

Two results on nodal sets of eigenfunctions of sub-Laplacians

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March 25, 2022

Abstract

We prove two results concerning the nodal sets of eigenfunctions of sub-Laplacians. The first one asserts the validity in this setting of Courant's theorem on the number of nodal domains of eigenfunctions. The second one is the $C\lambda^{-\frac{1}{2}}$ density (with respect to the sub-Riemannian distance) of the nodal sets of eigenfunctions with eigenvalue λ .

1 Introduction and main results

Let M be a smooth, connected, compact manifold of dimension N endowed with a smooth volume μ . Let X_1, \dots, X_m be smooth vector fields on M satisfying Hörmander's bracket-generating condition $\text{Lie}(X_1, \dots, X_m) = TM$ and let Δ be the sub-Laplacian

$$\Delta = -\sum_{i=1}^m X_i^* X_i$$

where $X_i^* = -X_i - \text{div}_\mu(X_i)$ is the adjoint of X_i in $L^2(M, \mu)$.

Sub-Laplacians are a natural generalization of the Euclidean Laplacian and the Laplace-Beltrami operator in Riemannian geometry. They have been studied a lot, one of the milestones in this field being the paper [9] where Hörmander proved their hypoellipticity.

Let Ω be a connected open subset of M . The operator $\Delta : C_c^\infty(\Omega) \rightarrow C_c^\infty(\Omega)$ is non-positive, symmetric and densely defined in $L^2(\Omega, \mu)$. In the sequel, we denote by $(\Delta_\Omega, \mathcal{D}(\Delta_\Omega))$ its Friedrichs extension (see Section 2 for reminders). When $\partial\Omega \neq \emptyset$, this corresponds to Dirichlet boundary conditions.

Proposition 1. *The selfadjoint operator $(-\Delta_\Omega, \mathcal{D}(\Delta_\Omega))$ has discrete point spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \rightarrow +\infty$ (with repetitions according to multiplicities). There exists an orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}}$ of $L^2(\Omega, \mu)$ such that for every $n \in \mathbb{N}$, $\varphi_n \in \mathcal{D}(\Delta_\Omega)$ and $-\Delta_\Omega \varphi_n = \lambda_n \varphi_n$.*

Our goal is to study the nodal sets of eigenfunctions of $-\Delta_\Omega$ (i.e., the set where an eigenfunction φ vanishes) and the nodal components (i.e., the connected components of $\{\varphi \neq 0\}$). The analogous problem for the Laplace-Beltrami operator on a Riemannian manifold has been investigated a lot, see [13] for a recent overview. Our results extend results which are well-known for eigenfunctions of Laplace-Beltrami operators.

Assumption 1. *We assume that one of the following holds:*

1. $N = 2$
2. *or the manifold M , the volume μ and the vector fields X_1, \dots, X_m are real-analytic.*

Under this assumption, it is known (see [6] and [16]) that any u satisfying $(\Delta_\Omega - \lambda)u = 0$ in Ω for some $\lambda \in \mathbb{R}$ and vanishing in a non-empty open subset of Ω vanishes in fact everywhere in Ω . This unique continuation property will be used in the proof of Theorem 1.

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Assumption 2. We assume that either $\Omega = M$, or the boundary of Ω is smooth and non-characteristic, meaning that for any $x \in \partial\Omega$, there exists $1 \leq i \leq m$ such that $X_i(x) \notin T_x \partial\Omega$.

Under this second assumption, the eigenfunctions of $-\Delta_\Omega$ are smooth up to the boundary $\partial\Omega$ (see [12, Theorem III, point (4)]), which will be important in the proofs.

We denote by $Z_f = \{f = 0\} \subset \bar{\Omega}$ the zero set of a function f . A nodal domain of f is a connected component of $\Omega \setminus Z_f$. Our main results generalize to the subelliptic setting Courant's theorem ([8]) and the density of the zero set of eigenfunctions; they seem to be the first results concerning nodal sets of sub-Laplacians.

Theorem 1. Under Assumptions 1 and 2, for any $n \in \mathbb{N}$, any eigenfunction of $-\Delta_\Omega$ with eigenvalue λ_n has at most n nodal domains.

Theorem 2. Under Assumption 2, there exists $C > 0$ depending only on Ω such that for any eigenpair λ, φ , the nodal set Z_φ intersects any sub-Riemannian ball of radius greater than $C\lambda^{-1/2}$.

The definition of sub-Riemannian balls will be recalled in Section 5, they are the natural extension of Riemannian balls to the present subelliptic setting.

Remark 2. Theorem 2 is optimal in the following sense. Consider the Baouendi-Grushin sub-Laplacian $\Delta_{\text{BG}} = \partial_x^2 + x^2 \partial_y^2$ on $(-1, 1)_x \times (\mathbb{R}/2\pi\mathbb{Z})_y$. If $k \in \mathbb{Z}$ and ψ_k is in the lowest energy space of the 1D operator $H_k = -\partial_x^2 + k^2 x^2$ on $(-1, 1)_x$, then $\Psi_k : (x, y) \mapsto \psi_k(x) \cos(ky)$ is an eigenfunction of $-\Delta_{\text{BG}}$ associated with eigenvalue $\mu_k = |k| + o(1)$ as $k \rightarrow +\infty$. From the knowledge of the geometry of the Grushin balls (see [4, Section 3.1]), we obtain that balls of radius $c\mu_k^{-\frac{1}{2}}$ for c small enough do not necessarily intersect the nodal set of Ψ_k , proving the sharpness of Theorem 2.

The organization of the paper is as follows. In Section 2, we prove Proposition 1. In Section 3, we prove Theorem 1. In Section 4, we introduce tools coming from sub-Riemannian geometry, namely the nilpotent approximation and the desingularization, which are used in the proof of Theorem 2 given in Section 5.

Acknowledgments. I thank Yves Colin de Verdière, Luca Rizzi, Valentina Franceschi, Hajer Bahouri and Suresh Eswarathasan for interesting discussions concerning this note.

2 Proof of Proposition 1

In this section, we prove Proposition 1.

