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NULL-CONTROLLABILITY PROPERTIES OF THE GENERALIZED TWO-DIMENSIONAL BAOUENDI-GRUSHIN EQUATION WITH NON-RECTANGULAR CONTROL SETS

JÉRÉMI DARDÉ, ARMAND KOENIG, AND JULIEN ROYER

ABSTRACT. We consider the null-controllability problem for the *generalized Baouendi-Grushin equation* $(\partial_t - \partial_x^2 - q(x)^2 \partial_y^2)f = \mathbf{1}_\omega u$ on a rectangular domain. Sharp controllability results already exist when the control domain ω is a vertical strip, or when $q(x) = x$. In this article, we provide upper and lower bounds for the minimal time of null-controllability for general q and non-rectangular control region ω . In some geometries for ω , the upper bound and the lower-bound are equal, in which case, we know the exact value of the minimal time of null-controllability.

Our proof relies on several tools: known results when ω is a vertical strip and cutoff arguments for the upper bound of the minimal time of null-controllability; spectral analysis of the harmonic oscillator $-\partial_x^2 + \nu^2 q(x)^2$ when $\operatorname{Re}(\nu) > 0$, pseudo-differential-type operators on polynomials and Runge's theorem for the lower bound.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

1.1. The Baouendi-Grushin equation. In this article, we study some controllability properties of the two-dimensional *generalized Baouendi-Grushin equation*.

Let L_-, L_+ and $I = (-L_-, L_+)$. Let $q \in C^0(\bar{I})$ such that $q(0) = 0$ and $q(x) \neq 0$ for all $x \in \bar{I} \setminus \{0\}$. We denote by \mathbb{T} the one-dimensional torus $\mathbb{R}/2\pi\mathbb{Z}$. For $T > 0$, $f_0 \in L^2(I \times \mathbb{T})$ and $F \in L^2((0, T); L^2(I \times \mathbb{T}))$ we consider the generalized Baouendi-Grushin equation

$$(1.1) \quad \begin{cases} (\partial_t - \partial_x^2 - q(x)^2 \partial_y^2)f(t, x, y) = F(t, x, y), & t \in (0, T), x \in I, y \in \mathbb{T} \\ f(t, x, y) = 0, & t \in (0, T), x \in \partial I, y \in \mathbb{T}, \\ f(0, x, y) = f_0, & x \in I, y \in \mathbb{T}. \end{cases}$$

Note that, because $q(0) = 0$, the equation degenerates on the vertical axis $\{0\} \times \mathbb{T}$. Nevertheless, the equation is well posed. Precisely, the Friedrichs extension (see [23, Section 4.3]) of the operator

$$f \in C_c^\infty(I \times \mathbb{T}) \mapsto -\partial_x^2 f - q(x)^2 \partial_y^2 f,$$

generates an analytic semigroup, which allows to define a solution of (1.1) in the sense of semigroup [12]. In our case, this solution is smooth in the following sense:

Proposition 1.1. *For any initial condition $f_0 \in L^2(I \times \mathbb{T})$ and any source term $F \in L^2((0, T); L^2(I \times \mathbb{T}))$, there exists a unique $f \in C^0([0, T]; L^2(I \times \mathbb{T})) \cap L^2((0, T); H)$ solution of (1.1), with*

$$H = \{f \in L^2(I \times \mathbb{T}), \partial_x f \in L^2(I \times \mathbb{T}), q \partial_y f \in L^2(I \times \mathbb{T})\}.$$

This result is proven in [7] in the case $q(x) = x^\gamma$ with $\gamma > 0$. The proof is easily generalized to our case of interest.

1.2. Control problem for the Baouendi-Grushin equation. Our study focus on internal null-controllability of the Baouendi-Grushin equation. More precisely, let $\omega \subset I \times \mathbb{T}$ be a non-empty open set and $u \in L^2((0, T); L^2(I \times \mathbb{T}))$. The controlled Baouendi-Grushin equation reads

$$(1.2) \quad \begin{cases} (\partial_t - \partial_x^2 - q(x)^2 \partial_y^2) f(t, x, y) = \mathbf{1}_\omega u(t, x, y), & t \in (0, T), x \in I, y \in \mathbb{T} \\ f(t, x, y) = 0, & t \in (0, T), x \in \partial I, y \in \mathbb{T}, \\ f(0, x, y) = f_0, & x \in I, y \in \mathbb{T}, \end{cases}$$

where f is the state of the system, and $\mathbf{1}_\omega u$ is the control supported in ω .

Definition 1.2 (Null-controllability). Let $T > 0$. The Bouendi-Grushin equation (1.2) is said to be null-controllable on ω in time T if, for any initial condition $f_0 \in L^2(I \times \mathbb{T})$, there exists $u \in L^2((0, T); L^2(I \times \mathbb{T}))$ such that the solution f of eq. (1.2) satisfies $f(T, \cdot, \cdot) = 0$ in $I \times \mathbb{T}$.

It is known that, contrary to usual non-degenerate parabolic equations like the heat equation, due to the degeneracy of q on $\{0\} \times \mathbb{T}$, the null-controllability properties of (1.2) strongly depend on the control set ω and the time horizon T . More precisely, for certain control sets ω , equation (1.2) is not null-controllable in any positive time T , whereas for other control sets ω a minimal time of null-controllability appears. We refer to the bibliographical comments, section 1.4 below, for a detailed description of the known results on the subject.

In the present paper, we aim to give precise null-controllability results for equation (1.2), for a large class of control sets ω and a large class of functions q .

1.3. Main results. Before presenting the main results of our study, we introduce the so-called *Agmon distance* of a point $x \in I$ to the origin, defined by:¹

$$(1.3) \quad d_{\text{Agm}} : x \in I \mapsto \int_0^x q(s) ds.$$

This quantity appears naturally in the definition of critical times of null-controllability for the Baouendi-Grushin equation.

1.3.1. Lack of null-controllability in small time for a class of control sets ω . Our main result is a negative result of null-controllability for small times. We show that if the control set ω stays at positive distance from a horizontal segment of the form $(a, b) \times \{y_0\}$, with $-L_- \leq a < 0 < b \leq L_+$ and $y_0 \in \mathbb{T}$, then equation (1.2) is not null-controllable on ω for time T smaller than a precisely given critical time.

To properly state the result, we need to introduce a modified version of the Agmon distance. More precisely, we set $\widetilde{d}_{\text{Agm}}(x) = d_{\text{Agm}}(x)$ for $x \in I$, and $\widetilde{d}_{\text{Agm}}(-L_-) = \widetilde{d}_{\text{Agm}}(L_+) = +\infty$.

Theorem 1.3. *Assume that $q \in C^2(\bar{I})$ is such that $q(0) = 0$, $q'(0) > 0$ and $q(x) \neq 0$ whenever $x \neq 0$. Let ω be an open subset of $I \times \mathbb{T}$. Assume that there exist $a \in [-L_-, 0)$, $b \in (0, L_+]$ and $y_0 \in \mathbb{T}$ such that*

$$\text{distance}((a, b) \times \{y_0\}, \omega) > 0.$$

Then, the generalized Baouendi-Grushin equation (1.2) is not null-controllable on ω in time T such that

$$T < \frac{1}{q'(0)} \min(\widetilde{d}_{\text{Agm}}(a), \widetilde{d}_{\text{Agm}}(b)).$$

It is a generalization of [20, theorem 3.3], where the result is proved for I symmetric with respect to the origin and, more restrictively, for $q(x) = x$. In this particular case, some spectral results used in the proof are obtained from explicit solutions of particular ordinary differential equations. These explicit formulae are not available in our general setting. This theorem is proved in section 3, with the spectral analysis done in section 4.

Therefore, a key step in our proof of theorem 1.3 is the study of spectral properties of the family of non-selfadjoint operators defined on $L^2(I)$ by

$$(1.4) \quad P_\nu : -\partial_x^2 + \nu^2 q(x)^2, \quad \text{Dom}(P_\nu) = H^2(I) \times H_0^1(I),$$

¹It corresponds to the usual definition of the Agmon distance given for example in [18, Eq. (6.3)], with $V = q^2$ and $E = 0$.

for $\operatorname{Re}(\nu) > 0$. More precisely, we obtain a localization of an eigenvalue of P_ν , as well as precise Agmon type estimates for an associated eigenfunction,² uniformly in $\nu = |\nu|e^{i\theta}$ with $|\nu|$ large enough and $0 \leq |\theta| \leq \theta_0$ for some $\theta_0 \in [0, \pi/2)$.

Note that [24] contains closely related spectral asymptotics. However, we cannot apply them directly since our domain has a boundary and we need uniform estimates with respect to the parameter ν .

1.3.2. *Precise critical time of null-controllability for a class of control sets ω .* With theorem 1.3 we can deduce in some setting the critical time for the null-controllability of (1.2).

We first mention a natural adaptation of [20, theorem 3.1]. It was actually claimed in [20, Remark 3.2], but the statement was imprecise if q is not odd. We take the opportunity to correct the statement:

Theorem 1.4. *Assume that $q \in C^3(\bar{I})$ is such that $q(0) = 0$ and $\inf_{\bar{I}} q' > 0$. Let ω be an open subset of $I \times \mathbb{T}$. Assume that there exists $\gamma = (\gamma_x, \gamma_y) \in C^0(\mathbb{T}, I \times \mathbb{T})$ that is not homotopic to a constant path and such that $\gamma(\mathbb{T}) \subset \omega$.*

The generalized Baouendi-Grushin equation is null-controllable on ω in time T such that

$$T > \frac{1}{q'(0)} \max \left(d_{\text{Agm}} \left(\min_{\mathbb{T}}(\gamma_x) \right), d_{\text{Agm}} \left(\max_{\mathbb{T}}(\gamma_x) \right) \right).$$

We detail the proof of this theorem in section 2. Combining theorems 1.3 and 1.4, we obtain the following result which gives the precise critical time of null-controllability of the generalized Baouendi-Grushin for a large class of control sets ω and functions q :

Theorem 1.5. *Let $q \in C^3(\bar{I})$ such that $q(0) = 0$ and $\inf_{\bar{I}} q > 0$. Let $\gamma_1, \gamma_2 \in C^0(\mathbb{T}, I)$ such that for every $y \in \mathbb{T}$, $\gamma_1(y) < \gamma_2(y)$. Let $\omega = \{(x, y) \in I \times \mathbb{T} : \gamma_1(y) < x < \gamma_2(x)\}$ and³*

$$T_* := \frac{1}{q'(0)} \max \left(d_{\text{Agm}} \left(-\max_{\mathbb{T}}(\gamma_2^-) \right), d_{\text{Agm}} \left(\max_{\mathbb{T}}(\gamma_1^+) \right) \right).$$

Then the generalized Baouendi-Grushin equation (1.2) is null-controllable on ω in any time $T > T_$, but it is not null-controllable on ω in time $T < T_*$.*

This theorem is proved in section 5.

1.3.3. *Comments.* Before proceeding further, we make some additional comments on our results.

- The assumptions regarding the function q in theorem 1.3 are slightly more general than in theorem 1.4. They seem also more natural in the context of our study. Therefore, we conjecture that theorem 1.4 holds for functions $q \in C^3(\bar{I})$ satisfying

$$q(0) = 0, \quad q'(0) > 0, \quad q(x) \neq 0 \text{ for all } x \in \bar{I}.$$

But up to our knowledge, this is still an open question.

- There are still numerous geometrical configurations not included in theorem 1.5. Nevertheless, in lots of situations, theorem 1.4 and theorem 1.3 give information about null-controllability properties. As an example, in the geometrical configuration described in fig. 1, combining theorem 1.4 and theorem 1.3, we obtain the existence of a critical time

$$T_* \in \left(\frac{d_{\text{Agm}}(-a)}{q'(0)}, \frac{d_{\text{Agm}}(-b)}{q'(0)} \right)$$

such that the Baouendi-Grushin equation is null-controllable on ω in time $T > T_*$, and is not null-controllable on ω in time $T < T_*$. In the two geometrical configurations presented in fig. 2, theorem 1.3 implies that the Baouendi-Grushin equation is not null-controllable on ω in time $T < d_{\text{Agm}}(a)/q'(0)$. Note that in these two configurations, the question of the null-controllability of (1.2) in ω for some time T large enough is still open.

1.4. Bibliographical comments.

²For $\nu \in \mathbb{R}_+^*$, we localize the smallest eigenvalue of P_ν , and for ν complex, we localize its analytic continuation.

³When f is a real valued function, we denote $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ its positive and negative part respectively.

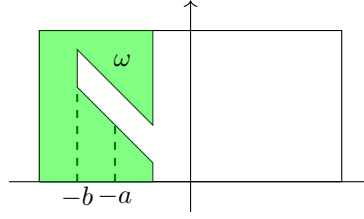


FIGURE 1. In this configuration, we obtain lower and upper bounds of the critical time of null-controllability

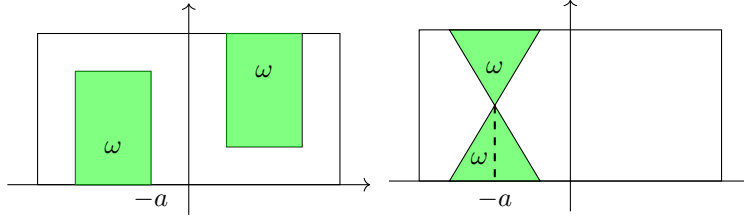


FIGURE 2. In these configurations, we obtain a lower bound on the critical time of null-controllability.

1.4.1. *On the Baouendi-Grushin equation.* The study of controllability properties of system (1.2) began with the pioneering work [7], where the authors study the null-controllability of the equation

$$(1.5) \quad \begin{cases} (\partial_t - \partial_x^2 - x^{2\gamma} \partial_y^2) f = 1_\omega u, & t \in (0, T), \quad x \in (-1, 1), \quad y \in \mathbb{T}, \\ f(t, x, y) = 0, & t \in (0, T), \quad x = \pm 1, \quad y \in \mathbb{T}. \end{cases}$$

They prove that in the case $\gamma \in (0, 1)$ (weak degeneracy), (1.2) is null-controllable for any control set ω and any time $T > 0$, whereas in the case $\gamma > 1$ (strong degeneracy), it is not null-controllable for any control set ω and any time $T > 0$, except if ω contains $\{0\} \times \mathbb{T}$ in which case it is null-controllable in any positive time T .

