\to 0}\limsup_{N\to+\infty} \mathrm{I} = 0 \tag{4.4}$$

and

$$\lim_{N\to+\infty} \mathrm{II} = 0 \tag{4.5}$$

for every $\varepsilon > 0$.

To check Eq. (4.4), we define

$$W_{t,\beta} = \left(W_{t,\beta}(1), W_{t,\beta}(2), \ldots, W_{t,\beta}(d)\right),$$

as the $((-\infty,0^-]\cup[0^+,+\infty))\times\mathbb{R}^{d-1}$-valued stochastic process, where $\{\{W_{t,\beta}(i)\}_{t\geq 0}\}_{1\leq i\leq d}$ are independent, $\{W_{t,\beta}(1)\}_{t\geq 0}$ is an independent copy of $B_{2t,\beta}$ introduced in the last subsection, and $W_{t,\beta}(i)$ is an independent copy of $\{B_{2t}\}_{t\geq 0}$ for $2\leq i\leq d$. We write $W_{t,\beta}$ as $W^u_{t,\beta}$ when $W_{0,\beta}=u$. By Eq. (4.1) and **Assumption (A)**,

$$\mathbb{E}[\eta_s(x)]=\mathbb{E}\left[\rho_0\left(\frac{X^{N,x}_{sN^2,\beta}}{N}\right)\right]\quad\text{and}\quad\mathbb{E}[\bar{\eta}^{\varepsilon N,+}_s(x)]=\frac{1}{|\Lambda^+_{x,\varepsilon N}|}\sum_{y\in\Lambda^+_{x,\varepsilon N}}\mathbb{E}\left[\rho_0\left(\frac{X^{N,y}_{sN^2,\beta}}{N}\right)\right]$$

for every $x\in\mathbb{Z}^d$. Hence, by Theorem 4.2,

$$\begin{aligned}\limsup_{N\to+\infty}\mathrm{I}=&\int_0^t\int_{u\in\mathbb{R}^{d-1}}H(0,u)\mathbb{E}\left[\rho_0\left(W^{(0^+,u)}_{s,\beta}\right)\right]duds\\&-\int_0^t\int_{u\in\mathbb{R}^{d-1}}H(0,u)\left\{\frac{1}{\varepsilon(2\varepsilon)^{d-1}}\int_{v\in[0,\varepsilon]\times[-\varepsilon,\varepsilon]^{d-1}}\mathbb{E}\left[\rho_0\left(W^{(0^+,u)+v}_{s,\beta}\right)\right]dv\right\}du\,ds,\end{aligned}$$

According to the definition of $W_{t,\beta}$,

$$\lim_{\substack{v_1\downarrow 0,\\ v_i\to 0\text{ for }2\leq i\leq d}}\mathbb{E}\left[\rho_0\left(W^{(0^+,u)+v}_{s,\beta}\right)\right]=\mathbb{E}\left[\rho_0\left(W^{(0^+,u)}_{s,\beta}\right)\right]$$

for every $u\in\mathbb{R}^{d-1}$ and hence Eq. (4.4) holds.

Now we check Eq. (4.5). According to the fact that the covariance operator is bilinear and $\eta_t=\tilde{\eta}_{tN^2}$, it is easy to check that Eq. (4.5) is a direct corollary of the following claim.

Claim. For every $s>0$,

$$\lim_{t\to+\infty}\sup_{N\geq 1,x,y\in Z^d}\mathrm{Cov}\left(\tilde{\eta}_t(x),\tilde{\eta}_s(y)\right)=0.\tag{4.6}$$

