



HAL
open science

Nonsmooth Modal Analysis of a Varying Cross-Sectional Area Bar in Unilateral Contact

David Urman, Mathias Legrand

► **To cite this version:**

David Urman, Mathias Legrand. Nonsmooth Modal Analysis of a Varying Cross-Sectional Area Bar in Unilateral Contact. 2022. hal-03619197v1

HAL Id: hal-03619197

<https://hal.science/hal-03619197v1>

Preprint submitted on 25 Mar 2022 (v1), last revised 12 Dec 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Nonsmooth Modal Analysis of a varying cross-sectional area bar in unilateral contact

David Urman, Mathias Legrand

March 24, 2022

Abstract

Nonsmooth modes of vibration allow for identification of resonant behaviours and attendant vibratory frequencies in structures prone to unilateral contact conditions on the boundary. The prominent approach for finding nonsmooth modes of vibration entails finding continua of periodic solutions to the system in question. In this paper, nonsmooth modes of a one-dimensional bar of varying cross-sectional area prone to unilateral contact with a rigid obstacle are determined. While numerical and analytical techniques were previously proposed, they were limited to constant cross section bar and could not be applied on the varying area bar for which the classical d'Alembert solution no longer exists. In this article, nonsmooth modal analysis of the varying area bar is performed via a novel treatment of the Signorini conditions within the finite element framework: the *nodal boundary method*. The nodal boundary method solves the Signorini problem by switching between two sets of shape functions describing either (1) inactive contact motion (motion away from the rigid obstacle) or (2) active contact motion (bar in contact with the rigid obstacle). In the proposed nodal boundary method, the motion of the contacting node does not participate in the resulting governing Ordinary Differential Equation (ODE). Instead, its motion is prescribed by the boundary conditions and is dictated by the motion of internal nodes. The nodal boundary method results in a discontinuous ODE in the internal nodes which can be solved both analytically and via numerical techniques. Solutions obtained by the nodal boundary method exhibit several advantages over existing numerical techniques: no chattering at contact, no penetration of the rigid obstacle, and existence of periodic solutions. Specifically, these periodic solutions are readily detectable via the shooting method with sequential continuation. The nodal boundary method is used successfully for the nonsmooth modal analysis of different models of the varying area bar. Besides, application of the nodal boundary method for nonsmooth modal analysis of the uniform area cantilever bar in Dirichlet or Robin boundary conditions is also demonstrated for comparison with existing literature.

Contents

1	Introduction	2
2	Problem statement	3
3	Finite-Element formulation	4
4	Nodal Boundary Method	5
4.1	Comment on application of NBM to other shape functions	6
4.2	Inactive contact motion	7
4.3	Active contact motion	8
4.4	NBM-FEM formulation of Signorini problem	9
4.5	Energy conservation properties of solutions to NBM-ODE	11
5	Nonsmooth modal analysis	12
5.1	Time-marching techniques and shooting method	12
5.2	Error estimation	13
5.3	Sequential continuation with correction	13

6	Results	14
6.1	Convergence of Crank-Nicolson and NBM	14
6.2	Nonsmooth modes	15
6.2.1	Forced-response curves	15
6.2.2	Constant cross-section and internally resonant bar	16
6.2.3	Constant cross-section bar with soft support	17
6.2.4	Varying area bar	18
7	Conclusion	20
8	Supplementary Material	21
A	Appendix	23
A.1	Proof of $\phi'_N(1) > 0$	23
A.2	Conservation of energy away from instant of switch	24
A.3	Derivation of energy jump at switch	25

1 Introduction

For vibrations of structures prone to unilateral contact, families of periodic solutions are indicative of resonance frequencies and behaviours and are, in fact, referred to as nonsmooth modes of vibration (NSMs) [1–4]. In essence, NSMs are families of periodic motions of the unforced and undamped investigated structure [5–8]. In the present work, we focus on nonsmooth modal analysis of the bar of varying (cross-section) area in unilateral contact with a rigid structure. The dynamics of the bar are formulated via the wave equation with Signorini conditions for treatment of unilateral contact conditions [9, 10]. While periodic solutions have been obtained analytically for the case of the bar with constant area [1, 11], the same analytical method cannot be generalized to the case of the bar in varying area as a closed-form solution to the wave equation with varying coefficients has yet been obtained. Therefore, periodic solutions to the Signorini problem must be obtained numerically. Recently, works by Yoong et al. showed that application of the Wave Finite Element method (WFEM) to the problem of the bar of constant area in unilateral contact produces energy-conserving schemes allowing for existence of periodic solutions [12, 13]. However, the conservation of energy in WFEM solutions does not persist for the case of the varying-area bar or two-dimensional Signorini problems [13]. Other than via the WFEM, the Signorini problem has been also solved via the finite element method (FEM) [9, 10, 14–16]. In the FEM, application the Signorini boundary conditions leads to an ill-posed problem if not supplied with an impact law or some relationship between the stress and displacement [14, p. 9]. Thus, multiple methods for implementation of Signorini contact have been conceived. Such schemes are those utilizing a Newtonian impact law relating the pre- and post-impact velocities of the contacting node in the form $v(t^+) = -ev(t^-)$ where $e \in [0, 1]$. However, these schemes generate non-physical chattering for energy conserving schemes ($e = 1$) and annihilate periodic solutions with non-zero pre-contact velocity when $e \neq 1$) [14]. Another common treatment of Signorini conditions is to approximate the contact-force using a displacement-dependent model which regularizes the non-penetration condition. For example, penalty methods use a non-linear and possibly smooth function for the contact force univoquely expressed in terms of the displacement [14, 17]. However, since the convergence of penalty methods depend on a penalty parameter reaching infinity, these method inevitably result in stiff ODE and are generally found insufficient in solving the Signorini problem accurately [13, 14]. Next, mass redistribution methods [15, 18, 19] aim to solve the issue of ill-posedness by eliminating the mass of nodes on the contact boundary such that at, every instance, prediction of the location the boundary nodes is obtained by solving the static Signorini problem [15]. Specifically, the mass redistribution method proposed in [15] represents a formulation culminating in a an energy-conserving ODE which allows for periodic solutions while eliminating chattering at contact. However, the scheme requires solving a constrained optimization problem to redistribute the mass of the contact nodes. This issue was later readdressed in [18] where the usage of specific quadrature rules has reduced the computational cost of the mass redistribution. Regardless, the mass redistribution method requires usage of a different mass matrix than this used in classical FEM models. At last, it is worth mentioning the Nitsche method in FEM which solves the Signorini boundary condition in a weak

sense [16]. While penetration is indeed allowed in the Nitsche method, it has been proven that the Signorini boundary conditions are satisfied as the number of elements increases [16].

In this work, we propose the *Nodal Boundary method* (NBM) for treatment of Signorini boundary conditions in FEM. Applied to the Signorini problem, this method allows for existence of periodic solutions while eliminating chattering. While, similarly to [15], this method requires redistribution of masses, it does not require solution of a constrained optimization problem nor usage of “non-traditional” quadrature rules (as done in [18]). Thus, the proposed method allows for implementation of mass redistribution to classical finite-element models with no significant addition to computational effort. At last, to find periodic solutions of the FEM-NBM reduced model, the shooting method together with the continuation method are used, similar to the procedure suggested in [7]. In particular, nonsmooth modal analysis of the varying area bar, which could not be approached via other techniques such as WFEM and Newton impact law, is performed in this article via the NBM.

In what follows, we detail the problem statement and its finite-element formulation in sections 2 and 3, respectively. Next, we present the nodal boundary method applied to the Signorini conditions in Section 4. Application of nonsmooth modal analysis (ie, detection of periodic solutions) the model is presented in Section 5. At last, the results Section 6 consists of: convergence analysis, presentation of modes and forced motion analysis.

2 Problem statement

We consider a bar illustrated in Figure 1. The displacement field of the bar is denoted $\bar{u}(\bar{x}, \bar{t})$, where \bar{x} and \bar{t}

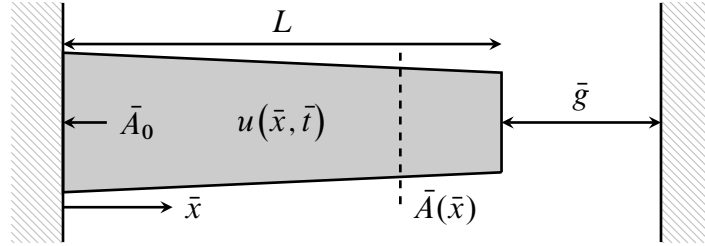


Figure 1: Bar of varying area prone to unilateral contact with a rigid wall.

represent the physical position and time, respectively. The bar is clamped to a wall at $\bar{x} = 0$ and is prone to unilateral contact with a rigid obstacle at its other end $\bar{x} = L$, where L denotes the length of the bar. At rest, the rigid obstacle is set at a distance \bar{g} from the bar.

Moreover, we consider the linear-elasticity framework for the deformation of the bar. Hence, the motion of the bar is described by the wave equation

$$\bar{u}_{tt}(x, t) = (A(x)\bar{u}_x(x, t))_x, \quad \forall x \in (0, 1), \quad t \in (0, \infty). \quad (1)$$

where subscripts denote partial differentiation with respect to the denoted variable. Furthermore, the following non-dimensional coordinates are introduced: $x = \bar{x}/L$ and $t = \bar{t}/c$. There, $c^2 = E\bar{A}_0/\rho$ with $E > 0$, $\rho > 0$ and $\bar{A}_0 > 0$ representing Young’s modulus, the density of the bar, and the physical area of the bar at $x = 0$, respectively. In turn, $A(x) > 0$ is a non-dimensional quantity representing the area variation in the bar. The physical cross-sectional area of the bar abides $\bar{A}(\bar{x}) = \bar{A}(xL) = A(x)\bar{A}_0$. Moreover, we use the following Dirichlet boundary condition at $x = 0$:

$$\bar{u}(0, t) = 0, \quad \forall t \in (0, \infty). \quad (2)$$

At $x = 1$ (ie, $\bar{x} = L$), we impose the Signorini boundary condition

$$0 \geq \bar{u}(1, t) - g \perp A(1)\bar{u}_x(1, t) \leq 0, \quad \forall t \in (0, \infty) \quad (3)$$

to describe contact of the bar with the rigid wall, where the non-dimensional gap distance $g = \bar{g}/L$ is introduced. Namely, the Signorini conditions (3), in the continuous setting, can be seen as a switching of boundary conditions at $x = 1$:

$$\text{Active contact conditions:} \quad \bar{u}(1, t) = g \quad \text{and} \quad A(1)\bar{u}_x(1, t) \leq 0 \quad (4)$$

$$\text{Inactive contact conditions:} \quad A(1)\bar{u}_x(1, t) = 0 \quad \text{and} \quad \bar{u}(1, t) \leq g. \quad (5)$$

For purposes of modal analysis, we are required to find the initial conditions $\bar{u}_0(x)$ and $\bar{v}_0(x)$ generating periodic solutions

$$\bar{u}(x, T) = u(x, 0) = \bar{u}_0(x), \quad \forall x \in [0, 1] \quad (6)$$

$$\bar{u}_t(x, T) = \bar{u}_t(x, 0) = \bar{v}_0(x), \quad \forall x \in [0, 1] \quad (7)$$

where T denotes the period of motion. Accordingly, to solve the problem numerically, we use the FEM with nodal boundary method (NBM) for treatment of boundary condition on the governing boundary value problem in Equations (1) to (5). Next, the shooting method and continuation are used to depict the continua of solutions answering Equations (6) and (7).

While the problem of modal analysis of the bar in unilateral contact with a constant cross-section, $A'(x) = 0$, has been studied before both analytically and numerically [1, 11, 12, 20], the same techniques could not be implemented for the case of $A'(x) \neq 0$. Namely, analytical techniques have relied on the exact solution to the wave equation to describe both inactive and active contact phases in a closed-form manner [1, 11, 20]. In contrast, the WFEM, in the case of the bar of uniform area, exhibits properties favoring the existence of periodic solutions such as: energy conservation and preservation of characteristic quantities [13]. However, the same properties are not exhibited for the case for the varying area bar as the upwind-flux, used in the WFEM, does not accurately solve for the transfer of quantities between elements [21, chapter 9]. In contrast, it will be shown in this manuscript that the NBM allows for detection of periodic solutions in the varying area bar by implementation of a Galerkin-Bubnov method and boundary shape functions to satisfy the inactive and active phases of the Signorini conditions.

