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THE SINGULAR SUPPORT OF SHEAVES IS γ-COISOTROPIC

STÉPHANE GUILLERMOU AND CLAUDE VITERBO

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1. Introduction

In [Vit92], a metric, denoted γ, was introduced on the set $\mathfrak{L}(T^*N)$ of Lagrangians Hamiltonianly isotopic to the zero section in T^*N, where N is a compact manifold and on $\mathfrak{D}\mathfrak{Ham}_c(T^*N)$ the group of Hamiltonian maps with compact support for $N = \mathbb{R}^n$ or T^n (and in [Vit06] it was extended to $\mathfrak{D}\mathfrak{Ham}_c(T^*N)$ for general compact N). The metric was generalized by Schwartz and Oh to general symplectic manifolds (M,ω) using Floer cohomology for the Hamiltonian case (see [Sch00; Oh05]), and by [Lec08] for the Lagrangian case[1]. The notion of γ-coisotropic sets in a symplectic manifold was defined in [Vit22]. A similar definition was due to Usher [Ush19] in the setting of Hofer distance under the name of "locally rigid". Notice that this yields a weaker notion : γ-coisotropic implies locally rigid in the sense of Usher.

In the context of sheaves on manifolds, there is a notion of coisotropic sets in the sense of Kashiwara-Schapira (see [KS90], definition 6.5.1, p. 271 and Definition 8.2), that the authors call **involutivity**. To avoid any confusion with other notions, we shall use the term cone-coisotropic for their definition as it is defined using the contingent and paratingent cones (see [Bou32]). With this notion, Kashiwara and Schapira proved

Theorem 1.1 ([KS90], theorem 6.5.4, p. 272)**.** *Given a sheaf $\mathscr{F} \in D^b(N)$, its singular support $SS(\mathscr{F})$ is cone-coisotropic.*

The aim of this paper is to prove that

Theorem 1.2. *Given a sheaf $\mathscr{F} \in D^b(N)$, its singular support $SS(\mathscr{F})$ is γ-coisotropic.*

Theorem 1.2 implies Theorem 1.1, since we shall prove the following connection between the two notions

Proposition 1.3. *Let V be a closed subset in (M,ω). If V is γ-coisotropic then it is cone-coisotropic.*

Remark 1.4. This is related to questions asked by Vichery in section 4 of [Vic13]. Our result, toghether with Proposition 1.3, stating that a γ-coisotropic set is cone-coisotropic, gives a more natural proof of the Kashiwara-Schapira theorem. Moreover, as opposed to the notion of cone-coisotropic, which is only invariant by C^1 symplectic diffeomorphisms, the notion of

[1]Assuming $[\omega]\pi_2(M,L) = 0$, $\mu_L\pi_2(M,L) = 0$, where μ_L is the Maslov class of L.

γ-coisotropic set is invariant by homeomorphisms preserving γ, in particular by symplectic homeomorphisms, i.e. homeomorphisms which are C^0 limits of symplectic diffeomorphisms. Along the way we prove a number of results relating the singular support and the spectral norm γ.

2. Notation, comments and acknowledgements

Since our results are local, we shall assume from now on that N is a compact manifold. The following definitions will turn out useful :

- The category $D(N)$ is the unbounded derived category of sheaves of k-vector spaces on N, for some given field k, but except in Appendix B the reader may assume we are dealing with its bounded version. For the operations on sheaves associated to a continuous map $f : M \longrightarrow N$ that is $f^{-1}, f_*, f^!, f_!$ and the two operations $\otimes, \mathscr{H}\mathrm{om}$ and their derived versions, $f^{-1}, Rf_*, Rf_!, f^!$ and $\otimes, \mathrm{R}\mathscr{H}\mathrm{om}$ on $D(N)$ we refer to [KS90].
- Choosing a real analytic structure on N we set $D_c(N)$ to be the category of constructible complexes $\mathscr{F}$, i.e. complexes such that, for some subanalytic stratification, the restriction of $\mathscr{F}$ to each stratum is locally constant and of finite rank.
- For $\mathscr{F} \in D(N)$, $SS(\mathscr{F})$ is the singular support (also called microsupport) defined by Kashiwara-Schapira in [KS90]. We set $SS^\bullet(\mathscr{F}) = SS(\mathscr{F}) \cap (T^*N \setminus 0_N)$.
- For an open set U in the symplectic manifold (M, ω) we denote by $\mathfrak{DHam}_c(U)$ the set of time one flows of Hamiltonian with compact support contained in U.

We thank Pierre Schapira for useful conversations. While writing this paper, we realized that some of the results in Subsection 6.3 were independently discovered by Asano and Ike in [AI22].

3. Reminders on the singular support of sheaves

Let N be a manifold, k a field and $D(N)$ the derived category of sheaves of k-vector spaces on N. For $\mathscr{F} \in D(N)$ we recall that $SS(\mathscr{F}) \subset T^*N$ is the closed conic subset defined by Kashiwara-Schapira as the closure of the set of points $(x;\xi)$ such that $(\mathrm{R}\Gamma_{\{f\geq 0\}}\mathscr{F})_x \neq 0$ for some function of class C^1 such that $f(x) = 0$ and $df_x = \xi$. It is called microsupport or singular support of $\mathscr{F}$.

The functor $\mathrm{R}\Gamma_{\{f\geq 0\}}$ does not commute with infinite direct sums and $(-)_x$ does not commute with infinite direct products. Here are two variations on the definition which have these commutation properties. For the statement of the lemmas we introduce the following definition.

Definition 3.1. *Let $\Omega \subset T^*N \setminus 0_N$ be an open conic subset. We call* Ω-lens *a locally closed subset Σ of N with the following properties: $\overline{\Sigma}$ is compact and there exists an open neighbourhood U of $\overline{\Sigma}$ and a function $g\colon U \times [0,1] \to \mathbb{R}$ of class C^1 such that*

(1) $dg_t(x) \in \Omega$ for all $(x,t) \in U \times [0,1]$, where $g_t = g|_{U\times\{t\}}$,
(2) $\{g_t < 0\} \subset \{g_{t'} < 0\}$ if $t \le t'$,
(3) the hypersurfaces $\{g_t = 0\}$ coincide on $U \setminus \overline{\Sigma}$,
(4) $\Sigma = \{g_1 < 0\} \setminus \{g_0 < 0\}$.

We recall that $SS^\bullet(k_{\{g_t<0\}}) = \{(x;\lambda dg_t(x));\ g_t(x) = 0,\ \lambda > 0\}$. Using the triangle $k_{\{g_0<0\}} \to k_{\{g_1<0\}} \to k_\Sigma \xrightarrow{+1}$ and the "triangle inequality" for the singular support (see [KS90]) we obtain $SS^\bullet(k_\Sigma) \subset \Omega$. The next lemma is

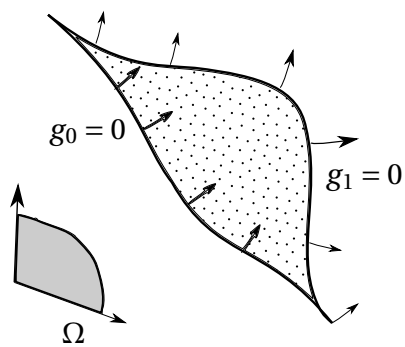

FIGURE 1. A lens. The arrows represent ∇g_1 and ∇g_0.

essentially a reformulation of the definition of the singular support.

Lemma 3.2. *Let $\mathscr{F} \in D(N)$ and let $\Omega \subset T^*N \setminus 0_N$ be an open conic subset. Then $SS(\mathscr{F}) \cap \Omega = \emptyset$ if and only if* $\mathrm{RHom}(k_\Sigma, \mathscr{F}) \simeq 0$ *for any Ω-lens Σ.*

Proof. (i) We first assume $SS(\mathscr{F}) \cap \Omega = \emptyset$. We recall that $\mathrm{RHom}(k_V, \mathscr{F}) \simeq \mathrm{R}\Gamma(V;\mathscr{F})$ for any open subset V of N. Using the notations in the definition of Ω-lens we have a distinguished triangle $k_{\{g_0<0\}} \to k_{\{g_1<0\}} \to k_\Sigma \xrightarrow{+1}$. Applying $\mathrm{RHom}(-, \mathscr{F}|_U)$ to this triangle we obtain the result, using the non-characteristic deformation lemma ([KS90, Prop. 2.7.2]) and the definition of the singular support.