Now we prove the claim. By Eq. (4.1) and **Assumption (A)**, for $s<t$,

$$\begin{aligned}\mathbb{P}\left(\tilde{\eta}_t(x)=1\right)\mathbb{P}\left(\tilde{\eta}_s(y)=1\right)&=\sum_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=u\right)\mathbb{P}\left(X^{N,y}_{s,\beta}=v\right)\rho_0\left(\tfrac{u}{N}\right)\rho_0\left(\tfrac{v}{N}\right)\\&=\sum_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=u\right)\mathbb{P}\left(\hat{X}^{N,y,t-s}_{t,\beta}=v\right)\rho_0\left(\tfrac{u}{N}\right)\rho_0\left(\tfrac{v}{N}\right)\\&=\sum_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=u,\hat{X}^{N,y,t-s}_{t,\beta}=v\right)\rho_0\left(\tfrac{u}{N}\right)\rho_0\left(\tfrac{v}{N}\right).\end{aligned}$$

Then, by Eq. (4.2),

$$\mathrm{Cov}\left(\tilde{\eta}_t(x),\tilde{\eta}_s(y)\right)=\mathrm{III}+\mathrm{IV}+\mathrm{V},$$

where

$$\mathrm{III}=\sum_{u\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=u,\tau^{N,\beta,t-s}_{x,y}\leq s\right)\rho_0\left(\tfrac{u}{N}\right),$$

$$\mathrm{IV}=-\sum_{u\in\mathbb{Z}^d}\sum_{v\neq u}\mathbb{P}\left(X^{N,x}_{t,\beta}=u,\hat{X}^{N,y,t-s}_{t,\beta}=v,\tau^{N,\beta,t-s}_{x,y}\leq s\right)\rho_0\left(\tfrac{u}{N}\right)\rho_0\left(\tfrac{v}{N}\right),$$

and

$$\mathrm{V}=-\sum_{u\in\mathbb{Z}^d}\mathbb{P}\left(X^{N,x}_{t,\beta}=\hat{X}^{N,y,t-s}_{t,\beta}=u\right)\rho_0^2\left(\tfrac{u}{N}\right).$$

Since $\tilde{\eta}_t(x),\tilde{\eta}_s(y)$ are positive correlated under **Assumption (A)** and $\mathrm{IV},\mathrm{V}\leq 0$,

$$0\leq\mathrm{Cov}\left(\tilde{\eta}_t(x),\tilde{\eta}_s(y)\right)\leq\mathrm{III}\leq\mathbb{P}\left(\tau^{N,\beta,t-s}_{x,y}<+\infty\right).$$

According to Markov property,

$$\mathbb{P}\left(\tau_{x,y}^{N,\beta,t-s}<+\infty\right)=\sum_{u\in\mathbb{Z}^d}\mathbb{P}\left(X_{t-s,\beta}^{N,x}=u\right)\mathbb{P}\left(\tau_{u,y}^{N,\beta,0}<+\infty\right).$$

For any $x=(x_1,x_2,\ldots,x_d)\in\mathbb{Z}^d$, we define $x^\perp=(x_2,\ldots,x_d)\in\mathbb{Z}^{d-1}$. Denote by $\{V_t\}_{t\geq 0}$ the symmetric simple random walk on $\mathbb{Z}^{d-1}$. For $\omega,\varpi\in\mathbb{Z}^{d-1}$, let

$$p_t(\omega,\varpi)=\mathbb{P}\left(V_t=\varpi\middle|V_0=\omega\right),\quad \Gamma(\omega)=\mathbb{P}\left(V_t=0\text{ for some }t\geq 0\middle|V_0=\omega\right).$$

According to the definition of $\{X_{t,\beta}^{N,x}\}_{t\geq 0}$, $\left\{\left(X_{t,\beta}^{N,x}-\hat{X}_{t,\beta}^{N,y,0}\right)^\perp\right\}_{t\geq 0}$ is a copy of $\{V_{2t}\}_{t\geq 0}$ with $V_0=(x-y)^\perp$. As a result, $\mathbb{P}\left(\tau_{u,y}^{N,\beta,0}<+\infty\right)\leq\Gamma\left((u-y)^\perp\right)$ and hence

$$\mathbb{P}\left(\tau_{x,y}^{N,\beta,t-s}<+\infty\right)\leq\sum_{u\in\mathbb{Z}^d}\mathbb{P}\left(X_{t-s,\beta}^{N,x}=u\right)\Gamma\left((u-y)^\perp\right).$$