We first recall briefly the classical Friedrichs extension construction. We denote by q_Ω the quadratic form on $C_c^\infty(\Omega)$ given by $q_\Omega(v, w) = (\Delta v, w)$ where (\cdot, \cdot) denotes the $L^2(\Omega, \mu)$ scalar product. It is closable and we denote by \hat{q}_Ω its closure. Explicitly, denoting by H the Hilbert space completion of $C_c^\infty(\Omega)$ with respect to the scalar product $(v, w)_H = (v, w) + q_\Omega(v, w)$, the inclusion map $\iota : C_c^\infty(\Omega) \rightarrow L^2(\Omega, \mu)$ extends by continuity to a linear map $\hat{\iota} : H \rightarrow L^2(\Omega, \mu)$. The quadratic form q_Ω also extends by continuity to a quadratic form \hat{q}_Ω over H , so that if \bar{v} and \bar{w} denote the equivalence classes of $\{v_n\}$ and $\{w_n\}$ in H , then $\hat{q}(\bar{v}, \bar{w}) = \lim_{n \rightarrow \infty} q_\Omega(v_n, w_n)$. One can check that $\hat{\iota}$ is injective, hence \hat{q}_Ω can be seen as a quadratic form on $L^2(\Omega, \mu)$, with domain $\mathcal{D}(\hat{q}_\Omega) = \hat{\iota}(H)$. More concretely, the domain $\mathcal{D}(\hat{q}_\Omega)$ consists of those $v \in L^2(\Omega, \mu)$ such that there exists $\{v_n\} \subset C_c^\infty(\Omega)$ such that $v_n \rightarrow v$ in $L^2(\Omega, \mu)$ and $q_\Omega(v_n - v_\ell) \rightarrow 0$ as $\ell, n \rightarrow \infty$.

Then the Friedrichs extension of $(\Delta, C_c^\infty(\Omega))$ is the operator $(\Delta_\Omega, \mathcal{D}(\Delta_\Omega))$ where

$$\mathcal{D}(\Delta_\Omega) = \{v \in \mathcal{D}(\hat{q}_\Omega) : \hat{q}_\Omega(v, \cdot) \text{ is } L^2(\Omega, \mu) \text{-continuous}\} \quad (1)$$

and $\Delta_\Omega v \in L^2(\Omega, \mu)$ is defined through the Riesz representation theorem by the relation $(\Delta_\Omega v, w) = \hat{q}_\Omega(v, w)$ for any $w \in \mathcal{D}(\hat{q}_\Omega)$ (note that $\mathcal{D}(\hat{q}_\Omega)$ is dense in $L^2(\Omega, \mu)$).

By the same procedure, for any open set $U \subset M$ (in particular $U = M$) we obtain the Friedrichs extension $(\Delta_U, \mathcal{D}(\Delta_U))$ of the non-positive symmetric operator $\Delta : C_c^\infty(U) \rightarrow C_c^\infty(U)$ (densely defined on $L^2(U, \mu)$).

Lemma 3. *Let $v \in \mathcal{D}(\widehat{q}_\Omega)$, and denote by \bar{v} its extension by 0 in $M \setminus \Omega$. Then $\bar{v} \in \mathcal{D}(\widehat{q}_M)$.*

Proof. Let $v_n \in C_c^\infty(\Omega)$ such that $v_n \rightarrow v$ in $L^2(\Omega, \mu)$ and $q_\Omega(v_n - v_\ell) \rightarrow 0$ as $n, \ell \rightarrow \infty$. Denote by $\bar{v}_n \in C_c^\infty(M)$ the extension of v_n by 0 in $M \setminus \Omega$. We have $\bar{v}_n \rightarrow \bar{v}$ in $L^2(M, \mu)$ and

$$q_M(\bar{v}_n - \bar{v}_\ell) = (\Delta(\bar{v}_n - \bar{v}_\ell), \bar{v}_n - \bar{v}_\ell) = (\Delta(v_n - v_\ell), v_n - v_\ell) = q_\Omega(v_n - v_\ell) \xrightarrow{n, \ell \rightarrow \infty} 0.$$

Thus $\bar{v} \in \mathcal{D}(\widehat{q}_M)$. □

Classical subelliptic estimates [9] imply that $(\Delta_M, \mathcal{D}(\Delta_M))$ has a compact resolvent. Using [15, Theorem XIII.64 p.245], this implies that

$$\{\bar{v} \in \mathcal{D}(\widehat{q}_M) \mid \|\bar{v}\|_{L^2(M, \mu)} \leq 1; \widehat{q}_M(\bar{v}) \leq b\}$$

is compact in the $L^2(M, \mu)$ topology for all $b \in \mathbb{R}$. Hence

$$\{v \in \mathcal{D}(\widehat{q}_\Omega) \mid \|v\|_{L^2(\Omega, \mu)} \leq 1; \widehat{q}_\Omega(v) \leq b\} \quad (2)$$

is compact in the $L^2(\Omega, \mu)$ topology for all $b \in \mathbb{R}$, since v can be extended to \bar{v} thanks to Lemma 3. Applying again [15, Theorem XIII.64 p.245], this time to $(\Delta_\Omega, \mathcal{D}(\Delta_\Omega))$, we obtain the existence of a complete orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ in $\mathcal{D}(\Delta_\Omega)$ so that $-\Delta_\Omega \varphi_n = \lambda_n \varphi_n$ with $0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq \dots \rightarrow +\infty$.

3 Proof of Theorem 1

We start the proof of Theorem 1 with the following integration by parts formula which follows from the definition of the quadratic form \widehat{q}_Ω recalled above:

Lemma 4. *If $u \in \mathcal{D}(\widehat{q}_\Omega)$, then $X_i u \in L^2(\Omega, \mu)$. Moreover if $u, v \in \mathcal{D}(\widehat{q}_\Omega)$, then*

$$\widehat{q}_\Omega(u, v) = - \sum_{i=1}^m \int_{\Omega} (X_i u)(X_i v) d\mu.$$

We denote by E_{λ_k} the eigenspace associated to the eigenvalue λ_k of $-\Delta_\Omega$.

Lemma 5 (Min-max principle). [15, Theorem XIII.1 p. 76]

1. $\varphi \in \mathcal{D}(\widehat{q}_\Omega) \setminus \{0\}$ belongs to E_{λ_1} if and only if it minimizes over $\mathcal{D}(\widehat{q}_\Omega) \setminus \{0\}$ the Rayleigh quotient

$$R(\varphi) = \frac{\sum_{i=1}^m \|X_i \varphi\|_{L^2(\Omega, \mu)}^2}{\|\varphi\|_{L^2(\Omega, \mu)}^2}. \quad (3)$$

In this case $R(\varphi) = \lambda_1$.

2. If $\varphi \in \mathcal{D}(\widehat{q}_\Omega) \setminus \{0\}$ is orthogonal to $E_{\lambda_1}, \dots, E_{\lambda_{k-1}}$ and $R(\varphi) = \lambda_k$, then $\varphi \in E_{\lambda_k}$.

The next two lemmas are classical in the Riemannian setting but their proofs require some care in the present sub-Riemannian (sR) context.