3 Finite-Element formulation

In order to apply FEM on the one-dimensional Signorini problem, the displacement within the bar is approximated by a series of piecewise Lagrange polynomials $\phi_i(x)$, $i = 0, 1, 2, \dots, N$, and corresponding nodal quantities u_i [22] located at the nodes $x_i = i/N$ for $i = 0, 1, 2, \dots, N$, as classically achieved. The approximation thus reads

$$\bar{u}(x, t) \approx \sum_{i=0}^N \phi_i(x) u_i(t) \equiv \mathbf{P}(x) \mathbf{u}(t), \quad \text{where } \phi_i(x_j) = \delta_{ij} \text{ and } u_i(t) \approx u(x_i, t) \quad (8)$$

with $\mathbf{u}(t)$ storing the time-domain nodal displacements and δ_{ij} denoting the Kronecker Delta. Furthermore, we introduced the vector quantity $\mathbf{P}(x) \equiv (\phi_0(x) \ \phi_1(x) \ \dots \ \phi_N(x))$ to simplify the representation of the finite-element in matrix form. The finite element applies to the weak form of PDE (1). It requires the definition of test functions $\mathbf{w}(t)$ corresponding to the nodal displacements $\mathbf{u}(t)$. In the NBM, the test functions $\mathbf{w}(t)$ will be subject to change through time (according to the phase of contact motion), while it is not the case for common representation in the FEM. This will be clarified in section 4. Hence, application of the finite element approximation on the weak form of PDE (1) with cantilever condition (2) ($u_0(t) = 0$ and such that $\phi_0(x)$ is omitted from $\mathbf{P}(x)$) yields

$$\mathbf{w}^\top(t) \mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{w}^\top(t) \mathbf{K} \mathbf{u}(t) - w_N(A(1)u_x(1, t)) = 0, \quad \forall \mathbf{w}(t) \quad (9)$$

where \mathbf{M} is the mass matrix, \mathbf{K} is the stiffness matrix, with respective entries

$$\mathbf{M} = \int_0^1 \mathbf{P}(x) \mathbf{P}(x) dx, \quad \mathbf{K} = \int_0^1 A(x) \mathbf{P}_x^\top(x) \mathbf{P}_x(x) dx. \quad (10)$$

While this notation is conventional in FE analysis, it is reminded here since the NBM formulation will rely on it significantly. Specifically, the nodal boundary method modifies the weak form (9) and the test functions $\mathbf{w}(t)$ actively participate in its formulation. We now introduce the nodal boundary method to treat the Signorini conditions (4) and (5) in the finite element framework. There, the displacement and the stress at $x = 1$ will be approximated using the nodal quantities

$$\bar{u}(1, t) \approx u_N(t), \quad A(1)\bar{u}_x(1, t) \approx A(1) \sum_{i=1}^N \phi_i'(1) u_i(t), \quad (11)$$

respectively. While the approximation $\bar{u}(1, t) \approx u_N(t)$ is equivalent to the one used in the classical FEM, the above stress approximation is non-traditional. Implementation of Neumann conditions in the classical FEM is done in a weak sense, and the motion of the nodes does not satisfy the Neumann condition for any grid. Instead, the error in answering the Neumann condition is reduced with increasing the number of elements or degree of polynomials. In the NBM, the Neumann conditions are imposed on the shape functions such that for any choice of number of elements or degree of polynomial. In fact, this is the key to the implementation of the NBM, as described below.

4 Nodal Boundary Method

In NBM, the FE approximation is obtained by plugging the approximations (11) into the complementarity conditions (3) such that

$$g \geq u_N(t) \perp A(1) \sum_{i=1}^N \phi'_i(1) u_i(t) \leq 0. \quad (12)$$

The main proposition in NBM is that condition (12) is solved by constructing shape functions capable of satisfying the inactive and active contact conditions. To do so, the boundary node $u_N(t)$ is isolated in (12) such that

$$g \geq u_N(t), \quad u_N(t) \leq - \sum_{i=1}^N \frac{\phi'_i(1)}{\phi'_N(1)} u_i(t), \quad (u_N(t) - g) \left(u_N(t) - \sum_{i=1}^N \frac{\phi'_i(1)}{\phi'_N(1)} u_i(t) \right) = 0. \quad (13)$$

Indeed, if (13) holds then (12) holds as well. By separating $u_N(t)$, we impose that $u_N(t)$ is no longer dictated by the ODE (9) but is dictated exclusively by condition (13). The following points introduce the remaining steps of the derivation (which will be elaborated in the upcoming sections):

1. To solve for the motion during inactive contact, the Signorini conditions (13) require that

$$u_N(t) = - \sum_{i=1}^{N-1} \frac{\phi'_i(1)}{\phi'_N(1)} u_i(t), \quad u_N(t) \leq g. \quad (14)$$

Substitution of the above into the FE approximation (8) effectively creates a family of shape functions that always satisfies the inactive contact conditions

$$\bar{u}(x, t) \approx \sum_{i=1}^{N-1} \left(\phi_i(x) - \phi_N(x) \frac{\phi'_i(1)}{\phi'_N(1)} \right) u_i(t). \quad (15)$$

In other words, any solution obtained using the above approximation satisfies the homogeneous Neumann condition taking place during inactive contact. In the FEM, all functions $\phi_i(x)$ have local support and are non-zero for elements containing the node $u_i(t)$. In the case of NBM, the principle of local support is followed as well, and only the shape functions at the element including the contact node $u_N(t)$ are affected by the approximation (15). In turn, the inactive contact inequality in (14) reads

$$u_N(t) \leq g \quad \Rightarrow \quad - \sum_{i=1}^{N-1} \frac{\phi'_i(1)}{\phi'_N(1)} u_i(t) \leq g. \quad (16)$$

2. To solve for the motion during active contact, we construct a family of shape functions that always answers

$$\bar{u}(L, t) \approx u_N(t) = g, \quad u_N(t) \leq - \sum_{i=1}^{N-1} \frac{\phi'_i(1)}{\phi'_N(1)} u_i(t). \quad (17)$$

Evidently, the set of shape functions answering this condition admits

$$\bar{u}(x, t) \approx \sum_{i=1}^{N-1} \phi_i(x) u_i(t) - \phi_N(x) g \quad (18)$$

Thus, any solution that is obtained using the above approximation satisfies the Dirichlet condition in (17). In turn, the active contact inequality in (17), under this approximation becomes

$$u_N(t) \leq - \sum_{i=1}^{N-1} \frac{\phi'_i(1)}{\phi'_N(1)} u_i(t) \Rightarrow - \sum_{i=1}^{N-1} \frac{\phi'_i(1)}{\phi'_N(1)} u_i(t) \geq g. \quad (19)$$

3. The different sets of shape functions described in (15) and (18) will lead to different sets of ODEs governing inactive and active contact motions, respectively. The Signorini conditions are then satisfied by switching between the two sets of functions according to inequalities (16) and (19), which are mutually exclusive, as expected from the Signorini conditions. At the moment of switch, the internal nodal displacements and velocities (internal nodes are those with indexed $i = 1, 2, \dots, N - 1$) are assumed to be continuous in time.
4. The NBM formulation results in an ODE, featuring discontinuous mass and stiffness matrices, which exhibits periodic solutions.

4.1 Comment on application of NBM to other shape functions

Although the NBM is derived in this manuscript using the classical FEM piecewise Lagrangian shape functions, this method can be also formulated using other shape functions. However, for other shape functions, precautions must be considered. One such precaution is that the shape functions must admit a stress approximation that is not always vanishing at the contacting end. To clarify, the NBM relies on the approximation of stress for the switching between active and inactive contact phases, as seen in Equation (12). In order to allow for a switching between contact phases, the shape function must be chosen such that the stress approximation at $x = 1$ does not exhibit $\phi'_i(1) = 0$ for all $i = 1, \dots, N$.

Here, the linear modes of the cantilever bar with uniform area

$$\phi_i(x) = \sin\left(\frac{(2i-1)\pi x}{2}\right), \quad i = 1, 2, \dots, N. \quad (20)$$

will be investigated as an example of a set of shape functions that cannot be used in NBM. These mode shapes bear the undesired property $\phi'_i(1) = 0, \forall i$. For such shape functions, the active contact phase cannot occur as can be seen by plugging Equation (20) into the Signorini conditions Equation (12)

$$g \geq u_N(t) \perp 0 \leq 0 \rightarrow g \geq u_N(t). \quad (21)$$

In simple terms, Equation (21) shows that the linear modes of the cantilever bar exhibit inactive contact motions exclusively and therefore a switch between contact phases cannot occur.

The choice of shape functions is crucial for other methods in contact dynamics and not only the NBM. In fact, for the shape functions (20), both the mass redistribution method [15] and Nitsche method would fail. The mass redistribution method would fail in the same fashion as NBM since it relies on strong enforcement of the Signorini boundary condition. In turn, the Nitsche method would not be able to approximate adequately the Signorini problem. This is illustrated in the remainder of this section. In the Nitsche method, the Signorini boundary conditions are enforced via the following approximation of the stress at the contact boundary

$$A(1)\bar{u}_x(1, t) \approx A(1) \sum_{i=1}^N \phi'_i(1) u_i(t) + \max\left(0, \gamma(u_N(t) - g) - A(1) \sum_{i=1}^N \phi'_i(1) u_i(t)\right), \quad \gamma > 0 \quad (22)$$

where γ is set to be constant [16]. The Signorini condition is then satisfied as $N \rightarrow \infty$

$$A(1)\bar{u}_x(1, t) = \lim_{N \rightarrow \infty} A(1) \sum_{i=1}^N \phi'_i(1) u_i(t) + \max\left(0, \gamma(u_N(t) - g) - A(1) \sum_{i=1}^N \phi'_i(1) u_i(t)\right) \quad (23)$$

$$A(1)\bar{u}_x(1, t) = A(1)\bar{u}_x(1, t) + \max(0, \gamma(\bar{u}(1, t) - g) - A(1)\bar{u}_x(1, t)), \quad \gamma > 0 \quad (24)$$

where the last term is equivalent to the Signorini condition (3). The convergence of the Nitsche method largely relies on the participation of the stress approximation in the right-hand side of Equation (23). This property is disrupted for the $\phi_i(x)$ in Equation (21). For these shape functions, the Nitsche stress approximation (22) reads

$$A(1)\bar{u}_x(1, t) \approx \gamma \max(0, u_N(t) - g). \quad (25)$$

Since the stress approximation is omitted from the right hand side of (25), the resulting approximation (25) is equivalent to a penalty force with penalty parameter γ [14]. The penalty method does not share the same convergence properties as Nitsche as can be seen that by taking the limit of Equation (25) as $N \rightarrow \infty$:

$$A(1)\bar{u}_x(1, t) = \lim_{N \rightarrow \infty} \gamma \max(0, u_N(t) - g) = \gamma \max(0, \bar{u}(1, t) - g). \quad (26)$$

This term, in contrast to Equation (24), is not equivalent to the Signorini conditions.

4.2 Inactive contact motion

The inactive contact motion condition (5) in the NBM framework is

$$\sigma(t) \equiv A(1) \sum_{i=1}^N \phi_i'(1) u_i(t) = 0, \quad u_N(t) \leq g. \quad (27)$$

where we use $\sigma(t)$ to denote the FE approximation of the stress at $x = 1$. For inactive contact motion, we impose that u_N satisfies equation (27) such that

$$u_N(t) = -\frac{1}{\phi_N'(1)} \sum_{i=1}^{N-1} \phi_i'(1) u_i(t) \equiv S(\mathbf{u}^o(t)), \quad \mathbf{u}^o(t) = (u_1(t) \ u_2(t) \ \dots \ u_{N-1}(t))^T \quad (28)$$

where $\mathbf{u}^o(t)$ gathers internal nodal displacements and the function S , acting on $\mathbf{u}^o(t)$, has been assigned to simplify the notation. Actually, S appears naturally throughout the derivation of both inactive and active contact motions in NBM. Indeed, this function is of integral importance to NBM and serves for multiple purposes: it defines the conditions for the switching between phases, the motion of the contact node during inactive contact phase, and the contact force applied on the bar during active contact phase. These roles of S are illustrated in Figure 2 and are explicitly explained later in Section 4.4. For the remainder of this article, the function S will be referred to as the *switching function*.

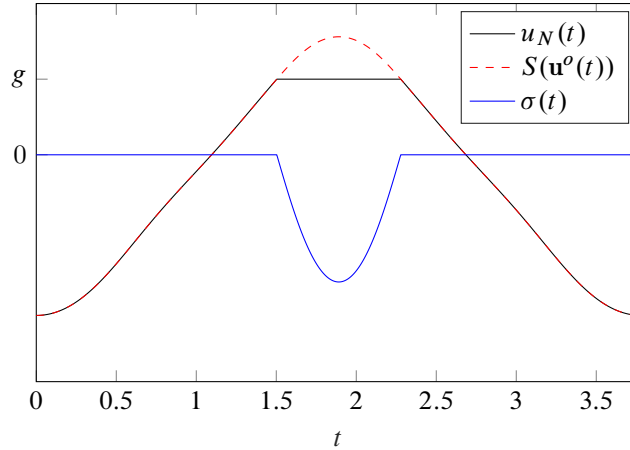


Figure 2: Switching function and associated quantities. The switching function dictates the phase of motion. For $S(\mathbf{u}^o(t)) \leq g$, inactive contact motion takes place, and active contact motion takes place otherwise. Moreover, the displacement at the contact boundary abides $u_N(t) = S(\mathbf{u}^o(t))$ during the inactive contact phase, $S(\mathbf{u}^o(t)) \leq g$, and the approximation of the stress at the contact boundary abides $\sigma(t) = A(1)\phi_N'(1)(g - S(\mathbf{u}^o(t)))$ during the active contact phase.

Following the substitution $u_N(t)$ in (28), $\mathbf{u}(t)$ can be related to $\mathbf{u}^o(t)$ via a linear operator $\mathbf{B} \in \mathbb{R}^{N \times N-1}$

$$\mathbf{u}(t) = \mathbf{B}\mathbf{u}^o(t) \quad B_{ij} = \begin{cases} \delta_{ij} & i = 1, \dots, N-1 \text{ and } j = 1, \dots, N-1, \\ -\frac{\phi_j'(1)}{\phi_N'(1)} & i = N \text{ and } j = 1, \dots, N-1. \end{cases} \quad (29)$$

This representation will be helpful when we introduce the ODE governing the inactive contact motion. Specifically, it allows representation of the approximation of inactive contact conditions to be

$$\bar{u}(x, t) \approx \mathbf{P}(x)\mathbf{B}\mathbf{u}^0(t), \quad u_N \leq g \quad (30)$$

which is equivalent to the presentation in (15).