(ii) We assume that there exists $(x_0;\xi_0) \in SS(\mathscr{F}) \cap \Omega$. Hence we can find $(x_1;\xi_1) \in \Omega$ (close to $(x_0;\xi_0)$) and a function $f\colon N \to \mathbb{R}$ of class C^1 such that $f(x_1) = 0$, $df(x_1) = \xi_1$ and $(\mathrm{R}\Gamma_{\{f\ge 0\}}\mathscr{F})_{x_1} \neq 0$. Now we can find a basis of open neighbourhoods of x_1, say U_n, $n \in \mathbb{N}$, such that $\Sigma_n := \{f \ge 0\} \cap U_n$ is an Ω-lens. Then there exists n such that $\mathrm{R}\Gamma(U_n; \mathrm{R}\Gamma_{\{f\ge 0\}}\mathscr{F}) \neq 0$ and the

result follows from

$$\begin{aligned}\mathrm{R}\Gamma(U_n;\mathrm{R}\Gamma_{\{f\geq 0\}}\mathscr{F}) &\simeq \mathrm{R}\Gamma(N;\mathrm{R}\mathscr{H}\mathrm{om}(k_{U_n},\mathrm{R}\mathscr{H}\mathrm{om}(k_{\{f\geq 0\}}\mathscr{F}))) \\ &\simeq \mathrm{R}\Gamma(N;\mathrm{R}\mathscr{H}\mathrm{om}(k_{\Sigma_n},\mathscr{F})) \simeq \mathrm{RHom}(k_{\Sigma_n},\mathscr{F})\end{aligned}$$

□

The functor $\mathrm{RHom}(k_\Sigma,-)$ commutes with direct products but not with direct sums. Here is a dual version of the previous lemma using the functor $\mathrm{R}\Gamma(N;k_\Sigma\otimes -)$ which commutes with direct sums.

Lemma 3.3. *Let $\mathscr{F}\in D(N)$ and let $\Omega\subset T^*N\setminus 0_N$ be an open conic subset. Then $SS(\mathscr{F})\cap\Omega=\emptyset$ if and only if $\mathrm{R}\Gamma(N;k_\Sigma\otimes\mathscr{F})\simeq 0$ for any Ω^a-lens Σ, where Ω^a is the antipodal set of Ω, that is, its image by $(x;\xi)\mapsto(x;-\xi)$.*

The first author thanks Pierre Schapira for a suggestion simplifying the next proof.

Proof. Let Σ be an Ω^a-lens. We write it as the difference of the two closed sets $\overline{\Sigma}$ and $Z=\overline{\Sigma}\setminus\Sigma$. We can find two families of open neighbourhoods of $\overline{\Sigma}$, say $\{U_n\}_{n\in\mathbb{N}}$, and Z, say $\{V_n\}_{n\in\mathbb{N}}$, such that, for all n, we have: $V_n\subset U_n$ and $U_n\setminus V_n$ is an Ω-lens. For each n we have the commutative diagram of restriction maps

$$\begin{tikzcd}\mathrm{R}\Gamma(U_n;\mathscr{F}) \arrow[r, "u_n"] \arrow[d] & \mathrm{R}\Gamma(V_n;\mathscr{F}) \arrow[d] \\ \mathrm{R}\Gamma(N;\mathscr{F}\otimes k_{\overline{\Sigma}}) \arrow[r, "u"] & \mathrm{R}\Gamma(N;\mathscr{F}\otimes k_Z)\end{tikzcd}$$

The cone of u is $\mathrm{R}\Gamma(N;k_\Sigma\otimes\mathscr{F})$, so we want to prove that u is an isomorphism, that is, u induces an isomorphism on all cohomology groups. The cone of u_n is $\mathrm{RHom}(k_{U_n\setminus V_n},F)$ and the previous lemma says that u_n is an isomorphism. Now the result follows from $H^i(N;\mathscr{F}\otimes k_{\overline{\Sigma}})\simeq\varinjlim_n H^i(U_n;\mathscr{F})$ and $H^i(N;\mathscr{F}\otimes k_Z)\simeq\varinjlim_n H^i(V_n;F)$, for all i (see [KS90, Rem. 2.6.6]). □

The previous two lemmas give the following proposition (see [KS90, Ex. 5.7]), away from the zero-section. To deal with the zero-section we use $SS(\mathscr{F})\cap 0_N=\mathrm{supp}(\mathscr{F})$.

Proposition 3.4. *Let $\mathscr{F}_n\in D(N)$, $n\in\mathbb{N}$, be given. Then $SS(\bigoplus_n\mathscr{F}_n)\subset\overline{\bigcup_n SS(\mathscr{F}_n)}$ and $SS(\prod_n\mathscr{F}_n)\subset\overline{\bigcup_n SS(\mathscr{F}_n)}$.*

4. Quantization of Hamiltonian isotopies

Everything in the next subsection is from [GKS] or [KS90]. Given φ, a homogeneous Hamiltonian map from $T^*N\setminus 0_N$ to $T^*N\setminus 0_N$, Guillermou, Kashiwara and Schapira associate a Kernel, that is an element of $D^b(N\times N)$, noted K_φ having the following properties[2]. First K_φ is a *quantization* of the homogeneous Lagrangian $\Lambda_\varphi = \{(z,\varphi(z)) \mid z\in T^*N\}\subset \overline{T^*N}\times T^*N$ in the sense that $SS^\bullet(K_\varphi)=\Lambda_\varphi$, where we set for short, for a sheaf $\mathscr{F}$ on a manifold N,

$$SS^\bullet(\mathscr{F}) = SS(\mathscr{F})\cap (T^*N\setminus 0_N)$$

Before we state the second property we recall the *composition* of sheaves:

Definition 4.1. *Let $\mathscr{F}\in D^b(M)$ and $K\in D^b(M\times N)$. We define*

$$K(\mathscr{F}) = \mathscr{F}\circ K = Rq_{N!}(K\otimes q_M^{-1}\mathscr{F})\in D^b(N),$$

where q_M, q_N are the projections of $M\times N$ on M and N respectively.

The second property is then :

Proposition 4.2 (See Proposition 7.1.2 (ii) in [KS90] and [GKS], formula (1.12))**.** *We have*

$$SS^\bullet(K_\varphi(\mathscr{F})) = \varphi(SS^\bullet(\mathscr{F}))$$

Now if φ is an exact non-homogeneous symplectic map from T^*M to T^*N (i.e. $PdQ-pdq$ is exact), we replace Λ_φ by the homogeneous lift of the graph of φ, say $\widehat{\Lambda}_\varphi\subset \overline{T^*M}\times T^*N\times T^*\mathbb{R}$, given by

$$\widehat{\Lambda}_\varphi = \left\{(q,-\tau p, Q(q,p), \tau P(q,p), F(q,p),\tau) \mid \varphi(q,p)=(Q,P), \tau>0\right\}$$

where $dF = PdQ-pdq$. Note that considering $\widehat{\Lambda}_\varphi$ as a correspondence in $\overline{T^*M}\times T^*N\times T^*\mathbb{R}$, if L is an exact Lagrangian in T^*M and

$$\widehat{L} = \left\{(q,\tau p, f_L(q,p),\tau) \mid (q,p)\in L, \tau\geq 0\right\}$$

we have $\widehat{\Lambda}_\varphi\circ\widehat{L} = \widehat{\varphi(L)}$.

From now on we assume $M=N$ and φ is a Hamiltonian map with compact support (which will always be the case in this paper), the existence of a quantization $\mathscr{K}_\varphi$ for $\widehat{\Lambda}_\varphi$ follows from the homogeneous case by considering $\widehat{\Lambda}_\varphi$ as a deformation of $\widehat{\Lambda}_{\mathrm{id}}$. Since $\widehat{\Lambda}_{\mathrm{id}}$ is quantized by $k_{\Delta_N\times[0,+\infty[}$ we conclude by Proposition 4.2. However we have to be careful that the singular support is slightly bigger than desired:

$$SS^\bullet(k_{\Delta_N\times[0,+\infty[}) = \widehat{\Lambda}_{\mathrm{id}}\cup\{(q,p,q,-p,t,0) \mid t\geq 0\}$$

[2]In fact in [GKS] the authors associate to a Hamiltonian isotopy φ^s a Kernel $\mathscr{K}_\Phi\in D^b(N\times N\times[0,1])$ such that $\mathscr{K}_{\Phi|N\times N\times\{s\}} = K_{\varphi^s}$.

but its intersection with $\{\tau > 0\}$ is equal to $\widehat{\Lambda}_{\mathrm{id}}$. We can find a homogeneous map ψ acting on $T^*(N^2 \times \mathbb{R}) \setminus 0_{N^2\times\mathbb{R}}$ such that ψ is the identity map near $\{\tau = 0\}$ and $\psi(\widehat{\Lambda}_{\mathrm{id}}) = \widehat{\Lambda}_\varphi$. We remark that φ can be lifted to a homogeneous map $\widehat{\varphi}$ acting on $T^*(N \times \mathbb{R}) \setminus \{\tau = 0\}$ (we homogenize the Hamiltonian function) but we cannot use $\mathrm{id}_{T^*N} \times \widehat{\varphi}$ for ψ, as would be natural, since it is not defined everywhere outside the zero section.