Without confusion, we also use $|\cdot|$ to denote the l^1-norm on $\mathbb{R}^{d-1}$. Then $\lim_{|\omega|\to+\infty}\Gamma(\omega)=0$ since $d-1\geq 3$. Therefore, for every $\delta>0$, there exists $M=M(\delta)$ such that $\Gamma(\omega)\leq\delta$ if $|\omega|>M$. As a result,

$$\mathbb{P}\left(\tau_{x,y}^{N,\beta,t-s}<+\infty\right)\leq\delta+\sum_{u:|(u-y)^\perp|\leq M}\mathbb{P}\left(X_{t-s,\beta}^{N,x}=u\right).$$

Since $\left\{\left(X_{t,\beta}^{N,x}\right)^\perp\right\}_{t\geq 0}$ is a copy of $\{V_t\}_{t\geq 0}$ with $V_0=x^\perp$,

$$\mathbb{P}\left(\tau_{x,y}^{N,\beta,t-s}<+\infty\right)\leq\delta+\mathbb{P}\left(|V_{t-s}-y^\perp|\leq M\middle|V_0=x^\perp\right)\leq\delta+(2M+1)^{d-1}p_{t-s}(0,0).$$

As a result,

$$\limsup_{t\to+\infty}\sup_{N\geq 1,x,y\in\mathbb{Z}^d}\mathrm{Cov}\left(\tilde{\eta}_t(x),\tilde{\eta}_s(y)\right)\leq\delta+(2M+1)^{d-1}\lim_{t\to+\infty}p_{t-s}(0,0)=\delta.$$

Since δ is arbitrary, Eq. (4.6) holds. As we have pointed out above, Eq. (4.6) implies Eq. (4.5) and the proof is completed. □

5. Tightness and Energy Estimates

In this section, we prove tightness of the sequence $\{Q^N\}_{N\geq 1}$ and an energy estimate for the densities of the limit points. The energy estimate is crucial to ensure the uniqueness of weak solutions to the hydrodynamic equations.

5.1. **Tightness.** The aim of this subsection is to prove the following tightness result.

Lemma 5.1 (Tightness)**.** *The sequence $\{Q^N\}_{N\geq 1}$ is tight. Moreover, any limit point of Q^N is concentrated on trajectories which are absolutely continuous with respect to the Lebesgue measure.*

Proof. The proof of tightness is standard, and so we only sketch the proof. The second statement in the lemma follows directly from the fact that $|\eta(x)|\leq 1$ for all x. To prove tightness, by [9, Chapter 4], we only need to prove for any $H\in C_c^2(\mathbb{R}^d)$,

$$\lim_{M\to\infty}\limsup_{N\to\infty}\mathbb{P}\Big(\sup_{0\leq t\leq T}|\langle\pi_t^N,H\rangle|\geq M\Big)=0 \tag{5.1}$$

and for every $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\Big(\sup_{0\leq t-s\leq\delta}\Big|\int_s^t\langle\pi_s^N,H\rangle ds\Big|\geq\varepsilon\Big)=0. \tag{5.2}$$

Eq. (5.1) follows from Chebyshev's inequality and the fact that $|\langle \pi_t^N, H\rangle| \leq C_H$ for some finite constant C_H. To prove Eq. (5.2), we only need to prove it separately for the martingale $M_t^N(H)$ defined in Eq. (3.1) and the terms (3.3)-(3.5). The martingale term satisfies (5.2) by Cauchy-Schwarz inequality and Eq. (3.2). Observe that all the terms (3.3)-(3.5) are bounded by C_H for some finite constant C_H, whence satisfy Eq. (5.2). This completes the proof. □

5.2. **Energy estimates.** In this subsection, we shall prove the following result.