Next, the inequality condition for inactive contact motion (27) (non-penetration) can be put in terms of the nodes $\mathbf{u}^o(t)$ via the switching function

$$u_N \leq g \Rightarrow S(\mathbf{u}^o) \leq g \quad (31)$$

To derive the ODE governing the inactive contact motion, we insert the inactive motion constraints (29) and (31) into the weak-form of the PDE, Equation (9),

$$\mathbf{w}^\top(t)(\mathbf{M}\mathbf{B}\ddot{\mathbf{u}}^o(t) + \mathbf{K}\mathbf{B}\mathbf{u}^o(t)) = 0 \quad (32)$$

$$u_N(t) = S(\mathbf{u}^o(t)), \quad \ddot{u}_N(t) = S(\ddot{\mathbf{u}}^o(t)), \quad S(\mathbf{u}^o(t)) \leq g. \quad (33)$$

In classical FEM, it is generally assumed Equation (32) is true for all $\mathbf{w}(t)$ and the subsequent omission of $\mathbf{w}(t)$ from the equation takes place. Here, however, omission of $\mathbf{w}(t)$ will lead to an over-defined system of ODEs ($N - 1$ variables in \mathbf{u}^o for N equations). To remedy this, we use the Galerkin-Bubnov method where we project the residual (the term multiplying \mathbf{w} in Equation (32)) on the same solution space used for $\mathbf{u}(t)$ [23]. This strategy has been proven successful in other applications and is commonly used when shape functions that satisfy the boundary conditions are involved in the approximation [23, 24]; [25, p. 300]. It is further noted that the strategy taken here in deriving the NBM has also been referred to as *basis recombination* [26, p. 112]. Following the Galerkin-Bubnov method, we project of the residual resulting from the approximation (30) on the composing trial functions. Under representation (9), this results effectively in modification of the test-function as follows

$$\mathbf{w}(t) = \mathbf{B}\mathbf{w}^o \quad (34)$$

where \mathbf{w}^o gathers all test function contributions corresponding to the internal nodes $\mathbf{u}^o(t)$. Then, substitution of (34) into (32) and omission of \mathbf{w}^o results in the reduced ODE (of $N - 1$ equations)

$$(\mathbf{M}_N\ddot{\mathbf{u}}^o(t) + \mathbf{K}_N\mathbf{u}^o(t)) = 0, \quad \mathbf{M}_N = \mathbf{B}^\top\mathbf{M}\mathbf{B}, \quad \mathbf{K}_N = \mathbf{B}^\top\mathbf{K}\mathbf{B}. \quad (35)$$

where the subscript N in \mathbf{M}_N (or \mathbf{K}_N) is used to denote the NBM coefficients corresponding to answering homogeneous *Neumann* conditions. We remark that, via the NBM, the displacement $u_N(t)$ has been effectively removed from the ODE such that neither the equation nor the inequality constraint in expression (35) includes u_N .

4.3 Active contact motion

The active contact condition in NBM, derived from Equation (13), reads

$$u_N(t) = g \text{ implying } \dot{u}_N(t) = 0 \text{ and } w_N(t) = 0 \quad (36)$$

Effectively, this approximation applies in the active contact phase and the velocity of the contact node is discontinuous at the moment of contact. Here, the test function vanishes on Dirichlet boundaries (ie, $w_N(t) = 0$ during active contact) which is common practice for FEM approximations [22, 23, 27]. Next, the complementarity condition (13) can be rewritten in terms of the switching function

$$A(1) \sum_{i=1}^N \phi_i'(L)u_i(t) \leq 0, \quad (37)$$

$$- \sum_{i=1}^{N-1} \frac{\phi_i'(1)}{\phi_N'(1)} u_i(t) \geq u_N(t), \quad (38)$$

$$S(\mathbf{u}^o(t)) \geq g. \quad (39)$$

Here, the transition between statements (37) and (38) requires that $\phi'_N(1) > 0$ holds. Otherwise, for example in the case $\phi'_N(1) \leq 0$, both inactive and active contact phases would have to occur for $S(\mathbf{u}^o(t)) < 0$, that is simultaneously, which disagrees with the mutual exclusivity of the inactive and active contact conditions in the complementarity conditions (3). Accordingly, for $\phi'_N(1) > 0$, we note that the active contact motion occurs for $S(\mathbf{u}^o(t)) > g$, and the inactive contact motion occurs for $S(\mathbf{u}^o(t)) < g$ such that both are mutually exclusive. Fortunately, for the case of Lagrangian polynomials used here, the statement $\phi'_N(1) > 0$ has been proven to hold for any number of elements. The proof can be found in Appendix A.1.

We continue with the substitution of expressions (36) and (39) into the FEM-ODE (9) where (36) is first recast in the matrix format

$$\mathbf{w}(t) = \mathbf{B}^d \mathbf{w}^o, \quad \mathbf{u}(t) = \mathbf{B}^d \mathbf{u}^o(t) + g \mathbf{b}^d, \quad S(\mathbf{u}^o(t)) \geq g \quad (40)$$

with the notations

$$B_{ij}^d = \begin{cases} \delta_{ij} & i = 1, \dots, N-1; j = 1, \dots, N-1 \\ 0 & i = N; j = 1, \dots, N-1 \end{cases} \quad \text{and } b_i^d = \begin{cases} 0 & i = 1, \dots, N-1 \\ 1 & i = N. \end{cases} \quad (41)$$

We then plug (40) into (9) to obtain an ODE in terms of \mathbf{u}^o multiplied by \mathbf{w}^o forming the scalar equation

$$(\mathbf{w}^o)^\top (\mathbf{M}_D \ddot{\mathbf{u}}^o(t) + \mathbf{K}_D \mathbf{u}^o(t) + g \mathbf{f}_D) = 0, \quad S(\mathbf{u}^o(t)) \geq g \quad (42)$$

where $\mathbf{M}_D = (\mathbf{B}^d)^\top \mathbf{M} \mathbf{B}^d$, $\mathbf{K}_D = (\mathbf{B}^d)^\top \mathbf{K} \mathbf{B}^d$, and $\mathbf{f}_D = (\mathbf{B}^d)^\top \mathbf{K} \mathbf{b}^d$, and the subscript D is used to denote the coefficients corresponding the ODE answering the nonhomogeneous *Dirichlet* conditions. Assuming equation (42) should be valid for all values \mathbf{w}^o , the following ODE formulation is obtained:

$$\mathbf{M}_D \ddot{\mathbf{u}}^o(t) + \mathbf{K}_D \mathbf{u}^o(t) + g \mathbf{f}_D = \mathbf{0}, \quad S(\mathbf{u}^o(t)) \geq g \quad (43)$$

Equation (43) is equivalent to the ODE obtained by application of classical Finite Element method on the clamped-clamped bar. Furthermore, the NBM formulation of the ODE for active contact is equivalent to this described by the basis recombination method for non-homogeneous boundary conditions [26, p. 112]. However, the NBM adds the restriction $S(\mathbf{u}^o(t)) > g$ to infer that the bar must be repulsed at all time throughout contact, as required by the active contact condition (4).

In the next section, we combine the ODEs corresponding to both active and inactive motions, Equation (35) and Equation (43) respectively, to construct the ODE approximation for the original Signorini problem.

4.4 NBM-FEM formulation of Signorini problem

The switching method [12, 28] is used for the enforcement of the Signorini conditions in NBM. In the switching method, the Signorini complementarity conditions are answered by alternating between the boundary conditions (4) and (5) both in the test and trial functions such that the inequality constraints are satisfied. In NBM, this translates to switching between Equation (35) and Equation (43), and the complete NBM-ODE reads

$$\begin{cases} \mathbf{M}_D \ddot{\mathbf{u}}^o(t) + \mathbf{K}_D \mathbf{u}^o(t) + g \mathbf{f}_D = \mathbf{0} & S(\mathbf{u}^o(t)) \leq g \\ \mathbf{M}_N \ddot{\mathbf{u}}^o(t) + \mathbf{K}_N \mathbf{u}^o(t) = \mathbf{0} & S(\mathbf{u}^o(t)) \geq g. \end{cases} \quad (44)$$

We note that at the moment of switch, denoted t_s such that $S(\mathbf{u}^o(t_s)) = g$, the resulting NBM-ODE (44) raises two conflicting definitions to the ODE. This conflict is resolved by extending the active and inactive contact NBM conditions as follows

$$\text{Active contact NBM: } S(\mathbf{u}^o(t)) > g, \text{ or } S(\mathbf{u}^o(t)) = g \text{ and } S(\dot{\mathbf{u}}^o(t^-)) > 0 \quad (45)$$

$$\text{Inactive contact NBM: } S(\mathbf{u}^o(t)) < g, \text{ or } S(\mathbf{u}^o(t)) = g \text{ and } S(\dot{\mathbf{u}}^o(t^-)) < 0 \quad (46)$$

which is mathematically sound since the velocity at the moment of switch indicates whether an active contact or an inactive contact occurs after the switch. Furthermore, note that the case of zero velocity $S(\dot{\mathbf{u}}^o(t_s^-)) = 0$ before contact is not investigated in definitions (45) and (46). Such solutions are referred to as grazing solutions and propose a challenge that is beyond the scope of this paper (the reader may refer to [10, p. 385] for some

of the intricacies involved with determining grazing periodic motions). Since grazing motions are excluded, the NBM is limited in its solutions. Thus, it is important to note that the NBM is not presented as a scheme for generation physically accurate simulations. Rather, the NBM is used for modal analysis and detection of periodic Signorini compliant non-grazing motions.

Next, at the instant of contact, we impose that the *internal* displacements $\mathbf{u}^o(t)$ and velocities $\dot{\mathbf{u}}^o(t_s)$ are continuous at switching instants

$$\text{Continuity of internal states: } \mathbf{u}^o(t^+) = \mathbf{u}^o(t^-), \dot{\mathbf{u}}^o(t^+) = \dot{\mathbf{u}}^o(t^-), \quad S(\mathbf{u}^o(t)) = g. \quad (47)$$

while only the contact node is characterized by discontinuous velocities, as will be shown later in Equation (51). The condition for continuity of the internal states (47) corresponds to common application of the Signorini conditions both numerically and analytically [1, 12, 14–16]. Thus, both internal displacements and internal velocities are assumed to be always continuous and only the acceleration $\ddot{\mathbf{u}}^o(t)$ is discontinuous at instants t_s (as can be deduced from Equation (44)). The discontinuity of *internal* accelerations at the moment of switch is a consequence of the NBM and is not expected in the true solution. However, solutions of the NBM still show good agreement with the true solution as $N \rightarrow \infty$, as illustrated in Section 6.

Given that the acceleration is discontinuous at the moment of switch, it is more convenient to represent the ODE (44) in terms of the acceleration at t^+

$$\ddot{\mathbf{u}}^o(t^+) = \begin{cases} -(\mathbf{M}_D)^{-1}(\mathbf{K}_D \mathbf{u}^o(t) + g \mathbf{f}_D) & \text{active contact NBM} \\ -(\mathbf{M}_N)^{-1}(\mathbf{K}_N \mathbf{u}^o(t)) & \text{inactive contact NBM} \end{cases} \quad (48)$$

where \mathbf{M}_D and \mathbf{M}_N are always invertible (the proof follows from \mathbf{M} being non-invertible and is not presented in this manuscript for sake of conciseness).

In turn, the approximation of $u(x, t)$ in NBM is defined via expressions (30) and (40)

$$\bar{u}(x, t) \approx \mathbf{P}(x) \mathbf{u}(t) = \begin{cases} \mathbf{P}(x)(\mathbf{B}^d \mathbf{u}^o(t) + g \mathbf{b}^d) & S(\mathbf{u}^o(t)) \geq g \\ \mathbf{P}(x) \mathbf{B} \mathbf{u}^o(t) & S(\mathbf{u}^o(t)) \leq g \end{cases} \quad (49)$$

At last, from expression (49), we obtain nonsmooth expressions for the displacement, velocity and stress at the contacting end (with strict inequality applied on the active contact condition)

$$\bar{u}(1, t) \approx u_N(\mathbf{u}^o(t)) = \begin{cases} g & S(\mathbf{u}^o(t)) \geq g \\ S(\mathbf{u}^o(t)) & S(\mathbf{u}^o(t)) \leq g \end{cases} \quad (50)$$

$$\bar{u}_t(1, t) \approx \dot{u}_N(\mathbf{u}^o(t), \dot{\mathbf{u}}^o(t)) = \begin{cases} 0 & S(\mathbf{u}^o(t)) > g \\ S(\dot{\mathbf{u}}^o(t)) & S(\mathbf{u}^o(t)) < g \end{cases} \quad (51)$$

$$A(1) \bar{u}_x(1, t) \approx \sigma(\mathbf{u}^o(t)) = \begin{cases} A(1) \phi'_N(1)(g - S(\mathbf{u}^o(t))) & S(\mathbf{u}^o(t)) \geq g \\ 0 & S(\mathbf{u}^o(t)) \leq g \end{cases} \quad (52)$$

We note that $\sigma(t)$ is continuous in the NBM formulation contrarily to the discontinuous behaviour of the true solution to the Signorini problem [1, 12, 20]. However, in section 6 it will be shown that the method still converges. Furthermore, it is important to note that schemes utilizing Newton's impact law with $e = 0$ are characterized by continuous contact pressure as well and yet show convergence to the true solution [14].

On another note, the acceleration $\ddot{u}_N(t)$ obtained by differentiating Equation (51) involves the Dirac-delta distribution at the instance of switch, which may affect the formulation of the NBM-ODE since it participates in its definition, see Equation (33). However, the influence of the Dirac-delta in the NBM formulation was not investigated in this article and is suppressed to simplify the formulation. Nevertheless, numerical experiments show that the NBM-ODE (48) admits solutions that converge to the true motion for large N . Such numerical experiment is explored in section Section 6.1.

The Signorini problem is hence formulated as the nonsmooth ODE (48) in $\mathbf{u}^o(t)$ with unique solutions given initial conditions $\mathbf{u}^o(0)$ and $\dot{\mathbf{u}}^o(0)$. Furthermore, the solutions generated by the NBM are characterized by *sticking phases*. Sticking phases are continuous intervals of time of non-zero measure where active contact motion occurs. To clarify, sticking phases stand in contrast to chattering exhibited by schemes utilizing a Newton

impact law with $e = 1$ [1]. This is a noteworthy property of this scheme since sticking phases occur in the true solution to the Signorini problem [1, 11, 12, 14].

Next, since periodic solutions require energy conservation, we investigate the energy conservation properties of solutions to the NBM-ODE.