Now we set

$$\mathscr{K}_\varphi = K_\psi(k_{\Delta_N \times [0,+\infty[})$$

and we find

$$SS^\bullet(\mathscr{K}_\varphi) = \widehat{\Lambda}_\varphi \cup \{(q, p, q, -p, t, 0) \mid t \geq 0\}$$

If we consider the whole isotopy φ_s and choose t_0 big enough so that $\widehat{\Lambda}_{\varphi_s}$ is contained in $T^*(N^2 \times]-t_0, t_0[)$ for all $s \in [0,1]$, then we see that $SS^\bullet(\mathscr{K}_{\varphi_s}) \cap T^*U$ is independent of s, where $U = N^2 \times (\mathbb{R} \setminus [-t_0, t_0])$. By [KS90, Prop. 5.4.5] we deduce that

$$\mathscr{K}_\varphi|_{N^2\times]-\infty,-t_0[} \simeq 0, \qquad \mathscr{K}_\varphi|_{N^2\times]t_0,+\infty[} \simeq k_{\Delta_N\times[0,+\infty[}|_{N^2\times]t_0,+\infty[}$$

In other words $\mathscr{K}_\varphi = \mathscr{K}_{\mathrm{id}} = k_{\Delta_N\times[0,+\infty[}$ outside of a compact set.

Remark 4.3. We could obtain the existence of $\mathscr{K}_\varphi$ when $M = N$ and φ has compact support without assuming that φ is Hamiltonian by adapting the result of [Gui12; Vit19] (see Theorem 5.1). In *loc. cit.* the existence of a quantization is proved for a Legendrian of J^1N which is the lift of a compact exact Lagrangian of T^*N. Here the graph of φ is not compact but coincides with the diagonal $T^*_{\Delta_N}N^2$ outside a compact set (if we choose $F = 0$ at infinity) so it should not be too difficult to extend the result in our situation.

Since we use $\mathscr{K}_\varphi$, defined over $N^2 \times \mathbb{R}$, instead of sheaf defined over $(N \times \mathbb{R})^2$, we change slightly the composition functor as follows, which becomes a mixture of $\circ$ and the convolution $*$ (see Section 5 for the definition)

Definition 4.4. *Let $\mathscr{F} \in D^b(M \times \mathbb{R})$ and $\mathscr{K} \in D^b(M \times N \times \mathbb{R})$. We define*

$$\mathscr{K}^{\circledast}(\mathscr{F}) = \mathscr{F} \circledast \mathscr{K} = Rs_! Rq_{2!}(q_{1,2}^{-1}\mathscr{K} \otimes q_1^{-1}\mathscr{F}) \in D^b(N \times \mathbb{R}),$$

where the maps $q_{1,2}\colon M \times N \times \mathbb{R}^2 \to M \times N \times \mathbb{R}$, $q_1\colon M \times N \times \mathbb{R}^2 \to M \times \mathbb{R}$, $s\colon M\times N\times\mathbb{R}^2 \to N\times\mathbb{R}$ are defined by $q_{1,2}(x, y, t_1, t_2) = (x, y, t_1)$, $q_1(x, y, t_1, t_2) = (x, t_2)$, $s(x, y, t_1, t_2) = (x, y, t_1 + t_2)$.

Now we obtain a non homogeneous version of Proposition 4.2.

Proposition 4.5. *For a compactly supported Hamiltonian map φ from T^*N to itself and any $\mathscr{F} \in D^b(N \times \mathbb{R})$ with $SS(\mathscr{F}) \subset \{\tau \geq 0\}$, we have $SS^\bullet(\mathscr{K}_\varphi^{\circledast}(\mathscr{F})) = \widehat{\varphi}(SS^\bullet(\mathscr{F}))$.*

In particular, for $L \in \mathscr{L}(T^*N)$ we have $SS^\bullet(\mathscr{K}_\varphi(\mathscr{F}_L)) = SS^\bullet(\mathscr{F}_{\varphi(L)})$ and, by uniqueness of the quantization (see Theorem 5.1, (4)), $\mathscr{K}_\varphi^{\circledast}(\mathscr{F}_L) \simeq \mathscr{F}_{\varphi(L)}$.

5. Spectral invariants for sheaves and Lagrangians

For M a symplectic manifold, (here we only need $M = T^*N$), if $\Lambda(M)$ is the bundle of Lagrangians subspaces of the tangent bundle to M, with fibre the Lagrangian Grassmannian $\Lambda(T_zM) \simeq \Lambda(n)$, we denote by $\widetilde{\Lambda}(M)$ the bundle induced by the universal cover $\widetilde{\Lambda}(n) \longrightarrow \Lambda(n)$.

When using coefficients others than $\mathbb{Z}/2\mathbb{Z}$ we assume we have a lifting $\widetilde{G}_L$ of the Gauss map $G_L : L \longrightarrow \Lambda(T^*N)$ given by $x \mapsto T_xL$ to $\widetilde{\Lambda}(T^*N)$. This is called a grading. Given a graded L, the canonical automorphism of the covering induces a new grading and we denote it as $L[1]$ (or $\widetilde{L}[1]$), and its k-th iteration as $L[k]$ (or $\widetilde{L}[k]$). The grading yields an absolute grading for the Floer homology[3] and hence for the complex of sheaves in the Theorem stated below. We refer to [Sei00] for more details on this, but point out that we shall never mention explicitly the grading.

An exact Lagrangian in T^*N is a pair (L, f_L) such that $df_L = \lambda_{|L}$ where $\lambda = pdq$ is the Liouville form. For an exact Lagrangian, a grading always exists since the obstruction to its existence is given by the Maslov class, and for exact Lagrangians in T^*N the Maslov class vanishes, as was proved by Kragh and Abouzaid (see [Kra13] and also the sheaf-theoretic proof by [Gui12]). When f_L is implicit we only write L, for example 0_N means $(0_N, 0)$. A Lagrangian brane is a triple $\widetilde{L} = (L, f_L, \widetilde{G}_L)$, where L is a compact Lagrangian. For c a real constant, we write $T_c\widetilde{L} = (L, f_L + c, \widetilde{G}_L)$. We also use the notation $\widetilde{L} + c$ or $L + c$ if the grading is irrelevant (as it will be most of the time).

Let L be a Lagrangian in T^*N and

$$\widehat{L} = \left\{(q, \tau p, f_L(q,p), \tau) \mid (q,p) \in L, \tau \geq 0\right\}$$

the homogenized Lagrangian in $T^*(N \times \mathbb{R})$.

Similarly for Hamiltonian maps in an exact manifold $(M, \omega = d\lambda)$ we look for pairs (φ, F) such that $\varphi^*\lambda - \lambda = dF$ on M.

To state the next result we recall the two adjoint functors, $*$ and $\mathscr{H}\mathrm{om}^*$, on the category $D(N \times \mathbb{R})$ introduced by Tamarkin in [Tam08]. First we set $s, q_1, q_2 : N \times (\mathbb{R})^2 \longrightarrow N \times \mathbb{R}$ given by $s(x, t_1, t_2) = (x, t_1 + t_2)$, $q_1(x, t_1, t_2) =$

[3]Without this extra piece of information, the Floer cohomology $FH^k(L_1, L_2)$ is generally defined only up to a shift in grading. However, when $L_2 = \varphi_H^1(L_1)$, there is an absolute grading, *a priori* depending on the choice of H. However in our situation we do not assume L_1 is Hamiltonianly isotopic to L_2, so the grading is required to have an absolute grading of the Floer cohomology.

(x,t_1), $q_2(x,t_1,t_2)=(x,t_2)$ and we define

$$\mathscr{F}*\mathscr{G}=Rs_!(q_1^{-1}\mathscr{F}\otimes q_2^{-1}\mathscr{G}),$$
$$\mathscr{H}\mathrm{om}^*(\mathscr{F},\mathscr{G})=Rq_{1*}\,\mathrm{R}\mathscr{H}\mathrm{om}(q_2^{-1}\mathscr{F},s^!\mathscr{G})$$

Then $\mathscr{H}\mathrm{om}^*$ is the adjoint of $*$ in the sense that

$$\mathrm{Mor}_{D(N\times\mathbb{R})}(\mathscr{F},\mathscr{H}\mathrm{om}^*(\mathscr{G},\mathscr{H}))=\mathrm{Mor}_{D(N\times\mathbb{R})}(\mathscr{F}*\mathscr{G},\mathscr{H})$$

We recall some properties of $\mathscr{H}\mathrm{om}^*$. We denote by $D_{\tau\geq 0}(N\times\mathbb{R})$ the subcategory of $D(N\times\mathbb{R})$ of objects with singular support contained[4] in $\{\tau\geq 0\}$. A base change formula gives, for any $c\in\mathbb{R}$, and $\mathscr{F},\mathscr{G}\in D^b(N\times\mathbb{R})$

$$\mathrm{R}\Gamma_{N\times\{-c\}}(N\times\mathbb{R};\mathscr{H}\mathrm{om}^*(\mathscr{F},\mathscr{G}))\simeq\mathrm{RHom}(\mathscr{F},T_{c*}(\mathscr{G})),\tag{5.1}$$

with $T_c\colon N\times\mathbb{R}\to N\times\mathbb{R}$, $(x,t)\mapsto(x,t+c)$. Note that $\widehat{T_cL}=T_c\widehat{L}$.