Lemma 5.2 (Energy estimate)**.** *Fix $\beta \geq 1$. Any limit Q^* of the sequence $\{Q^N\}_{N\geq 1}$ is concentrated on paths $\pi_t(du) = \rho(t,u)du$ such that $\rho(t,u) \in L^2([0,T], \mathcal{H}^1(\mathbb{R}^d \backslash \{u_1 = 0\}))$.*

By Lemma 5.1,

$$Q^*(\pi_t(du) = \rho(t,u)\, du \text{ for all } 0 \leq t \leq T) = 1.$$

By [7, Section 5.3], to prove Lemma 5.2, we only need to prove the following result.

Lemma 5.3. *For any $M > 1$, there exists a constant $K = K(M) < +\infty$ such that*

$$E_{Q^*}\left[\sup_H \left(\int_0^T \int_{\mathbb{R}^d} (\partial_{u_1} H)(s,u)\rho(s,u)duds - \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} H^2(s,u)duds\right)\right] \leq K,$$

where the supremum is carried over all functions $H \in C^{0,1}([0,T]\times\mathbb{R}^d)$ with compact support contained in $[0,T]\times([-M,M]^d \backslash \{|u_1| \leq \frac{1}{M}\})$.

Based on the above lemma, the proof of Lemma 5.2 follows from the same analysis as that given in the proof of [7, Lemma 5.7], where a crucial step is the utilization of Riesz's representation theorem, the details of which we omit here.

To prove Lemma 5.3, we need the following lemma.

Lemma 5.4.

$$Q^*\left(\rho(t,u) = E_{Q^*}[\rho(t,u)] \text{ for all } 0 \leq t \leq T\right) = 1.$$

Proof of Lemma 5.4. Since Q^* is concentrated on càdlàg paths, we only need to show that

$$Q^*\left(\pi_t(H) = E_{Q^*}[\pi_t(H)]\right) = 1 \tag{5.3}$$

for every $0 < t \leq T$ and $H \in C_c(\mathbb{R}^d)$. To prove Eq. (5.3), we claim that

$$\lim_{N\to+\infty} \mathrm{Var}\left(\pi_t^N(H)\right) = 0 \tag{5.4}$$

for every $0 < t \leq T$ and $H \in C_c(\mathbb{R}^d)$.

We first use Eq. (5.4) to prove Eq. (5.3) and then check Eq. (5.4). Note that $\pi_\cdot$ is the weak limit of a subsequence of $\{\pi_\cdot^N\}_{N\geq 1}$ but we still write this subsequence as $\{\pi^N\}_{N\geq 1}$ for simplicity. Since $\eta_t(x) \leq 1$,

$$|\pi_t^N(H)| \leq C\|H\|_\infty$$

for every $N \geq 1$, where $\|H\|_\infty = \sup_{u\in\mathbb{Z}^d} |H(u)|$ and $C < +\infty$ is a constant only depending on the compact support of H. As a result, by dominated convergence theorem,

$$\lim_{N\to+\infty} \mathbb{E}[\pi_t^N(H)] = E_{Q^*}[\pi_t(H)]. \tag{5.5}$$

Then, Fatou's lemma and Eq. (5.4) imply that

$$\mathrm{Var}_{Q^*}\left(\pi_t(H)\right) \leq \liminf_{N\to+\infty} \mathbb{E}\left[\left(\pi_t^N(H)\right)^2\right] - \left(E_{Q^*}[\pi_t(H)]\right)^2$$

$$= \lim_{N\to+\infty}\left(\mathbb{E}\left[\left(\pi_t^N(H)\right)^2\right] - \left(\mathbb{E}[\pi_t^N(H)]\right)^2\right) = \lim_{N\to+\infty}\mathrm{Var}(\pi_t^N(H)) = 0$$

and hence Eq. (5.3) holds.