4.5 Energy conservation properties of solutions to NBM-ODE

Solutions to the NBM-ODE (48) are equipped with the energy metric

$$2E(t) = \int_0^1 \bar{u}_t(x, t)^2 + A(x)\bar{u}_x(x, t)^2 dx \approx \dot{\mathbf{u}}^\top(t)\mathbf{M}\dot{\mathbf{u}}(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) \quad (53)$$

and exhibit the following properties

1. The ODE preserves energy for $S(\mathbf{u}^o(t)) > g$ and $S(\mathbf{u}^o(t)) < g$, away from instants t_s such that $S(\mathbf{u}^o(t_s)) = g$.
2. At a time instant t_s , where a transition between active and inactive contact occurs (namely, at an instant where $S(\mathbf{u}^o(t_s)) = g$ and $S(\dot{\mathbf{u}}^o(t_s)) \neq 0$), an instantaneous change in energy ΔE occurs

$$\Delta E = E(t_s^+) - E(t_s^-) = -|S(\dot{\mathbf{u}}^o(t_s))| \left(\frac{1}{2} M_{NN} S(\dot{\mathbf{u}}^o(t_s)) + \sum_{j=1}^{N-1} M_{Nj} \dot{u}_j(t_s) \right). \quad (54)$$

The energy after transition may either decrease ($\Delta E < 0$), increase ($\Delta E > 0$) or be conserved ($\Delta E = 0$).

For proofs, section A.2 establishes lemmas regarding statement 1 and Section A.3 details the proof for statement 2.

An immediate consequence of statements 1 and 2 is that solutions of the NBM-ODE may exhibit a periodic energy evolution in time. Indeed, this property is favourable for detection of periodic solutions, and it is expected that the NBM can be used for detection periodic solutions to the Signorini problem. From numerical experiments, such periodic solutions to the NBM were found. An example of a periodic solution is shown in Figure 3 from which it is clear that the NBM allows for existence of periodic solutions with sticking phases. In contrast,

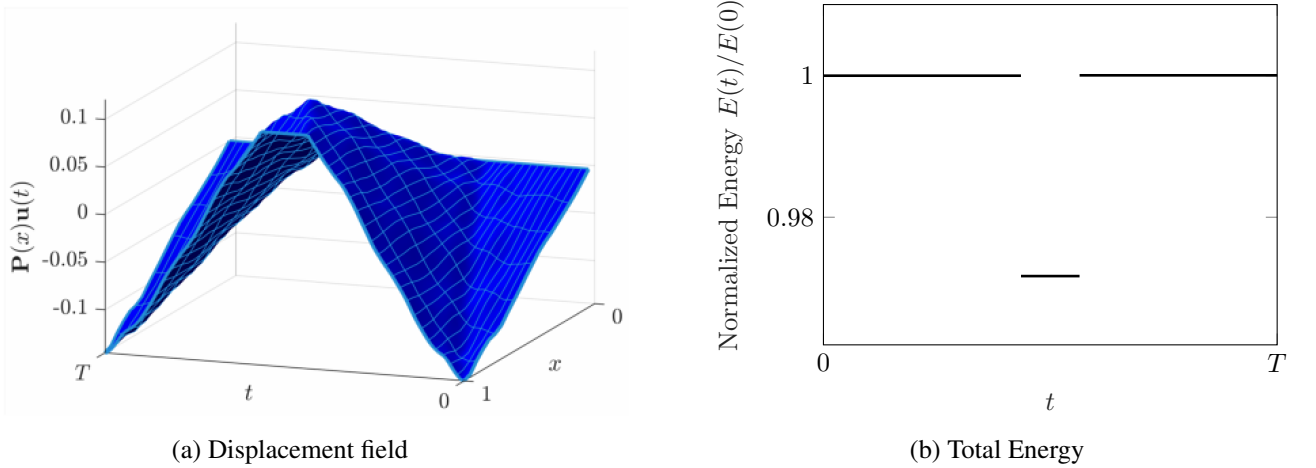


Figure 3: NBM periodic solution for the varying area bar (corresponding to the model investigated in Figure 4) of 10 elements and quadratic Lagrangian shape functions ($N = 20$). Note that the total energy is dissipated at the moment of contact but is completely regained at the end of the contact phase. Due to this characteristic, the NBM allows for periodic solutions with sticking phases.

solutions obtained via Nitsche method exhibit sticking phases only at convergence (ie, for high number of elements) [16] and solutions of scheme with Newton impact law $e = 1$ exhibit chattering [14]. While solutions to Newton's impact law with $e = 0$ exhibit sticking phases, they also exhibit energy dissipation for non-zero impact velocity [14] and therefore do not allow for periodic solutions. The NBM, in comparison to Newton's impact law $e = 0$, allows for regain of energy in transition from active to inactive contact.

Indeed, the existence of periodic solutions in NBM relates largely to the fact that energy can be regained throughout the motion, as seen in Figure 3. Although, it is important to note that the concept of “gain of energy” in the NBM is used in a numerical sense rather than a physical sense. For Newton’s impact law $e = 0$, the loss of energy can be seen as physical dissipation of energy from the system, while in the NBM there is no physical justification for a gain in energy. To clarify, the difference in energy (whether loss or gain) relates to the use of different sets of shape functions to describe the motion and not due to physical assumptions. For the true solution, at the instance of switch, $\bar{u}_t(x, t_s^+) = \bar{u}_t(x, t_s^-)$ holds true for $x \in (0, 1)$ [1, 11, 20]. However, for the approximation of the same motion in NBM, the approximations of $\bar{u}_t(x, t_s^+)$ and $\bar{u}_t(x, t_s^-)$ involve two different sets of shape functions, described in Equations (16) and (19), and the switch is accompanied by a difference in kinetic energy (54). Thus, the gain or loss in energy is a numerical phenomenon resulting from switching between two distinct sets of shape functions in both the trial and test functions. Along the same line, the terms dissipation and gain of energy will be used to describe the evolution of the energy metric in time rather than implying any physical energy transfer in or out of the system. Moreover, in the NBM, it was evident from numerical experiments that the energy jump ΔE diminishes for large N . Although, an analytical proof for this statement is not presented in this article. To conclude, the behaviour of the energy metric in the NBM should be understood as a consequence of the Galerkin-Bubnov method rather than as a physical imposition on the system (as done by implementing a Newton impact law with $e = 0$, for example).

5 Nonsmooth modal analysis

In the preceding sections, the NBM and FEM were used to approximate the solution to the initial boundary value problem exposed in Equation (1) to Equation (4). To solve for the remaining conditions, we require that $\mathbf{u}^o(t)$ and $\dot{\mathbf{u}}^o(t)$ are periodic, see Equation (6) and Equation (7),

$$\mathbf{u}^o(0) = \mathbf{u}^o(T) \tag{55}$$

$$\dot{\mathbf{u}}^o(0) = \dot{\mathbf{u}}^o(T). \tag{56}$$

In this article, we attempt to find such solutions and corresponding period using the shooting method. Moreover, continuation is used for detection of nonsmooth modes, ie families of periodic solutions [7].

5.1 Time-marching techniques and shooting method

In the shooting method [7], we find periodic solutions by aiming for a set of initial conditions, \mathbf{u}_0^o and \mathbf{v}_0^o , generating a periodic solutions, ie

$$\begin{aligned} \mathbf{u}^o(T) &= \mathbf{u}^o(0) \equiv \mathbf{u}_0^o \\ \dot{\mathbf{u}}^o(T) &= \dot{\mathbf{u}}^o(0) \equiv \mathbf{v}_0^o \end{aligned} \tag{57}$$

where $\mathbf{u}^o(t)$ is subject to the ODE (48).

In order to solve for the shooting equations, we solve for $\mathbf{u}^o(T)$ and $\dot{\mathbf{u}}^o(T)$ numerically via a numerical time marching scheme. For example, a Crank-Nicolson (CN) scheme (or Newmark scheme with $\beta = 1/2$ and $\gamma = 1/4$ [29]) may be used. While the CN scheme is considered energy stable for finite element schemes [29], it is an implicit algorithm and requires the implementation of a root solving algorithm. To implement the time-marching techniques, we shall first convert the NBM-ODE (48) into a system of first order ODEs $\dot{\mathbf{q}}(t) = \mathbf{G}(\mathbf{q}(t))$ with

$$\mathbf{G}(\mathbf{q}(t)) = \begin{cases} \begin{pmatrix} \mathbf{v}^o(t) \\ \mathbf{M}_D^{-1}(\mathbf{g}\mathbf{f}_D - \mathbf{K}_D\mathbf{u}^o(t)) \end{pmatrix} & \text{Active contact NBM} \\ \begin{pmatrix} \mathbf{v}^o(t) \\ -\mathbf{M}_N^{-1}\mathbf{K}_N\mathbf{u}^o(t) \end{pmatrix} & \text{Inactive contact NBM} \end{cases} \quad \text{and} \quad \mathbf{q}(t) = \begin{pmatrix} \mathbf{u}^o(t) \\ \mathbf{v}^o(t) \end{pmatrix}. \tag{58}$$

For application of the time-marching scheme, a discretization of the time span $t \in [0, T]$ into N_t steps is considered with the time-step $\Delta t = T/N_t$. In the time-marching scheme, the quantities $\mathbf{q}_i \approx \mathbf{q}(t_i)$ are solved iteratively for $i = 1, \dots, N_t$. After N_t steps, the state at the end of the period is obtained $\mathbf{q}(T) \approx \mathbf{q}_{N_t}$.

Indeed, the approximated state \mathbf{q}_{N_t} is a function of T and \mathbf{q}_0 . Substitution of the approximation $\mathbf{q}_{N_t}(\mathbf{q}_0, T)$ in Equation (57) reads

$$\mathbf{F}(\mathbf{q}_0, T) = \mathbf{q}_{N_t}(T, \mathbf{q}_0) - \mathbf{q}_0 = \mathbf{0}. \quad (59)$$

This problem constitutes an under-defined system of equations with $2(N - 1)$ equations for $2(N - 1) + 1$ variables. This gives rise to a continuum of solutions parameterized in T , ie $(\mathbf{q}_0(T), T)$. This continuum of solutions then directly corresponds to families of periodic solutions, otherwise known as nonsmooth modes (NSMs) [1].

5.2 Error estimation

For cases where the solution $\bar{u}(x, t)$ exists, a proper error estimate of the NBM solution is the L_2 norm $\|\mathbf{P}(x)\mathbf{u}(t) - \bar{u}(x, t)\|_2$ where $\mathbf{P}(x)\mathbf{u}(t)$ is defined in Equation (49). However, in the absence of closed-form solution, such as the periodic solutions sought for the varying-area bar, other error metrics are needed. While the error in the residual resulting from the approximation constitutes a common error metric for the Galerkin-Bubnov method [23, 30], the residual of the PDE (1) under the NBM-FEM approximation requires knowledge of $\phi_i''(x)$ for $x \in [0, 1]$. However, since $\phi_i(x)$ is described using the piecewise Lagrangian used in FE framework, the double derivative of $\phi_i(x)$ is not defined on element boundaries. Therefore, to quantify the error, the *residual estimator* [30, p. 93]

$$R(t) = h^4 \sum_{j=1}^{N_e} \int_{\mathcal{E}_j} \left(\sum_{i=1}^N \phi_i(x) \ddot{u}_i(t) - \phi_i''(x) u_i(t) \right)^2 dx \quad (60)$$

is used, where h describes the length of the element, N_e denotes the total number of elements and \mathcal{E}_j is the domain of the element j excluding its boundaries such that any given $\phi_i''(x)$ is defined everywhere in \mathcal{E}_j , and the boundary of the elements are excluded from the error metric (60). It is noted, that while the error metric effectively excludes points of discontinuity, it evaluates the accuracy of the solution for ranges where the approximation of $\bar{u}(x, t)$ is clearly defined. Thus, it is considered a proper metric for the evaluation of the solution's accuracy. For the NBM, we must evaluate the integral of $R(t)$ for $t \in [0, T]$. Since the acceleration $\ddot{\mathbf{u}}(t)$ is discontinuous at the moment of switch t_s , we define the *residual error* by excluding instances of discontinuity, similarly to the residual estimator,

$$R_\epsilon = \int_0^{t_s^{1-}} R(t) dt + \int_{t_s^{(N_s-1)+}}^T R(t) dt + \sum_{j=1}^{N_s} \int_{t_s^{j+}}^{t_s^{(j+1)-}} R(t) dt \quad (61)$$

where t_s^j denote various distinct instants of switch through the motion and N_s defines the total instances of switch in $[0, T]$. We duly note that the residual error does not take into account the discontinuities in time, and may be an inaccurate error metric for the Signorini problem. However, the metric R_ϵ does define the quality of the approximation of inactive and active phases of motion by their respective shape functions. Thus, R_ϵ is useful in determining the accuracy in the approximation of the active and inactive contact phases. Nevertheless, if the exact solution to the problem is known, the more accurate error norm $\|u(x, t) - \mathbf{P}(x)\mathbf{u}(t)\|_2$ will be used instead of R_ϵ .

5.3 Sequential continuation with correction

As evident from literature on modal analysis of the Signorini problem, NSM can be described by via a continuum of solutions $(\mathbf{q}_0(T), T)$ on a closed interval of periods, referred to as a backbone curve [1, 11, 12, 31]. This continuum of solutions can be found by applying sequential continuation on the system of equations (59) [12, 31]. Applying sequential continuation to the NBM numerically, it has been noticed that the sequential continuation for curves of $N \geq 10$ faces difficulties in obtaining solutions on the backbone curve $(\mathbf{q}_0(T), T)$. To distinguish between curves approximated using different N , we will denote a solution continuum as follows: $(\mathbf{q}_0(T), T)^N$.

For large N , it is more efficient to obtain solutions by first finding solution with a low N and then apply a shooting algorithm while recursively increasing the number of nodes for each point in the low- N curve. We

refer to this method as *sequential continuation with correction* (SCC). The steps of sequential continuation with correction are:

1. Obtain the nonlinear normal mode with low N (for instance, $N = 4$), stored as the series $(\mathbf{q}_0^k, T^k)^4$, $k = 1, 2, \dots, N_c$.
2. For the point $i = 1$ in the series $(\mathbf{q}_0^1, T^1)^N$, perform shooting on a system with higher number of nodes N^+ . First, interpolate $(\mathbf{q}_0^1)^N$ using the shape functions $\mathbf{P}(x)$ to obtain an initial guess for the desired N^+ approximation.
3. Solve the shooting equations with N^+ nodes with period T^1 and obtain $(\mathbf{q}_0^1, T^1)^{N^+}$.
4. Steps 2 and 3 above may be repeated for higher number of nodes while keeping the period T^1 constant.
5. Repeat 2, 3 and 4 for all points in the series discovered in step 1, ie $k = 2, 3, \dots, N_c$.