For $\mathscr{H}$ with singular support in $\{\tau\geq 0\}$, we have a morphism

$$\mathrm{R}\Gamma_{N\times\{b\}}(N\times\mathbb{R};\mathscr{H})\to\mathrm{R}\Gamma_{N\times[a,b]}(N\times\mathbb{R};\mathscr{H})\xleftarrow{\sim}\mathrm{R}\Gamma_{N\times\{a\}}(N\times\mathbb{R};\mathscr{H})$$

for any $a\leq b$. We remark that the cone of this morphism is $\mathrm{R}\Gamma_{N\times[a,b[}(N\times\mathbb{R};\mathscr{H})$. Using (5.1) we deduce a morphism, say u, from $\mathrm{RHom}(\mathscr{F},T_{c*}(\mathscr{G}))$ to $\mathrm{RHom}(\mathscr{F},T_{d*}(\mathscr{G}))$ for any $c\leq d$. Taking $\mathscr{F}=\mathscr{G}$, $c=0$ the image of $\mathrm{id}_{\mathscr{G}}$ gives

$$\tau_{0,d}\colon\mathscr{G}\to T_{d*}(\mathscr{G})\qquad\text{for any } d\geq 0\tag{5.2}$$

(we come back to this in §6). Then u is nothing but the composition with $T_{c*}(\tau_{0,d-c})$.

According to [Gui12] and [Vit19] we have

Theorem 5.1. *To each $L\in\mathscr{L}(T^*N)$ we can associate $\mathscr{F}_L\in D^b(N\times\mathbb{R})$ such that*

(1) $SS(\mathscr{F}_L)=\widehat{L}$.
(2) $\mathscr{F}_L$ is simple (cf. [KS90, Def. 7.5.4]), $\mathscr{F}_L=0$ near $N\times\{-\infty\}$ and $\mathscr{F}_L=k_{N\times\mathbb{R}}$ near $N\times\{+\infty\}$.
(3) We have an isomorphism

$$FH^\bullet(L_0,L_1;a,b)=H^*_{N\times[a,b[}\big(N\times\mathbb{R};\mathscr{H}om^*(\mathscr{F}_{L_0},\mathscr{F}_{L_1})\big)$$

(4) $\mathscr{F}_L$ is unique satisfying properties (1) and (2).
(5) There is a natural product map

$$\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_2})\otimes\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_2},\mathscr{F}_{L_3})\longrightarrow\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_3})$$

[4]This category contains the "Tamarkin category" which is the left orthogonal of $D_{\tau\leq 0}(N\times\mathbb{R})$. The sheaves $\mathscr{F}_L$ belongs to the Tamarkin category. The category $D_{\tau\geq 0}(N\times\mathbb{R})$ is stable by $*$ and $\mathscr{H}\mathrm{om}^*$. The Tamarkin category is stable by $*$ but not $\mathscr{H}\mathrm{om}^*$: typically when N is a point we have $\mathscr{H}\mathrm{om}^*(k_{[a,\infty[},k_{[b,\infty[})\simeq k_{]-\infty,b-a[}[1]$.

inducing in cohomology a map

$$\begin{array}{c} H^*_{N\times[\lambda,+\infty[}(N\times\mathbb{R};\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_2}))\otimes H^*_{N\times[\mu,+\infty[}(N\times\mathbb{R};\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_2},\mathscr{F}_{L_3})) \\ \Big\downarrow \cup_* \\ H^*_{N\times[\lambda+\mu,+\infty[}(N\times\mathbb{R};\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_3})) \end{array}$$

that coincides through the above identifications with the triangle product in Floer cohomology.

Remarks 5.2.

(1) Note that we have

$$\mathscr{F}_{T_cL} = T_{c*}(\mathscr{F}_L)$$

(2) In case L has a Generating Function Quadratic at infinity, $S(x,\xi)$ defined on the vector bundle $E \xrightarrow{\pi} N$, we can take for $\mathscr{F}_L$ the complex $R(\pi\times \mathrm{id}_{\mathbb{R}})_*(k_{U_S})$ where $U_S = \{(x,\xi,t) \mid S(x,\xi)\le t\}$.

5.1. **Definition of spectral invariants.** Now we can give a sheafy definition of the spectral invariants introduced in [Vit92]. The next formulation appears in [Vic13] (see also [Vit22]).

In fact the invariants can be defined for objects $\mathscr{F}$ of $D_{\tau\ge 0}(N\times\mathbb{R})$ satisfying:

$$\mathscr{F}|_{N\times]-\infty,-t_0[}\simeq 0,\quad \mathscr{F}|_{N\times]t_0,+\infty[}\simeq k_{N\times\mathbb{R}}|_{N\times]t_0,+\infty[}\quad \text{for some } t_0. \tag{5.3}$$

We have seen that $\mathscr{F}_L$ satisfies (5.3) for any $L\in\mathscr{L}(T^*N)$. For $\mathscr{F}_1$, $\mathscr{F}_2\in D_{\tau\ge 0}(N\times\mathbb{R})$ satisfying (5.3), $\mathscr{H}\mathrm{om}^*(\mathscr{F}_1,\mathscr{F}_2))$ is $k_{N\times\mathbb{R}}[1]$ near $N\times\{-\infty\}$ and 0 near $N\times\{+\infty\}$. Using 5.1 and $H^0_{N\times\{c\}}(N\times\mathbb{R};k_{N\times\mathbb{R}}[1])\simeq k$, we deduce a natural morphism $\mathscr{F}_1\to T_{c*}(\mathscr{F}_2)$, for any c big enough, which restricts to $\mathrm{id}\colon k_{N\times\mathbb{R}}\to k_{N\times\mathbb{R}}$ near $N\times\{+\infty\}$. More generally $H^*_{N\times\{c\}}(N\times\mathbb{R};k_{N\times\mathbb{R}}[1])\simeq H^*(N;k_N)$ and any class $\alpha\in H^d(N;k_N)$ yields a morphism

$$u(\alpha,\mathscr{F}_1,\mathscr{F}_2,c)\colon \mathscr{F}_1\to T_{c*}(\mathscr{F}_2)[d],$$

for c large enough, which restricts to $\alpha\colon k_{N\times\mathbb{R}}\to k_{N\times\mathbb{R}}[d]$ near $N\times\{+\infty\}$ (using $\mathrm{Mor}(k_N,k_N[d])\simeq H^d(N;k_N)$). The cup product corresponds to the composition, and for c (and t_0) large enough this is summarized by the following diagram

$$\begin{array}{ccc} H^*(N,k)\simeq R\Gamma(N,k_N) & \xrightarrow{\simeq} & \mathrm{RHom}(\mathscr{F}_1,T_{c*}\mathscr{F}_2) \\ \Big\downarrow\simeq & & \Big\downarrow\simeq \\ \mathrm{RHom}(k_{N\times]t_0,+\infty[},k_{N\times]t_0,+\infty[}) & \xrightarrow{\simeq} & \mathrm{RHom}\big(\mathscr{F}_{1|N\times]t_0,+\infty[},T_{c*}\mathscr{F}_{2|N\times]t_0,+\infty[}\big) \end{array}$$

Definition 5.3. *For $\mathscr{F}_1, \mathscr{F}_2 \in D_{\tau\geq 0}(N\times\mathbb{R})$ satisfying* (5.3) *and $\alpha \in H^d(N;k_N)$ we set*

$$c(\alpha,\mathscr{F}_1,\mathscr{F}_2) = -\inf\{c;\ \exists u\colon \mathscr{F}_1 \to T_{c*}(\mathscr{F}_2)[d],\ u|_{N\times]t_0,+\infty[} = \alpha,\ \textit{for } t_0 \gg 0\}$$

(Here t_0 is big enough so that both $\mathscr{F}_1$ and $T_{c*}(\mathscr{F}_2)$ satisfy (5.3)*). We also set $c_-(\mathscr{F}_1,\mathscr{F}_2) = c(1,\mathscr{F}_1,\mathscr{F}_2)$, if N is oriented with fundamental class μ_N, $c_+(\mathscr{F}_1,\mathscr{F}_2) = c(\mu_N,\mathscr{F}_1,\mathscr{F}_2)$ and $\gamma(\mathscr{F}_1,\mathscr{F}_2) = c_+(\mathscr{F}_1,\mathscr{F}_2) - c_-(\mathscr{F}_1,\mathscr{F}_2)$.*

For $L_1, L_2 \in \mathscr{L}(T^*N)$ we set $c(\alpha,L_1,L_2) = c(\alpha,\mathscr{F}_{L_1},\mathscr{F}_{L_2})$ and define $c_\pm(L_1,L_2)$, $\gamma(L_1,L_2)$ in the same way. Finally we set $c_\pm(L) = c_\pm(0_N,L)$.