More surprisingly, in the case $\gamma = 1$, which corresponds to (1.2) with $q(x) = x$, and for $\omega = (a, b) \times \mathbb{T}$, with $0 < a < b$, there exists a critical time $T_* \geq \frac{a^2}{2}$ such that (1.2) is null controllable on ω in time T , for every $T > T_*$, and is not null controllable on ω in time T , for every $T < T_*$. It is also proved that if $\gamma = 1$ and ω contains the vertical line $\{0\} \times \mathbb{T}$, equation (1.5) is null controllable in any time $T > 0$. Such a minimal time of null-controllability would not be surprising for equations with finite speed of propagation, such as the wave equation [3], but the Baouendi-Grushin equation has a *infinite* speed of propagation.

Many works followed, trying to characterize precisely the critical time T_* , and to generalize the result to different geometrical settings and different functions q . The first exact characterization of T_* is given in [10] in the case $q(x) = x$ and with two symmetric vertical strips as control set, that is $\omega = (-1, -a) \times (a, 1)$, $a \in (0, 1)$. Using the transmutation method and sideways energy estimates, the authors prove that eq. (1.2) is null-controllable in ω in any time $T > \frac{a^2}{2}$, and is not null-controllable in ω in any time $T < \frac{a^2}{2}$.

When ω is a vertical strip of the form $(a, b) \times \mathbb{T}$, with $a > 0$, as in [7], the precise value of the critical time T_* was obtained independently in the works [1, 8]. More precisely, in [1], using new estimates for biorthogonal sequences to real exponentials and the moments method, the authors prove that in the case $q(x) = x$, the critical time is $\frac{a^2}{2}$. In [8], with a function q satisfying the assumptions of theorem 1.5, the author use a Carleman strategy to obtain that (1.2) is null controllable on ω in any

time $T > T_*$, and not null-controllable on ω in any time $T < T_*$, with

$$T_* = \frac{d_{\text{Agm}}(a)}{q'(0)}.$$

All the strategies developed in [7, 10, 1, 8], although very different, rely on a Fourier expansion of system (1.2) with respect to the y -variable and the study of the obtained family of one dimensional parabolic equation in the variables t, x . As a consequence, the control set ω has to be a vertical strip, which seems to be an important restriction of the proposed methods. Nevertheless, in [20], the authors generalize the positive null-controllability results obtained in [8] to a large class of control sets: with a control set ω and a function q given as in theorem 1.5, system (1.2) is null controllable in any time $T > T_*$, with

$$T_* = \max(d_{\text{Agm}}(-\sup(\gamma_2^-)), d_{\text{Agm}}(\sup(\gamma_1^+))).$$

In the specific case $q(x) = x$, they also prove that if there exist $a, b \in I$, $a < 0 < b$, and $y_0 \in \mathbb{T}$ such that $\text{distance}((a, b) \times \{y_0\} \cap \omega) > 0$, then (1.2) is not null controllable in time $T < \min(a^2, b^2)/2$, whereas if there is $y_0 \in \mathbb{T}$ such that $\text{distance}(I \times \{y_0\} \cap \omega) > 0$, then (1.2) is not null controllable on ω in any positive time T . Theorem 1.5 is the generalization of this result to a wider class of functions q .

To end this overview on controllability issues for the parabolic Baouendi-Grushin equation, we point out that partial controllability results are known in some multidimensional configurations [8] while precise results are known for cascade systems of two-dimensional Baouendi-Grushin equations with one control, in the case $q(x) = x$ [1].

1.4.2. *Some related problems.* Let us briefly mention the literature on related problems, in several directions: other degenerate parabolic equations, minimal time of null controllability for parabolic systems, and other type of degenerate equations.

Since the pioneering works [21, 22] on the null-controllability of the one-dimensional heat equation, the null-controllability of non-degenerate parabolic equations has been extensively studied. The null-controllability of degenerate parabolic equations is a more recent subject of study. The case of a degeneracy at the boundary of the domain is now well-understood [14] (see also the references therein).

When the degeneracy occurs in the domain, we lack for the moment a general theory, and equations are studied case by case. The two-dimensional Baouendi-Grushin equations is arguably the simplest and best understood equation of that type. Very similar results, including a minimal time of null-controllability for quadratic degeneracy, have been observed for the heat equation on the Heisenberg group [5, 8], and the Kolmogorov equation [4, 6, 9, 16, 26].

The related problem of *approximate controllability* for degenerate parabolic equations has been studied in a somewhat general framework [29].

A minimal time of null-controllability might also appear for the heat equation with punctual control [19] and for systems of parabolic equations, degenerate or not [2, 11].

Finally, let us mention that the subelliptic wave equation is not controllable [30], and that the Grushin-Schrödinger equation has a minimal time of controllability [13, 31].

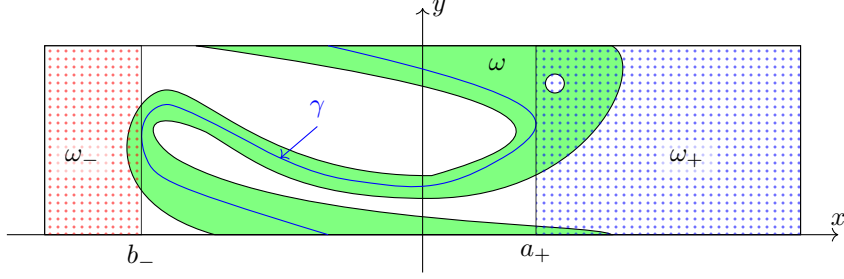
2. NULL-CONTROLLABILITY IN LARGE TIME

In this section, we prove theorem 1.4. The idea of the proof is to use known controllability results for equation (1.2) when the control set is a vertical strip combined with cutoff arguments. More precisely, we recall the following result [8, theorem 1.4].⁴

Proposition 2.1. *Assume that q satisfies the assumptions of theorem 1.4. Let $\omega = (a, b) \times \mathbb{T}$, with $-L_- \leq a < b \leq L_+$. Then*

- if $0 < a$, the Baouendi-Grushin equation is null-controllable on ω in time $T > d_{\text{Agm}}(a)/q'(0)$,
- if $b < 0$, the Baouendi-Grushin equation is null-controllable on ω in time $T > d_{\text{Agm}}(b)/q'(0)$,
- if $a < 0 < b$, the Baouendi-Grushin equation is null-controllable on ω in time $T > 0$.

⁴The reference [8] states the result with a control on the boundary. But cutoff arguments allows to construct control on vertical strips from boundary controls, as in [20, Appendix A].

FIGURE 3. Definition of ω_- (red) and ω_+ (blue).

Proof of theorem 1.4. Let b_-, a_+ in I be such that $b_- < \inf_{\mathbb{T}}(\gamma_x) \leq \sup_{\mathbb{T}}(\gamma_x) < a_+$. We set $\omega_- = (-L_-, b_-)$ and $\omega_+ = (a_+, L_+)$ (see fig. 3). Proposition 2.1 implies that for any time

$$T > \frac{1}{q'(0)} \max(d_{\text{Agm}}(b_-), d_{\text{Agm}}(a_+)),$$

the Baouendi-Grushin equation is null-controllable on ω_- and on ω_+ in time T . Consequently, for any initial condition $f_0 \in L^2(I \times \mathbb{T})$, there exist u_- and u_+ such that f_- and f_+ solutions of

$$\begin{cases} (\partial_t - \partial_x^2 - q(x)^2 \partial_y^2) f_{\pm}(t, x, y) = \mathbb{1}_{\omega_{\pm}} u_{\pm}(t, x, y), & t \in (0, T), x \in I, y \in \mathbb{T}; \\ f_{\pm}(t, x, y) = 0, & t \in (0, T), x \in \partial I, y \in \mathbb{T}; \\ f_{\pm}(0, x, y) = f_0(x, y), & x \in I, y \in \mathbb{T}; \end{cases}$$

satisfy $f_{\pm}(T, \cdot, \cdot) = 0$ in $I \times \mathbb{T}$.

By definition of γ , ω_+ and ω_- are included in two distinct connected components of $(I \times \mathbb{T}) \setminus \gamma$. We denote Υ_+ (resp. Υ_-) the connected component of $(I \times \mathbb{T}) \setminus \gamma$ containing ω_+ (resp. ω_-). Let $\chi \in C^\infty(I \times \mathbb{T})$ be such that $\chi \equiv 1$ in $\Upsilon_+ \setminus \bar{\omega}$ and $\chi \equiv 0$ in $(I \times \mathbb{T}) \setminus \bar{\Upsilon}_+ \cup \bar{\omega}$. In particular, $\chi \equiv 0$ in Υ_- and $\text{supp}(\partial_x \chi) \subset \omega$, $\text{supp}(\partial_y \chi) \subset \omega$. Define $f = \chi f_- + (1 - \chi) f_+$. It is easily verified that f satisfies

$$\begin{cases} (\partial_t - \partial_x^2 - q(x)^2 \partial_y^2) f(t, x, y) = \mathbb{1}_{\omega} u(t, x, y), & t \in (0, T), x \in I, y \in \mathbb{T}; \\ f(t, x, y) = 0, & t \in (0, T), x \in \partial I, y \in \mathbb{T}; \\ f(0, x, y) = f_0(x, y), & x \in I, y \in \mathbb{T}; \\ f(T, x, y) = 0, & x \in I, y \in \mathbb{T}; \end{cases}$$

with a source term $u \in L^2((0, T); L^2(I \times \mathbb{T}))$. As a consequence, the Baouendi-Grushin is null controllable on ω in time T .

As this holds for any $T > \max(d_{\text{Agm}}(b_-), d_{\text{Agm}}(a_+))/q'(0)$, for any $b_- < \inf_{\mathbb{T}}(\gamma_x)$, for any $a_+ > \sup_{\mathbb{T}}(\gamma_x)$, the result follows. \square

3. LACK OF NULL-CONTROLLABILITY

In this section, we will prove the following case of theorem 1.3.

Theorem 3.1. *Assume that $q \in C^2(\bar{I})$ is such that $q(0) = 0$, $q'(0) > 0$ and $q(x) \neq 0$ whenever $x \neq 0$. Let ω be an open subset of $I \times \mathbb{T}$. Assume that there exist $a \in (-L_-, 0)$ and $y_0 \in \mathbb{T}$ such that*

$$\text{distance}((a, L_+) \times \{y_0\}, \omega) > 0.$$

Then, the generalized Baouendi-Grushin equation (1.2) is not null-controllable on ω in time $T < d_{\text{Agm}}(a)/q'(0)$.

Remark 3.2. By changing x in $-x$, I in $-I$ and q in $-q$, this theorem implies that if $b \in (0, L_+)$ and if $\text{distance}((-L_-, b) \times \{y_0\}, \omega) > 0$, then the generalized Baouendi-Grushin equation (1.2) is not null-controllable on ω in time $T < d_{\text{Agm}}(b)/q'(0)$.

To completely prove theorem 1.3, there is two more cases:

- distance $((a, b) \times \{y_0\}, \omega) > 0$ with $-L_- < a < 0 < b < L_+$, lack of null-controllability in time $T < \min(d_{\text{Agm}}(a), d_{\text{Agm}}(b))/q'(0)$;
- distance $((-L_-, L_+) \times \{y_0\}, \omega) > 0$, lack of null-controllability in any time $T > 0$.

The proofs of these cases are a minor modification of the one of theorem 3.1. We mention in footnotes the most important modifications and we leave the details to the reader.

Under the hypotheses of this theorem, there exists a neighborhood W_0 of y_0 such that (see fig. 4) $\omega \cap ([a, L_+] \times W_0) = \emptyset$. We may assume that W_0 is a closed interval with non-empty interior. To prove theorem 3.1, we assume without loss of generality that ω is the complement of the rectangle $[a, L_+] \times W_0$:

$$(3.1) \quad \omega = (I \times \mathbb{T}) \setminus ([a, L_+] \times W_0).$$

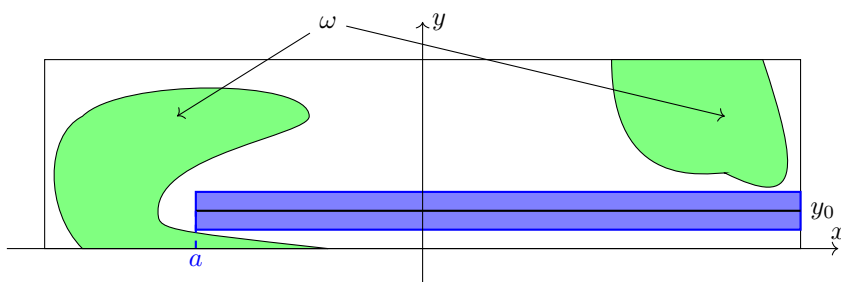


FIGURE 4. In green, the domain ω . If we have a horizontal segment that stays at positive distance from ω , we can thicken it into a rectangle that is disjoint from ω .

3.1. Observability inequality. Using standard duality arguments (see [15, theorem 2.44]), the null-controllability of the generalized Baouendi-Grushin equation (1.2) is equivalent to the following *observability inequality*: there exists $C > 0$ such that for every $g_0 \in L^2(I \times \mathbb{T})$, the solution g of

$$(3.2) \quad \begin{cases} (\partial_t - \partial_x^2 - q(x)^2 \partial_y^2)g(t, x, y) = 0, & t \in (0, T), x \in I, y \in \mathbb{T}; \\ g(t, x, y) = 0, & t \in (0, T), x \in \partial I, y \in \mathbb{T}; \\ g(0, x, y) = g_0(x, y) & (x, y) \in I \times \mathbb{T}, \end{cases}$$

satisfies

$$(3.3) \quad \|g(T, \cdot, \cdot)\|_{L^2(I \times \mathbb{T})}^2 \leq C \|g\|_{L^2((0, T) \times \omega)}^2.$$

Therefore, to prove theorem 3.1, our strategy is to construct a counterexample to this observability inequality.

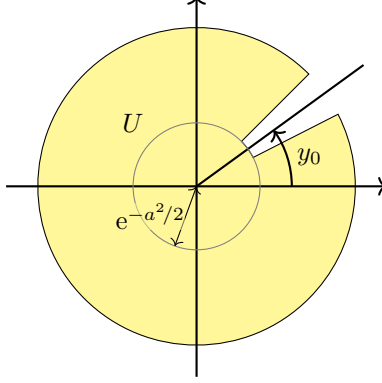
3.2. Model case. We start with a model equation, that we study to showcase the main ideas of the proof of theorem 3.1 without some of the more technical aspects. Consider the Baouendi-Grushin equation on $\mathbb{R} \times \mathbb{T}$:

$$(3.4) \quad (\partial_t - \partial_x^2 - x^2 \partial_y^2)g(t, x, y) = 0.$$

Let $\omega \subset \mathbb{R} \times I$ be open. We say that Baouendi-Grushin equation (3.4) is observable on ω in time $T > 0$ if there exists $C > 0$ such that for all g solution of (3.4), the following observability inequality holds:

$$(3.5) \quad \|g(T, \cdot, \cdot)\|_{L^2(\mathbb{R} \times \mathbb{T})}^2 \leq C \|g\|_{L^2((0, T) \times \omega)}^2.$$

We prove the following theorem.