Lemma 6. *Let $D \subset M$ be a connected open set with $\partial D \neq \emptyset$. Then $\lambda_1(D) > 0$, and $\lambda_1(D)$ is a simple eigenvalue. Moreover, any corresponding eigenvector does not change sign in D .*

Proof. Assume for the sake of a contradiction that $\lambda_1(D) = 0$ and let u be an eigenfunction $\Delta u = 0$. Then $(\Delta u, u) = 0$ hence by definition $\widehat{q}_D(u, u) = 0$, which implies $\|X_i u\|_{L^2(D, \mu)} = 0$ for any i thanks to Lemma 4. But thanks to hypoelliptic regularity [9] we know that $u \in C^\infty(D)$ (a priori not up to the boundary if D is arbitrary) hence $X_i u \equiv 0$ in D . Then $[X_{i_1}, [X_{i_2}, \dots]] \dots u \equiv 0$ for any bracket of the vector fields, hence by the Hörmander bracket-generating condition u is constant in D . The only constant which belongs to $D(\Delta_D)$ is 0 hence a contradiction.

Let $u_0 \neq 0$ be in the first eigenspace of Δ_D . Then $|u_0| \in \mathcal{D}(\widehat{q}_\Omega)$ and for any i ,

$$X_i|u_0| = \begin{cases} X_i u_0 & \text{a.e. in } \{u_0 > 0\} \\ 0 & \text{a.e. in } \{u_0 = 0\} \\ -X_i u_0 & \text{a.e. in } \{u_0 < 0\}. \end{cases}$$

The proof of this fact is classical and consists in approximating $|u_0|$ by $f_\varepsilon \circ u$ where $f_\varepsilon(z) = (z^2 + \varepsilon^2)^{1/2} - \varepsilon$. Then one has $R(|u_0|) = R(u_0)$ where the Rayleigh quotient R is defined in (3). According to Lemma 5, this implies that $|u_0|$ is also in the first eigenspace of Δ_D , which concludes the proof. Hence $u_0^+ = (u_0 + |u_0|)/2$ and $u_0^- = (|u_0| - u_0)/2$ are also in the first eigenspace. If for both of them there exists a set on which they are positive, we get a contradiction with the unique continuation of eigenfunctions (implied by Assumption 1). Hence $u_0^+ \equiv 0$ or $u_0^- \equiv 0$, which implies the lemma. \square

Lemma 7. *Let $u \in \mathcal{D}(\Delta_\Omega)$ satisfying $-\Delta_\Omega u = \lambda u$ in Ω . Let D be a nodal domain of u . Then the restriction \underline{u} of u to D belongs to $\mathcal{D}(\Delta_D)$, and it is an eigenfunction of the Dirichlet problem in D , associated to the smallest eigenvalue $\lambda = \lambda_1(D)$.*

Proof. Thanks to Assumption 2, u is smooth up to the boundary of Ω (see [12, Theorem III, point (4)]), hence \underline{u} is smooth up to the boundary of D . Without loss of generality, we assume that \underline{u} is non-negative. We follow the proof of Lemma 2.0.1 in [2], which does not use any regularity on the boundary of the nodal domain.

We fix a Riemannian structure on M , which induces a distance dist_R and a gradient ∇_R . Let $\chi_n \in C_c^\infty(\overline{D})$ be a cut-off function such that

- $\chi_n(x) = 1$ for $\text{dist}_R(x, \partial D) \geq 1/n$
- $\chi_n(x) = 0$ for $\text{dist}_R(x, \partial D) \leq 1/(2n)$
- $\|\nabla_R \chi_n\|_{L^\infty} \lesssim n$
- $\|\nabla_R^2 \chi_n\|_{L^\infty} \lesssim n^2$

(the existence of χ_n is shown in [2]). Since X_1, \dots, X_m are smooth, this implies that $\|X_i \chi_n\|_{L^\infty} \lesssim n$ and $\|X_i^* \chi_n\|_{L^\infty} \lesssim n$ for any i , and $\|\Delta \chi_n\|_{L^\infty} \lesssim n^2$.

We set $u_n = \chi_n \underline{u}$. We have $u_n \rightarrow \underline{u}$ in $L^2(D, \mu)$. Let us prove that $q_D(u_n - u_\ell) \rightarrow 0$ as $\ell, n \rightarrow \infty$. To simplify notations, we set $\alpha = \alpha_{n, \ell} = \chi_n - \chi_\ell$. We have

$$\begin{aligned} |q_D(u_n - u_\ell)| &= \int_D \sum_{i=1}^m (X_i(u_n - u_\ell))^2 d\mu = \int_D \sum_{i=1}^m \alpha u X_i^* ((X_i \alpha) u + \alpha (X_i u)) d\mu \\ &= - \int_D \alpha u^2 \Delta \alpha d\mu - \int_D \alpha^2 u \Delta u d\mu \\ &\quad - \int_D \left(\sum_{i=1}^m (X_i \alpha) (X_i^* u) \alpha u + (X_i^* \alpha) (X_i u) \alpha u \right) d\mu \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4}$$

We show that $I_j \rightarrow 0$ for $j = 1, 2, 3$, as $n, \ell \rightarrow +\infty$ (recall that the dependence of α in n, ℓ is omitted in the notations). We denote by A_n the support of $\nabla \chi_n$, in particular $\text{vol}(A_n) \rightarrow 0$. We have

$$\left| \int_D \alpha u^2 \Delta \alpha d\mu \right| \leq C \ell^2 \int_{A_\ell} u^2 d\mu + C n^2 \int_{A_n} u^2 d\mu. \tag{5}$$

We use the fact that u is smooth up to the boundary to get that $u^2 \leq C \ell^{-2}$ in A_ℓ and similarly $u^2 \leq C n^{-2}$ in A_n . Hence

$$|I_1| = \left| \int_D \alpha u^2 \Delta \alpha d\mu \right| \leq C(\text{vol}(A_\ell) + \text{vol}(A_n)) \xrightarrow{n, \ell \rightarrow +\infty} 0.$$

Then, we have

$$\left| \int_D \alpha^2 u \Delta u \, d\mu \right| = \lambda \int_D \alpha^2 u^2 \, d\mu \leq C(\text{vol}(A_\ell) + \text{vol}(A_n)) \int_D u^2 \, d\mu \xrightarrow{n, \ell \rightarrow +\infty} 0.$$

One can also check that

$$I_3 = -\frac{1}{2} \int_D u^2 \left(\sum_{i=1}^m \alpha X_i^2 \alpha + (X_i \alpha)^2 + \alpha (X_i^* \alpha)^2 + (X_i^* \alpha)^2 + \alpha^2 (\text{div}_\mu(X_i))^2 \right) d\mu$$

and once again

$$|I_3| \leq C\ell^2 \int_{A_\ell} u^2 \, d\mu + Cn^2 \int_{A_n} u^2 \, d\mu \leq C(\text{vol}(A_\ell) + \text{vol}(A_n)) \xrightarrow{n, \ell \rightarrow +\infty} 0.$$

All in all, $|q_D(u_n - u_\ell)| \rightarrow 0$ as $n, \ell \rightarrow +\infty$.