It is not difficult to see that

$$c_-(\mathscr{F}_1,\mathscr{F}_2) = \inf\{c(\alpha,\mathscr{F}_1,\mathscr{F}_2);\ \alpha \in H^*(N;k_N)\}$$

and

$$c_+(\mathscr{F}_1,\mathscr{F}_2) = \sup\{c(\alpha,\mathscr{F}_1,\mathscr{F}_2);\ \alpha \in H^*(N;k_N)\}$$

Indeed, a class α induces by multiplication $v\colon \mathscr{F}_2 \to \mathscr{F}_2[d]$, $v|_{N\times]t_0,+\infty[} = \alpha$; now if we have $u\colon \mathscr{F}_1 \to T_{c*}(\mathscr{F}_2)[0]$, $u|_{N\times]t_0,+\infty[} = 1$, then $v\circ u\colon \mathscr{F}_1 \to T_{c*}(\mathscr{F}_2)[d]$ satisfies $v\circ u|_{N\times]t_0,+\infty[} = \alpha$ so $c(1,\mathscr{F}_1,\mathscr{F}_2) \leq c(\alpha,\mathscr{F}_1,\mathscr{F}_2)$.

Using the construction of the sheaf $\mathscr{F}_L$ by using Floer cohomology it is easy to see that the present definition of $c(\alpha,\mathscr{F}_{L_1},\mathscr{F}_{L_2})$ coincides with the definition of $c(\alpha,L_1,L_2)$ using Floer cohomology. In particular, we have that $c_+(\mathscr{F}_{L_1},\mathscr{F}_{L_2}) = -c_-(\mathscr{F}_{L_2},\mathscr{F}_{L_1})$ and this can be extended to the general case of $\mathscr{F}_1,\mathscr{F}_2$ using [Vit19].

We have given the definition using the right hand side of (5.1). We can also rewrite it with the left hand side as follows. Let us set $\mathscr{H} = Rt_*\mathscr{H}\mathrm{om}^*(\mathscr{F},\mathscr{G}))$, where $t\colon N\times\mathbb{R}\to\mathbb{R}$ is the projection. Then $\mathscr{H}$ contains all the information about the spectral invariants. More precisely (5.1) is isomorphic to $\mathrm{R}\Gamma_{\{-c\}}(\mathscr{H})$ and $\mathscr{H}$ has its singular support in $\{\tau \geq 0\}$ so we have the morphisms

$$\mathrm{R}\Gamma(N;k_N) \simeq \mathrm{R}\Gamma_{\{-t_0\}}(\mathbb{R};\mathscr{H}) \xrightarrow{\sim} \mathrm{R}\Gamma_{[-t_0,-c]}(\mathbb{R};\mathscr{H}) \leftarrow \mathrm{R}\Gamma_{\{-c\}}(\mathbb{R};\mathscr{H}) \tag{5.4}$$

for $t_0 \gg 0$ and any $c \leq t_0$. To find the spectral invariants we look for the infimum on the c such that a given class is in the image of the composition of this morphisms.

Remark 5.4. We can see that the infimum in Definition 5.3 is actually a minimum, at least when $\mathscr{F}$ is of the form $\mathscr{F}_L$. Indeed, this follows either by identifying the spectral invariants of the Definition to those obtained by using Floer cohomology, or by appealing to the theory of persistence modules and barcodes (see Appendix A Proposition A.1).

We notice the following immediate consequence of the definitions

Proposition 5.5. *There exists a morphism $u : \mathscr{F}_{L_1} \longrightarrow \mathscr{F}_{L_2}$ such that $u = \mathrm{Id}$ at $+\infty$ if and only if $c_-(L_1, L_2) \geq 0$. In particular there is a morphism $k_{N\times[0,+\infty[} \longrightarrow \mathscr{F}$ if and only if $c_-(L) \geq 0$.*

5.2. **The case of Hamiltonian maps.** If φ is a Hamiltonian map of $T^*\mathbb{R}^n$ with compact support, we define the spectral invariants of φ as those of its graph, say $\Gamma_\varphi \subset \overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n$, compactified in the following way: we choose a symplectomorphism $\psi\colon \overline{T^*\mathbb{R}^n} \times T^*\mathbb{R}^n \xrightarrow{\sim} T^*\Gamma_{\mathrm{id}}$ such that $\psi(\Gamma_{\mathrm{id}}) = 0_{\Gamma_{\mathrm{id}}}$; then $\psi(\Gamma_\varphi)$ coincides with the zero-section outside a compact set and we can compactify it as Γ^c_φ in the cotangent bundle of the sphere. In particular $\Gamma^c_{\mathrm{id}} = 0_{S^{2n}}$ and the invariants of φ, $c_+(\varphi)$, $c_-(\varphi)$, are defined as those of $(\Gamma^c_\varphi, 0_{S^{2n}})$. This is well-defined since another choice of ψ would give a pair $(\Gamma'^c_\varphi, 0_{S^{2n}})$ symplectomorphic to $(\Gamma^c_\varphi, 0_{S^{2n}})$.

On the other hand we have seen in §4 that φ has an associated sheaf $\mathscr{K}_\varphi \in D(\mathbb{R}^{2n+1})$. In the next paragraph we explain how we can turn the above definition of $c_+(\varphi)$, $c_-(\varphi)$ into Definition 5.6 which uses $\mathscr{K}_\varphi$, $\mathscr{K}_{\mathrm{id}}$ and not the intermediate symplectomorphism ψ.

Equivalence with Definition 5.6. We first remark that we don't really need to compactify Γ_{id} into the sphere to obtain the invariants. Let D_1 be the subcategory of $D_{\tau\geq 0}(\Gamma_{\mathrm{id}} \times \mathbb{R})$ of complexes isomorphic to $k_{\Gamma_{\mathrm{id}}\times[c,+\infty[}$ (for some c) and D_2 be the subcategory of $D_{\tau\geq 0}(S^{2n} \times \mathbb{R})$ of complexes isomorphic to $k_{S^{2n}\times[c,+\infty[}$ near $\{\infty\} \times [c, +\infty[$. Then the compactification and the restriction map induce functors $\alpha\colon D_1 \to D_2$ and $\beta\colon D_2 \to D_1$ such that $\beta \circ \alpha = \mathrm{id}$, $\alpha(\tau_{a,b}(\mathscr{F})) = \tau_{a,b}(\alpha(\mathscr{F}))$. We deduce that $c_-(\alpha(\mathscr{F}_1), \alpha(\mathscr{F}_2)) = c_-(\mathscr{F}_1, \mathscr{F}_2)$ (we remark that c_- makes sense even in the non compact case). In particular

$$c_-(\varphi) = c_-(\mathscr{F}_{\Gamma^c_\varphi}, \mathscr{F}_{0_{S^{2n}}}) = c_-(\mathscr{F}_{\Gamma_\varphi}, \mathscr{F}_{0_{\mathbb{R}^{2n}}}) \tag{5.5}$$

Now the symplectomorphism ψ induces an equivalence of categories between $\mathscr{K}^{\circledast}_\psi\colon D_{\tau\geq 0}(\mathbb{R}^{2n+1}) \xrightarrow{\sim} D_{\tau\geq 0}(\Gamma_{\mathrm{id}} \times \mathbb{R})$ which commutes with the translation T_{c*} in the last variables. We have $\mathscr{K}^{\circledast}_\psi(\mathscr{K}_\varphi)) \simeq \mathscr{F}_{\Gamma_\varphi}$ and we obtain from (5.5) that the spectral invariant $c_-(\varphi)$ we have defined above coincides with the one in Definition 5.6 below. To obtain c_+ we use $c_+(\mathscr{F}_1, \mathscr{F}_2) = -c_-(\mathscr{F}_2, \mathscr{F}_1)$.

The existence of $\mathscr{K}_\psi$ doesn't really follows from §4 because ψ is not compactly supported. But the choice of ψ is irrelevant and we can choose $\psi\colon T^*\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ given by $\psi(q_1, q_2, p_1, p_2) = (q_1, p_2, p_1 + p_2, q_1 - q_2)$. In this case we can give a direct construction of $\mathscr{K}_\psi$ as follows. Let us define $\Sigma^+ \subset \mathbb{R}^{4n+1}$ by $\Sigma^+ = \{(q_1, q_2, Q_1, Q_2, t) \mid Q_1 = q_1,\ t \geq (q_1 - q_2)Q_2\}$, whose boundary is the projection of the lift $\widehat{\Lambda}_\psi$ of the graph of ψ to the base. Then we can check that $\mathscr{K}_\psi = k_{\Sigma^+}$ is the quantization of ψ.