Now we check Eq. (5.4). According to an analysis similar with that leading to Eq. (4.6),

$$\begin{aligned}\mathrm{Var}\left(\pi_t^N(H)\right) &\le \frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d}\left|H\left(\frac{x}{N}\right)\right|\left|H\left(\frac{y}{N}\right)\right|\mathrm{Cov}(\eta_t(x),\eta_t(y))\\ &\le \frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d}\left|H\left(\frac{x}{N}\right)\right|\left|H\left(\frac{y}{N}\right)\right| P\left(\tau_{x,y}^{N,\beta,0}<+\infty\right)\\ &\le \frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d}\left|H\left(\frac{x}{N}\right)\right|\left|H\left(\frac{y}{N}\right)\right|\Gamma\left((y-x)^{\perp}\right).\end{aligned}$$

Since $d-1\ge 3$, for any $\varepsilon>0$, there exists $M=M(\varepsilon)<+\infty$ that $\Gamma(u)<\varepsilon$ when $|u|>M$. Since H is with compact support, there exists $C=C(H)\in[1,+\infty)$ that $H(u)=0$ when $|u_1|>C$. As a result, for sufficiently large N, the last formula is bounded by

$$\begin{aligned}&\frac{\varepsilon}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\sum_{y:|(y-x)^{\perp}|>M}\left|H\left(\frac{x}{N}\right)\right|\left|H\left(\frac{y}{N}\right)\right| + \frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\sum_{y:|(y-x)^{\perp}|\le M,\|y_1\|\le CN}\left|H\left(\frac{x}{N}\right)\right|\left|H\left(\frac{y}{N}\right)\right|\\ &\qquad\le C_H\varepsilon + C_H(2M+1)^{d-1}N^{1-d}.\end{aligned}$$

This is enough to prove Eq. (5.4). □

At last, we prove Lemma 5.3.

Proof of Lemma 5.3. By Lemma 5.4, we only need to show that

$$\sup_H\left(\int_0^T\int_{\mathbb{R}^d}(\partial_{u_1}H)(s,u)E_{Q^*}[\rho(s,u)]duds - \frac{1}{2}\int_0^T\int_{\mathbb{R}^d}H^2(s,u)duds\right) < +\infty \tag{5.6}$$

for each $M>0$, where the supremum is carried over all functions $H\in C^{0,1}([0,T]\times\mathbb{R}^d)$ with compact support contained in $[0,T]\times([-M,M]^d\backslash\{|u_1|\le\frac{1}{M}\})$.

Since Q^* is any weak limit of the sequence $\{Q^N\}_{N\ge1}$ along some subsequence, which we still denote by $\{Q^N\}_{N\ge1}$ for simplicity,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}(\partial_{u_1}H)(s,u)E_{Q^*}[\rho(s,u)]duds &= \lim_{\delta\to0}\lim_{N\to+\infty}\int_0^T\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\frac{H\left(s,\frac{x}{N}+\delta e_1\right)-H\left(s,\frac{x}{N}\right)}{\delta}\mathbb{E}[\eta_s(x)]ds\\ &= \lim_{\delta\to0}\lim_{N\to+\infty}\int_0^T\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}H\left(s,\tfrac{x}{N}\right)\frac{\mathbb{E}[\eta_s(x-N\delta e_1)]-\mathbb{E}[\eta_s(x)]}{\delta}ds,\end{aligned}$$

where $e_1=(1,0,\dots,0)$. Above, the first identify follows from Eq. (5.5) and the second from summation by parts. By Eq. (4.1),

$$\mathbb{E}[\eta_s(x)] = \mathbb{E}\left[\rho_0\left(\tfrac{X_{sN^2,\beta}^{N,x}}{N}\right)\right].$$

By Theorem 4.2,

$$\int_0^T\int_{\mathbb{R}^d}(\partial_{u_1}H)(s,u)E_{Q^*}[\rho(s,u)]duds = \lim_{\delta\to0}\int_0^T\int_{\mathbb{R}^d}H(s,u)\frac{\mathbb{E}[\rho_0(W_{s,\beta}^{u-\delta e_1})]-\mathbb{E}[\rho_0(W_{s,\beta}^u)]}{\delta}duds, \tag{5.7}$$

where $W_{s,\beta}$ is defined as in the proof of Lemma 4.1.