Figures 4 and 5 are used to illustrate the method.

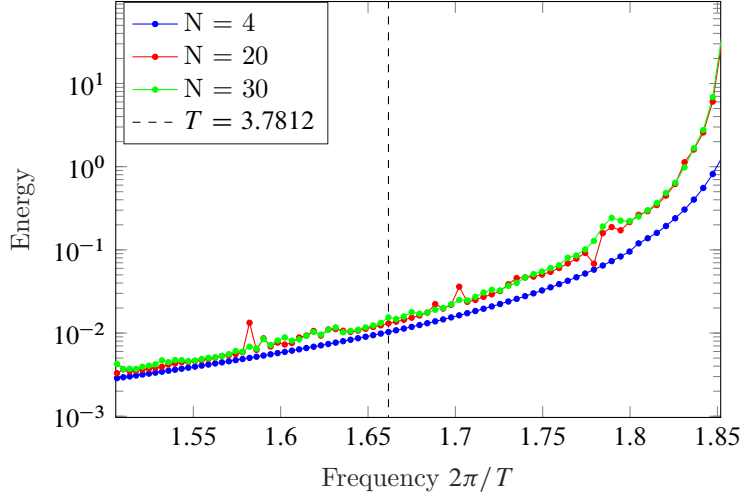


Figure 4: Sequential Continuation with Correction: every point represents a periodic solution of specific frequency and energy. The curve for $N = 4$ is found using sequential continuation. For each point on this curve, a shooting algorithm is applied to obtain a periodic solution in higher N and the same frequency. Dotted line relates to Figure 5 depicting solutions along this line. The results in this figure and in Figure 5 were obtained using CN and NBM with $g = 0.1$ and $A(x) = 1 - x/2$.

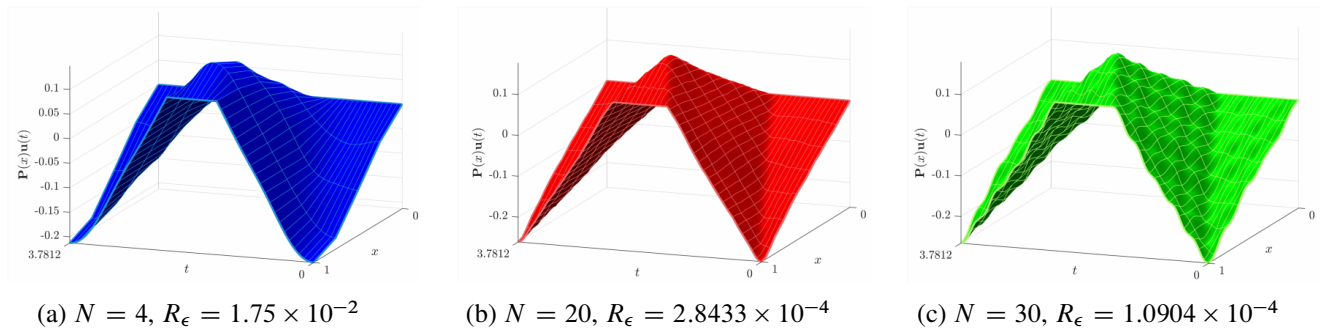


Figure 5: Periodic displacement field corresponding to Figure 4, for period $T = 3.7812$. With higher number of nodes, the obtained displacement field is more accurate as evident by the lower R_ϵ value.

6 Results

6.1 Convergence of Crank-Nicolson and NBM

In this section, we verify the validity of the NBM for the cantilever bar of uniform area, ie $A(x) = 1, \forall x \in [0, 1]$ and $g = 0.1$. For this model, analytical solutions and nonsmooth modal analysis results are available [1, 11, 13].

We compare the exact solution with its NBM approximation for $T = 3.5$. The corresponding NSM displacement field reads [11]

$$\bar{u}(x, t) = f(t + x) - f(t - x) \quad \text{with} \quad f(s) = 0.1 \begin{cases} -s & s \in [-1, 1], \\ s - 2 & s \in [1, 2.5], \\ 3 - s & s \in [2.5, 4.5]. \end{cases} \quad (62)$$

The NBM model is assigned the initial conditions generating the exact periodic solutions. The exact initial conditions are discretized and their values are taken at loci x_i corresponding to the NBM-FEM nodes $u_i(t)$

$$u_i^o(0) = f(x_i) - f(-x_i) = -0.2x_i \quad \text{and} \quad v_i^o(0) = f'(x_i) - f'(-x_i) = 0 \quad (63)$$

inserted in the implicit CN time-marching with $N_t = 2000$ steps and $\Delta t = 1.75 \cdot 10^{-3}$. The error used in the convergence analysis is expressed in the L_2 -norm

$$\|\mathbf{P}(x)\mathbf{u}(t) - \bar{u}(x, t)\|_2 = \frac{1}{T} \sqrt{\int_0^1 \int_0^T (\bar{u}(x, t) - \sum_{i=1}^{N-1} \phi_i(x)u_i^o(t) + \phi_N(x)u_N(\mathbf{u}^o(t)))^2 dt dx} \quad (64)$$

where $u_N(\mathbf{u}^o(t))$ is defined in Equation (50). From Figure 6, it is noted that, for any shape function polynomial degree, convergence is approximately of first order in terms of the number of elements. Indeed, since the exact

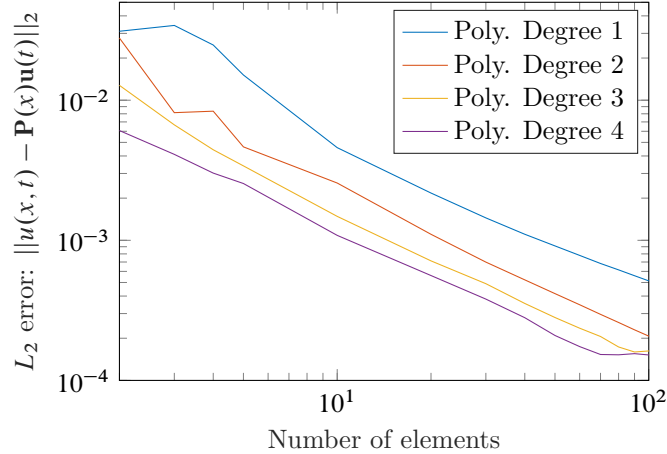


Figure 6: Error in NBM approximation. Curves denote different order of Lagrangian shape function in the FE approximation.

solution is only once piecewise differentiable as evident from Equation (62), the order of convergence of the FE methods is limited to first order [22, p. 117]. A sample of the solution for $N = 200$ is illustrated in Figure 7.

6.2 Nonsmooth modes

In this section, we shall use the nonsmooth modal analysis techniques developed in Section 5 to perform nonsmooth modal analysis for three variations of the bar: the internally resonant cantilever uniform area bar, the uniform area cantilever bar with soft support, and the varying area cantilever bar. To verify the validity of the nonsmooth modes presented here, we will compare each nonsmooth mode with its corresponding forced-response diagram since it is expected that the backbone branch will align with the frequency and energy at resonance [7, 13]. All results in Section 6.2 are generated for a gap distance $g = 0.001$ to comply with the models investigated in [13, 31]. Furthermore, all NSMs were generated via FE models of 20 elements and quadratic shape functions for which the backbone branches were depicting resonant points sufficiently.

6.2.1 Forced-response curves

The forced response-curves are generated by solving the equation

$$u_{tt}(x, t) + cu_t(x, t) + (A(x)u_x(x, t))_x = \bar{F} \cos(\omega t) \quad (65)$$

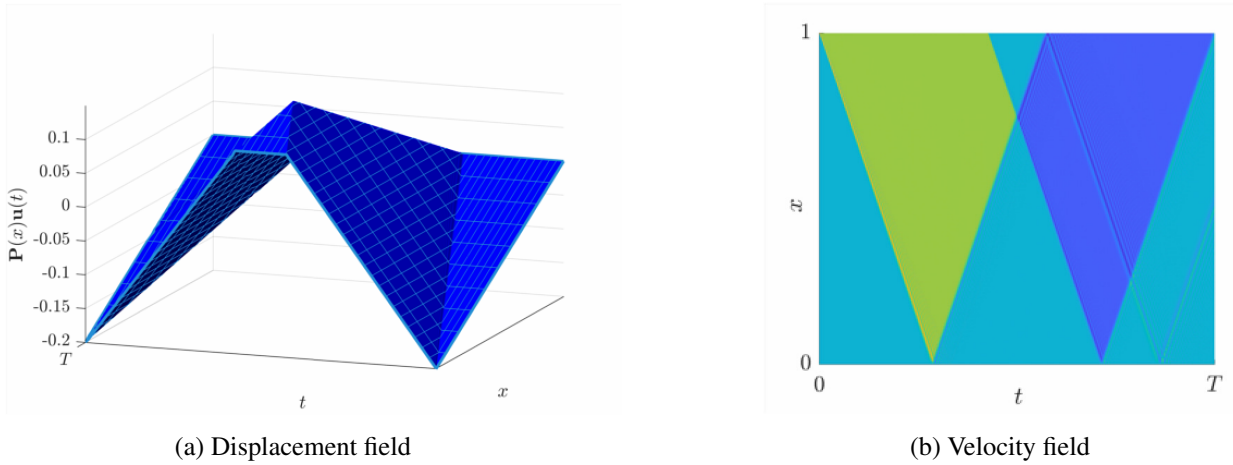


Figure 7: NBM solution emerging from initial conditions (63) for 100 elements and quadratic shape functions ($N = 200$). We note a small disturbance in the velocity field Figure 7(b). This disturbance in the velocity field seems to travel along characteristic lines and diminishes progressively as higher number of elements and degree of polynomials are used.

where c , \bar{F} and ω denote the damping coefficient, forcing amplitude and the forcing frequency, respectively. This governing PDE is complemented with the Signorini boundary conditions (5) and (4) as well as the boundary conditions imposed at $x = 0$ by the model in questions. The resulting Signorini problem is then solved for $c = 0.1, 0.2, \dots, 0.7$ and for frequencies ω within the range of the detected NSM. For each set of values c and ω , we record the sum of kinetic and potential energies of the structure at steady state to plot the forced-response diagrams.

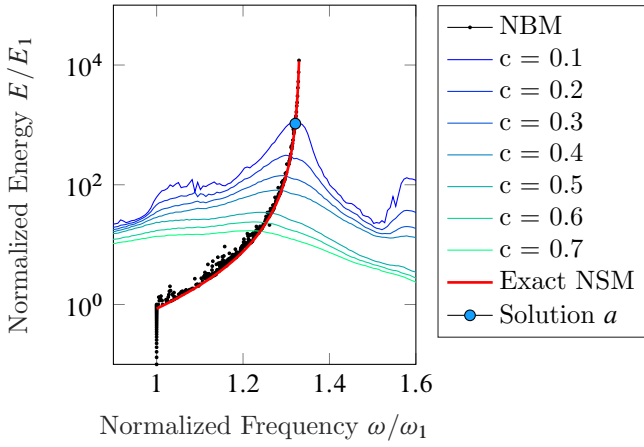
In practice, it is assumed a steady state is reached as $t \rightarrow \infty$. Although we often expect a forced motion to reach a periodic steady state, for some frequencies, quasi-periodic or chaotic solutions take place [12]. Thus, to obtain the forced-response curve, the Signorini problem is solved until a periodic motion is obtained or until the energy's mean value throughout a forcing period is sufficiently stable.

All forced-response curves were obtained using the FEM framework with 20 elements and quadratic shape functions and solved via SICONOS [32]. SICONOS uses a Moreau-Jean scheme to implement the Newton-impact law in the resulting system of ODEs. Here, the Newton-impact law is applied on a classical FEM approximation of the model in question, that is without application of NBM. The coefficient of restitution used to generate the forced responses in Section 6.2.2 to Section 6.2.4 is $e = 0$. It is important to note that forced-response diagrams for the models have been also obtained using SICONOS with $e = 1$, Nitsche's method, and NBM. Since all the forced-response curves yielded very similar results, we chose to present just those obtained for SICONOS with $e = 0$ for sake of conciseness.

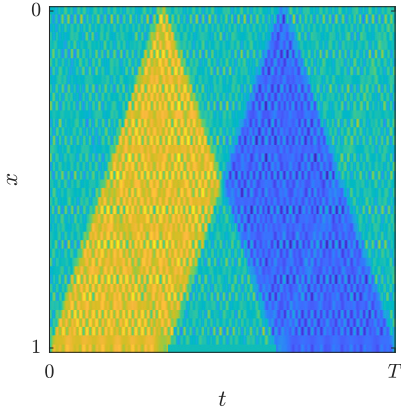
6.2.2 Constant cross-section and internally resonant bar

The internally resonant bar of uniform area $A(x) = 1$ is an example that has been investigated both numerically [12, 31] and analytically [1, 11, 20]. The system is known to manifest an intricate modal space consisting of families of iso-periodic periodic solutions in a dense set of periods and families of periodic solutions of the same frequency and energy.