Hence $\underline{u} \in \mathcal{D}(\widehat{q}_D)$. Then we have to check that $|\widehat{q}_D(\underline{u}, v)| \leq C(\underline{u}, v)_{L^2(D, \mu)}$. It is sufficient to check it for $v \in C_c^\infty(D)$ and then extend it by density to $\mathcal{D}(\widehat{q}_D)$. Let $v \in C_c^\infty(D)$. We have

$$\begin{aligned} \widehat{q}_D(\underline{u}, v) &= - \lim_{n \rightarrow +\infty} \int_D \sum_{i=1}^m X_i(\chi_n u) X_i v \, d\mu = \lim_{n \rightarrow +\infty} (\Delta u_n, v)_{L^2(D, \mu)} \\ &= (\Delta \underline{u}, v)_{L^2(D, \mu)} = -\lambda(\underline{u}, v)_{L^2(D, \mu)} \end{aligned}$$

since $u_n = \underline{u}$ on $\text{Supp}(v)$ for n sufficiently large. Hence $\underline{u} \in \mathcal{D}(\Delta_D)$.

Assume for the sake of a contradiction that $\lambda > \lambda_1(D) > 0$. Let us denote by $u_0 \geq 0$ an element of the first eigenspace of $-\Delta_D$ which exists thanks to Lemma 6. Then according to Lemma 1, u_0 and \underline{u} are orthogonal for the $L^2(D, \mu)$ scalar product. But both functions are non-negative, and thanks to the unique continuation property of eigenfunctions which is satisfied thanks to Assumption 1, we know that u_0 and \underline{u} cannot vanish on any open set. This gives $u_0 = 0$ or $\underline{u} = 0$. Both possibilities are impossible since $\lambda > \lambda_1(D) > 0$, hence the contradiction. \square

To prove Theorem 1, we follow the arguments of [8, Chapter VI (p. 453-454)] (see also [5, Appendix D]). Suppose that $u \in E_{\lambda_k}$ has at least $(k+1)$ nodal domains D_1, \dots, D_{k+1} . We also assume $\lambda_{k-1} < \lambda_k$. For $1 \leq i \leq k$, we denote by u_i the restriction of u to D_i , which lies in the first eigenspace of the Dirichlet problem in D_i according to Lemma 7. In particular its Rayleigh quotient $R(u_i)$ is equal to λ_k due to Point 1. of Lemma 5. We extend u_i by 0 in $\Omega \setminus D_i$, and we still denote by u_i this extension, which belongs to $\mathcal{D}(\widehat{q}_\Omega)$ according to Lemma 3. We can determine a_i such that $f = \sum_{i=1}^k a_i u_i$ is orthogonal in $L^2(\Omega, \mu)$ to the $(k-1)$ first eigenfunctions $\varphi_1, \dots, \varphi_{k-1}$ of $-\Delta_\Omega$ on Ω . We have $R(f) = \lambda_k$, hence f is an eigenfunction for λ_k according to the min-max principle (Point 2 of Lemma 5). But f vanishes in the open set D_{k+1} in contradiction with the unique continuation property of eigenfunctions which is satisfied thanks to Assumption 1.

Remark 8. *Without assuming Assumption 1, counterexamples to the unique continuation property are known, although for a perturbation of Δ , and not directly for Δ (see [3]).*

4 Sub-Riemannian tools

This section introduces the tools of sub-Riemannian (sR) geometry which will be needed in the proof of Theorem 2. For a more comprehensive introduction to sR geometry, we refer to [1] and [4].

SR metric and sR balls. The sR metric is defined for $q \in M$, $v \in T_q M$ as

$$g_q(v) = \inf \left\{ \sum_{i=1}^m u_i^2, \quad v = \sum_{i=1}^m u_i X_i(q) \right\}$$

and it is finite for $v \in \mathcal{D} = \text{Span}(X_1, \dots, X_m)$. It induces a notion of distance $d : M \times M \rightarrow \mathbb{R}_+$, and the distance between two points is always finite thanks to Hörmander's bracket generating condition (Chow-Rashevsky's theorem). The sR balls are then defined as

$$B_r(q) = \{q' \in M, d(q, q') < r\} \quad (6)$$

for $q \in M$ and $r > 0$.

SR flag. We define the sR flag as follows: we define $\mathcal{D}^0 = \{0\}$, $\mathcal{D}^1 = \mathcal{D}$, and, for any $j \geq 1$, $\mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}, \mathcal{D}^j]$. For any $q \in M$, this gives a flag

$$\{0\} = \mathcal{D}_q^0 \subset \mathcal{D}_q^1 \subset \dots \subset \mathcal{D}_q^{r-1} \subsetneq \mathcal{D}_q^{r(q)} = T_q M.$$

The integer $r(q)$ is called the *non-holonomic order* of \mathcal{D} at q .

For $i \in \{0, \dots, r(q)\}$, we set $n_i(q) = \dim \mathcal{D}_q^i$. The sequence $(n_i(q))_{0 \leq i \leq r(q)}$ is called the *growth vector* at point q . The number

$$\mathcal{Q}(q) = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q)), \quad (7)$$

is generically the Hausdorff dimension of the metric space induced by the sub-Riemannian distance on M (see [14]).

We define a non-decreasing sequence of weights $w_i(q)$. Given any $i \in \{1, \dots, N\}$, there exists a unique $j \in \{1, \dots, N\}$ such that $n_{j-1}(q) + 1 \leq i \leq n_j(q)$. We set $w_i(q) = j$. Roughly speaking, $w_i(q)$ is the minimal length of the brackets of X_1, \dots, X_m needed to generate i independent directions at q .

4.1 Nilpotentization

The aim of the following paragraphs is to introduce a system of local coordinates, called privileged coordinates, in which it is natural to write Taylor expansion of vector fields defined on the sR manifold. In particular, the first order in the Taylor approximation is called the nilpotent approximation of the vector field.

Non-holonomic orders. The *non-holonomic order* of a smooth germ of function is

$$\text{ord}_q(f) = \min\{p \in \mathbb{N} : \exists i_1, \dots, i_p \in \{1, \dots, m\} \text{ such that } (X_{i_1} \dots X_{i_p} f)(q) \neq 0\}$$

where we adopt the convention that $\min \emptyset = +\infty$.