For all $\beta \geq 1$, we claim that there exists $C = C(\beta, \rho_0, M) < +\infty$ that

$$\left| \frac{\mathbb{E}[\rho_0(W^{u-\delta e_1}_{s,\beta})] - \mathbb{E}[\rho_0(W^{u}_{s,\beta})]}{\delta} \right| \leq C \tag{5.8}$$

for every (u, s, δ) satisfying $|u_1| \in \left[\frac{1}{M}, M\right], 0 \leq s \leq T$ and $\delta < \frac{1}{2M}$. This is enough to prove Eq. (5.6) since by Cauchy-Schwarz inequality, we may bound the right-hand side of (5.7) by

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^d} H^2(s,u)duds + \frac{C^2}{2}T(2M)^d.$$

It remains to prove Eq. (5.8). Without loss of generality, we assume that $u_1 \in \left[\frac{1}{M}, M\right]$ since the other part can be checked in the same way. If $\beta > 1$, since $u_1, u_1 - \delta > 0$ when $\delta < \frac{1}{2M}$,

$$W^u_{s,\beta} = \left(|B^{u_1}_{2s}(1)|, B^{u_2}_{2s}(2), \ldots, B^{u_d}_{2s}(d)\right), \quad W^{u-\delta e_1}_{s,\beta} = \left(|B^{u_1-\delta}_{2s}(1)|, B^{u_2}_{2s}(2), \ldots, B^{u_d}_{2s}(d)\right)$$

according to the definition of $W_{s,\beta}$ given in Section 4, where $\{B_t(i) : t \geq 0\}_{1\leq i\leq d}$ are independent copies of standard Brownian motions and $B^a_t(i)$ means $B_0(i) = a$ for $1 \leq i \leq d$. By translation invariance,

$$B^{u_1}_{2s}(1) =_d B^0_{2s}(1) + u_1, \quad B^{u_1-\delta}_{2s}(1) =_d B^0_{2s}(1) + u_1 - \delta.$$

By Lagrange's mean value theorem,

$$\left| \frac{\mathbb{E}[\rho_0(W^{u-\delta e_1}_{s,\beta})] - \mathbb{E}[\rho_0(W^{u}_{s,\beta})]}{\delta} \right| \leq \|\partial_{u_1}\rho_0\|_\infty$$

and hence Eq. (5.8) holds for $\beta > 1$.

When $\beta = 1$,

$$W^u_{s,\beta} = \left(B^{u_1}_{2s,1}, B^{u_2}_{2s}(2), \ldots, B^{u_d}_{2s}(d)\right), \quad W^{u-\delta e_1}_{s,\beta} = \left(B^{u_1-\delta}_{2s,1}, B^{u_2}_{2s}(2), \ldots, B^{u_d}_{2s}(d)\right),$$

where $\{B_{t,1}\}_{t\geq 0}$ is the snapping out Browning motion with parameter 2α. For any $u \in \mathbb{R}^d, v \in \mathbb{R}^{d-1}$, let $f_{u^\perp}(t, v)$ be the density function of $(B^{u_2}_t(2), B^{u_3}_t(3), \ldots, B^{u_d}_t(d))$ at v, then

$$\frac{\mathbb{E}[\rho_0(W^{u-\delta e_1}_{s,\beta})] - \mathbb{E}[\rho_0(W^{u}_{s,\beta})]}{\delta} = \int_{\mathbb{R}^{d-1}} \frac{\mathbb{E}\left[\rho_0\left(B^{u_1-\delta}_{2s,1}, v\right)\right] - \mathbb{E}\left[\rho_0\left(B^{u_1}_{2s,1}, v\right)\right]}{\delta} f_{u^\perp}(2s, v)dv. \tag{5.9}$$