FIGURE 5. The domain U .

Theorem 3.3. *let $a > 0$, $W_0 \subset \mathbb{T}$ a closed interval with non-empty interior and*

$$(3.6) \quad \omega = (\mathbb{R} \times \mathbb{T}) \setminus ([-a, a] \times W_0).$$

Let $T > 0$ such that

$$T < \frac{a^2}{2}.$$

The Baouendi-Grushin equation (3.4) is not observable on ω in time T .

Before going into the proof, let us examine some solutions of the Baouendi-Grushin equation that are concentrated around $x = 0$. Taking the n -th Fourier coefficient in y of g , which we will denote by $\hat{g}(t, x, n)$, we get

$$(3.7) \quad (\partial_t - \partial_x^2 + n^2 x^2) \hat{g}(t, x, n) = 0.$$

Thus, the Baouendi-Grushin equation is transformed into a family of parabolic equations $(\partial_t + H_n)g_n = 0$, where H_n is the *harmonic oscillator* $-\partial_x^2 + n^2 x^2$. The spectral properties of the harmonic oscillator are well-known (see, e.g., [23, §1.3] or appendix A), and in particular the first eigenvalue is $|n|$ with associated eigenfunction $\varphi_n(x) = (|n|/\pi)^{1/4} e^{-|n|x^2/2}$. Thus, if $(a_n)_{n>0}$ is a complex-valued sequence with only a finite number of nonzero terms, the function g defined by

$$(3.8) \quad g(t, x, y) := \sum_{n>0} a_n e^{iny - nx^2/2 - nt}$$

is a solution of the Baouendi-Grushin equation (3.4). We will look for a counterexample of the observability inequality (3.5) in this class of functions.

This solution can be written as $g(t, x, y) = g_{\text{pol}}(e^{iy-t-x^2/2})$ with

$$(3.9) \quad g_{\text{pol}}(z) := \sum_{n>0} a_n z^n.$$

We will use the fact that g is a polynomial in $z = e^{iy-t-x^2/2}$ to rewrite the observability inequality we want to disprove as an inequality on polynomials. More precisely, we have the following estimate.

Lemma 3.4. *Assume that the observability inequality for the Baouendi-Grushin equation (3.5) holds. Let $U \subset \mathbb{C}$ be defined by (see fig. 5)*

$$U = D(0, e^{-a^2/2}) \cup \{z \in \mathbb{C} : |z| < 1, \arg(z) \notin W_0\}.$$

Then, there exists $C > 0$ such that for every polynomial $p \in \mathbb{C}[X]$,

$$\|p\|_{L^2(D(0, e^{-T}))} \leq C \|p\|_{L^\infty(U)}.$$

Proof. Step 1: Observability inequality. Let $p(z) = \sum_{n \geq 0} a_n z^n$ a polynomial and set $g_{\text{pol}}(z) = zp(z) = \sum_{n > 0} a_{n-1} z^n$. The discussion above shows that g defined by

$$g(t, x, y) = g_{\text{pol}}(e^{iy-t-x^2/2})$$

is a solution of the Baouendi-Grushin equation (3.4). The observability inequality on this class of functions reads

$$(3.10) \quad \int_{\mathbb{R} \times \mathbb{T}} |g(T, x, y)|^2 dx dy \leq C \int_{[0, T] \times \omega} |g_{\text{pol}}(e^{iy-t-x^2/2})|^2 dt dx dy.$$

Step 2: Left-hand side of the observability inequality (3.10). Since the functions $\psi_n: (x, y) \mapsto e^{iny-nx^2/2}$ are orthogonal in $L^2(\mathbb{R} \times \mathbb{T})$, the left-hand side can be rewritten as

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{T}} |g(T, x, y)|^2 dx dy &= \int_{\mathbb{R} \times \mathbb{T}} \left| \sum_{n > 0} a_{n-1} e^{-nT} \psi_n(x, y) \right|^2 dx dy \\ &= \sum_{n > 0} |a_{n-1}|^2 e^{-2nT} \|\psi_n\|_{L^2}^2 \\ &= \sum_{n > 0} \frac{2\pi^{3/2}}{\sqrt{n}} |a_{n-1}|^2 e^{-2nT}, \end{aligned}$$

Elementary computations in polar coordinates prove that the function $z \mapsto z^n$ are orthogonal in $L^2(D(0, R), m)$, where m is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$, and that

$$\|z^n\|_{L^2(D(0, R), m)}^2 = \frac{\pi R^{2n+2}}{n+1}.$$

Thus,

$$\begin{aligned} \|p\|_{L^2(D(0, e^{-T}))}^2 &= \sum_{n \geq 0} \frac{\pi}{n+1} |a_n|^2 e^{-2(n+1)T} \\ &\leq \sum_{n > 0} \frac{\pi}{\sqrt{n}} |a_{n-1}|^2 e^{-2nT} \\ (3.11) \quad &= \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R} \times \mathbb{T}} |g(T, x, y)|^2 dx dy. \end{aligned}$$

Step 3: Right-hand side of the observability inequality (3.10). We write the right-hand side of the observability inequality by making the change of variables $(x, z) = (x, e^{-t+iy-x^2/2})$. We have $dx dm(z) = |z|^2 dt dx dy$. Thus, if we denote by $\Omega \subset \mathbb{R} \times \mathbb{C}$ the image of $(0, T) \times \omega$ by this change of variables, we have

$$\begin{aligned} \int_{(0, T) \times \omega} |g_{\text{pol}}(e^{-t+iy-x^2/2})|^2 dt dx dy &= \int_{\Omega} |g_{\text{pol}}(z)|^2 |z|^{-2} dx dm(z) \\ (3.12) \quad &= \int_{\Omega} |p(z)|^2 dx dm(z). \end{aligned}$$

By definition, $\Omega = \bigcup_{x \in \mathbb{R}} (\{x\} \times \mathcal{D}_x)$, where

$$\mathcal{D}_x = \{e^{-t+iy-x^2/2}, t \in (0, T), (x, y) \in \omega\}.$$

We claim that for every $0 < t < T$ and $x \in \mathbb{R}$, $\mathcal{D}_x \subset U$ (this is the reason we defined U this way). Indeed, if $-a < x < a$, and $(x, y) \in \omega$, then, by definition of ω as the complement of a $[-a, a] \times W_0$ (eq. (3.6)), we necessarily have $y \notin W_0$. It follows that $e^{-t+iy-x^2/2} \in U$. In the case $x \notin [-a, a]$, we have $x^2/2 > a^2/2$. Then, $|e^{-t+iy-x^2/2}| < e^{-a^2/2}$. It follows again that $e^{-t+iy-x^2/2} \in U$. Thus, using

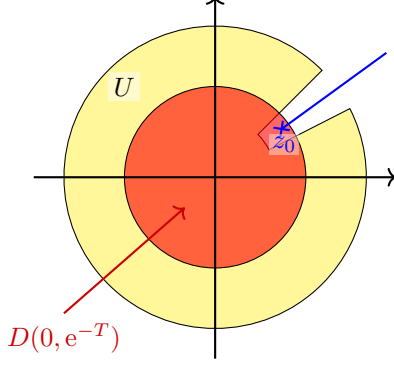


FIGURE 6. When the disk $D(0, e^{-T})$ (in red) is not included in U , we can find holomorphic functions that are small in U but arbitrarily big in $D(0, e^{-T})$. For instance, we can construct with Runge's theorem a sequence of polynomials that converges to $z \mapsto (z - z_0)^{-1}$ away from the blue line.

Hölder inequality in eq. (3.12),

$$\begin{aligned} \int_{(0,T) \times \omega} |g_{\text{pol}}(e^{-t+iy-x^2/2})|^2 dt dx dy &= \int_{x \in \mathbb{R}} \int_{\mathcal{D}_x} |p(z)|^2 dm(z) dx \\ &\leq \int_{x \in \mathbb{R}} m(\mathcal{D}_x) \|p\|_{L^\infty(U)}^2 dx. \end{aligned}$$

Since $\mathcal{D}_x \subset D(0, e^{-x^2/2})$, $m(\mathcal{D}_x) \leq \pi e^{-x^2}$. Hence

$$(3.13) \quad \int_{(0,T) \times \omega} |g_{\text{pol}}(e^{-t+iy-x^2/2})|^2 dt dx dy \leq \sqrt{\pi} \|p\|_{L^\infty(U)}^2.$$

Now, plugging the lower-bound of the left-hand side (3.11) and the upper bound of the right-hand side (3.13) into the observability inequality (3.10), we obtain a constant $C > 0$ such that

$$\|p\|_{L^2(D(0, e^{-T}))}^2 \leq C \|p\|_{L^\infty(U)}^2. \quad \square$$

Proof of theorem 3.3. To disprove the inequality, we only have to disprove the inequality on polynomials given by the previous lemma 3.4. If $T < a^2/2$, the disk $D(0, e^{-T})$ is not included in U (see fig. 6). For instance, if $\varepsilon > 0$ is small enough, $z_0 = e^{iy_0 - T - \varepsilon}$ is not in U . In fact, the half-line $z_0[1, +\infty)$ stays at positive distance from U (see fig. 6). Then, according to Runge's theorem [32, theorem 13.9], there exists a sequence of polynomials (p_k) that converges uniformly on every compact of $\mathbb{C} \setminus z_0[1, +\infty)$ to $z \mapsto (z - z_0)^{-1}$.

Since \bar{U} is a compact subset of $\mathbb{C} \setminus z_0[1, +\infty)$, the sequence p_k stay uniformly bounded on U , i.e., $\sup_k \|p_k\|_{L^\infty(U)} < +\infty$. But $z_0 \in D(0, e^{-T})$, and therefore $\|(z - z_0)^{-1}\|_{L^2(D(0, e^{-T}))} = +\infty$. Thanks to Fatou's lemma, this proves that $\|p_k\|_{L^2(D(0, e^{-T}))} \rightarrow +\infty$ as $k \rightarrow +\infty$.

We have proved that (p_k) is a counterexample to the inequality of lemma 3.4, which concludes the proof of theorem 3.3. \square

3.3. From the model case to the generalized Baouendi-Grushin equation. If we consider the generalized Baouendi-Grushin equation (3.2) and take the n -th Fourier coefficient in y of g , we get

$$(3.14) \quad (\partial_t - \partial_x^2 + n^2 q(x)^2) \hat{g}(t, x, n) = 0.$$

For $n \geq 0$, let P_n be the unbounded operator on $L^2(I)$ with domain $\text{Dom}(P_n) := H^2(I) \cap H_0^1(I)$ defined by $P_n g = (-\partial_x^2 + n^2 q^2)g$. We will denote λ_n the first eigenvalue of P_n and by φ_n a corresponding eigenfunction. Remark that we do not require φ_n to be normalized in $L^2(I)$ (this will make some definitions slightly easier later on). Then, we will look for a counterexample of the observability inequality (3.3) with solutions of the generalized Baouendi-Grushin equation (3.2) of the form

$$(3.15) \quad g(t, x, y) := \sum_n a_n \varphi_n(x) e^{iny - \lambda_n t}.$$

Heuristically, this should work because we expect the eigenfunctions φ_n to be localized around $x = 0$ as $n \rightarrow +\infty$, in which case the operator $-\partial_x^2 + n^2 q^2$ looks like $-\partial_x^2 + n^2 q'(0)^2 x^2$, and the eigenvalue

and eigenfunctions look like $\lambda_n \approx nq'(0)$ and $\varphi_n(x) \approx (n/\pi)^{1/4}e^{-nq'(0)x^2/2}$. So the solutions g defined above look like the solutions used to treat the model case (eq. (3.8)), up to a factor $q'(0)$. In fact, a better approximation of φ_n would be the so-called WKB approximation⁵

$$(3.16) \quad \begin{aligned} \varphi_n(x) &\approx (n/\pi)^{1/4}c_0(x)e^{-n d_{\text{Agm}}(x)}, \\ c_0'(x) &= \frac{q'(0) - d_{\text{Agm}}'(x)}{2d_{\text{Agm}}'(x)}c_0(x), \\ c_0(0) &= 1. \end{aligned}$$

Thus, we have

$$(3.17) \quad g(t, x, y) \approx c_0(x) \sum_n a_n e^{n(iy - q'(0)t - d_{\text{Agm}}(x))},$$

I.e., g can almost be written as $g(t, x, y) \approx c_0(x)g_{\text{pol}}(e^{iny - tq'(0) - d_{\text{Agm}}(x)})$, where g_{pol} is the polynomial

$$g_{\text{pol}}(z) := \sum_n a_n z^n.$$

Let us write this in a exact way. Consider $\tilde{\varphi}_n(x) := n^{1/4}e^{-nq'(0)x^2/2}$ and let Π_n be the spectral projection associated to the eigenvalue λ_n of P_n . We define⁶

$$(3.18) \quad \varphi_n := \Pi_n \tilde{\varphi}_n.$$

Let $\varepsilon \in (0, 1)$ (that we will later choose close to 0) and set

$$(3.19) \quad \gamma_{t,x}(n) := e^{-t(\lambda_n - q'(0)n)} \varphi_n(x) e^{n d_{\text{Agm}}(x)(1-\varepsilon)}.$$

Then, with the choice of φ_n in eq. (3.18), the solution g defined in eq. (3.15) can be written as

$$(3.20) \quad g(t, x, y) = \sum_n a_n \gamma_{t,x}(n) e^{n(iy - q'(0)t - (1-\varepsilon)d_{\text{Agm}}(x))}.$$

In some sense, this formula tells us that g can be written as “pseudo-differential-type” operator applied to the “model solution” $g_{\text{pol}}(e^{iy - q'(0)t - (1-\varepsilon)d_{\text{Agm}}(x)})$. To successfully adapt the strategy used for the model Baouendi-Grushin equation, we need some continuity estimates for these “pseudo-differential-type” operators. We claim that the following estimate holds.