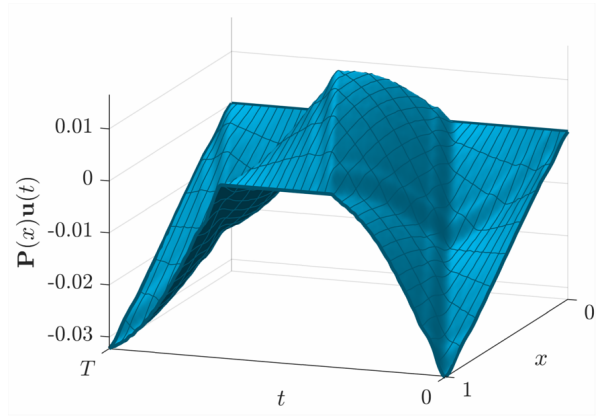
In Figure 8(a), we compare the curve obtained by NBM to the curve of piecewise-linear analytical solutions obtained in [13]. It is clear that the NSM obtained from NBM lays closely to the exact piecewise-linear NSM obtained in literature [1, 11, 13]. We note that there exist other exact periodic solutions on top of the NSM curve as was concluded in [11]. It is known, from exact solutions and numerical solutions, that the spectrum of the internally resonant bar consists of iso-periodic nonsmooth modes existing as lines above the exact NSM branch in the frequency-energy diagram [11]. Therefore, the group of solutions found by NBM may be considered numerically accurate due to the existence of solutions above the Exact NSM curve. Nevertheless, we note that the NBM backbone curve does cross all points of resonance in the forced-response diagram. This shows that the NBM is useful for the modal analysis of the Signorini problem.



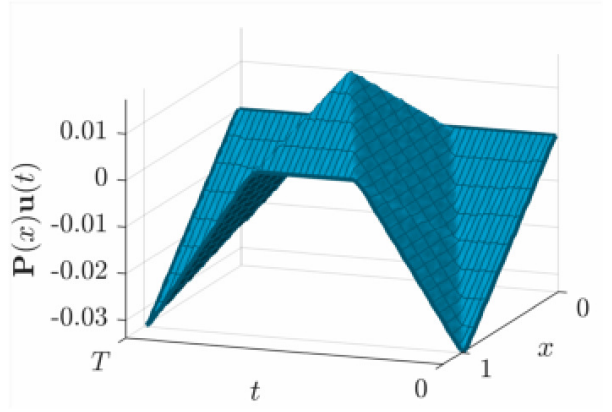
(a) Forced response curves



(c) Velocity field of solution a , autonomous motion generated using NBM with CN time-marching



(b) Displacement field of solution a , forced and damped motion generated by SICONOS with $e = 0$



(d) Displacement field of solution a , autonomous motion using NBM with CN time-marching

Figure 8: NSM of the internally resonant bar detected by the NBM. ($\bar{F} = 0.05$). Exact NSM is the piecewise-linear mode detected for the internally resonant bar in [1, 13]. $\omega_1 = \pi/2$ and $E_1 \approx 5.9 \times 10^{-7}$. We note here that the velocity field Figure 8(c) involves porous oscillations which are common to methods in the FE framework. However, a heat-map representation of the velocity field is useful in demonstrating that the found solution follows (approximately) the characteristic lines, exhibited by the exact solution [13].

6.2.3 Constant cross-section bar with soft support

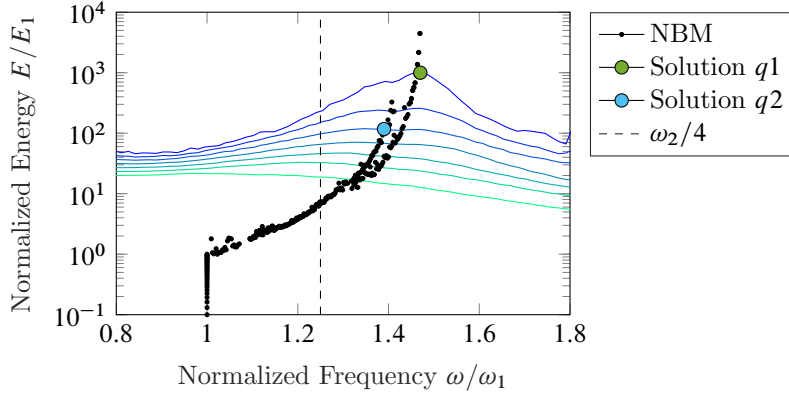
Similarly to [13, 31], the bar with soft support features a uniform cross-sectional area $A(x) = 1$ where the homogeneous Dirichlet condition at $x = 0$ is replaced with the Robin condition

$$ku(0, t) = u_x(0, t), \quad k = \bar{k}L/(EA_0) \quad (66)$$

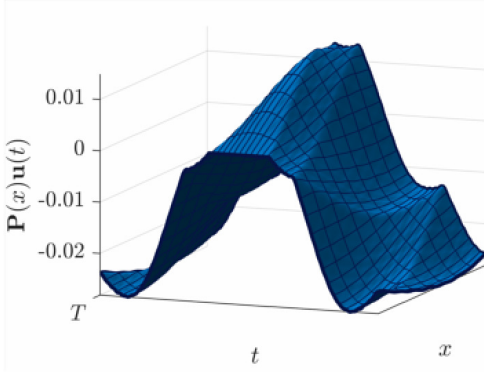
where \bar{k} is the physical stiffness coefficient of the spring. The NBM was used to handle the Signorini boundary condition at $x = 1$ while the soft support condition (66) was treated using the classical finite element technique.

For this experiment, we set $k = 0.5$ to replicate the results in [13, 31]. Corresponding results are shown in Figure 9. Again, as in the case of the internally resonant bar, the alignment between occurrences of resonance and the NSM detected by NBM is clear. The motions obtained in the NBM analysis are similar to those obtained in [31] for the same values. Specifically, the displacement fields depicted in Figure 9(b) and Figure 9(c) are similar to those presented in [31, (a) and (b) in Figure 11].

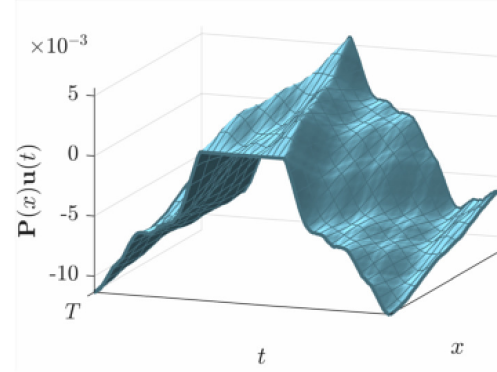
In Figure 9(a), it seems apparent that both displacements relate to two different branches of the solution. Moreover, the forced response curves of less damped motions have two peaks which may indicate the existence of two distinct NSM branches. Here, the branch corresponding to solution q_2 has been detected until a maximal energy point. Backward sequential continuation has then revealed a distinct a set of points to which the solution q_1 belongs. These points seem to consist of a curve and the origin of this curve coincides with a subharmonic 4 of the second fundamental frequency ω_2 . This coincidence with the subharmonic $\omega_2/4$ may suggest the existence



(a) Forced response curves



(b) Displacement field of solution q_1 , autonomous NBM motion



(c) Displacement field of solution q_2 , autonomous NBM motion

Figure 9: NSM analysis of the bar with soft support with $k = 0.5$ detected by the NBM. Forced response curves with $\bar{F} = 0.025$ and $c = 0.1, 0.2, \dots, 0.7$. First natural frequency $\omega_1 \approx 0.65$, second natural frequency $\omega_2 \approx 3.3$ and grazing energy $E_1 \approx 5.9 \times 10^{-7}$.

of an internal resonance in the proximity of both curves. Further attempts using sequential continuation to reveal the internal resonance between the curves were not successful. Indeed, the use of sequential continuation prevents us from confirming confidently the existence of the two distinct branches since the method does not distinguish between branches belonging to different continua [33]. In order to affirm this hypothesis, a continuation method capable of resolving internal resonances is required.

A method generally used for detection of internal resonances is the pseudo-arclength continuation [7]. However, the use of pseudo-arclength continuation relies on the tangent to the backbone curve to formulate the next solution along the curve [33]. Due to the nonsmoothness of the motions in NSMs, such tangent cannot be formulated at every point on the branch. In fact, pseudo-arclength continuation was applied in [31, p. 9] for detection of nonsmooth modes. While some continuous sections of the NSM were detected, pseudo-arclength continuation has failed to reveal internal resonances and could not reveal backbone curves for long ranges of frequencies [31, p. 10].

6.2.4 Varying area bar

While modal analysis of the bar of uniform area has been the subject of both analytical or numerical analysis [11, 13, 31], these relied on knowledge of the d'Alembert function or Green's function for this purpose. Here, the NBM allows for modal analysis of the varying-area bar as it relies on Lagrangian shape-functions approximation to determine the behaviour of the bar during inactive and active contact phase. For the varying area bar in Figure 1, several area functions $A(x)$ were considered. In order to simplify the discussion for the remainder of

this section, the following terminology is introduced to distinguish between the investigated models:

$$\text{heav-bar } A(x) = 1 - 0.5\Theta(x - 0.5) \quad (67)$$

$$\text{lin-bar } A(x) = 1 - x/2 \quad (68)$$

$$\text{quad-bar } A(x) = 0.5x(2 - x) \quad (69)$$

where $\Theta(x)$ denotes the Heaviside function, and the heav-bar hence exhibits two cross-sectional areas: \bar{A}_0 for $\bar{x} \in [0, 0.5L]$ and $0.5\bar{A}_0$ for $\bar{x} \in (0.5L, 1]$. The corresponding NSM is illustrated in Figure 10(a). The detected

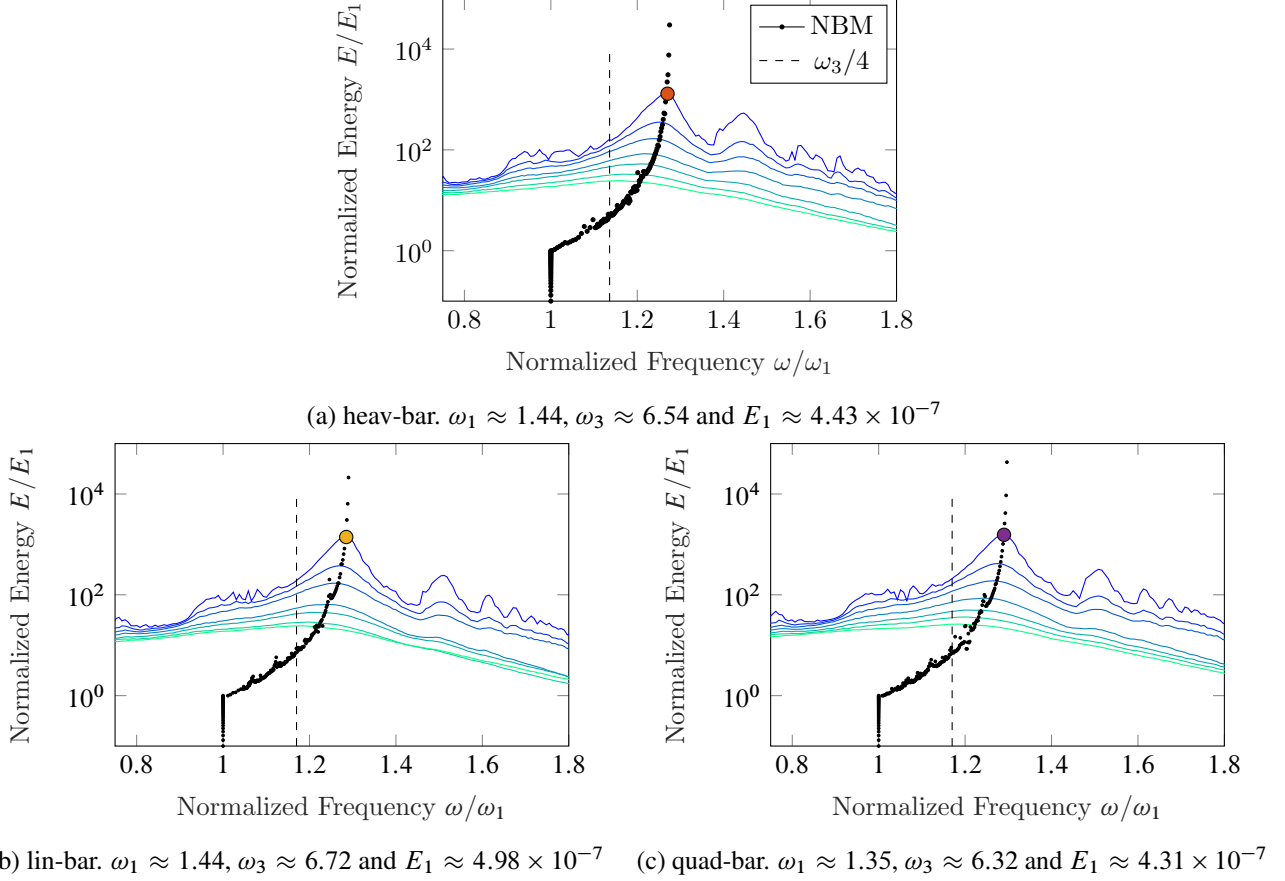


Figure 10: Backbone and forced response curves ($\bar{F} = 0.05$) for varying area bar. NSMs detected by the NBM with 20 elements of quadratic shape functions. The depicted points in sub-figures (a), (b) and (c) correspond respectively to solutions *b*, *c*, and *d* investigated in Figure 11(b).

NSMs of the lin- and quad-bars are illustrated in Figure 10(b) and Figure 10(c), respectively. The results derive from the NBM do coincide the resonant points in the forced-response diagrams. Furthermore, it is noted that all models consist of bars with decreasing areas such that $A(0) = 1$ and $A(1) = 0.5$.

The effect of area variation on the modal space of the bar in unilateral contact is of interest. In contrast to the conclusions from the bar with soft support in Section 6.2.3, no internal resonances were detected for the cases of the varying area bar around the subharmonic $\omega_3/4$ or other subharmonics within spectrum of the backbone curve. Thus, Figure 11 illustrates the backbone curves of all varying area models and that of the uniform area investigated in Section 6.2.2. In Figure 11(a), the backbones of the varying area bars exhibit higher energies for the same normalized frequencies when compared to the uniform area bar. In other terms, the behaviour of the varying area bars can be characterized as “softer” in relation to the uniform area bar. Along the same line, it is noted that while the lin- and quad-bars exhibit a similar stiffening pattern, the heav-bar is characterized by the softest stiffening. Thus, it is indicative that the varying area function affects the stiffening behavior and, in turn, the range size of resonant frequencies.