This justifies that both definitions coincide for a Hamiltonian map of T^*S^n obtained from a compactly supported map of $T^*\mathbb{R}^n$. In the next paragraph we see that the case of a Hamiltonian map of T^*N with support in a Darboux chart is reduced to this case.

Support in a Darboux chart. Let $S_\varphi(q,P,\xi)$ be a generating function quadratic at infinity for Γ_φ over $\Delta_{\mathbb{R}^{2n}}$, that is,

$$\begin{cases} p-P=\frac{\partial S_\varphi}{\partial q}(q,P;\xi) \\ Q-q=\frac{\partial S_\varphi}{\partial P}(q,P;\xi) \\ \frac{\partial S_\varphi}{\partial \xi}(q,P;\xi)=0 \end{cases}$$

Since φ is compactly supported, so is S_φ in the q,P variables, that is, $S_\varphi(q,P;\xi)=B(\xi)$ where B is a non-degenerate quadratic form, for $|q|+|P|$ large enough. We can then extend S_φ to a generating function quadratic at infinity on a bundle over S^{2n} (see [Vit92]), and $c_\pm(\varphi)$ are defined as

$$c_+(\varphi)=\inf\{t \mid T\cup\mu_{S^{2n}}\neq 0 \text{ in } H^*(S^t,S^{-\infty})\},$$
$$c_-(\varphi)=\inf\{t \mid T\cup 1\neq 0 \text{ in } H^*(S^t,S^{-\infty})\},$$

where T is the Thom class associated to the negative bundle of B and $S^{-\infty}$ means S^{-c} for c large enough. Note that these coincide with the spectral invariants defined using Floer cohomology (see [Vit95]).

Now if φ is defined on T^*N but has support in a Darboux chart, it can be considered either as a compact supported Hamiltonian map on T^*N or on $\mathbb{R}^{2n}$ and we can define the spectral invariants in two ways (using Floer cohomology). By an easy pseudoconvexity argument (see Appendix D) we can see that these two definitions of $c_\pm(\varphi)$ do coincide.

Sheaf definition of spectral invariants for a Hamiltonian map. Now we can give another formulation of the definition of $c_\pm(\varphi)$. Let N be a compact manifold and let φ be a Hamiltonian map of T^*N with compact support. We consider its homogenized graph $\widehat{\Lambda}_\varphi\subset\overline{T^*N}\times T^*N\times T^*\mathbb{R}$ as in §4. We have associated to φ its quantization $\mathscr{K}_\varphi$, with singular support $\widehat{\Lambda}_\varphi$, and it coincides with $\mathscr{K}_{\mathrm{id}}=k_{\Delta_N\times[0,+\infty[}$ outside a compact subset of $N^2\times\mathbb{R}$. Now $\mathscr{H}\mathrm{om}^*(\mathscr{K}_{\mathrm{id}},\mathscr{K}_\varphi)$ has similar properties as $\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_2})$, namely, it is $k_{\Delta_N\times\mathbb{R}}[1]$ near $N^2\times\{-\infty\}$ and 0 near $N^2\times\{+\infty\}$. As in the case of compact Lagrangians we have $H^*_{N^2\times\{c\}}(N^2\times\mathbb{R};k_{\Delta_N\times\mathbb{R}}[1])\simeq H^*(N;k_N)$ and any class $\alpha\in H^d(N;k_N)$ yields a morphism

$$u(\alpha,\varphi,c)\colon \mathscr{K}_{\mathrm{id}}\to T_{c*}(\mathscr{K}_\varphi)[d],$$

for c big enough, which restricts to $\alpha\colon k_{\Delta_N\times\mathbb{R}}\to k_{\Delta_N\times\mathbb{R}}[d]$ near $N\times\{+\infty\}$ (using $\mathrm{Hom}(k_{\Delta_N},k_{\Delta_N}[d])\simeq H^d(N;k_N)$.

Definition 5.6. *For a compactly supported Hamiltonian map φ of T^*N and $\alpha \in H^d(N;k_N)$ we set*

$$c(\alpha,\varphi) = -\inf\{c;\ \exists u\colon \mathscr{K}_{\mathrm{id}} \to T_{t*}(\mathscr{K}_\varphi)[d],\ u|_{N^2\times]t_0,+\infty[} = \alpha,\ \textit{for } t_0 \gg 0\}$$

(Here t_0 is big enough so that over $N^2\times]t_0,+\infty[$ we have $T_{c*}(\mathscr{K}_\varphi) \simeq k_{N\times\mathbb{R}}$.) We also set $c_-(\varphi) = c(1,\varphi)$, $c_+(\varphi) = c(\mu_N,\varphi)$ and $\gamma(\varphi) = c_+(\varphi) - c_-(\varphi)$. If ψ is another compactly supported Hamiltonian map, we set $\gamma(\varphi,\psi) = \gamma(\varphi\psi^{-1})$.

We have $c_-(\varphi) = \inf\{c(\varphi,\alpha);\ \alpha \in H^\star(N;k_N)\}$ and $c_+(\varphi) = \sup\{c(\varphi,\alpha);\ \alpha \in H^\star(N;k_N)\}$.

Since $\Gamma(\varphi)$ coincides with the zero section outside a compact set, we have $c_-(\varphi) \le 0 \le c_+(\varphi)$ as in [Vit92] and $c_+(\varphi) = -c_-(\varphi^{-1})$.

Note the following analogue of Proposition 5.5

Lemma 5.7. *Let φ be a compact supported Hamiltonian map. Then there is a morphism $\mathscr{K}_{\mathrm{id}} \longrightarrow \mathscr{K}_\varphi$ in $D^b(N\times N\times\mathbb{R})$ equal to* Id *at $+\infty$ if and only if $c_-(\varphi) = 0$. Analogously there is a morphism $\mathscr{K}_\varphi \longrightarrow \mathscr{K}_{\mathrm{id}}$ in $D^b(N\times N\times\mathbb{R})$ if and only if $c_+(\varphi) = 0$.*

5.3. **Defining γ-coisotropic subsets.** For an open set U in the symplectic manifold (M,ω) we denote by $\mathfrak{DHam}_c(U)$ the set of time one flows of Hamiltonian with compact support contained in U.

Definition 5.8 (see definition 6.1 in [Vit22])**.** *Let V be a set in (M,ω). We say that V is γ-***coisotropic at** *$x \in V$ if, for all pairs of balls $B(x,\eta) \subset B(x,\varepsilon)$ with $0 < \eta < \varepsilon$, there is a $\delta > 0$ such that, for all φ in $\mathfrak{DHam}_c(B(x,\varepsilon))$ and satisfying $\varphi(V) \cap B(x,\eta) = \emptyset$, we have $\gamma(\varphi) > \delta$.*

*We shall say that V is γ-***coisotropic** *if it is non-empty and γ-coisotropic at each $x \in V$.*

We refer to [Vit22], in particular section 6, for properties and examples of γ-coisotropic subsets. In particular we remind the reader of the following result from [Vit22]

Proposition 5.9. *Let φ be a homeomorphism preserving γ, for example a C^0-limit of Hamiltonian maps (usually called Hamiltonian homeomorphisms) and V a γ-coisotropic set. Then, the image by φ of V is a γ-coisotropic set.*

Note that the completion of $\mathscr{L}(T^*N)$ for γ is denoted by $\widehat{\mathscr{L}}(T^*N)$. In [Vit22] the γ-support of an element L in $\widehat{\mathscr{L}}(T^*N)$ was defined, and it was proved that it must be a γ-coisotropic subset.

6. The γ_g-metric on sheaves

We shall define the analogue of the γ distance between sheaves. It is also a sheaf analogue of the barcode distance (see [KS18] or [AI20]) and relies on the morphism τ introduced by Tamarkin (see [Tam08] or [GKS, Prop. 4.8]) that we have already encountered in (5.2).

To define τ in general we recall that the subcategory $D_{\tau\geq 0}(N\times\mathbb{R})$ of $D(N\times\mathbb{R})$ can be characterized as follows (see [KS90, Prop. 5.2.3, 3.5.4]). Let P be the endofunctor of $D(N\times\mathbb{R})$ defined by

$$P(\mathscr{F}) = Rs_*(\mathscr{F}\boxtimes k_{[0,+\infty[})$$

with $s\colon N\times\mathbb{R}^2\to N\times\mathbb{R}$, $(x,t_1,t_2)\mapsto(x,t_1+t_2)$ (this is almost $\mathscr{F}*k_{N\times[0,+\infty[}$ up to replacing from $Rs_!$ to Rs_*). Then the natural morphism $k_{[0,+\infty[}\to k_{\{0\}}$ induces a morphism $P(\mathscr{F})\to\mathscr{F}$ and $D_{\tau\geq 0}(N\times\mathbb{R})$ is formed by the $\mathscr{F}$ such that $P(\mathscr{F})\to\mathscr{F}$ is an isomorphism. Now we have in general $Rs_*(\mathscr{F}\boxtimes k_{\{c\}})\simeq T_{c*}(\mathscr{F})$, where $T_c(x,t)=(x,t+c)$ and we obtain in the same way, using the morphism $k_{[0,+\infty[}\to k_{\{c\}}$, a morphism $P(\mathscr{F})\to T_{c*}(\mathscr{F})$ for any $c\geq 0$. In particular we have a morphism of endofunctors defined on $D_{\tau\geq 0}(N\times\mathbb{R})$ for any $c\geq 0$

$$\tau_{0,c}(\mathscr{F})\colon \mathscr{F}\to T_{c*}(\mathscr{F})$$

We shall call this morphism the Tamarkin morphism.