Lemma 3.5. *Let $T > 0$ and $\varepsilon > 0$. Define $\gamma_{t,x}$ as in eq. (3.19). Let $\gamma_{t,x}(z\partial_z)$ be the operator on polynomials defined by*

$$\gamma_{t,x}(z\partial_z) \left(\sum a_n z^n \right) = \sum \gamma_{t,x}(n) a_n z^n.$$

Let X be a compact subset of \mathbb{C} . Let V be an open star-shaped with respect to 0 neighborhood of X . There exist $C > 0$ and $N \in \mathbb{N}$ such that for every polynomial $p \in \mathbb{C}[X]$ with a zero of order N at 0 and for every $0 < t < T$ and $x \in I$,

$$\|\gamma_{t,x}(z\partial_z)(p)\|_{L^\infty(X)} \leq C \|p\|_{L^\infty(V)}.$$

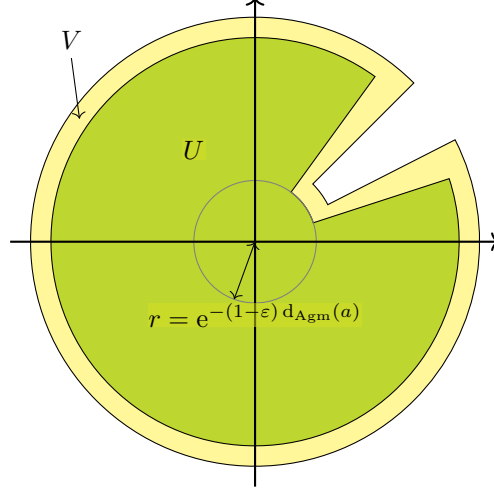
As $\gamma_{t,x}(n)$ is related to the eigenvalues and eigenfunctions of $-\partial_x^2 + n^2q(x)^2$, proving this lemma requires a spectral analysis of this operator. What is more surprising is that we actually need a spectral analysis of $-\partial_x^2 + \nu^2q(x)^2$ when ν is not necessarily real, meaning we have to do some *non selfadjoint* spectral analysis. We will prove lemma 3.5 in section 4.3 with the spectral analysis done in the rest of section 4 and a general estimate on operators on polynomials [27, theorem 18].

We will also use the relatively elementary bounds on λ_n and $\|\varphi_n\|_{L^2(I)}$ given by the following proposition.

⁵We write here the first term in the WKB expansion of eigenfunctions, and only in dimension 1, because it is enough for our purposes. But such a construction can be refined with more terms and in higher dimension [18, Chapter 3 & Chapter 6, theorem A.3].

Also, in the differential equation that defines c_0 , there is a division by $d_{\text{Agm}}'(x)$, which is equal to 0 at $x = 0$. But the numerator is also 0 at $x = 0$, and simple Taylor expansions at $x = 0$ proves that the quotient appearing in the differential equation for c_0 is actually well-defined at $x = 0$.

⁶We could also have chosen $\tilde{\varphi}_n$ to be the WKB expansion defined previously, which would be a better approximation of the eigenfunction. But since we are projecting on the actual eigenfunction afterwards, this is not necessary.

FIGURE 7. The domains U and V .

Proposition 3.6. *In the limit $n \rightarrow +\infty$, $\lambda_n = nq'(0) + o(n)$. Moreover, there exist $c > 0$ and $N \geq 0$ such that for every $n \geq N$, $\|\varphi_n\|_{L^2(I)} \geq c$.*

This proposition is standard, at least in spirit, and follows from the fact that the first eigenfunction of P_n looks like $e^{-nq'(0)x^2/2}$ when $n \rightarrow +\infty$ (see, e.g., [18, theorem 4.23 & Eq. (4.20)]), and we provide a proof in section 4.2.

With these two estimates, we prove the following version of lemma 3.4 adapted for the generalized Bouendi-Grushin equation.

Lemma 3.7. *Assume that the observability inequality (3.3) for the generalized Baouendi-Grushin equation holds. Let $\varepsilon > 0$ and let $U \subset \mathbb{C}$ be defined by (see fig. 7)⁷*

$$U = D(0, e^{-(1-\varepsilon)d_{\text{Agm}}(a)}) \cup \{z \in \mathbb{C} : |z| < 1, \arg(z) \notin W_0\}.$$

Let V be a neighborhood of \bar{U} that is star-shaped with respect to 0. Then, there exist $C > 0$ and $N \in \mathbb{N}$ such that for every polynomial $p \in \mathbb{C}[X]$ with a zero of order N at 0, we have

$$\|p\|_{L^2(D(0, e^{-q'(0)T(1+\varepsilon)})} \leq C\|p\|_{L^\infty(V)}.$$

Proof. The proof mostly follows the one of lemma 3.4, but with the error term $\gamma_{t,x}$ which will be handled by lemma 3.5. Let $N > 0$ as in proposition 3.6 and lemma 3.7. Let $p(z) = \sum_{n \geq N} a_n z^n$ a polynomial. The discussion above shows that g defined by

$$\begin{aligned} (3.21) \quad g(t, x, y) &= \sum_{n \geq N} a_n \varphi_n(x) e^{-\lambda_n t + iny} \\ &= \gamma_{t,x}(z \partial_z)(p)(e^{iy - q'(0)t - (1-\varepsilon)d_{\text{Agm}}(x)}) \end{aligned}$$

⁷In the variant of theorem 3.1 where $\text{distance}((a, b) \times \{y_0\}, \omega) > 0$ mentioned in remark 3.2, we have to add $D(0, e^{-(1-\varepsilon)d_{\text{Agm}}(b)})$ to U . In the variant where $\text{distance}(I \times \{y_0\}, \omega) > 0$, U is only the pacman $\{z \in \mathbb{C} : \arg(z) \notin W_0\}$. Their proofs are minor adaptations and are left to the reader.

is a solution of the Baouendi-Grushin equation (3.2).

Step 1: Left-hand side of the observability inequality (3.3). Since the functions $\psi_n: (x, y) \mapsto \varphi_n(x)e^{iny}$ are orthogonal, the left-hand side can be rewritten as

$$\begin{aligned} \int_{I \times \mathbb{T}} |g(T, x, y)|^2 dx dy &= \int_{I \times \mathbb{T}} \left| \sum_{n \geq N} a_n e^{-\lambda_n T} \psi_n(x, y) \right|^2 dx dy \\ &= \sum_{n \geq N} |a_n|^2 e^{-2\lambda_n T} \|\psi_n\|_{L^2(I \times \mathbb{T})}^2, \end{aligned}$$

using the lower bounds on $\|\varphi_n\|$ given by proposition 3.6, we get that $\|\psi_n\|_{L^2} \geq c > 0$ for $n \geq N$. Thus,

$$\int_{I \times \mathbb{T}} |g(T, x, y)|^2 dx dy \geq c \sum_{n \geq N} |a_n|^2 e^{-2\lambda_n T}.$$

Now, thanks to the asymptotics for λ_n given by proposition 3.6, there exists $C_\varepsilon > 0$ such that $\lambda_n \leq nq'(0)(1 + \varepsilon) + C_\varepsilon$. Thus,

$$\int_{I \times \mathbb{T}} |g(T, x, y)|^2 dx dy \geq ce^{-2TC_\varepsilon} \sum_{n \geq N} |a_n|^2 e^{-2nq'(0)T(1+\varepsilon)}.$$

As in the proof of lemma 3.4, we denote by m the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$, the functions $z \mapsto z^n$ are orthogonal on $L^2(D(0, R), m)$ and $\|z^n\|_{L^2(D(0, R), m)}^2 = \pi R^{2n+2}/(n+1)$. Thus,

$$\begin{aligned} \|p\|_{L^2(D(0, e^{-q'(0)T(1+\varepsilon)}))}^2 &= \sum_{n \geq N} \frac{\pi}{n+1} |a_n|^2 e^{-2(n+1)q'(0)T(1+\varepsilon)} \\ &\leq \sum_{n \geq N} \pi e^{-2q'(0)T(1+\varepsilon)} |a_n|^2 e^{-2nq'(0)T(1+\varepsilon)} \\ (3.22) \qquad &\leq C \int_{I \times \mathbb{T}} |g(T, x, y)|^2 dx dy. \end{aligned}$$

Step 2: Right-hand side of the observability inequality (3.3). We make the analogous change of variables as in the model case, but adapted to our case, i.e., $(x, z) = (x, e^{-q'(0)t+iy-(1-\varepsilon)d_{\text{Agm}}(x)})$. We have $dx dm(z) = q'(0)|z|^2 dt dx dy$. Thus, if we denote by $\Omega \subset I \times \mathbb{C}$ the image of $(0, T) \times \omega$ by this change of variables, which is a subset of $I \times D(0, 1)$, we have

$$\int_{(0, T) \times \omega} |g(t, x, y)|^2 dt dx dy = \frac{1}{q'(0)} \int_{\Omega} |\gamma_{t,x}(z\partial_z)(p)(z)|^2 |z|^{-2} dx dm(z).$$

We kept for simplicity the notation $\gamma_{t,x}$ but of course t is now a function of (x, z) . For any $0 < t < T$ and $(x, y) \in \omega$, we have $|e^{-t+iy-d_{\text{Agm}}(x)}| \geq e^{-T-\max(d_{\text{Agm}}(-L_-), d_{\text{Agm}}(L_+))}$. Thus, in the previous integral, we can bound $|z|^{-2} \leq C$ for some $C > 0$:

$$\begin{aligned} \int_{(0, T) \times \omega} |g(t, x, y)|^2 dt dx dy &\leq C \int_{\Omega} |\gamma_{t,x}(z\partial_z)(p)(z)|^2 dx dm(z) \\ &\leq C \sup_{(x,z) \in \Omega} |\gamma_{t,x}(z\partial_z)(p)(z)|^2. \end{aligned}$$

As in the model case, if $(x, z) \in \Omega$, then $z \in U$. Indeed, let $(x, z) \in \Omega$, i.e., $z = e^{-t+iy-d_{\text{Agm}}(x)(1-\varepsilon)}$ with $(x, y) \in \omega$. If $a \leq x < L_+$, then, by definition of ω as the complement of $[a, L_+) \times W_0$ (eq. (3.1)), we necessarily have $y \notin W_0$. It follows that $z \in U$. In the case $x < a < 0$, since d_{Agm} is decreasing on $[-L_-, 0]$, we have $|z| = e^{-t-d_{\text{Agm}}(x)(1-\varepsilon)} < e^{-d_{\text{Agm}}(a)(1-\varepsilon)}$. It follows again that $z \in U$. Thus,

$$\int_{(0, T) \times \omega} |g(t, x, y)|^2 dt dx dy \leq C \sup_{(t,x) \in (0, T) \times I} \|\gamma_{t,x}(z\partial_z)(p)\|_{L^\infty(U)}^2$$

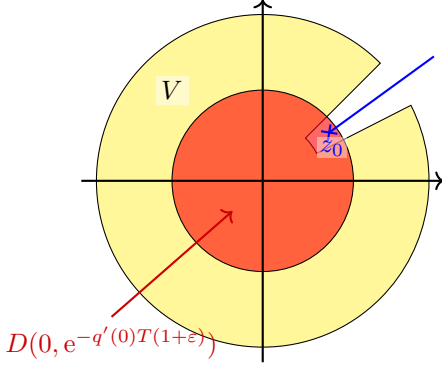


FIGURE 8. When the disk $D(0, e^{-q'(0)T(1+\epsilon)})$ (in red) is not included in U , we can find holomorphic functions that are small in U but arbitrarily big in $D(0, e^{-q'(0)T(1+\epsilon)})$. For instance, we can construct with Runge's theorem a sequence of polynomials that converges to $z \mapsto z^{N+1}(z - z_0)^{-1}$ away from the blue line.

Now, we use the operator estimate of lemma 3.5, which gives

$$(3.23) \quad \int_{(0,T) \times \omega} |g(t, x, y)|^2 dt dx dy \leq C \|p\|_{L^\infty(V)}^2.$$

Step 3: Conclusion. Now, plugging the lower-bound of the left-hand side (3.22) and the upper bound of the right-hand side (3.23) into the observability inequality (3.3), we get

$$\|p\|_{L^2(D(0, e^{-q'(0)T(1+\epsilon)})}^2 \leq C \|p\|_{L^\infty(V)}^2. \quad \square$$

Proof of theorem 3.1. As in the model case, we end the proof of non-null controllability by disproving the inequality on polynomial given by the previous lemma 3.7. If

$$q'(0)T(1 + \varepsilon) < d_{\text{Agm}}(a)(1 - \varepsilon),$$

the disk $D(0, e^{-q'(0)T(1+\varepsilon)})$ is not included⁸ in \bar{U} , and we can choose a compact neighborhood V of \bar{U} that is star-shaped with respect to 0 and such that $D(0, e^{-q'(0)T(1+\varepsilon)})$ is not included in \bar{V} (see fig. 8). Choose some $z_0 \in D(0, e^{-q'(0)T(1+\varepsilon)})$ that is not in \bar{V} . Since V is star-shaped with respect to 0, the half-line $z_0[1, +\infty)$ stays at positive distance from V . Then, according to Runge's theorem [32, theorem 13.9], there exists a sequence of polynomials $(\tilde{p}_k)_k$ that converges uniformly on every compact of $\mathbb{C} \setminus z_0[1, +\infty)$ to $z \mapsto (z - z_0)^{-1}$.

Set $p_k(z) := z^{N+1}\tilde{p}_k$. We prove that (p_k) is a counterexample to the inequality of lemma 3.7 with the same method as in the model case. Since \bar{V} is a compact subset of $\mathbb{C} \setminus z_0[1, +\infty)$, p_k stays uniformly bounded on V , i.e., $\sup_k \|p_k\|_{L^\infty(V)} < +\infty$. But $z_0 \in D(0, e^{-q'(0)T(1+\varepsilon)})$, and therefore $\|z^{N+1}(z - z_0)^{-1}\|_{L^2(D(0, e^{-q'(0)T(1+\varepsilon)})} = +\infty$. Thanks to Fatou's lemma, this proves that $\|p_k\|_{L^2(D(0, e^{-q'(0)T(1+\varepsilon)})} \rightarrow +\infty$ as $k \rightarrow +\infty$.

We have proved that the inequality of lemma 3.7 does not hold, which implies that the observability inequality (3.3) does not hold either, which in turn implies that the generalized Baouendi-Grushin equation is not null-controllable. This holds for any $\varepsilon > 0$ and any T such that $(1 + \varepsilon)q'(0)T < d_{\text{Agm}}(a)(1 - \varepsilon)$. Thus, the generalized Baouendi-Grushin equation is not null-controllable if $T < d_{\text{Agm}}(a)/q'(0)$. \square

4. SPECTRAL ANALYSIS

As explained in section 3.3, we need some spectral properties on the operator $P_\nu = -\partial_x^2 + \nu^2 q(x)^2$ with Dirichlet boundary conditions on I (defined precisely in eq. (1.4)). We start with an asymptotic of the first eigenvalue, and in following subsection, we prove some Agmon-type upper bound for the associated eigenfunctions.