Moreover, comparison of the varying-area bar NSMs with respect to their true (not normalized) frequencies in Figure 11(b) shows that the area variation shifts the backbone curves towards lower frequency ranges. This can be explained by the amount of total mass removed from the bar. For all varying area models, the total mass

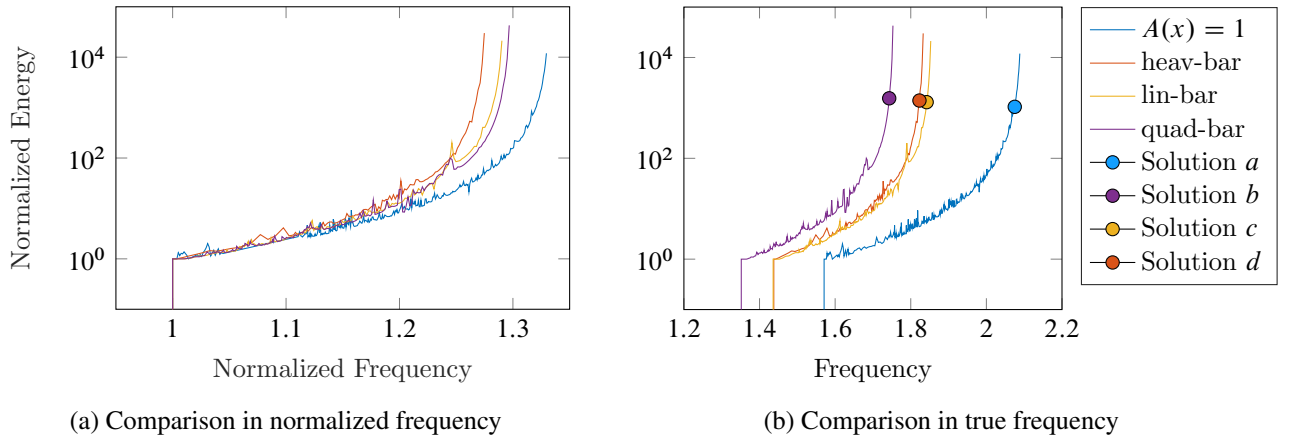


Figure 11: Backbone curves of different area cantilever bars corresponding to Figure 8(a) and Figure 10.

of the bar, $m_{\text{tot}} = \int \rho \bar{A}(x) dx$, is lower than that of the uniform area bar. While the heav-bar and lin-bar exhibit a 1/4 decrease in total mass, the quad-area bar exhibits a greater 1/3 decrease. Thus, a greater backward shift in the backbone curves is noticed when the removed mass is greater. Indeed, this is expected considering the fundamental physics of mechanical oscillators: with all other parameters being constant, the resonant frequency of the oscillator reduces with the reduction of mass. At last, Figure 12 entails NSM motion of similar energy. It is noted that the NSM motions of the varying area bar models while qualitatively similar to the NSM motions of the uniform area bar depicted in Figure 8(d) and 8(c), exhibit piecewise nonlinear displacement fields in space-time rather than piecewise-linear displacement field of the uniform area bar. Furthermore, the amplitudes of motion are greater in the varying area model which can be explained by diminishing kinetic energy due to loss of total mass and the consequent increase in potential energy. While the results presented here are novel, a more in depth and more conclusive analysis in regard to the effect of different area variation is a subject of future research with the NBM.

7 Conclusion

The nodal boundary method, for treatment of Signorini boundary conditions in the framework of FEM, was presented. The method was developed for nonsmooth modal analysis purposes entailing the detection of periodic solutions to the autonomous Signorini problem. Compared to application of Newton impact law in FEM or WFEM, the resulting ODE from the NBM formulation for the varying area bar allows for existence of periodic solutions with a continuous sticking phase at contact.

The NBM assumes different approximations of the contacting nodal displacement $u_N(t)$ during inactive and active contact phases. The state $u_N(t)$ is dictated by (1) boundary conditions and (2) nodes that are not prone to contact (internal nodes). While the treatment of active contact is done similarly to classical FEM (clamped condition at end of bar), in the treatment of inactive contact, the homogeneous Neumann boundary condition is enforced in a strong sense such that the approximation of the contact stress exactly vanishes, that is $\sigma_N(t) = 0$, throughout the entire inactive contact duration. The two associated approximations of the quantity $u_N(t)$ can be seen as constituting two distinct sets of shape functions. The residuals for the inactive and active motion approximations are then projected onto their respective set of shape functions to form two distinct ODEs. The Signorini problem is then solved by switching between the sets of shape functions both in trial and test functions. At last, the full NBM-ODE consists of a nonsmooth set of equations which can be solved either analytically or numerically.

Moreover, nonsmooth modal analysis via NBM resulted in valid backbone curves aligning with resonances of forced-response diagrams. These results were obtained for three cases: the cantilever bar of uniform area, the cantilever bar with soft support, and the cantilever bar of varying area.

The results presented for the uniform area bar and the bar with soft support have agreed with previous research on the topic. Furthermore, the NBM has allowed for characterization of two distinct NSMs for the bar with soft support. The two distinct curves seem to relate via an internal resonance as one of the curves originates from a sub-harmonic of the motion. However, affirmation of this result could not be achieved with

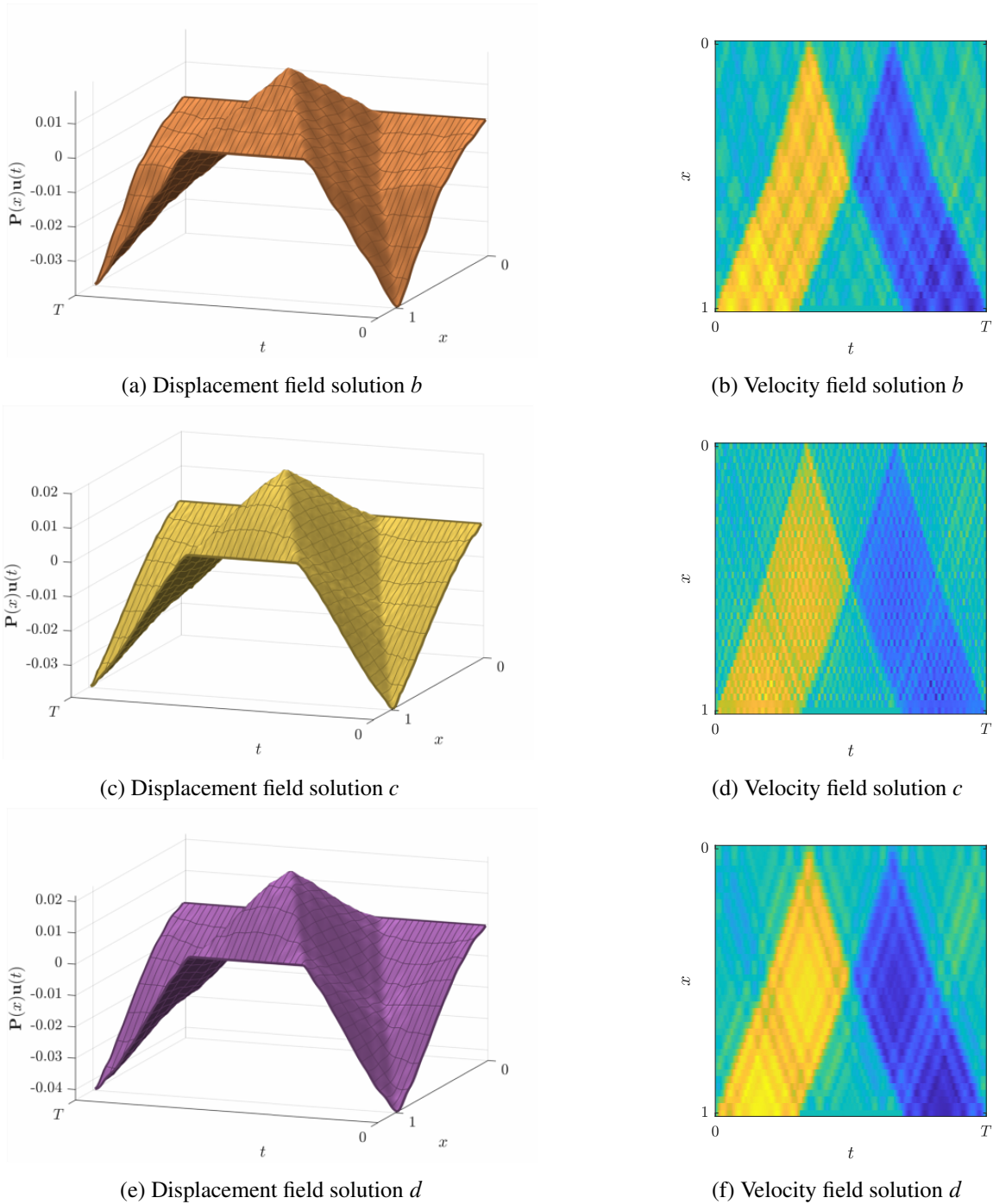


Figure 12: Comparison of selected autonomous NBM motions from the backbone curves of all varying-area bars. The locations of the selected motions on the FEP is noted in Figure 10 and Figure 11(b).



sequential continuation and the detection of internal resonances in nonsmooth modes is subject for future research. Furthermore, application of the NBM to discover the modal space of the bar of varying area in unilateral contact has been proven successful and the results show good agreement with the forced-response curves.

8 Supplementary Material

Excerpts of scripts and algorithms used to perform the analysis and generate this manuscript are available on Zenodo [34].

References

- [1] A. Thorin and M. Legrand. Nonsmooth Modal Analysis: From the Discrete to the Continuous Settings. In *Advanced Topics in Nonsmooth Dynamics: Transactions of the European Network for Nonsmooth Dynamics*, pages 191–234. Springer, 2018. [\[DOI\]](#), [\[HAL\]](#) .
- [2] M. Legrand, S. Junca, and S. Heng. Nonsmooth Modal Analysis of a N -Degree-of-Freedom System Undergoing a Purely Elastic Impact Law. *Communications in Nonlinear Science and Numerical Simulation*, 45:190–219, 2017. [\[DOI\]](#), [\[HAL\]](#) .
- [3] E.H. Moussi, S. Bellizzi, B. Cochelin, and I. Nistor. Nonlinear Normal Modes of a Two Degrees-of-Freedom Piecewise Linear System. *Mechanical Systems and Signal Processing*, 64-65:266–281, 2015. [\[DOI\]](#), [\[HAL\]](#) .
- [4] S. Peter, F. Schreyer, and R. Leine. A Method for Numerical and Experimental Nonlinear Modal Analysis of Nonsmooth Systems. *Mechanical Systems and Signal Processing*, 120:793–807, 2019. [\[DOI\]](#).
- [5] G. Kerschen. Definition and Fundamental Properties of Nonlinear Normal Modes. In *Modal Analysis of Nonlinear Mechanical Systems*, volume 555, pages 1–46. Springer, 2014. [\[DOI\]](#).
- [6] S. Shaw. Invariant Manifold Representations of Nonlinear Modes of Vibration. In G. Kerschen, editor, *Modal Analysis of Nonlinear Mechanical Systems*, volume 555, pages 47–74. Springer, 2014. [\[DOI\]](#).
- [7] M. Peeters, R. Vigi  , G. S  randour, G. Kerschen, and J.-C. Golinval. Nonlinear Normal Modes, Part II: Toward a practical computation using numerical continuation techniques. *Mechanical Systems and Signal Processing*, 23(1):195–216, 2009. [\[DOI\]](#), [\[HAL\]](#) .
- [8] G. James, V. Acary, and F. P  rignon. Periodic Motions of Coupled Impact Oscillators. In R.I. Leine, V. Acary, and O. Br  uls, editors, *Advanced Topics in Nonsmooth Dynamics: Transactions of the European Network for Nonsmooth Dynamics*, pages 93–134. Springer, 2018. [\[DOI\]](#), [\[HAL\]](#) .
- [9] V. Yastrebov. *Numerical Methods in Contact Mechanics*. John Wiley & Sons, 2013. [\[DOI\]](#).
- [10] B. Brogliato. *Nonsmooth Mechanics: Models, Dynamics and Control*, volume 2. Springer, 1999. [\[DOI\]](#).
- [11] D. Urman, M. Legrand, and S. Junca. D’Alembert Function for Exact Non-Smooth Modal Analysis of the Bar in Unilateral Contact. *Nonlinear Analysis: Hybrid Systems*, 43:101115, 2021. [\[DOI\]](#),[\[HAL\]](#) .
- [12] C. Yoong, A. Thorin, and M. Legrand. Nonsmooth Modal Analysis of an Elastic Bar Subject to a Unilateral Contact Constraint. *Nonlinear Dynamics*, 91(4):2453–2476, 2018. [\[DOI\]](#), [\[HAL\]](#) .
- [13] C. Yoong. *Nonsmooth Modal Analysis of a Finite Linear Elastic Bar Subject to Unilateral Contact Constraint*. PhD thesis, McGill University, 2018. [\[URL\]](#).
- [14] D. Doyen, A. Ern, and S. Piperno. Time-Integration Schemes for the Finite Element Dynamic Signorini Problem. *SIAM Journal on Scientific Computing*, 33(1):223–249, 2011. [\[DOI\]](#), [\[HAL\]](#) .
- [15] H. Khenous, P. Laborde, and Y. Renard. Mass Redistribution Method for Finite Element Contact Problems in Elastodynamics. *European Journal of Mechanics - A/Solids*, 27(5):918 – 932, 2008. [\[DOI\]](#), [\[HAL\]](#) .
- [16] F. Chouly, M. Fabre, P. Hild, R. Mlika, J. Pousin, and Y. Renard. An Overview of Recent Results on Nitsche’s Method for Contact Problems. In *Geometrically Unfitted Finite Element Methods and Applications*, pages 93–141. Springer, 2017. [\[DOI\]](#), [\[HAL\]](#) .
- [17] P. Wriggers and G. Zavarise. *Computational Contact Mechanics*, chapter 6. Wiley, 2004. [\[DOI\]](#).
- [18] C. Hager, S. H  ber, and B. Wohlmuth. A Stable Energy-Conserving Approach for Frictional Contact Problems Based on Quadrature Formulas. *International Journal for Numerical Methods in Engineering*, 73(2):205–225, 2008. [\[DOI\]](#).
- [19] C. Hager and B. Wohlmuth. Analysis of a Space-Time Discretization for Dynamic Elasticity Problems Based on Mass-Free Surface Elements. *SIAM Journal on Numerical Analysis*, 47(3):1863–1885, 2009. [\[DOI\]](#), [\[HAL\]](#) .
- [20] C. Bertrand. Periodic Solutions of a One-Dimensional Elastic Bar Subject to a Unilateral Constraint. Technical report, ENTPE, 2020. [\[HAL\]](#) .
- [21] R. J. LeVeque. *Finite Volume Methods for Hyperbolic Problems*, volume 31. Cambridge university press, 2002. [\[DOI\]](#).
- [22] T. Hughes. *Finite Element Method - Linear Static and Dynamic Finite Element Analysis*. Dover Publications, 2000.
- [23] V. Kryszko, J. Awrejcewicz, and G. Narkaitis. Nonlinear Vibration and Characteristics of Flexible Plate-Strips with Non-Symmetric Boundary Conditions. *Communications in Nonlinear Science and Numerical Simulation*, 11(1):95–124, 2006. [\[DOI\]](#).
- [24] Y. Modarres-Sadeghi and M. Pa  doussis. Nonlinear Dynamics of Extensible Fluid-Conveying Pipes, Supported at Both Ends. *Journal of Fluids and Structures*, 25(3):535–543, 2009. [\[DOI\]](#).
- [25] M. Pa  doussis. *Pipes Conveying Fluid: Nonlinear and Chaotic Dynamics*, volume 1 of *Fluid-Structure Interactions*, chapter 5, pages 277–414. Academic Press, 1998. [\[DOI\]](#).
- [26] J. Boyd. *Chebyshev and Fourier Spectral Methods*, volume 49. Springer, 2 edition, 2001.
- [27] L. Evans. *Partial Differential Equations*. American Mathematical Society, 2010.
- [28] J. Aitchison and M. Poole. A Numerical Algorithm for the Solution of Signorini Problems. *Journal of Computational and Applied Mathematics*, 94(1):55–67, 1998. [\[DOI\]](#).
- [29] S. Krenk. Energy Conservation in Newmark Based Time Integration Algorithms. *Computer Methods in Applied Mechanics and Engineering*, 195(44):6110–6124, 2006. [\[DOI\]](#).