The case of $D_{\tau\geq 0}(N\times\mathbb{R})$ is in fact sufficient for this paper but we can consider a slightly more general case without modifying the proofs. Let $g\colon T^*N\setminus 0_N\to\mathbb{R}$ be a 1-homogeneous autonomous Hamiltonian function and let $D_{g\geq 0}(N)$ be the full subcategory of $D(N)$ formed by the $\mathscr{F}$ with $SS^\bullet(\mathscr{F})\subset\{g\geq 0\}$. We let $\varphi^t=\varphi^t_g\colon T^*N\setminus 0_N\to T^*N\setminus 0_N$ be the corresponding homogeneous flow. We set for short $K^a_\varphi(\mathscr{F})=\mathscr{F}\circ K_{\varphi^a}$.

We consider the lifting of the graph of the whole isotopy φ^t to Lagrangian in $T^*(N\times N\times\mathbb{R})$ given by

$$\big\{(q,p,Q_t(q,p),P_t(q,p),t,g(q,p))\big\},$$

where $\varphi^t(q,p)=Q_t(q,p),P_t(q,p))$ and the associated kernel $K_\varphi\in D(N^2\times\mathbb{R})$. Then $K_\varphi(\mathscr{F})\in D(N\times\mathbb{R})$ is such that $K_\varphi(\mathscr{F})|_{N\times\{a\}}=K^a_\varphi(\mathscr{F})$. Now, if $\mathscr{F}\in D_{g\geq 0}(N)$, then $K_\varphi(\mathscr{F})\in D_{\tau\geq 0}(N\times\mathbb{R})$. Indeed a point of $SS^\bullet(K_\varphi(\mathscr{F}))$ is written (q,p,Q,P,t,τ) with $(q,p)\in SS^\bullet(\mathscr{F})$ and $\tau=g(q,p)$. We thus have the Tamarkin morphism $\tau_{0,c}(K_\varphi(\mathscr{F}))\colon K_\varphi(\mathscr{F})\to T_{c*}(K_\varphi(\mathscr{F}))$ for $c\geq 0$. Restricting to $N\times\{a\}$ and setting $b=a+c$ we obtain

$$\tau_{a,b}(\mathscr{F})\colon K^a_\varphi(\mathscr{F})\to K^b_\varphi(\mathscr{F}),\qquad \text{for } a\leq b. \tag{6.1}$$

The morphisms $\tau_{a,b}$ are invariant by K^t_φ, for any t, and compatible with composition:

$$\begin{aligned} K^t_\varphi(\tau_{a,b}(\mathscr{F})) &= \tau_{a+t,b+t}(K^t_\varphi(\mathscr{F})), \\ \tau_{a,c}(\mathscr{F}) &= \tau_{b,c}(\mathscr{F}) \circ \tau_{a,b}(\mathscr{F}), \quad \text{for } a \le b \le c \end{aligned} \tag{6.2}$$

Examples 6.1.

(1) Let $N = \mathbb{R}$ and g be the norm given by the standard metric, so $g(t,\tau) = |\tau|$. Then $\varphi^a(t,\tau) = (t + a\frac{\tau}{|\tau|}, \tau)$. As a result $K_{\varphi^a}(k_{[x,y[}) = k_{[x+a,y+a[}$ and $SS(K_{\varphi^a}) = \left\{(t, t + a\frac{\tau}{|\tau|}, \tau, \tau)\right\} \subset T^*(\mathbb{R}^2)$.

(2) In the case of $N \times \mathbb{R}$ and $g = \tau$ we recover the case we introduced in the first paragraph (and originally due to Tamarkin).

(3) If g is non negative we have $D_{g\ge 0}(N) = D(N)$. Typically g is the norm given by a Riemannian metric and φ_g is the normalized geodesic flow. In this case, denoting by $B(x,r)$ the open ball at x of radius r, we find

$$K^a_\varphi(k_{B(x,r)}) \simeq \begin{cases} k_{B(x,r+a)} & \text{for } -r < a < r_{inj} - r, \\ k_{\overline{B(x,-r-a)}}[-\dim N] & \text{for } -r_{inj} - r < a \le -r, \end{cases}$$

and

$$K^a_\varphi(k_{\overline{B(x,r)}}) \simeq \begin{cases} k_{\overline{B(x,r-a)}} & \text{for } r - r_{inj} \le a \le r, \\ k_{B(x,a-r)}[\dim N] & \text{for } r < a < r + r_{inj}, \end{cases}$$

where r_{inj} is the injectivity radius.

6.1. **The γ_g-topology.** The following definition is the analogue of the spectral metric (in the case $g = \tau$) (and a sheaf analogue of the barcode distance). It is introduced in [KS18] or [AI20] (up to a small modification).

Definition 6.2. *Let $g\colon T^*N \setminus 0_N \to \mathbb{R}$ be a 1-homogeneous autonomous Hamiltonian function and $\varphi = \varphi^t_g$ its flow. For $\mathscr{F}, \mathscr{G} \in D_{g\ge 0}(N)$, we let $\gamma_g(\mathscr{F}, \mathscr{G}) \in [0,\infty]$ be the infimum of the $c \ge 0$ for which there exist $a, b \ge 0$ such that $a + b = c$ and such that there are morphisms*

$$u\colon \mathscr{F} \to K^a_\varphi(\mathscr{G}), \qquad v\colon \mathscr{G} \to K^b_\varphi(\mathscr{F})$$

satisfying $K^a_\varphi(v) \circ u = \tau_{0,a+b}(\mathscr{F})$ and $K^b_\varphi(u) \circ v = \tau_{0,a+b}(\mathscr{G})$.

The authors thank N. Vichery for the next remark.

Remark 6.3. The case $D_{g\ge 0}(N)$ is not really more general than the case $D_{\tau\ge 0}(N \times \mathbb{R})$. Indeed the functor $K_\varphi(-)\colon D(N) \to D(N \times \mathbb{R})$ is fully faithful and we have $T_{a*}(K_\varphi(-)) \simeq K_\varphi(K^a_\varphi(-))$. Hence, for $\mathscr{F}, \mathscr{G} \in D_{g\ge 0}(N)$ we have $\gamma_g(\mathscr{F}, \mathscr{G}) = \gamma_\tau(\mathscr{K}_\varphi(\mathscr{F}), \mathscr{K}_\varphi(\mathscr{G}))$.

The definition in [KS18] requires $a = b$ which leads to a bigger distance, say γ'_g, with $\gamma_g \le \gamma'_g \le 2\gamma_g$. The definition in [AI20] only asks $K^a_\varphi(v) \circ u = \tau_{0,a+b}(\mathscr{F})$ and $K^b_\varphi(u') \circ v' = \tau_{0,a+b}(\mathscr{G})$, for two possibly different morphisms u', v'. This leads to a smaller distance but we don't know if it is equivalent to γ_g. Lemma 6.8 below says that γ_g is equivalent to $\inf\{\gamma_g(\mathscr{C}(u), 0);\, u\colon \mathscr{F} \to \mathscr{G}\}$, where $\mathscr{C}(u)$ denotes the cone of u.

Lemma 6.4. *For $\mathscr{F}, \mathscr{G}, \mathscr{H} \in D_{g\ge 0}(N)$ we have*

(1) $\gamma_g(\mathscr{F}, \mathscr{G}) = \gamma_g(K^t_\varphi(\mathscr{F}), K^t_\varphi(\mathscr{G})) = \gamma_g(\mathscr{G}, \mathscr{F})$.
(2) $\gamma_g(\mathscr{F}, \mathscr{H}) \le \gamma_g(\mathscr{F}, \mathscr{G}) + \gamma_g(\mathscr{G}, \mathscr{H})$.
(3) $\gamma_g(0, \mathscr{F}) = \inf\{t \ge 0;\, \tau_{0,t}(\mathscr{F}) = 0\}$.
(4) If $\mathscr{F}$ satisfies $\gamma_g(\mathscr{F}, 0) = 0$, then $\mathscr{F} \simeq 0$.
(5) For any distinguished triangle $\mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \xrightarrow{+1}$, we have

$$\gamma_g(0, \mathscr{G}) \le \gamma_g(0, \mathscr{F}) + \gamma_g(0, \mathscr{H})$$

(6) For composable morphisms u, v with cones $\mathscr{C}(u)$, $\mathscr{C}(v)$, $\mathscr{C}(v \circ u)$, we have $\gamma_g(0, \mathscr{C}(v \circ u)) \le \gamma_g(0, \mathscr{C}(u)) + \gamma_g(0, \mathscr{C}(v))$.