⁸In the variant of theorem 3.1 where $\text{distance}((a, b) \times \{y_0\}, \omega) > 0$ mentioned in remark 3.2, taking into account footnote 7, the condition becomes $q'(0)T(1 + \varepsilon) < \min(d_{\text{Agm}}(a), d_{\text{Agm}}(b))(1 - \varepsilon)$. In the variant where $\text{distance}(I \times \{y_0\}, \omega) > 0$, again taking into account footnote 7, $D(0, e^{-q'(0)T(1+\varepsilon)})$ is never included in U .

For $\theta_0 \in [0, \frac{\pi}{2})$, we set

$$(4.1) \quad \Sigma_{\theta_0} = \{z \in \mathbb{C} : |z| \geq 1, |\arg(z)| \leq \theta_0\}.$$

4.1. The first eigenvalue and corresponding spectral projection. We denote by H_β the non-selfadjoint harmonic oscillator $-\partial_x^2 + \beta^2 x^2$ on \mathbb{R} . We refer to Appendix A for the precise definition and properties of H_β .

In this paragraph we prove that the operator P_ν has an eigenvalue close to the first eigenvalue of the model operator $H_{q'(0)\nu}$, and that the corresponding spectral projection is also a perturbation of the spectral projection of $H_{q'(0)\nu}$. See Proposition 4.2.

For this we first prove that the resolvent of P_ν is a perturbation of the resolvent of $H_{q'(0)\nu}$, in the sense that the difference between these two resolvents is smaller than the resolvent of $H_{q'(0)\nu}$.

Notice that the resolvents of P_ν and $H_{q'(0)\nu}$ are not defined on the same space. We denote by $\mathbf{1}_I$ the operator which maps $u \in L^2(\mathbb{R})$ to its restriction on I : $\mathbf{1}_I u = u|_I \in L^2(I)$. Then $\mathbf{1}_I^*$ maps a function $v \in L^2(I)$ to its extension by 0 on \mathbb{R} .

Proposition 4.1. *Let $\gamma \in \mathbb{R}$ and $\varepsilon > 0$. For $\nu \in \Sigma_{\theta_0}$ we set*

$$(4.2) \quad \mathcal{Z}_\nu = \{z \in \mathbb{C} : |z| \leq \gamma|\nu|, \text{distance}(z, \sigma(H_{q'(0)\nu})) \geq \varepsilon|\nu|\}.$$

Then there exists $\nu_0 \geq 1$ such that for $\nu \in \Sigma_{\theta_0}$ with $|\nu| \geq \nu_0$ and $z \in \mathcal{Z}_\nu$ we have $z \in \rho(P_\nu)$, and

$$\sup_{z \in \mathcal{Z}_\nu} \|(P_\nu - z)^{-1} - \mathbf{1}_I(H_{q'(0)\nu} - z)^{-1}\mathbf{1}_I^*\|_{\mathcal{L}(L^2(I))} = \underset{\nu \in \Sigma_{\theta_0}}{o} \left(\frac{1}{|\nu|} \right).$$

Notice that for $\nu \in \Sigma_{\theta_0}$ we have $\mathcal{Z}_\nu = |\nu| \mathcal{Z}_{\nu/|\nu|}$. The spectral properties of the model operator $H_{q'(0)\nu}$ are given in proposition A.1. We deduce in particular that there exists $C > 0$ such that

$$(4.3) \quad \forall \nu \in \Sigma_{\theta_0}, \forall z \in \mathcal{Z}_\nu, \quad \|(H_{q'(0)\nu} - z)^{-1}\| \leq \frac{C}{|\nu|}.$$

Proof. For $\nu \in \Sigma_{\theta_0}$ and $z \in \mathcal{Z}$ we set

$$R_\nu(z) = \mathbf{1}_I(H_{q'(0)\nu} - z)^{-1}\mathbf{1}_I^* \in \mathcal{L}(L^2(I)).$$

Let

$$\rho \in \left] \frac{1}{3}, \frac{1}{2} \right[.$$

We consider a cut-off function $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ supported in $[-2, 2]$ and equal to 1 on $[-1, 1]$. Then for $\nu \in \Sigma_{\theta_0}$ and $x \in \bar{I}$ we set

$$\chi_\nu(x) = \chi(|\nu|^\rho x).$$

Step 1: Approximation close to $x = 0$. We first prove that if $|\nu|$ is large enough then $R_\nu(z)\chi_\nu(P_\nu - z) - \chi_\nu$ extends to a bounded operator on $L^2(I)$ for all $z \in \mathcal{Z}_\nu$, and

$$(4.4) \quad \sup_{z \in \mathcal{Z}_\nu} \|R_\nu(z)\chi_\nu(P_\nu - z) - \chi_\nu\|_{\mathcal{L}(L^2(I))} \xrightarrow{|\nu| \rightarrow +\infty} 0.$$

Here and everywhere below it is implicitly understood that ν always belongs to Σ_{θ_0} .

Let $u \in \text{Dom}(P_\nu)$. We have $\chi_\nu u \in \text{Dom}(P_\nu)$ and, if $|\nu|$ is large enough, $\mathbf{1}_I^* \chi_\nu u$ belongs to $\text{Dom}(H_\nu)$. For $x \in \bar{I}$ we set

$$r(x) = q(x)^2 - q'(0)^2 x^2,$$

so that

$$R_\nu(z)(P_\nu - z)\chi_\nu u = \chi_\nu u + \nu^2 R_\nu(z)r\chi_\nu u.$$

Then, denoting $[\chi_\nu, P_\nu]$ the commutator of χ_ν and P_ν , we have $[\chi_\nu, P_\nu] = [\partial_x^2, \chi_\nu] = \chi_\nu'' + 2\chi_\nu' \partial_x$, hence

$$(4.5) \quad \begin{aligned} R_\nu(z)\chi_\nu(P_\nu - z)u &= R_\nu(z)(P_\nu - z)\chi_\nu u + R_\nu(z)[\chi_\nu, P_\nu]u \\ &= \chi_\nu u + \nu^2 R_\nu(z)r\chi_\nu u - R_\nu(z)\chi_\nu'' u + 2R_\nu(z)(\chi_\nu' u)'. \end{aligned}$$

Since $|r(x)\chi_\nu(x)| \lesssim |\nu|^{-3\rho}$, we have by eq. (4.3)

$$\|\nu^2 R_\nu(z)r\chi_\nu\|_{\mathcal{L}(L^2(I))} \lesssim |\nu|^{2-1-3\rho} \xrightarrow{|\nu| \rightarrow +\infty} 0.$$

Similarly,

$$\|R_\nu(z)\chi_\nu''\|_{\mathcal{L}(L^2(I))} \lesssim |\nu|^{2\rho-1} \xrightarrow{|\nu| \rightarrow +\infty} 0.$$

For the last term in eq. (4.5) we recall that for $v \in L^2(\mathbb{R})$ we have

$$(4.6) \quad \left\| \partial_x (H_{q'(0)\bar{\nu}} - \bar{z})^{-1} v \right\|_{L^2(\mathbb{R})}^2 + \bar{\nu}^2 q'(0)^2 \left\| x (H_{q'(0)\bar{\nu}} - \bar{z})^{-1} v \right\|_{L^2(\mathbb{R})}^2 \\ = \left\langle v, (H_{q'(0)\bar{\nu}} - \bar{z})^{-1} v \right\rangle_{L^2(\mathbb{R})} + \bar{z} \left\| (H_{q'(0)\bar{\nu}} - \bar{z})^{-1} v \right\|_{L^2(\mathbb{R})}^2.$$

We multiply by $e^{i\theta}$ and take the real part. This gives, uniformly in $\nu \in \Sigma_{\theta_0}$ and $z \in \mathcal{Z}_\nu$,

$$(4.7) \quad \left\| \partial_x (H_{q'(0)\bar{\nu}} - \bar{z})^{-1} \right\|_{\mathcal{L}(L^2(\mathbb{R}))}^2 \lesssim \frac{1}{|\nu|} \quad \text{and} \quad \left\| x (H_{q'(0)\bar{\nu}} - \bar{z})^{-1} \right\|_{\mathcal{L}(L^2(\mathbb{R}))}^2 \lesssim \frac{1}{|\nu|^3}.$$

Taking the adjoint in the first inequality gives for $|\nu|$ large enough

$$\|R_\nu(z)\partial_x(\chi'_\nu u)\|_{L^2(I)} \leq \left\| (H_{q'(0)\nu} - z)^{-1} \partial_x(\mathbb{1}_I \chi'_\nu u) \right\|_{L^2(\mathbb{R})} \\ \lesssim |\nu|^{-\frac{1}{2}} \|\mathbb{1}_I \chi'_\nu u\|_{L^2(\mathbb{R})} \\ \lesssim |\nu|^{\rho-\frac{1}{2}} \|u\|_{L^2(I)},$$

and eq. (4.4) follows.

Step 2: Approximation away from $x = 0$. There exists $c_0 \in]0, 1]$ such that for all $x \in I$ we have

$$|q(x)| \geq c_0 |x|.$$

On $L^2(I)$ we consider the operator

$$\tilde{P}_\nu = P_\nu + \nu^2 \mathbb{1}_{[-|\nu|^{-\rho}, |\nu|^{-\rho}]},$$

(with domain $\text{Dom}(\tilde{P}_\nu) = H^2(I) \cap H_0^1(I)$). Since $q(x)^2 + \mathbb{1}_{[-|\nu|^{-\rho}, |\nu|^{-\rho}]} \geq c_0^2 |\nu|^{-2\rho}$ on I we have for $u \in \text{Dom}(\tilde{P}_\nu)$

$$\text{Re}(e^{-i\theta} \langle \tilde{P}_\nu u, u \rangle) \geq \cos(\theta) c_0^2 |\nu|^{2-2\rho} \|u\|_{L^2(I)}^2.$$

By the Lax-Milgram Theorem and elliptic regularity we obtain that any $z \in \mathcal{Z}_\nu$ is in the resolvent set of \tilde{P}_ν with

$$(4.8) \quad \|(\tilde{P}_\nu - z)^{-1}\|_{\mathcal{L}(L^2(I))} \leq \frac{|\nu|^{2\rho-2}}{\cos(\theta)c_0^2}.$$

Then we have

$$(\tilde{P}_\nu - z)^{-1}(1 - \chi_\nu)(P_\nu - z)u = (\tilde{P}_\nu - z)^{-1}(1 - \chi_\nu)(\tilde{P}_\nu - z)u \\ = (\tilde{P}_\nu - z)^{-1}(\tilde{P}_\nu - z)(1 - \chi_\nu)u + (\tilde{P}_\nu - z)^{-1}[1 - \chi_\nu, \tilde{P}_\nu]u \\ = (1 - \chi_\nu)u - 2(\tilde{P}_\nu - z)^{-1}(\chi'_\nu u)' + (\tilde{P}_\nu - z)^{-1}\chi''_\nu u.$$

As above we estimate

$$\|(\tilde{P}_\nu - z)^{-1}\chi''_\nu\|_{\mathcal{L}(L^2(I))} \lesssim |\nu|^{4\rho-2} \xrightarrow{|\nu| \rightarrow +\infty} 0$$

and

$$\|(\tilde{P}_\nu - z)^{-1}(\chi'_\nu u)'\|_{L^2(I)} \lesssim |\nu|^{\rho-1} \|\chi'_\nu u\|_{L^2(I)} \lesssim |\nu|^{2\rho-1} \|u\|_{L^2(I)}.$$

This proves that

$$(4.9) \quad \sup_{z \in \mathcal{Z}_\nu} \left\| (\tilde{P}_\nu - z)^{-1}(1 - \chi_\nu)(P_\nu - z) - (1 - \chi_\nu) \right\|_{\mathcal{L}(L^2(I))} \xrightarrow{|\nu| \rightarrow +\infty} 0.$$

Step 3: Conclusion. For $\nu \in \Sigma_{\theta_0}$ and $z \in \mathcal{Z}_\nu$ we set

$$Q_\nu(z) = R_\nu(z)\chi_\nu + (\tilde{P}_\nu - z)^{-1}(1 - \chi_\nu).$$

By eq. (4.4) and eq. (4.9) we have for $u \in D(P_\nu)$

$$\begin{aligned} & \|Q_\nu(z)(P_\nu - z)u - u\|_{L^2(I)} \\ & \leq \|R_\nu(z)\chi_\nu(P_\nu - z)u - \chi_\nu u\|_{L^2(I)} + \|(\tilde{P}_\nu - z)^{-1}(1 - \chi_\nu)(P_\nu - z)u - (1 - \chi_\nu)u\|_{L^2(I)} \\ & = \underset{|\nu| \rightarrow +\infty}{o}(1) \|u\|_{L^2(I)}. \end{aligned}$$

This proves that for $|\nu|$ large enough, the operator $Q_\nu(z)(P_\nu - z) = 1 + (Q_\nu(z)(P_\nu - z) - 1)$ extends to a bounded operator on $L^2(I)$, which is invertible with inverse bounded uniformly in ν , and

$$(4.10) \quad \left\| (Q_\nu(z)(P_\nu - z))^{-1} - 1 \right\|_{\mathcal{L}(L^2(I))} \xrightarrow{|\nu| \rightarrow +\infty} 0.$$

In particular, $(P_\nu - z)$ is injective. Since it has compact resolvent, it is boundedly invertible, and

$$(P_\nu - z)^{-1} = (Q_\nu(z)(P_\nu - z))^{-1}Q_\nu(z).$$

We get

$$(4.11) \quad \begin{aligned} (P_\nu - z)^{-1} - R_\nu(z) &= (Q_\nu(z)(P_\nu - z))^{-1}(\tilde{P}_\nu - z)^{-1}(1 - \chi_\nu) \\ &\quad - (Q_\nu(z)(P_\nu - z))^{-1}R_\nu(z)(1 - \chi_\nu) \\ &\quad + \left((Q_\nu(z)(P_\nu - z))^{-1} - 1 \right) R_\nu(z). \end{aligned}$$

For the first term we use eq. (4.8). For the third we use eq. (4.10) and eq. (4.3). Finally, for the second term we observe that on $\text{supp}(1 - \chi_\nu)$ we have $|x| \gtrsim |\nu|^{-\rho}$ so for $u \in L^2(I)$ we have by the second inequality of eq. (4.7)

$$\left\| (1 - \chi_\nu) \mathbb{1}_I (H_{q'(0)\bar{\nu}} - \bar{z})^{-1} \mathbb{1}_I^* u \right\|_{L^2(I)}^2 \lesssim |\nu|^{2\rho} \|x(H_{q'(0)\bar{\nu}} - \bar{z})^{-1} \mathbb{1}_I^* u\|_{L^2(\mathbb{R})}^2 \lesssim |\nu|^{2\rho-3} \|u\|_{L^2(I)}^2.$$