The *non-holonomic order* of a smooth germ of vector field X at q , denoted by $\text{ord}_q(X)$, is the real number

$$\text{ord}_q(X) = \sup\{\sigma \in \mathbb{R} : \text{ord}_q(Xf) \geq \sigma + \text{ord}_q(f), \quad \forall f \in C^\infty(q)\}.$$

For example, there holds $\text{ord}_q([X, Y]) \geq \text{ord}_q(X) + \text{ord}_q(Y)$ and $\text{ord}_q(fX) \geq \text{ord}_q(f) + \text{ord}_q(X)$. As a consequence, every X which has the property that $X(q') \in \mathcal{D}_q^i$ for any q' in a neighborhood of q is of non-holonomic order $\geq -i$.

Privileged coordinates. Locally around any $q \in M$, it is possible to define a system of so-called *privileged coordinates* of M (see [4]).

A family (Z_1, \dots, Z_N) of N vector fields is said to be *adapted* to the sR flag at q if it is a frame of $T_q M$ at q and if $Z_i(q) \in \mathcal{D}_q^{w_i(q)}$ for any $i \in \{1, \dots, N\}$. In other words, for any $i \in \{1, \dots, r(q)\}$, the vectors $Z_1, \dots, Z_{n_i(q)}$ at q span \mathcal{D}_q^i .

A system of *privileged coordinates* at q is a system of local coordinates (x_1, \dots, x_N) verifying

$$\text{ord}_q(x_i) = w_i, \quad \text{for } 1 \leq i \leq N. \quad (8)$$

In particular, privileged coordinates satisfy $\partial_{x_i} \in \mathcal{D}_q^{w_i(q)} \setminus \mathcal{D}_q^{w_i(q)-1}$ at q , meaning that privileged coordinates are adapted to the flag.

For example, if (Z_1, \dots, Z_N) is an adapted frame at q , it is proved in [10, Appendix B] that the inverse of the local diffeomorphism

$$(x_1, \dots, x_n) \mapsto \exp(x_1 Z_1) \circ \dots \circ \exp(x_N Z_N)(q)$$

defines privileged coordinates at q (called exponential coordinates of the second kind).

Dilations. Fix $q \in M$. For every $\varepsilon \in \mathbb{R} \setminus \{0\}$, the dilation $\delta_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is defined by

$$\delta_\varepsilon(x) = (\varepsilon^{w_1(q)} x_1, \dots, \varepsilon^{w_N(q)} x_N)$$

for every $x = (x_1, \dots, x_N)$ - we omit the dependance in q in the notation. A dilation δ_ε acts also on functions and vector fields on \mathbb{R}^N by pull-back: $\delta_\varepsilon^* f = f \circ \delta_\varepsilon$ and $\delta_\varepsilon^* X$ is the vector field such that $(\delta_\varepsilon^* X)(\delta_\varepsilon^* f) = \delta_\varepsilon^*(Xf)$ for any $f \in C^1(\mathbb{R}^N)$. In particular, for any vector field X of non-holonomic order k , there holds $\delta_\varepsilon^* X = \varepsilon^{-k} X$.

Nilpotent approximation. Fix a system of privileged coordinates $\psi_q = (x_1, \dots, x_N) : U \rightarrow \mathbb{R}^N$ defined in a neighborhood U of q . Coming back to the vector fields X_1, \dots, X_m , we write the Taylor expansion

$$X_i(x) \sim \sum_{\alpha, j} a_{\alpha, j} x^\alpha \partial_{x_j}. \quad (9)$$

For a sequence of integers $\alpha = (\alpha_1, \dots, \alpha_N)$, we define the weighted degree of $x^\alpha = x_1^{\alpha_1} \dots x_N^{\alpha_N}$ to be $w(\alpha) = w_1(q)\alpha_1 + \dots + w_N(q)\alpha_N$. Since $X_i \in \mathcal{D}$, its non-holonomic order is necessarily -1 , hence there holds $w(\alpha) \geq w_j(q) - 1$ if $a_{\alpha, j} \neq 0$. Therefore, we may write X_i as a formal series

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \dots$$

where $X_i^{(k)}$ is a homogeneous vector field of degree k , meaning that

$$\delta_\varepsilon^*(\psi_q)_* X_i^{(k)} = \varepsilon^k (\psi_q)_* X_i^{(k)}.$$

We set

$$\widehat{X}_i^q = (\psi_q)_* X_i^{(-1)}, \quad 1 \leq i \leq m$$

which is a vector field on \mathbb{R}^N . Then \widehat{X}_i^q is homogeneous of degree -1 with respect to dilations, meaning that $\delta_\varepsilon^* \widehat{X}_i^q = \varepsilon^{-1} \widehat{X}_i^q$ for $\varepsilon \neq 0$. Moreover,

$$\widehat{X}_i^q = \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_\varepsilon^*(\psi_q)_* X_i$$

in the C^∞ topology (all derivatives uniformly converge on compact subsets). For $\varepsilon > 0$ small enough we have

$$X_i^\varepsilon := \varepsilon \delta_\varepsilon^*(\psi_q)_* X_i = \widehat{X}_i^q + \varepsilon R_{i, \varepsilon}^q \quad (10)$$

where $R_{i,\varepsilon}^q$ depends smoothly on ε for the C^∞ topology (see also [1, Lemma 10.58]). An important property is that $(\widehat{X}_1^q, \dots, \widehat{X}_m^q)$ generates a nilpotent Lie algebra of step $r(q)$ (see [10, Proposition 2.3]).

Finally, the nilpotent approximation of X_1, \dots, X_m at q is then defined as $\widehat{M}^q \simeq \mathbb{R}^N$ endowed with the vector fields $\widehat{X}_1^q, \dots, \widehat{X}_m^q$. The nilpotent approximation depends a priori on the initial choice of privileged coordinates. An explicit example of computation of nilpotent approximation is given in [10, Example 2.8].

The nilpotent approximation of a measure μ on M at $q \in M$ is the measure on \mathbb{R}^N

$$\widehat{\mu}^q = \lim_{\varepsilon \rightarrow 0} r^{-\mathcal{Q}(q)} \delta_\varepsilon^* \mu \quad (11)$$

where the convergence is understood in the vague topology. It is important to note that $\widehat{\mu}^q$ is proportional to the Lebesgue measure.

4.2 Desingularization

Regular and singular points. We say that $q \in M$ is *regular* if the growth vector $(n_i(q'))_{0 \leq i \leq r(q')}$ at q' is constant for q' in a neighborhood of q . Otherwise, q is said to be *singular*. If any point $q \in M$ is regular, we say that the structure is *quiregular*.