As we have recalled in Section 4,

$$\begin{aligned}
\mathbb{E}[\rho_0(B^{u_1}_{2s,1}, v)] &= \mathbb{E}\left[\left(\frac{1 + e^{-2\alpha L_{2s}}}{2}\right)\rho_0\left(|B^{u_1}_{2s}|, v\right) + \left(\frac{1 - e^{-2\alpha L_{2s}}}{2}\right)\rho_0\left(-|B^{u_1}_{2s}|, v\right)\right] \\
&= \mathbb{E}\left[\left(\frac{1 + e^{-2\alpha L_{2s}(-u_1)}}{2}\right)\rho_0\left(|B^0_{2s} + u_1|, v\right) + \left(\frac{1 - e^{-2\alpha L_{2s}(-u_1)}}{2}\right)\rho_0\left(-|B^0_{2s} + u_1|, v\right)\right]
\end{aligned}$$

where L_t is the local time of $\{B_t\}_{t\geq 0}$ at point 0 and $L_t(a)$ is the local time of $\{B_t\}_{t\geq 0}$ at point a. Note that in the above equations we utilize the fact that B^a_t hits 0 if and only if B^0_t hits $-a$ when we write B^a_t as $B^0_t + a$ in the sense of coupling. Similarly, we may write $\mathbb{E}[\rho_0(B^{u_1-\delta}_{2s,1}, v)]$ as

$$\mathbb{E}\left[\left(\frac{1 + e^{-2\alpha L_{2s}(-u_1+\delta)}}{2}\right)\rho_0\left(|B^0_{2s} + u_1 - \delta|, v\right) + \left(\frac{1 - e^{-2\alpha L_{2s}(-u_1+\delta)}}{2}\right)\rho_0\left(-|B^0_{2s} + u_1 - \delta|, v\right)\right].$$

Hence,

$$\mathbb{E}\left[\rho_0\left(B^{u_1-\delta}_{2s,1}, v\right)\right] - \mathbb{E}\left[\rho_0\left(B^{u_1}_{2s,1}, v\right)\right] = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = -\mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_{2s}(-u_1)}}{2}\right)\rho_0\left(|B^0_{2s}+u_1|,v\right)\right] + \mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_{2s}(-u_1+\delta)}}{2}\right)\rho_0\left(|B^0_{2s}+u_1-\delta|,v\right)\right]$$

and

$$\mathrm{II} = -\mathbb{E}\left[\left(\frac{1-e^{-2\alpha L_{2s}(-u_1)}}{2}\right)\rho_0\left(-|B^0_{2s}+u_1|,v\right)\right] + \mathbb{E}\left[\left(\frac{1-e^{-2\alpha L_{2s}(-u_1+\delta)}}{2}\right)\rho_0\left(-|B^0_{2s}+u_1-\delta|,v\right)\right].$$

We only calculate I since II can be calculated in the same way. For any $a \neq 0$, we define $\tau_a = \inf\{t : \ B^0_t = a\}$. Let $p(t,a)$ be the density function of τ_a, then by strong Markov property,

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_{2s}(-u_1)}}{2}\right)\rho_0\left(|B^0_{2s}+u_1|,v\right)\right] &= \mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_{2s}(-u_1)}}{2}\right)\rho_0\left(|B^0_{2s}+u_1|,v\right)\mathbf{1}_{\{\tau_{-u_1}\leq 2s\}}\right] \\ &\quad + \mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_{2s}(-u_1)}}{2}\right)\rho_0\left(|B^0_{2s}+u_1|,v\right)\mathbf{1}_{\{\tau_{-u_1}>2s\}}\right] \\ &= \int_0^{2s} p(\theta,-u_1)\mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_{2s-\theta}}}{2}\right)\rho_0\left(|B^0_{2s-\theta}|,v\right)\right]d\theta + \mathbb{E}\left[\rho_0\left(|B^0_{2s}+u_1|,v\right)\mathbf{1}_{\{\tau_{-u_1}>2s\}}\right],\end{aligned}$$

where we utilize the facts that $L_t(a) = 0$ when $t < \tau_a$ and that $(L_t(-a), B^{-a}_t + a)$ and (L_t, B^0_t) have the same distribution. As a result,

$$\mathrm{I} = \mathrm{III} + \mathrm{IV},$$

where

$$\mathrm{III} = \int_0^{2s}\left(p(\theta,-u_1+\delta) - p(\theta,-u_1)\right)\mathbb{E}\left[\left(\frac{1+e^{-2\alpha L_{2s-\theta}}}{2}\right)\rho_0\left(|B^0_{2s-\theta}|,v\right)\right]d\theta$$

and

$$\mathrm{IV} = \mathbb{E}\left[\rho_0\left(|B^0_{2s}+u_1-\delta|,v\right)\mathbf{1}_{\{\tau_{-u_1+\delta}>2s\}}\right] - \mathbb{E}\left[\rho_0\left(|B^0_{2s}+u_1|,v\right)\mathbf{1}_{\{\tau_{-u_1}>2s\}}\right].$$