Taking the adjoint gives

$$\left\| \mathbb{1}_I H_{q'(0)\nu}^{-1} \mathbb{1}_I^* (1 - \chi_\nu) \right\|_{\mathcal{L}(L^2(I))} = \underset{\nu \rightarrow +\infty}{o} \left(\frac{1}{|\nu|} \right),$$

and the conclusion follows from eq. (4.11). \square

Proposition 4.2. *For $\nu \in \Sigma_{\theta_0}^*$ large enough there exists an eigenvalue λ_ν of P_ν such that*

$$|\lambda_\nu - q'(0)\nu| = \underset{|\nu| \rightarrow +\infty}{o}(|\nu|).$$

If we denote by $\Pi_\nu \in \mathcal{L}(L^2(I))$ the associated spectral projector, then Π_ν has rank one. Moreover, if we denote the spectral projection associated to the eigenvalue $q'(0)\nu$ of the complex harmonic oscillator $H_{q'(0)\nu}$ by $\Pi_{q'(0)\nu}^H \in \mathcal{L}(L^2(\mathbb{R}))$, we have

$$\left\| \Pi_\nu - \mathbb{1}_I \Pi_{q'(0)\nu}^H \mathbb{1}_I^* \right\|_{\mathcal{L}(L^2(I))} \xrightarrow[\nu \in \Sigma_{\theta_0}]{|\nu| \rightarrow +\infty} 0.$$

Proof. Let ν_0 be given by Proposition 4.1 for $\varepsilon = 1$ and $\gamma = 2q'(0)$. For $\nu \in \Sigma_{\theta_0}$ with $|\nu| \geq \nu_0$ we set

$$B_\nu = \frac{1}{2i\pi} \int_{|z - q'(0)\nu| = |\nu|} (P_\nu - z)^{-1} dz.$$

By Proposition 4.1 we have

$$\left\| B_\nu - \mathbb{1}_I \Pi_{q'(0)\nu}^H \mathbb{1}_I^* \right\|_{\mathcal{L}(L^2(I))} \xrightarrow{|\nu| \rightarrow +\infty} 0.$$

In particular, there exists $\nu_1 \geq \nu_0$ such that if $|\nu| \geq \nu_1$ we have

$$\left\| B_\nu - \mathbb{1}_I \Pi_{q'(0)\nu}^H \mathbb{1}_I^* \right\|_{\mathcal{L}(L^2(I))} \leq \frac{1}{2}.$$

By [25, §I.4.6] this implies that

$$\dim(\text{Ran}(B_\nu)) = \dim(\text{Ran}(\Pi_{q'(0)\nu}^{\text{H}})) = 1.$$

This proves that for $|\nu| \geq \nu_1$ the operator P_ν has a unique simple eigenvalue λ_ν in the disk $D(q'(0)\nu, |\nu|)$, and B_ν is the corresponding spectral projection Π_ν . It remains to prove the estimate on λ_ν . For this we reproduce the same argument with any $\varepsilon \in]0, 1]$. Then, given $\varepsilon \in]0, 1]$, there exists $\nu_\varepsilon \geq \nu_1$ such that the projection

$$\frac{1}{2i\pi} \int_{|z - q'(0)\nu| = \varepsilon|\nu|} (P_\nu - z)^{-1} dz$$

is well defined and has rank 1. This implies that λ_ν belongs to $D(q'(0)\nu, \varepsilon|\nu|)$ when $|\nu| \geq \nu_\varepsilon$ and concludes the proof. \square

4.2. Agmon Estimates of Eigenfunctions.

Proposition 4.3. *Let $\varepsilon > 0$. For $\text{Re}(\nu) > 0$ such that proposition 4.2 ensures the existence of the eigenvalue λ_ν close to $q'(0)\nu$, let φ_ν be an eigenfunction associated to λ_ν . For $\nu = |\nu|e^{i\theta}$, set*

$$\mu_\nu := \frac{\text{Re}(e^{-i\theta}\lambda_\nu)}{\cos(\theta)}.$$

Let $K \in C^2(I; \mathbb{R})$ and $w_\nu := e^{\nu K(x)}\varphi_\nu(x)$. The following Agmon equality holds:

$$\|w'_\nu\|_{L^2(I)}^2 + \int_I (|\nu|^2(q(x)^2 - K'^2(x)) - \mu_\nu) |w_\nu(x)|^2 dx = 0.$$

Proof. Let $\nu = |\nu|e^{i\theta}$. The proof follows the usual Agmon's equality strategy. We have $\varphi_\nu = e^{-\nu K}w_\nu$. Thus, using Leibniz' formula,

$$\begin{aligned} 0 &= (P_\nu - \lambda_\nu)\varphi_\nu \\ &= -(e^{-\nu K})''w_\nu - 2(e^{-\nu K})'w'_\nu - e^{-\nu K}w''_\nu + (\nu^2q^2 - \lambda_\nu)e^{-\nu K}w_\nu \\ &= (-w''_\nu + 2\nu K'w'_\nu + (-\nu^2K'^2 + \nu K'' + \nu^2q^2 - \lambda_\nu)w_\nu) e^{-\nu K}. \end{aligned}$$

Thus,

$$(4.12) \quad -w''_\nu + 2\nu K'w'_\nu + (-\nu^2K'^2 + \nu K'' + \nu^2q^2 - \lambda_\nu)w_\nu = 0.$$

Multiplying by $e^{-i\theta}\overline{w_\nu}$, integrating and taking the real part, we get

$$0 = \underbrace{-\text{Re} \int_I e^{-i\theta} w''_\nu \overline{w_\nu}}_{I_1} + \underbrace{|\nu| \text{Re} \int_I (2K'w'_\nu \overline{w_\nu} + K''|w_\nu|^2)}_{I_2} + \underbrace{\int_I (\cos(\theta)|\nu|^2(q^2 - K'^2) - \text{Re}(e^{-i\theta}\lambda_\nu)) |w_\nu|^2}_{I_3}.$$

Integrating by parts in I_1 , we have $I_1 = \cos(\theta)\|w'_\nu\|_{L^2(I)}^2$. Also integrating by parts in I_2 , we get

$$\begin{aligned} I_2 &= |\nu| \int_I (K'(|w_\nu|^2)' + K''|w_\nu|^2) \\ &= |\nu| \int_I (-K''|w_\nu|^2 + K''|w_\nu|^2) \\ &= 0. \end{aligned}$$

Thus,

$$0 = I_1 + I_3 = \cos(\theta)\|w'_\nu\|_{L^2(I)}^2 + \int_I (\cos(\theta)|\nu|^2(q^2 - K'^2) - \text{Re}(e^{-i\theta}\lambda_\nu)) |w_\nu|^2,$$

which is the claimed estimate (multiplied by $\cos(\theta)$). \square

We will use proposition 4.3 with $K = (1 - \varepsilon)d_{\text{Agm}}$, where d_{Agm} defined in eq. (1.3). Up to this point, we assumed φ_ν to be an eigenfunction of P_ν associated to the first eigenvalue, but we did not specified how it is normalized. We do this in the following definition, which is the natural extension of the definition of φ_n when $n \in \mathbb{N}$ (eq. (3.18))

Definition 4.4. For $\operatorname{Re}(\nu) > 0$ let $\tilde{\varphi}_\nu \in L^2(I)$ be defined by

$$\tilde{\varphi}_\nu(x) := \nu^{1/4} e^{-q'(0)\nu x^2/2}.$$

Set also

$$\varphi_\nu := \Pi_\nu(\tilde{\varphi}_\nu),$$

where Π_ν is the spectral projector for P_ν associated with λ_ν .

Proposition 4.5. *Let φ_ν as in definition 4.4. Let $\theta_0 \in (0, \pi/2)$. There exists $C = C_{\theta_0} > 0$ such that for every $\nu \in \Sigma_{\theta_0}$ with $|\nu| > 1$,*

$$\begin{aligned} \|\tilde{\varphi}_\nu\|_{L^2(I)} &\leq C; \\ \|\varphi_\nu\|_{L^2(I)} &\leq C. \end{aligned}$$

Proof. Step 1: Estimation on $\tilde{\varphi}_\nu$. Since $\tilde{\varphi}_\nu$ is a restriction of $\nu^{1/4} e^{-\nu q'(0)x^2/2}$ on \mathbb{R} , we have

$$(4.13) \quad \|\tilde{\varphi}_\nu\|_{L^2(I)}^2 \leq |\nu|^{1/2} \int_{\mathbb{R}} e^{-2\operatorname{Re}(\nu)q'(0)x^2/2} dx = \sqrt{\frac{\pi|\nu|}{q'(0)\operatorname{Re}(\nu)}} \leq \sqrt{\frac{\pi}{q'(0)\cos(\theta_0)}}.$$

Step 2: Estimation on φ_ν . Using the notations of proposition 4.2, we have

$$(4.14) \quad \|\varphi_\nu\|_{L^2(I)} \leq \left(\|\mathbf{1}_I \tilde{\Pi}_{q'(0)\nu} \mathbf{1}_I^* \|_{\mathcal{L}(L^2(I))} + \|\Pi_\nu - \mathbf{1}_I \tilde{\Pi}_{q'(0)\nu} \mathbf{1}_I^* \|_{\mathcal{L}(L^2(I))} \right) \|\tilde{\varphi}_\nu\|_{L^2(I)}.$$

Moreover, according to proposition 4.2, for $|\nu|$ large enough in Σ_{θ_0} ,

$$\|\Pi_\nu - \mathbf{1}_I \tilde{\Pi}_{q'(0)\nu} \mathbf{1}_I^* \|_{\mathcal{L}(L^2(I))} \leq 1.$$

Since the left-hand side is continuous in ν , we have that for $|\nu| > 1$ and $\nu \in \Sigma_{\theta_0}$,

$$\|\Pi_\nu - \mathbf{1}_I \tilde{\Pi}_{q'(0)\nu} \mathbf{1}_I^* \|_{\mathcal{L}(L^2(I))} \leq C_{\theta_0}.$$

Finally, according to proposition A.1, $\|\tilde{\Pi}_{q'(0)\nu}\|_{\mathcal{L}(L^2)}$ stays bounded for $\nu \in \Sigma_{\theta_0}^*$, thus

$$\|\mathbf{1}_I \tilde{\Pi}_{q'(0)\nu} \mathbf{1}_I^* \|_{\mathcal{L}(L^2(I))} + \|\Pi_\nu - \mathbf{1}_I \tilde{\Pi}_{q'(0)\nu} \mathbf{1}_I^* \|_{\mathcal{L}(L^2(I))} \leq C_{\theta_0}.$$

Plugging this fact into eq. (4.14) and combined with the fact that $\|\tilde{\varphi}_\nu\|_{L^2(I)}$ is bounded (eq. (4.13)), we get the claimed estimate. \square

Corollary 4.6. *Let φ_ν as in definition 4.4. Let $\theta_0 \in (0, \pi/2)$ and $\varepsilon \in (0, 1)$. There exists $C = C_{\theta_0, \varepsilon} > 0$ such that for every $\nu \in \Sigma_{\theta_0}$,*

$$\begin{aligned} \int_I |\varphi_\nu(x) e^{\nu(1-\varepsilon) d_{\text{Agm}}(x)}|^2 dx &\leq C|\nu|. \\ \|\varphi_\nu e^{\nu(1-\varepsilon) d_{\text{Agm}}(x)}\|_{L^\infty(I)} &\leq C|\nu|. \end{aligned}$$

Proof. Let $w_\nu(x) := \varphi_\nu(x) e^{\nu(1-\varepsilon) d_{\text{Agm}}(x)}$. According to the Agmon equality of proposition 4.3, we have

$$(4.15) \quad \|w'_\nu\|_{L^2(I)}^2 + \int_I (|\nu|^2 q(x)^2 (1 - (1 - \varepsilon)^2) - \mu_\nu) |w_\nu(x)|^2 dx = 0.$$

Step 1: First inequality. Let $K > 0$. We claim that if $x \in I$, $\nu \in \Sigma_{\theta_0}$ and $|\nu| d_{\text{Agm}}(x) > K$, then $|\nu|^2 q(x)^2 > cK|\nu|$ for some c depending on q , but not on $x \in I$, $\nu \in \Sigma_{\theta_0}$ nor K .

Indeed, routine computations show that in the limit $x \rightarrow 0$, $q(x)^2 \sim q'(0)^2 x^2$ and $d_{\text{Agm}}(x) \sim q'(0)x^2/2$. Thus, if $c_1 > 0$ is small enough, for every $|x| \leq c_1$, $q(x)^2 \geq q'(0)^2 x^2/2$ and $d_{\text{Agm}}(x) \leq q'(0)x^2$. Moreover, since $q(x) = 0$ only at $x = 0$, for every $|x| > c_1$, $q(x)^2 \geq c_2 > 0$. Then, if $d_{\text{Agm}}(x)|\nu| \geq K$, we consider two cases: $|x| \leq c_1$ and $|x| > c_1$. In the second case, $q(x)^2 \geq c_2$ thus $|\nu|^2 q(x)^2 \geq c_2 |\nu|^2$. In the case $|x| \leq c_1$, we have $q(x)^2 \geq q'(0)^2 x^2/2 \geq q'(0) d_{\text{Agm}}(x)/2$, thus, $|\nu|^2 q(x)^2 \geq |\nu| K q'(0)/2$, proving the claim.