Desingularization of vector fields. When $q \in M$ is a singular point, it is possible to lift locally in a neighborhood U of q the vector fields X_1, \dots, X_m to vector fields $\widetilde{X}_1, \dots, \widetilde{X}_m$ on $\widetilde{U} = U \times \mathbb{R}^K$, so that the lift $\widetilde{q} = (q, 0)$ of q is a regular point in \widetilde{U} , and many properties of the vector fields are preserved.

Lemma 9. [10, Lemma 2.5 and Theorem 2.9] *Let q be a point in M . Then there exists $K \in \mathbb{N}$, a neighborhood $U \subset M$ of q , coordinates (x, y) on $\widetilde{U} = U \times \mathbb{R}^K$, and smooth vector fields on \widetilde{U} ,*

$$\widetilde{X}_i(x, y) = X_i(x) + \sum_{j=1}^K b_{ij}(x, y) \partial_{y_j}, \quad i = 1, \dots, m,$$

such that

- $\widetilde{X}_1, \dots, \widetilde{X}_m$ satisfy Hörmander's bracket generating condition in \widetilde{U} ;
- every \widetilde{p} in \widetilde{U} is regular;
- denoting by $\pi : \widetilde{M} \rightarrow M$ the canonical projection, and by \widetilde{d} the sub-Riemannian distance defined by $\widetilde{X}_1, \dots, \widetilde{X}_m$ on \widetilde{U} , we have $\pi_* \widetilde{X}_i = X_i$, and for $p \in U$ and ε small enough,

$$B(p, \varepsilon) = \pi \left(B^{\widetilde{d}}((p, 0), \varepsilon) \right). \quad (12)$$

5 Proof of Theorem 2

The proof of Theorem 2 splits into two steps. The first one consists in proving an upper bound on the first eigenvalue of the Dirichlet sub-Laplacian in a sR ball whose radius tends to 0. This upper bound is uniform in the center of the ball when this center is a regular point. The second step is to use Lemma 9 (i.e., a desingularization) to conclude.

5.1 The first eigenvalue of the sub-Laplacian in a small sR ball

We fix $q \in M$ and we take a chart $\psi_q : U \rightarrow \mathbb{R}^N$ of privileged coordinates at q , with $\psi_q(q) = 0$. Pushing forward to \mathbb{R}^N the vector fields X_i (resp. the measure μ), we can consider them as vector fields (resp. a measure) on U . In the sequel, we also push-forward the distance, the balls etc. In particular the push-forward of the ball $B_r(q)$ defined in (6) is denoted by $B_{r,q} \subset \mathbb{R}^N$, and it is centered at $0 \in \mathbb{R}^N$. For that to make sense, we require $0 < r < r(q)$ where $r(q)$ is the maximal radius such that $B_r(q) \subset U$.

The nilpotent approximations of X_1, \dots, X_m (resp. of μ) at q are vector fields \widehat{X}_i^q (resp. a measure $\widehat{\mu}^q$) in \mathbb{R}^N (not only near 0, since we can extend them by homogeneity). They induce a nilpotentized sR distance and nilpotentized sR balls $\widehat{B}_{r,q}$ (with center $0 \in U$).

We denote by $\Delta_{r,q}$ the Dirichlet sub-Laplacian in $B_{r,q}$, constructed with the vector fields X_1, \dots, X_m and the measure μ . For $r \leq r(q)$, we denote by L_r^2 the space $L_r^2 = L^2(B_{r,q}, \mu)$. We also fix $u_1 \in C_c^\infty(\widehat{B}_{1/2,q})$ such that $u_1(0) \neq 0$. Finally, we set

$$u_r(x) = r^{-\mathcal{Q}(q)/2} u_1(\delta_{1/r} x)$$

and we have $u_r \in C_c^\infty(\widehat{B}_{r/2,q})$.

Lemma 10. *There exists $c(q) > 0$ such that for any $r \leq r(q)$ and any $1 \leq i \leq m$,*

$$\|\widehat{X}_i^q u_r\|_{L_r^2} \leq c(q) r^{-1} \|u_r\|_{L_r^2} \quad (13)$$

Remark 11. *Note that the norms involved in (13) are the L_r^2 norms and not the $L^2(\widehat{B}_{r,q}, \widehat{\mu}^q)$ norms, which would have been more “natural” to state an inequality like (13).*

Proof. First, the ball-box theorem ([4], [10, Corollary 2.1]) yields the existence of $0 < \alpha \leq 1$ (depending on q) such that

$$B_{\alpha r, q} \subset \widehat{B}_{r, q} \subset B_{\alpha^{-1} r, q}. \quad (14)$$

We use the fact that $u_r(x) = r^{-\mathcal{Q}(q)/2} u_1(\delta_{1/r} x)$ and the homogeneity in r of \widehat{X}_i^q , $\widehat{B}_{r,q}$ and $\widehat{\mu}^q$. Due to (11) this implies the following two convergences

$$\begin{aligned} r \|\widehat{X}_i^q u_r\|_{L^2(\widehat{B}_{r,q}, \mu)} &= r^{-\mathcal{Q}(q)/2} \|\widehat{X}_i^q u_1\|_{L^2(\widehat{B}_{1,q}, \delta_r^* \mu)} \xrightarrow{r \rightarrow 0} \|\widehat{X}_i^q u_1\|_{L^2(\widehat{B}_{1,q}, \widehat{\mu}^q)} \\ \|u_r\|_{L^2(\widehat{B}_{\alpha^2 r, q}, \mu)} &= r^{-\mathcal{Q}(q)/2} \|u_1\|_{L^2(\widehat{B}_{\alpha^2, q}, \delta_r^* \mu)} \xrightarrow{r \rightarrow 0} \|u_1\|_{L^2(\widehat{B}_{\alpha^2, q}, \widehat{\mu}^q)}. \end{aligned}$$

Taking the ratio of the two convergences (justified by the fact that $u_1(0) \neq 0$ hence the last limit is $\neq 0$), we obtain

$$\|\widehat{X}_i^q u_r\|_{L^2(\widehat{B}_{r,q}, \mu)} \leq c(q) r^{-1} \|u_r\|_{L^2(\widehat{B}_{\alpha^2 r, q}, \mu)}. \quad (15)$$