It is shown in [2, Chapter 8] that

$$p(\theta,a) = \frac{a}{\sqrt{2\pi}}e^{-\frac{a^2}{2\theta}}\theta^{-\frac{3}{2}}.$$

As a result, there exists $C_1 = C_1(M) < +\infty$ independent of $0 \leq s \leq T$ that

$$\int_0^r \left|\frac{d}{da}p(\theta,a)\right|d\theta < C_1$$

for any $0 \leq r \leq 2T$ and $\frac{1}{2M} \leq a \leq M$. Then, by Lagrange's mean value theorem, for $\delta < \frac{1}{2M}$,

$$\frac{|\mathrm{III}|}{\delta} \leq C_1\|\rho_0\|_\infty$$

Now we deal with IV. Since $\tau_{-u_1+\delta} < \tau_{-u_1}$ for $\{B^0_t\}_{t\geq 0}$,

$$\mathrm{IV} = \mathrm{V} + \mathrm{VI},$$

where

$$\mathrm{V} = -\mathbb{E}\left[\rho_0\left(|B^0_{2s}+u_1-\delta|,v\right)\mathbf{1}_{\{\tau_{-u_1}>2s>\tau_{-u_1+\delta}\}}\right]$$

and

$$\mathrm{VI} = \mathbb{E}\left[\left(\rho_0\left(|B^0_{2s}+u_1-\delta|,v\right) - \rho_0\left(|B^0_{2s}+u_1|,v\right)\right)\mathbf{1}_{\{\tau_{-u_1+\delta}>2s\}}\right].$$

According to the expression of V and the definition of $p(t,a)$,

$$|\mathrm{V}| \leq \|\rho_0\|_\infty\int_0^{2s}\left|p(\theta,-u_1+\delta) - p(\theta,-u_1)\right|d\theta$$

and hence $\frac{|\mathrm{V}|}{\delta} \le C_1\|\rho_0\|_\infty$ when $\delta < \frac{1}{2M}$ by Lagrange's mean value theorem. Since $||x|-|y|| \le |x-y|$, by Lagrange's mean value theorem,

$$\frac{|\mathrm{VI}|}{\delta} \le \|\partial_{u_1}\rho_0\|_\infty.$$

As a result,

$$\frac{|\mathrm{I}|}{\delta} \le \frac{|\mathrm{III}|+|\mathrm{IV}|}{\delta} \le 2C_1\|\rho_0\|_\infty + \|\partial_{u_1}\rho_0\|_\infty \tag{5.10}$$

when $\delta < \frac{1}{2M}$. Note that Eq. (5.10) still holds when $|\mathrm{I}|$ is replaced by $|\mathrm{II}|$ according to the same analysis. Therefore,

$$\left|\frac{\mathbb{E}\Big[\rho_0\left(B^{u_1-\delta}_{2s,1}, v\right)\Big] - \mathbb{E}\Big[\rho_0\left(B^{u_1}_{2s,1}, v\right)\Big]}{\delta}\right| \le 2\big(2C_1\|\rho_0\|_\infty + \|\partial_{u_1}\rho_0\|_\infty\big)$$

when $\delta < \frac{1}{2M}$ and hence Eq. (5.8) follows from Eq. (5.9) when $\beta = 1$. This completes the proof. □

Acknowledgments. Xue thanks the financial support from the National Natural Science Foundation of China with grant numbers 11501542. Zhao thanks the financial support from the ANR grant MICMOV (ANR-19-CE40-0012) of the French National Research Agency (ANR).

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