Thus, if K is large enough, then for every $x \in I$ such that $d_{\text{Agm}}(x)|\nu| > K$

$$|\nu|^2 q(x)^2 (1 - (1 - \varepsilon)^2) - \mu_\nu \geq c|\nu|,$$

where we used the fact that $\mu_\nu \sim q'(0)|\nu|/\cos(\arg(\nu))$ because $\lambda_\nu = q'(0)\nu + o(\nu)$. Then, splitting the integral in the Agmon equality (4.15) into a part for $|\nu| d_{\text{Agm}}(x) < K$ and $|\nu| d_{\text{Agm}}(x) > K$, we get

$$\begin{aligned} c|\nu| \|w_\nu\|_{L^2(|\nu| d_{\text{Agm}}(x) > K)}^2 &\leq \int_{|\nu| d_{\text{Agm}}(x) < K} (\mu_\nu - |\nu|^2 q(x)^2 (1 - (1 - \varepsilon)^2)) |w_\nu(x)|^2 dx \\ &\leq C|\nu|^2 \|w_\nu\|_{L^2(|\nu| d_{\text{Agm}}(x) < K)}^2. \end{aligned}$$

We rewrite this as

$$\|w_\nu\|_{L^2(|\nu| d_{\text{Agm}}(x) > K)}^2 \leq C|\nu| \|w_\nu\|_{L^2(|\nu| d_{\text{Agm}}(x) < K)}^2.$$

Adding $\|w_\nu\|_{L^2(|\nu| d_{\text{Agm}}(x) < K)}^2$ on each side, this proves that

$$\|w_\nu\|_{L^2(I)}^2 \leq C|\nu| \|w_\nu\|_{L^2(|\nu| d_{\text{Agm}}(x) < K)}^2.$$

Using the definition of w_ν , we see that for $|\nu| d_{\text{Agm}}(x) < K$, $|w_\nu(x)| \leq e^K |\varphi_\nu(x)|$. Thus, using also the property $\|\varphi_\nu\|_{L^2(I)} \leq C$ (proposition 4.5)

$$\begin{aligned} \|w_\nu\|_{L^2(I)}^2 &\leq C|\nu|^2 \|\varphi_\nu\|_{L^2(|\nu| d_{\text{Agm}}(x) < K)}^2 \\ &\leq C|\nu|. \end{aligned}$$

Step 2: Second inequality. We again use Agmon's equality (4.15) to get

$$\|w'_\nu\|_{L^2(I)}^2 \leq \mu_\nu \|w_\nu\|_{L^2(I)}^2 \leq C|\nu| \|w_\nu\|_{L^2(I)}^2 \leq C|\nu|^2.$$

The claimed estimate then follows from Sobolev's embedding of $H^1(I)$ into $L^\infty(I)$. \square

We also prove the lower bound of proposition 3.6 for φ_n when $n \geq 0$ is large enough.⁹

Proof of proposition 3.6. The part about λ_n was already proven in proposition 4.2. By definition of φ_n , we have

$$\varphi_n = \Pi_n \tilde{\varphi}_n = \mathbf{1}_I \Pi_{q'(0)n}^H \mathbf{1}_I^* \tilde{\varphi}_n + (\Pi_n - \mathbf{1}_I \Pi_{q'(0)n}^H \mathbf{1}_I^*) \tilde{\varphi}_n.$$

According to proposition 4.2, we have

$$\Pi_n - \mathbf{1}_I \Pi_{q'(0)n}^H \mathbf{1}_I^* \xrightarrow{n \rightarrow +\infty} 0.$$

Moreover, denoting by $\varphi_{\beta,1}^H(x) = (\text{Re}(\beta)/\pi)^{1/4} e^{-\beta x^2}$ the first eigenvector of H_β , we have for $\beta > 0$, $\Pi_\beta^H = \langle \varphi_{\beta,1}, \cdot \rangle \varphi_{\beta,1}^H$. Thus,

$$\langle \varphi_{q'(0)n,1}^H, \mathbf{1}_I^* \tilde{\varphi}_n \rangle = \sqrt{n} \left(\frac{q'(0)}{\pi} \right)^{1/4} \int_I e^{-nq'(0)x^2} dx.$$

The integral above is on I , but if we integrate on \mathbb{R} instead, we only add a small error term. Thus,

$$\langle \varphi_{q'(0)n,1}^H, \mathbf{1}_I^* \tilde{\varphi}_n \rangle = \left(\frac{\pi}{q'(0)} \right)^{1/4} + o_{n \rightarrow +\infty}(1).$$

Hence,

$$\varphi_n = \left(\frac{\pi}{q'(0)} \right)^{1/4} \mathbf{1}_I \varphi_{q'(0)n,1}^H + o_{n \rightarrow +\infty}(\|\tilde{\varphi}_n\|_{L^2(I)} + 1).$$

Since $\|\tilde{\varphi}_n\|_{L^2(I)}$ is bounded (proposition 4.5) and since $\|\mathbf{1}_I \varphi_{q'(0)n,1}^H\|_{L^2(I)} = 1 + o(1)$ (thanks to similar computations as above), this proves the claimed lower bound. \square

⁹This theorem actually holds if ν ranges in Σ_{θ_0} by using the expression of Π_β^H . We don't need this, so we refrain from doing so.

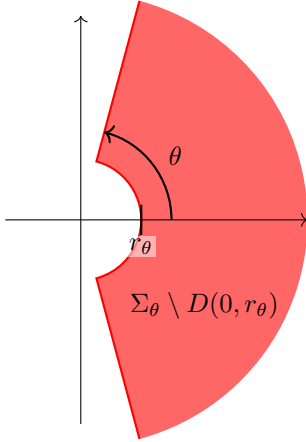


FIGURE 9. An example of a set $\Sigma_\theta \setminus D(0, r_\theta) \subset \Omega$. The angle θ is allowed to be arbitrarily close to $\pi/2$, but then, the radius r_θ of the disk we avoid may blow up arbitrarily fast. For the Ω we will consider, the corresponding r_θ does blow up when $\theta \rightarrow \pi/2$ (at least, we cannot exclude that it blows up).

4.3. Estimate for some pseudo-differential type operators on polynomials. In this section, we use the spectral analysis of the operator P_ν to deduce the operator estimate of lemma 3.5. In order to do that, we need some definitions and theorems about a general class of operators on polynomials. The following comes from [27, definition 9 & theorem 18].

Definition 4.7. Let Ω be an open subset of \mathbb{C} such that for every $\theta \in (0, \pi/2)$, there exists $r_\theta \geq 0$ such that $\Sigma_\theta \setminus D(0, r_\theta) \subset \Omega$ (see fig. 9).

We denote by $\mathcal{S}(\Omega)$ the set of functions γ holomorphic on Ω that have sub-exponential growth on each $\Sigma_\theta \cap \Omega$, i.e., for each $\theta \in (0, \pi/2)$ and $\delta > 0$, we have

$$p_{\theta, \delta}(\gamma) := \sup_{z \in \Sigma_\theta \cap \Omega} |\gamma(z)e^{-\delta|z|}| < +\infty.$$

We endow $\mathcal{S}(\Omega)$ with the topology defined by the seminorms $p_{\theta, \delta}$ for all $\theta \in (0, \pi/2)$ and $\delta > 0$.

For the next theorem, if U is an open subset of \mathbb{C} , we denote the set of bounded holomorphic functions on U that have a zero of order n_0 at 0 by $\mathcal{O}_{n_0}^\infty(U)$. We endow $\mathcal{O}_{n_0}^\infty(U)$ with the L^∞ -norm.

Theorem 4.8. Let $\Omega \subset \mathbb{C}$ as in definition 4.7 and set $n_0 = \min \min\{n \in \mathbb{N} : [n, +\infty) \subset \Omega\}$. Let γ in $\mathcal{S}(\Omega)$ and $\gamma(z\partial_z)$ be the operator on polynomials with a zero of order n_0 at 0, defined by:

$$\gamma(z\partial_z) \left(\sum_{n \geq n_0} a_n z^n \right) = \sum_{n \geq n_0} \gamma(n) a_n z^n.$$

Let U be a bounded open subset of \mathbb{C} . Let V be a neighborhood of \bar{U} that is star shaped with respect to 0. Then there exists $C > 0$ such that for all polynomials p with a zero of order n_0 at 0,

$$\|\gamma(z\partial_z)(p)\|_{L^\infty(U)} \leq C \|p\|_{L^\infty(V)}.$$

Moreover, the constant C above can be chosen continuously in $\gamma \in \mathcal{S}(\Omega)$: the map $\gamma \in \mathcal{S}(\Omega) \mapsto \gamma(z\partial_z)$ is continuous from $\mathcal{S}(\Omega)$ to $\mathcal{L}(\mathcal{O}_{n_0}^\infty(V), \mathcal{O}_{n_0}^\infty(U))$.

We now have all the pieces needed to prove lemma 3.5.

Proof of lemma 3.5. Let $\theta_0 \in (0, \pi/2)$. According to proposition 4.2 there exists r_{θ_0} such that if $|\nu| > r_{\theta_0}$ and $|\arg(\nu)| < \theta_0$, then there exists a unique eigenvalue λ_ν of P_ν close to $q'(0)\nu$. Moreover, this eigenvalue is algebraically and geometrically simple.

Set $\Omega = \bigcup_{\theta_0 \in (0, \pi/2)} \Sigma_{\theta_0} \setminus D(0, r_{\theta_0})$. Notice that by definition, Ω satisfies the property of definition 4.7. For $\nu \in \Omega$, we define

$$\gamma_{t,x}(\nu) := e^{-t(\lambda_\nu - q'(0)\nu)} \varphi_\nu(x) e^{\nu d_{\text{Agm}}(x)(1-\varepsilon)},$$

which is the natural extension of the definition of $\gamma_{t,x}(n)$ when $n \in \mathbb{N}$.

Step 1: The family $(\gamma_{t,x})_{t,x}$ is a bounded family¹⁰ of $\mathcal{S}(\Omega)$. According to proposition 4.2, λ_ν is algebraically simple on Ω . Thus, according to analytic perturbation theory (see, e.g., [25, Chapter VII, §1]), λ_ν and the associated spectral projector are holomorphic in $\nu \in \mathbb{C}$, $\operatorname{Re}(\nu) > 0$. Since $\varphi_\nu = \Pi_\nu(\nu^{1/4}e^{-\nu q'(0)x^2/2})$, φ_ν is holomorphic in $\nu \in \Omega$. We conclude that $\gamma_{t,x}(\nu)$ is holomorphic in $\nu \in \Omega$.

We still have to prove that $(\gamma_{t,x})_{t,x}$ is a bounded family of $\mathcal{S}(\Omega)$. Let us set

$$\begin{aligned}\eta_t(\nu) &:= e^{-t(\lambda_\nu - q'(0)\nu)}; \\ \zeta_x(\nu) &:= \varphi_\nu(x)e^{\nu d_{\operatorname{Agm}}(x)(1-\varepsilon)},\end{aligned}$$

and prove that both of the families $(\eta_t)_{0 < t < T}$ and $(\zeta_x)_{x \in I}$ are bounded in $\mathcal{S}(\Omega)$. It is easy to see that $(\gamma_1, \gamma_2) \in (\mathcal{S}(\Omega))^2 \mapsto \gamma_1 \gamma_2 \in \mathcal{S}(\Omega)$ is bounded, so this will prove the claim.

Let $\theta_0 \in (0, \pi/2)$ and $\delta > 0$. According to proposition 4.2, we have in the limit $|\nu| \rightarrow +\infty$, $\nu \in \Sigma_{\theta_0}$, $\lambda_\nu - \nu q'(0) = o(\nu)$. Thus, for ν large enough in Σ_{θ_0} ,

$$|\eta_t(\nu)e^{-\delta|\nu|}| = |e^{t o(|\nu|) - \delta|\nu|}| < e^{TC_{\theta_0, \delta}}.$$

Thus, $(\eta_t)_{0 < t < T}$ is a bounded family of $\mathcal{S}(\Omega)$.

Similarly, according to corollary 4.6, we have for any $x \in I$, and ν large enough in Σ_{θ_0} ,

$$|\zeta_x(\nu)e^{-\delta|\nu|}| \leq C.$$

This prove that $(\zeta_x)_{x \in I}$ is a bounded family of $\mathcal{S}(\Omega)$.

Step 2: Conclusion. Let $n_0 = \min\{n \in \mathbb{N} : [n, +\infty) \subset \Omega\}$.¹¹ Let U be a bounded open neighborhood of X such that $\bar{U} \subset V$. Then, the sets U and V satisfy the hypotheses of theorem 4.8. Hence, according to theorem 4.8, there exists $C > 0$ such that for every polynomials p with a zero of order n_0 at 0, and for every $x \in I$ and $0 < t < T$,

$$\|\gamma_{t,x}(z\partial_z)(p)\|_{L^\infty(U)} \leq C\|p\|_{L^\infty(V)}. \quad \square$$

5. CRITICAL TIME OF NULL-CONTROLLABILITY FOR SOME DOMAINS

In this section, we prove theorem 1.5.

Proof of theorem 1.5. Set $a_- = -\max(\gamma_2^-)$ and $a_+ = \max(\gamma_1^+)$. Denote by $y_- \in \mathbb{T}$ and $y_+ \in \mathbb{T}$ points where these maxima are reached.

Step 1: Lower bound of the minimal time. For this step, we only have to treat the case $T_* > 0$. In this case, either $a_- < 0$ or $a_+ > 0$. If $a_- < 0$, for any $a_- < a < 0$, the segment $[a, L_+] \times \{y_-\}$ stays at positive distance of ω , and thanks to theorem 3.1, the generalized Baouendi-Grushin equation (3.2) is not null-controllable on ω in time $T < d_{\operatorname{Agm}}(a)/q'(0)$. Similarly, if $a_+ > 0$, for any $0 < a < a_+$, the segment $[-L_-, a] \times \{y_+\}$ stays at positive distance from ω , and the generalized Baouendi-Grushin equation is not null-controllable in time $T < d_{\operatorname{Agm}}(a)/q'(0)$.

This holds for any $a_- < a < 0$ and $0 < a < a_+$, thus the generalized Baouendi-Grushin equation (3.2) is not null-controllable in time $T < T_*$.

Step 2: Upper bound of the minimal time. Let $\varepsilon > 0$ small enough so that $\gamma_2 - \gamma_1 > \varepsilon$. Let $\tilde{\gamma}_1 = \max(\gamma_1, a_- - \varepsilon)$ and $\tilde{\gamma}_2 = \min(\gamma_2, a_+ + \varepsilon)$. By using the information $\gamma_2 - \gamma_1 > \varepsilon$, $a_- \leq \gamma_2$, $\gamma_1 \leq a_+$ and by looking at the different cases, we readily get $\tilde{\gamma}_2 - \tilde{\gamma}_1 \geq \varepsilon$. Then, we define the path

$$\gamma = (\gamma_x, \gamma_y): s \in \mathbb{T} \mapsto \left(\frac{\tilde{\gamma}_1(s) + \tilde{\gamma}_2(s)}{2}, s \right).$$

¹⁰Let us recall that if E is locally convex vector space whose topology is generated by a family $(p_\iota)_\iota$ of seminorms, a subset X of E is bounded if and only if for every ι , the set $\{p_\iota(x), x \in X\}$ is a bounded subset of \mathbb{R} .

¹¹In fact, we can be more precise in the the construction of Ω and ensure that $\mathbb{R}_+ \subset \Omega$, in which case $n_0 = 0$. Indeed, the spectral theory of compact operators proves that for every $\nu \in \mathbb{C}$, the spectrum of P_ν is a discrete sequence of eigenvalues. The uniqueness of the solution of Cauchy problems for ODEs proves that when $\nu > 0$, these eigenvalues are actually geometrically and algebraically simple. Finally, perturbation theory proves that the first eigenvalue in holomorphic on the neighborhood of \mathbb{R}_+ . We do not need this, so we do not detail this.