Using (14) (for $\alpha^{-1} r$ and αr instead of r), we obtain

$$\|\widehat{X}_i^q u_r\|_{L_r^2} \leq \|\widehat{X}_i^q u_r\|_{L^2(\widehat{B}_{\alpha^{-1} r, q}, \mu)} \leq c(q) r^{-1} \|u_r\|_{L^2(\widehat{B}_{\alpha r, q}, \mu)} \leq c(q) r^{-1} \|u_r\|_{L_r^2}$$

which implies the lemma. \square

Corollary 12. *For any $q \in M$, there exists $c(q) > 0$ such that for any sufficiently small r , there holds $\lambda_1(B_r(q)) \leq c(q) r^{-2}$.*

Proof. We fix $q \in M$ and $1 \leq i \leq m$. According to (14), we know that $u_{\alpha r}$ is in the domain of the quadratic form defining $\Delta_{r,q}$. We write $X_i = \widehat{X}_i^q + R_i^q$ as in (10). Then, R_i^q has only homogeneous of order ≥ 0 components, implying that $\delta_r^* R_i^q u_{\alpha r}$ is uniformly bounded in r . Thus,

$$\|R_i^q u_{\alpha r}\|_{L_r^2}^2 \leq C(q) \|u_{\alpha r}\|_{L_r^2}^2. \quad (16)$$

Hence,

$$\begin{aligned} \|X_i u_{\alpha r}\|_{L_r^2}^2 &\leq 2 \left(\|\widehat{X}_i^q u_{\alpha r}\|_{L_r^2}^2 + \|R_i^q u_{\alpha r}\|_{L_r^2}^2 \right) \\ &\leq (c(q) r^{-2} + C(q)) \|u_{\alpha r}\|_{L_r^2}^2 \\ &\leq c'(q) r^{-2} \|u_{\alpha r}\|_{L_r^2}^2 \end{aligned}$$

where in the second inequality we used Lemma 10. By the min-max principle (Lemma 5), we get the result. \square

Lemma 13. *When q is regular, the constant $c(q)$ in Corollary 12 can be taken uniform in a small neighborhood of q .*

Proof. We use the fact that taking a nilpotent approximation is a “uniform” procedure near a regular point (but it is not uniform near a singular point). This fact is described in Section 2.2.2 in [10], and it mainly relies on the property that q being regular, there exists a frame Y_1, \dots, Y_N which is an adapted frame at every point q' in a neighborhood V of q . This yields a continuously varying system of privileged coordinates in V , and a “continuous” nilpotent approximation in V (see Definition 2.9 in [10]).

This implies that the constant α in (14) can be taken uniform over $q' \in V$. Note that $u_r \in C_c^\infty(\widehat{B}_{r/2,q}) \subset C_c^\infty(\widehat{B}_{r,q'})$ for $q' \in V$ (possibly made slightly smaller) by continuity - at this point we should recall that $\widehat{B}_{r/2,q}$ and $\widehat{B}_{r,q'}$ are both centered at $0 \in \mathbb{R}^N$. Then, the convergence (11) is also uniform over V , so the inequality (15) remains true in V with a uniform constant $c(V)$: for any $q' \in V$,

$$\|\widehat{X}_i^{q'} u_r\|_{L^2(\widehat{B}_{r,q',\mu})} \leq c(V) r^{-1} \|u_r\|_{L^2(\widehat{B}_{\alpha^2 r,q',\mu})}.$$

All in all, (13) is seen to hold at any $q' \in V$, with a constant $c(q')$ uniform over $q' \in V$.

Following Definition 2.9 in [10], we know that $q' \mapsto X_i(q') - \widehat{X}_i^{q'}(q') = R_i^{q'}$ is continuous in $q' \in V$. This implies that the constant $C(q)$ in (16) can be taken uniform over V . More precisely, this means that there exists $C(V) > 0$ such that for any $r > 0$ and any $q' \in V$,

$$\|R_i^{q'} u_{\alpha r, q'}\|_{L^2_{r,q'}}^2 \leq C(V) \|u_{\alpha r, q'}\|_{L^2_{r,q'}}^2 \quad (17)$$

where we have indicated that both the eigenfunction and the L^2 space also depend on the point q' (and the constant α is uniform over V). This implies that the constant $c(q)$ in Corollary 12 can be taken uniform over V , which proves Lemma 13. \square

Remark 14. *Corollary 12 and Lemma 13 establish upper bounds on the first Dirichlet eigenvalue. Note that lower bounds on the first Neumann eigenvalue were established in [11], this is equivalent to Poincaré’s inequality.*

5.2 End of the proof of Theorem 2

Let us finish the proof of Theorem 2.

We assume that φ_λ is an eigenfunction of $-\Delta_\Omega$ not belonging to the first eigenspace E_{λ_1} . We denote by D_j its nodal domains. According to Lemma 7, the restriction of φ_λ to each domain D_j is an eigenfunction of the Dirichlet sub-Laplacian Δ_{D_j} , it belongs to its first eigenspace, and $\lambda_1(D_j) = \lambda$ for each D_j . Now, if $x \in \Omega$ and $d(q, Z_{\varphi_\lambda}) > r$, where d is the sub-Riemannian distance (either in M or in Ω , this does not change anything here), then $B_r(q) \subset D_j$ for some j . By the min-max principle, it implies that $\lambda = \lambda_1(D_j) \leq \lambda_1(B_r(q))$. But $\lambda_1(B_r(q)) \leq c(q)r^{-2}$ thanks to Corollary 12, hence $r \leq c(q)\lambda^{-1/2}$.

If q is regular, using Lemma 13, we obtain that the constant $c(q')$ above is in fact uniform for q' in a neighborhood of q . Hence any sR centered in a neighborhood V of q and of radius $\geq c'(V)\lambda^{-1/2}$ will intersect Z_{φ_λ} , which concludes the proof of the theorem “locally near q ” in this case.

If $q \in \Omega$ is a singular point, the idea is to desingularize the vector fields at q thanks to Lemma 9 in order to be able to apply the regular case that we just addressed. Following the notations of Lemma 9, we consider

$$\widetilde{X}_i(x, y) = X_i(x) + \sum_{j=1}^K b_{ij}(x, y) \partial_{y_j}, \quad i = 1, \dots, m,$$

for $(x, y) \in U \times \mathbb{R}^K$.

It is tempting to consider the sub-Laplacian

$$\tilde{\Delta} = - \sum_{i=1}^m \tilde{X}_i^* \tilde{X}_i$$

on $U \times \mathbb{R}^K$ (the adjoint being computed with respect to $\tilde{\mu} = \mu \otimes \mathcal{L}_{\mathbb{R}^K}$, where $\mathcal{L}_{\mathbb{R}^K}$ is the Lebesgue measure on \mathbb{R}^K). But this does not work for our purposes, and instead we will build a sub-Laplacian $\tilde{\Delta}^c$ satisfying the following key properties:

- It is defined on the compact manifold $\Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K$.
- In a neighborhood of $(q, 0)$ we have $\tilde{\Delta}^c = \tilde{\Delta}$.
- The vector fields defining $\tilde{\Delta}^c$ satisfy Hörmander's bracket generating condition everywhere in $\Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K$.