- [30] E. Süli. *Finite Element Methods for Partial Differential Equations*. Oxford University Computing Laboratory Oxford, 2002.
- [31] T. Lu and M. Legrand. Nonsmooth Modal Analysis via the Boundary Element Method for One-Dimensional Bar Systems. *Nonlinear Dynamics*, October 2021. [DOI], [HAL] .
- [32] V. Acary, O. Bonnefon, M. Brémond, O. Huber, F. Pérignon, and S. Sinclair. An Introduction to Siconos. Technical Report RT-0340, INRIA, 2019. [HAL] .
- [33] E. Allgower and K. Georg. *Numerical Continuation Methods: an Introduction*, volume 13. Springer, 2012. [DOI].
- [34] D. Urman and Legrand M. MATLAB Scripts for Nonsmooth Modal Analysis of the bar via Nodal Boundary Method, 2022. [DOI].

A Appendix

A.1 Proof of $\phi'_N(1) > 0$

Lemma 1. *In the context of classical finite elements for the PDE (1) with Lagrangian shape functions $\phi_i(x)$ for $i = 1, 2, \dots, N$ consisting of uniformly spaced nodes on loci $x_i = i/N$, the value of $\phi'_N(1)$ depends on the order of shape functions only.*

Proof. The proof follows from the construction of shape functions in the FEM. In the classical finite element, the structure is divided into elements and each element consists of a set of shape functions that are Lagrange polynomials. These shape functions are local to the element and their definition depends only on the order of the polynomial chosen for this specific element [22]. Thus, the value $\phi'_N(1)$ is dependent only on the order of shape functions used. \square

This lemma will be necessary in the generalization of the theorem below to any number of elements and any order of shape functions.

Theorem 2. *For the classical FE approximation for the PDE (1) with Lagrangian shape functions $\phi_i(x)$ $i = 1, 2, \dots, N$ based on uniformly spaced nodes $x_i = i/N$, the statement*

$$\phi'_N(1) > 0 \tag{70}$$

always holds.

Proof. The proof consists of first proving Inequality (70) for the case of a single element by inspecting the exact expression of the Lagrangian function. Then, the proof is expanded to any number of elements and/or shape functions by virtue of Lemma 1.

We start by approximating the bar's displacement using a single element. It follows, then, that $N \geq 1$ stands for the order of the shape function such that

$$\phi_i(x) = \prod_{\substack{0 \leq m \leq N \\ m \neq i}} \frac{x - x_m}{x_i - x_m}, \quad x \in [0, 1]. \tag{71}$$

Then, for $\phi_N(x)$, we can simplify the expression using $x_i = i/N$ to simplify the denominator

$$\phi_N(x) = \prod_{m=0}^{N-1} \frac{x - x_m}{x_N - x_m} = N^N \prod_{m=0}^{N-1} \frac{x - x_m}{N - m}. \tag{72}$$

Then, we take the derivative of $\phi_N(x)$ to obtain

$$\phi'_N(x) = N^N \sum_{j=0}^{N-1} \frac{1}{N - j} \prod_{\substack{0 \leq m \leq N-1 \\ m \neq j}} \frac{x - x_m}{N - m}. \tag{73}$$

To show $\phi'_N(1) > 0$, we simply evaluate every term in the expression

$$\begin{aligned}\phi'_N(1) &= N^N \sum_{j=0}^{N-1} \frac{1}{N-j} \prod_{\substack{0 \leq m \leq N-1 \\ m \neq j}} \frac{1-x_m}{N-m} = N^N \sum_{j=0}^{N-1} \frac{1}{N-j} \prod_{\substack{0 \leq m \leq N-1 \\ m \neq j}} \frac{1}{N} \frac{N-m}{N-m} \\ &= \left(\frac{N^N}{N^{N-1}} \right) \sum_{j=0}^{N-1} \frac{1}{N-j} = N \sum_{j=1}^{N-1} \frac{1}{N-j} = N \sum_{j=1}^N \frac{1}{j}.\end{aligned}\tag{74}$$

We note that all the quantities presented here are exclusively positive and that their sum and product will also be positive such that $\phi'_N(1) > 0$ holds. By virtue of Lemma 1, we may conclude that if $\phi'_N(1) > 0$ holds for a single element, it will hold for any number of elements. \square

A.2 Conservation of energy away from instant of switch

We introduce two lemmas on the energy conservation during active and inactive contact motions, away from the moment of switch (t_s such that $S(\mathbf{u}^o(t_s)) = g$).

Lemma 3. *For the FEM-NBM ODE (35) developed for a cantilever bar in inactive contact conditions, the energy (involving boundary nodes and non reduced matrices) term*

$$2E(t) = \ddot{\mathbf{u}}^\top(t) \mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{u}^\top(t) \mathbf{K} \mathbf{u}(t)\tag{75}$$

is conserved, that is $\dot{E}(t) = 0$ for $S(\mathbf{u}^o(t)) < g$.

Proof. This lemma is proven by developing the term \dot{E} from Equation (75) and plugging the solution of Equation (35).

First, we differentiate equation (75) with respect to t , and for symmetric \mathbf{M} and \mathbf{K} (as they are in the given in the FE formulation [22]) we obtain

$$2\dot{E}(t) = \ddot{\mathbf{u}}^\top(t) (\mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{K} \mathbf{u}(t)).\tag{76}$$

Next, we note that for inactive contact $S(\mathbf{u}^o(t)) < g$, the NBM formulation of u_N in (28) admits $\dot{u}_N = S(\dot{\mathbf{u}}^o(t))$. Under this restriction, the relationship between $\dot{\mathbf{u}}(t)$ and $\dot{\mathbf{u}}^o(t)$ can be described via $\dot{\mathbf{u}}(t) = \mathbf{B} \dot{\mathbf{u}}^o(t)$. Next, substitution of the previous identity into expression (76) admits $\dot{E}(t) = \dot{\mathbf{u}}^{o\top}(t) (\mathbf{M}_N \ddot{\mathbf{u}}^o(t) + \mathbf{K}_N \mathbf{u}^o(t)) = 0$, for $S(\mathbf{u}^o(t)) < g$ where the last equality completes this proof by virtue of (35). \square

Lemma 4. *For the FEM-NBM ODE (43) developed for a cantilever bar with non-homogeneous Dirichlet conditions, the energy metric (75) (similar to the energy metric used in lemma 3) is conserved, that is $\dot{E}(t) = 0$ for $S(\mathbf{u}^o(t)) > g$.*

Proof. This theorem is proven by developing the term $\dot{E}(t)$ from equation (75) and plugging the solution of equation (43). We note that application of (36) implies that

$$\dot{u}_N(t) = 0.\tag{77}$$

We can conclude that $\mathbf{u}(t)$ belongs to the same space as \mathbf{w} in (40), ie

$$\dot{\mathbf{u}}(t) = \mathbf{B}^d \dot{\mathbf{u}}^o(t).\tag{78}$$

Thus, we plug-in the identities (40) and (78) into (76) such that $\dot{E}(t) = \dot{\mathbf{u}}^{o\top}(t) (\mathbf{M}_D \ddot{\mathbf{u}}^o(t) + \mathbf{K}_D \mathbf{u}^o(t) + g \mathbf{f}_D) = 0$, for $S(\mathbf{u}^o(t)) > g$ where the last equality is obtained by virtue of (43). \square

A.3 Derivation of energy jump at switch

The term ΔE in Equation (54) is derived for the case for the transition between inactive and active contact. This procedure can be then replicated for the case of transition between active and inactive contact. The latter will not be detailed in this paper but rather its conclusion will be presented here, culminating in term (54).

We assume that the state of switch is denoted by $\mathbf{u}^o(t_s)$. At time t_s , we expect that $S(\mathbf{u}^o(t_s)) \neq 0$ such that a transition between inactive and active contact occurs. At the time t_s , for the switch from active to inactive contact to occur, $\mathbf{u}^o(t_s)$ satisfies $S(\mathbf{u}^o(t_s)) = g$, $S(\dot{\mathbf{u}}^o(t_s)) < 0$ where it is presumed that $S(\dot{\mathbf{u}}^o(t_s)) < 0$ to guarantee a transition to inactive contact (ie, the stress at the wall will become positive immediately after t_s). Just before the moment of switch, t_s^- , the system is established to be in active contact, ie $u_N(t_s^-) = g$ and $\dot{u}_N(t_s^-) = 0$. The energy of the system at this stage can be described as follows

$$2E(t_s^-) = [\dot{\mathbf{u}}^o(t_s)^\top \quad 0] \begin{bmatrix} \mathbf{M}_{oo} & \mathbf{M}_{oN} \\ \mathbf{M}_{oN}^\top & M_{NN} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}^o(t_s) \\ 0 \end{bmatrix} + [\mathbf{u}^o(t_s)^\top \quad g] \begin{bmatrix} \mathbf{K}_{oo} & \mathbf{K}_{oN} \\ \mathbf{K}_{oN}^\top & K_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{u}^o(t_s) \\ g \end{bmatrix} \quad (79)$$

where \mathbf{M} and \mathbf{K} are shown as block matrices to simplify the presentation of later calculations. Next, the energy of the system at t_s^+ , presuming that the bar is in inactive contact motion, is given as follows

$$2E(t_s^+) = [\dot{\mathbf{u}}^o(t_s)^\top \quad S(\dot{\mathbf{u}}^o(t_s))] \begin{bmatrix} \mathbf{M}_{oo} & \mathbf{M}_{oN} \\ \mathbf{M}_{oN}^\top & M_{NN} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}^o(t_s) \\ S(\dot{\mathbf{u}}^o(t_s)) \end{bmatrix} + \dots \quad (80)$$

$$[\mathbf{u}^o(t_s)^\top \quad g] \begin{bmatrix} \mathbf{K}_{oo} & \mathbf{K}_{oN} \\ \mathbf{K}_{oN}^\top & K_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{u}^o(t_s) \\ g \end{bmatrix}, \quad S(\dot{\mathbf{u}}^o(t_s)) < 0$$

Then, the term ΔE is given as follows (where calculations where omitted for sake of conciseness)

$$\Delta E = E(t_s^+) - E(t_s^-) = S(\dot{\mathbf{u}}^o(t_s)) [\mathbf{M}_{oN}^\top \dot{\mathbf{u}}^o(t_s) + \frac{1}{2} M_{NN} S(\dot{\mathbf{u}}^o(t_s))], \quad S(\dot{\mathbf{u}}^o(t_s)) < 0. \quad (81)$$

Similarly, if we would have started our derivation with the assumption that t_s denotes a switch from inactive to active contact, we would have reached the conclusion

$$\Delta E = -S(\dot{\mathbf{u}}^o(t_s)) [\mathbf{M}_{oN}^\top \dot{\mathbf{u}}^o(t_s) + \frac{1}{2} M_{NN} S(\dot{\mathbf{u}}^o(t_s))], \quad S(\dot{\mathbf{u}}^o(t_s)) > 0. \quad (82)$$

We can therefore generalize and say that

$$\Delta E = -|S(\dot{\mathbf{u}}^o(t_s))| [\mathbf{M}_{oN}^\top \dot{\mathbf{u}}^o(t_s) + \frac{1}{2} M_{NN} S(\dot{\mathbf{u}}^o(t_s))]. \quad (83)$$

This term is presented using the elements of \mathbf{M} in Equation (54).