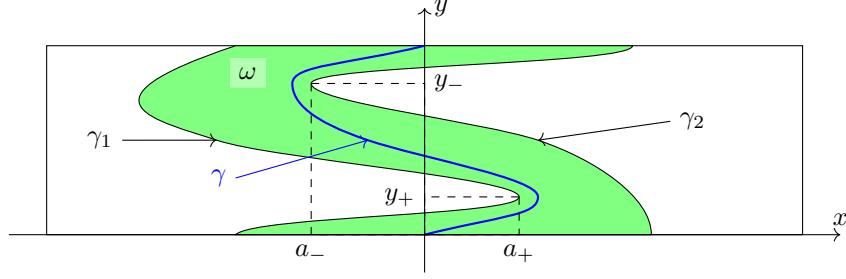


FIGURE 10. In green, the domain ω . At $y = y_-$, the function γ_2^- takes its maximum a_- . Then, the interval $(a_-, L_+) \times \{y_-\}$ is disjoint from $\bar{\omega}$. So, the Grushin equation is not null-controllable in time $T < d_{\text{Agm}}(a_-)/q'(0)$. Similarly, the interval $(-L_-, a_+) \times \{y_+\}$ is disjoint from $\bar{\omega}$. So, the Grushin equation is not null-controllable in time $T < d_{\text{Agm}}(a_+)/q'(0)$. Also, if we take a path γ (here in blue) that is close to the boundary of ω around $y = y_-$ and $y = y_+$, then, we can apply theorem 1.4, and the Grushin equation is null-controllable in time $T > \max(d_{\text{Agm}}(a_-), d_{\text{Agm}}(a_+))/q'(0)$.

This path satisfies $\gamma_1 + \varepsilon/2 \leq \gamma_x \leq \gamma_2 - \varepsilon/2$, so $\gamma(\mathbb{T}) \subset \omega$. Moreover, it is not homotopic to a constant path, because it is homotopic to $s \in \mathbb{T} \mapsto (0, s)$. Moreover, this path satisfies

$$a_- - \frac{\varepsilon}{2} \leq \gamma_x \leq a_+ + \frac{\varepsilon}{2}.$$

Thus, theorem 1.4 implies that the generalized Baouendi-Grushin equation (3.2) is null-controllable in time $T > \max(d_{\text{Agm}}(a_+ + \varepsilon/2), d_{\text{Agm}}(a_- - \varepsilon/2))/q'(0)$. As this holds for every $\varepsilon > 0$ small enough, the result follows. \square

APPENDIX A. NON-SELFADJOINT HARMONIC OSCILLATORS

Let \mathbb{C}_+ be the complex right half-plane $\{\beta \in \mathbb{C} : \text{Re}(\beta) > 0\}$. Let $\beta \in \mathbb{C}_+$. We discuss in this appendix the basic properties of the non-selfadjoint harmonic oscillator (or Davies operator) defined on $L^2(\mathbb{R})$ by

$$(A.1) \quad H_\beta = -\partial_x^2 + \beta^2 x^2.$$

More precisely, we set

$$\text{Dom}(H_\beta) = \{u \in L^2(\mathbb{R}) : -\partial_x^2 u + \beta^2 x^2 u \in L^2(\mathbb{R})\},$$

where $-\partial_x^2 u + \beta^2 x^2 u$ is understood in the sense of distributions, and we define H_β by (A.1) on $\text{Dom}(H_\beta)$. This defines an unbounded operator on $L^2(\mathbb{R})$. When $\beta = 1$ we recover the usual harmonic oscillator.

The spectral properties of the Davies operator has been studied extensively (see, among others, [17, §14.5], [23, §14.4], [28], and the references therein), and the properties stated in this appendix are standard, at least in spirit. Nevertheless, for the reader convenience, we collect and prove the properties needed in our study.

Proposition A.1. *Let $\beta \in \mathbb{C}_+$.*

- (i) *The operator H_β is closed with compact resolvent.*
- (ii) *The adjoint of H_β is $H_\beta^* = H_{\bar{\beta}}$.*
- (iii) *The spectrum of H_β is given by the sequence of (geometrically and algebraically) simple eigenvalues $(\lambda_{\beta,k}^H)_{k \in \mathbb{N}^*}$ where*

$$\lambda_{\beta,k}^H = (2k - 1)\beta.$$

An eigenfunction associated to $\lambda_{\beta,k}^H$ is given by

$$\varphi_{\beta,k}^H(x) = h_{k-1}(\sqrt{\beta}x)e^{-\frac{\beta x^2}{2}}, \quad \text{where } h_{k-1}(x) = (-1)^{k-1} e^{x^2} \partial_x^{k-1}(e^{-x^2})$$

is the $(k-1)$ -th Hermite polynomial.

- (iv) Let $\theta_0 \in [0, \frac{\pi}{2})$. Let $\gamma > 0$, $\varepsilon > 0$ and \mathcal{Z}_β as defined by (4.2). Then there exists $C > 0$ such that if $\arg(\beta) \leq \theta_0$ and $z \in \mathcal{Z}_\beta$ then we have

$$(A.2) \quad \|(H_\beta - z)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \frac{C}{|\beta|}.$$

- (v) We denote by Π_β^H the spectral projection of H_β associated to the eigenvalue $\lambda_{\beta,1}^H = \beta$. Then we have

$$(A.3) \quad \|\Pi_\beta^H\|_{\mathcal{L}(L^2(\mathbb{R}))} = \sqrt{\frac{|\beta|}{\operatorname{Re}(\beta)}} = \frac{1}{\sqrt{\cos(\arg(\beta))}}.$$

Proof. Step 1: H_β has compact resolvent. The closedness of H_β is clear. Let $\theta = \arg(\beta) \in [-\theta_0, \theta_0]$. For $u \in \operatorname{Dom}(H_\beta)$ we have

$$(A.4) \quad \langle e^{-i\theta} H_\beta u, u \rangle_{L^2(\mathbb{R})} = e^{-i\theta} \|u'\|_{L^2(\mathbb{R})}^2 + |\beta|^2 e^{i\theta} \|xu\|_{L^2(\mathbb{R})}^2,$$

so $e^{-i\theta} H_\beta$ is sectorial with angle θ . In particular, $(e^{-i\theta} H_\beta + 1)$ is injective. Now let

$$B^1(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : xu \in L^2(\mathbb{R})\}.$$

This is a Hilbert space for the natural norm

$$\|u\|_{B^1(\mathbb{R})}^2 = \|u\|_{H^1(\mathbb{R})}^2 + \|xu\|_{L^2(\mathbb{R})}^2.$$

For $u, v \in B^1(\mathbb{R})$ we set

$$Q_\beta(u, v) = e^{-i\theta} \langle u', v' \rangle_{L^2(\mathbb{R})} + |\beta|^2 e^{i\theta} \langle xu, xv \rangle_{L^2(\mathbb{R})} + \langle u, v \rangle_{L^2(\mathbb{R})}.$$

Let $f \in L^2(\mathbb{R})$. By the Lax-Milgram Theorem, there exists a unique $u \in B^1(\mathbb{R})$ such that $Q_\beta(u, v) = \langle f, v \rangle_{L^2(\mathbb{R})}$ for all $v \in B^1(\mathbb{R})$. In the sense of distributions we have $e^{-i\theta}(-u'' + \beta^2 x^2 u) + u = f \in L^2(\mathbb{R})$, so $u \in \operatorname{Dom}(H_\beta)$ and $(e^{-i\theta} H_\beta + 1)u = f$. This proves that $e^{i\theta}$ belongs to the resolvent set of H_β .

Finally, taking the real part of (A.4) gives for $u \in \operatorname{Dom}(H_\beta)$

$$\|u'\|_{L^2(\mathbb{R})}^2 + |\beta|^2 \|xu\|_{L^2(\mathbb{R})}^2 \leq \frac{\operatorname{Re} \langle e^{-i\theta} H_\beta u, u \rangle}{\cos(\theta)} \leq \frac{1}{2 \cos(\theta)} \left(\|H_\beta u\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2 \right)$$

We deduce that $\operatorname{Dom}(H_\beta)$ is compactly embedded in $L^2(\mathbb{R})$. Since the resolvent set of H_β is not empty, it has compact resolvent. In particular, its spectrum consists of isolated eigenvalues of finite multiplicities.

Step 2: $(H_\beta)^ = H_{\bar{\beta}}$.* Let $v \in \operatorname{Dom}(H_{\bar{\beta}})$. For $u \in \operatorname{Dom}(H_\beta)$ we have $\langle H_\beta u, v \rangle = \langle u, H_{\bar{\beta}} v \rangle$, so $\operatorname{Dom}(H_{\bar{\beta}}) \subset \operatorname{Dom}(H_\beta^*)$ and $H_\beta^* = H_{\bar{\beta}}$ on $\operatorname{Dom}(H_{\bar{\beta}})$. Now let $v \in \operatorname{Dom}(H_\beta^*)$ and $f = H_\beta^* v$. Then $v \in L^2(\mathbb{R})$ and in the sense of distributions we have $-v'' + \bar{\beta}^2 x^2 v = f \in L^2(\mathbb{R})$, so $v \in \operatorname{Dom}(H_{\bar{\beta}})$. This proves that $\operatorname{Dom}(H_\beta^*) \subset \operatorname{Dom}(H_{\bar{\beta}})$, and hence $H_\beta^* = H_{\bar{\beta}}$.

Step 3: Eigenvalues and eigenfunctions of H_β , completeness of the eigenfunctions. For $k \in \mathbb{N}^*$ we have $\varphi_{b,k}^H \in \operatorname{Dom}(H_\beta)$ and it is classical computation that for $\beta = 1$, (see, e.g., [23, §1.3])

$$H_1 \varphi_{1,k}^H = \lambda_{1,k}^H \varphi_{1,k}^H$$

Noticing that $\varphi_{\beta,k}^H(x) = \varphi_{1,k}(\sqrt{\beta}x)$, routine computations show that

$$H_\beta \varphi_{\beta,k}^H = \lambda_{\beta,k}^H \varphi_{\beta,k}^H.$$

Then $\lambda_{\beta,k}^H$ is an eigenvalue of H_β and $\varphi_{\beta,k}^H$ is a corresponding eigenfunction.

Let $u \in \text{span}(\varphi_{\beta,k}^H)_{k \in \mathbb{N}^*}^\perp$. Then for all polynomial p we have $\int_{\mathbb{R}} u(x)p(\sqrt{\beta}x)e^{-\beta x^2} dx = 0$. For $\xi \in \mathbb{R}$ we set $F(\xi) = \int_{\mathbb{R}} e^{ix\xi} u(x)e^{-\beta x^2} dx$. Then F is analytic and $F^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. This implies that $u(x) = 0$ for almost all $x \in \mathbb{R}$, so the family $(\varphi_{\beta,k}^H)_{k \in \mathbb{N}^*}$ is complete.

Step 4: Resolvent estimate. Let $\theta_0 \in (0, \frac{\pi}{2})$. The map

$$(\theta, \zeta) \mapsto (H_{e^{i\theta}} - e^{i\theta} \zeta)^{-1}$$

is continuous and hence bounded on the compact $[-\theta_0, \theta_0] \times \mathcal{Z}_1$, so eq. (A.2) holds if $|\beta| = 1$.

For $\rho > 0$ we consider on $L^2(\mathbb{R})$ the unitary operator Θ_ρ such that for $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$(A.5) \quad (\Theta_\rho u)(x) = \rho^{\frac{1}{2}} u(\rho x).$$

We observe that

$$\Theta_{|\beta|^{\frac{1}{2}}}^{-1} H_\beta \Theta_{|\beta|^{\frac{1}{2}}} = |\beta| H_\omega,$$

and we deduce (A.2) in the general case.

Step 5: Spectral projector. In the integral $\langle \varphi_{\beta,k}^H, \varphi_{\beta,1}^H \rangle = \int_{\mathbb{R}} h_{k-1}(\sqrt{\beta}x)e^{-\beta x^2} dx$, we make the change of variables and integration path $x' = \sqrt{\beta}x$, which can be justified thanks to the gaussian decay of the integrand, and we find

$$\langle \varphi_{\beta,k}^H, \varphi_{\beta,1}^H \rangle = \frac{1}{\sqrt{\beta}} \int_{\mathbb{R}} h_{k-1}(x)e^{-x^2} dx = \frac{1}{\sqrt{\beta}} \langle \varphi_{1,k}^H, \varphi_{1,1}^H \rangle.$$

Since the functions $(\varphi_{1,k}^H)$ are eigenfunctions of the self-adjoint operator H_1 associated to different eigenvalues, we get that for $k \neq 1$, $\langle \varphi_{1,k}^H, \varphi_{1,1}^H \rangle = 0$. Moreover, with a direct computations, $\langle \varphi_{1,1}^H, \varphi_{1,1}^H \rangle = \sqrt{\pi}$:

$$\langle \varphi_{\beta,k}^H, \varphi_{\beta,1}^H \rangle = \begin{cases} \sqrt{\frac{\pi}{\beta}} & \text{if } k = 1; \\ 0 & \text{if } k \neq 1. \end{cases}$$

Thus, we can define

$$\tilde{\Pi}_\beta u = \frac{\langle \varphi_{\beta,1}^H, u \rangle}{\langle \varphi_{\beta,1}^H, \varphi_{\beta,1}^H \rangle} \varphi_{\beta,1}^H.$$

Then, for $k \in \mathbb{N}^*$

$$\tilde{\Pi}_\beta \varphi_{\beta,k}^H = \begin{cases} 1 & \text{if } k = 1; \\ 0 & \text{if } k \neq 1. \end{cases}$$

According to step 3, the family $(\varphi_{\beta,k}^H)_k$ is complete, hence, by density, $\tilde{\Pi}_\beta$ is indeed the spectral projection Π_β^H .

Since $\varphi_{\beta,1}^H(x) = e^{-\frac{\beta x^2}{2}}$, we can compute

$$\|\varphi_{\beta,1}^H\|_{L^2(\mathbb{R})}^2 = \|\varphi_{\beta,1}^H\|_{L^2(\mathbb{R})}^2 = \sqrt{\frac{\pi}{\text{Re}(\beta)}} \quad \text{and} \quad \langle \varphi_{\beta,1}^H, \varphi_{\beta,1}^H \rangle = \sqrt{\frac{\pi}{\beta}},$$

and (A.3) follows. \square

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