The construction of $\tilde{\Delta}^c$ is achieved through cut-offs and extensions of the vector fields \tilde{X}_i . In the sequel, $\mathbb{R}/2\pi\mathbb{Z}$ is identified with $[-\pi, \pi)$ (with periodic boundary).

Fix $\delta > 0$ small and $V_1 \Subset V_2 \Subset U$ neighborhoods of q . We consider the following cut-off functions:

- $\chi_0 : \Omega \rightarrow \mathbb{R}^+$ a smooth function which is equal to 1 in V_2 and 0 in $\Omega \setminus U$.
- $\chi_1 : (\mathbb{R}/2\pi\mathbb{Z})^K \rightarrow \mathbb{R}^+$ a smooth function which is equal to 1 for $|y| \leq 2\delta$ and equal to 0 for $|y| \geq 3\delta$.
- $\chi_2 : \Omega \rightarrow \mathbb{R}^+$ a smooth function which is equal to 1 on $\Omega \setminus V_2$ and equal to 0 in V_1 .
- $\chi_3 : (\mathbb{R}/2\pi\mathbb{Z})^K \rightarrow \mathbb{R}^+$ a smooth function which is equal to 0 for $|y| \leq \delta$ and 1 for $|y| \geq 2\delta$.

Note that χ_1 and χ_3 are periodic. We consider the vector fields

$$\tilde{X}_i^c(x, y) = X_i + \sum_{j=1}^K \chi_0(x) \chi_1(y) b_{ij}(x, y) \partial_{y_j}$$

on $\Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K$ and the sub-Laplacian on $\Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K$ defined by

$$\tilde{\Delta}^c = - \sum_{i=1}^m (\tilde{X}_i^c)^* \tilde{X}_i^c + (\chi_2(x)^2 + \chi_3(y)^2) \sum_{j=1}^K \partial_{y_j}^2.$$

Let $\varphi_\lambda(x)$ be an eigenfunction of $-\Delta_\Omega$, with eigenvalue λ . We consider $\psi_\lambda : \Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K \rightarrow \mathbb{R}$ defined by $\psi_\lambda(x, y) = \varphi_\lambda(x)$. This is an eigenfunction of $-\tilde{\Delta}^c$ with eigenvalue λ . We apply Step 1 to $\tilde{\Delta}^c$, which has all required properties: it is defined on a compact manifold, it is subelliptic, the vector fields defining it are regular at $(q, 0)$ (since they coincide with \tilde{X}_i near $(q, 0)$), and the non-characteristic boundary condition is verified on $\partial(\Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K)$.

Therefore, there exists $c(\tilde{q}) > 0$ uniform near $(q, 0)$ (for $\tilde{q} \in \tilde{V} \subset \Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K$) and independent of λ such that

$$Z_{\psi_\lambda} \text{ intersects any sR ball of radius } \geq c(\tilde{V}) \lambda^{-1/2} \text{ centered at a point } \tilde{q} \in \tilde{V}. \quad (18)$$

These sR balls are computed with the vector fields defining $\tilde{\Delta}^c$, and these vector fields coincide near $(q, 0)$ with \tilde{X}_i ; hence it is equivalent to compute the sR balls with the vector fields \tilde{X}_i since we are considering small balls near $(q, 0)$, with radius much smaller than δ . By the projection property (12), since $Z_{\psi_\lambda} = Z_{\varphi_\lambda} \times (\mathbb{R}/2\pi\mathbb{Z})^K$, we finally obtain that

$$Z_{\varphi_\lambda} \text{ intersects any sR ball of radius } \geq c(\tilde{V}) \lambda^{-1/2} \text{ centered at a point } \pi(\tilde{q}) \in \pi(\tilde{V}) \quad (19)$$

where $\pi : \Omega \times (\mathbb{R}/2\pi\mathbb{Z})^K$ is the canonical projection. The constant involved in (19) is thus uniform in a neighborhood of q . Using the compactness of $\bar{\Omega}$ we obtain the result.

References

- [1] Andrei Agrachev, Davide Barilari and Ugo Boscain. *A comprehensive introduction to sub-Riemannian geometry*. Cambridge University Press, 2019.
- [2] Colette Anné. Bornes sur la multiplicité. *Prépublications EPFL*, mars 1992.
- [3] Hajer Bahouri. Non prolongement unique des solutions d’opérateurs “somme de carrés”. *Annales de l’institut Fourier*, 1986, tome 36, no 4, p. 137-155.
- [4] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, pages 1–78. Springer, 1996.
- [5] Pierre Bérard and Daniel Meyer. Inégalités isopérimétriques et applications. *Annales scientifiques de l’École Normale Supérieure*, 1982, vol. 15, no 3, p. 513-541.
- [6] Jean-Michel Bony. Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. *Annales de l’institut Fourier*, 1969, vol. 19, no 1, p. 277-304.
- [7] Marco Carfagnini and Maria Gordina. Dirichlet sub-Laplacians on homogeneous Carnot groups: spectral properties, asymptotics and heat content. *Arxiv preprint arXiv:2202.10410*.
- [8] Richard Courant and David Hilbert. *Methods of mathematical physics*. John Wiley & Sons, 2008.
- [9] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 1967, vol. 119, p. 147-171.
- [10] Frédéric Jean. *Control of nonholonomic systems: from sub-Riemannian geometry to motion planning*. Springer, 2014.
- [11] David Jerison. The Poincaré inequality for vector fields satisfying Hörmander’s condition. *Duke Mathematical Journal*, 1986, vol. 53, no 2, p. 503-523.
- [12] Joseph J. Kohn and Louis Nirenberg. Non-coercive boundary value problems. *Communications on Pure and Applied Mathematics*, 1965, vol. 18, no 3, p. 443-492.
- [13] Alexander Logunov and Eugenia Malinnikova. Review of Yau’s conjecture on zero sets of Laplace eigenfunctions. In: *Current Developments in Mathematics*, 2018, vol. 2018, no 1, p. 179-212.
- [14] John Mitchell. On Carnot-Carathéodory metrics. *Journal of Differential Geometry*, 1985, vol. 21, no 1, p. 35-45.
- [15] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.
- [16] Kinji Watanabe. Sur l’unicité du prolongement des solutions des équations elliptiques dégénérées. *Tôhoku Math. J.* 1982, vol. 34, no 2, p. 239-249.