Proof. (1) follows from (6.2) and (3) follows from the definition of γ_g. The triangular inequality (2) is proved in [AI20, Prop. 4.11].

Let us prove (4). By Remark 6.3 we can assume that $N = M \times \mathbb{R}$ and $g = \tau$. Let $x_0 \in M$ be given and let $i\colon \{x_0\} \times \mathbb{R} \to M \times \mathbb{R}$ be the inclusion. We have $\tau_{a,b}(i^{-1}\mathscr{F}) = i^{-1}(\tau_{a,b}(\mathscr{F}))$ and it follows that $\gamma_g(i^{-1}\mathscr{F}, i^{-1}\mathscr{G}) \le \gamma_g(\mathscr{F}, \mathscr{G})$. If $\mathscr{F} \ne 0$, there exists x_0 such that $i^{-1}(\mathscr{F}) \ne 0$. Hence we can assume that $N = \mathbb{R}$. We can also take $g = |\tau|$ (it is differentiable on $T^*\mathbb{R} \setminus 0_{\mathbb{R}}$) instead of $g = \tau$. The corresponding τ morphisms, say $\tau^g_{a,b}$, coincide with $\tau_{a,b}$ on $D_{\tau\ge 0}(\mathbb{R})$ but they are defined on $D(\mathbb{R})$. Up to shifting $\mathscr{F}$ in degrees and translating, we can assume that $H^0\mathscr{F}_0 \ne 0$. Hence there exists $u \in H^0(]-\varepsilon, \varepsilon[; \mathscr{F})$ with $u_0 \ne 0$. Interpreting u as a morphism and chosing a map $v : \mathscr{F}_0 \longrightarrow k$ non-vanishing on the germ defined by u, we get that the composition of v and u; $k_{]-\varepsilon,\varepsilon[} \xrightarrow{u} \mathscr{F} \xrightarrow{v} k_{\{0\}}$ is the canonical morphism. Now the morphisms $\tau^g_{0,\delta}(-)$, for $0 < \delta < \varepsilon$ give the commutative diagram (using Example 6.1 (3)) where the vertical arrows are given by $\tau^g_{0,\delta}$

$$\begin{tikzcd}
k_{]-\varepsilon,\varepsilon[} \arrow[r, "u"] \arrow[d] & \mathscr{F} \arrow[r, "v"] \arrow[d] & k_{\{0\}} \arrow[d, "w"'] \\
k_{]-\varepsilon+\delta,\varepsilon-\delta[} \arrow[r] & T_{\delta *}(\mathscr{F}) \arrow[r] & k_{]-\delta,\delta[}[1]
\end{tikzcd}$$

We have $w \circ v \circ u \ne 0$ and we deduce that $\tau^g_{0,\delta}(\mathscr{F}) \ne 0$, as required.

Let us prove (5) (a similar but more detailed argument will be given in the proof of Lemma 6.8-(i)). If $\tau_{0,a}(\mathscr{F}) = 0$, then $\tau_{0,a}(\mathscr{G}) \circ u = K^a_\varphi(u) \circ$

$\tau_{0,a}(\mathscr{F}) = 0$ and $\tau_{0,a}(\mathscr{G})$ factorizes through v. In the same way, if $\tau_{0,b}(\mathscr{H}) = 0$, then $\tau_{0,b}(\mathscr{G})$ factorizes through $K^b_\varphi(v)$. Hence $\tau_{0,a+b}(\mathscr{G})$ factorizes through $K^b_\varphi(v) \circ K^b_\varphi(u) = 0$. We conclude with (3).

We deduce (6) from (5) since, by the octaedron axiom, the three cones appear in a distinguished triangle. □

If N is real analytic and $\mathscr{F}, \mathscr{G}$ are constructible and satisfy $\gamma_g(\mathscr{F}, \mathscr{G}) = 0$, then $\mathscr{F} \simeq \mathscr{G}$ (see [PSW] and also Proposition B.7 for limits of constructible sheaves). However γ_g is only a pseudo-metric as shown by Berkouk and Ginot (see [BG18], proposition 6.9) and the following example.

Example 6.5. We define two sheaves on the real line $\mathscr{F} = \bigoplus_{x\in\mathbb{Q}} k_{[x,\infty[}$ and $\mathscr{G} = \bigoplus_{x\in\mathbb{Q}^\times} k_{[x,\infty[}$, where $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$. We remark that $K^t_\varphi(\mathscr{F}) \simeq T_{t*}(\mathscr{F})$, where $T_t(x) = x + t$. For a given $\varepsilon > 0$ we choose a bijection $f\colon \mathbb{Q} \to \mathbb{Q}^\times$ such that $x \le f(x) \le x + \varepsilon$, for all $x \in \mathbb{Q}$, and we define $u = \bigoplus_{x\in\mathbb{Q}} u_x\colon \mathscr{F} \to \mathscr{G}$, $v = \bigoplus_{x\in\mathbb{Q}^\times} v_x\colon \mathscr{G} \to T_{\varepsilon*}\mathscr{F}$ where $u_x\colon k_{[x,\infty[} \to k_{[f(x),\infty[}$ and $v_x\colon k_{[x,\infty[} \to k_{[f^{-1}(x)+\varepsilon,\infty[}$ are the natural morphisms. Using u, v we see that $\gamma_g(\mathscr{F}, \mathscr{G}) \le \varepsilon$. Hence $\gamma_g(\mathscr{F}, \mathscr{G}) = 0$.

Definition 6.6. *We denote by γ_g the topology associated to the pseudo-distance γ_g. In other words a sequence $(\mathscr{F}_j)_{j\ge1}$ in $D(N)$ γ_g-converges to $\mathscr{F}$ if and only if* $\lim_j \gamma_g(\mathscr{F}_j, \mathscr{F}) = 0$.

Remark 6.7. In view of the Lemma 6.8, a sequence $(\mathscr{F}_n)_{n\in\mathbb{N}}$ converges to $\mathscr{F}$ if and only if there exist a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 and morphisms $u_n\colon \mathscr{F}_n \to K^{\varepsilon_n}_\varphi(\mathscr{F})$ such that $\gamma(0, \mathscr{C}(u_n))$ converges to 0.

Part (2) in the next lemma is analogue to [AI20, Lem. 4.14].

Lemma 6.8. *Let $u\colon \mathscr{F} \to \mathscr{F}'$ be a morphism in $D(N)$ and let $\mathscr{C}$ be the cone of u.*

(1) If there exists $v\colon \mathscr{F}' \to K^\varepsilon_\varphi(\mathscr{F})$ such that $v \circ u = \tau_{0,\varepsilon}$ and $K^\varepsilon_\varphi(u) \circ v = \tau_{0,2\varepsilon}$, then $\gamma(\mathscr{C}, 0) \le 2\varepsilon$.

(2) If $\gamma(\mathscr{C}, 0) < \varepsilon$, then there exists $v\colon \mathscr{F}' \to K^{2\varepsilon}_\varphi(\mathscr{F})$ such that $v \circ u = \tau_{0,2\varepsilon}$ and $K^{2\varepsilon}_\varphi(u) \circ v = \tau_{0,4\varepsilon}$.

Proof. (1) We consider the morphism of triangles given by the functorial morphism (6.1),

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "u"] \arrow[d] & \mathscr{F}' \arrow[r, "\alpha"] \arrow[d] \arrow[dl, dotted, "v"'] \arrow[dr, "f"] & \mathscr{C} \arrow[r, "\beta"] \arrow[d] \arrow[dr, "g"] & \mathscr{F}[1] \arrow[d] \\
K^\varepsilon_\varphi(\mathscr{F}) \arrow[r] & K^\varepsilon_\varphi(\mathscr{F}') \arrow[r] & K^\varepsilon_\varphi(\mathscr{C}) \arrow[r] & K^\varepsilon_\varphi(\mathscr{F})[1]
\end{tikzcd}$$

Here all small triangles commute (which defines f and g). We have $f = K^{\varepsilon}_{\varphi}(\alpha)\circ\tau_{0,\varepsilon}(\mathscr{F}') = K^{\varepsilon}_{\varphi}(\alpha\circ u)\circ v = 0$. In the same way g also vanishes. Since $\tau_{0,\varepsilon}(\mathscr{C})\circ\alpha = f$ vanishes, $\tau_{0,\varepsilon}(\mathscr{C})$ factorizes through β as $\tau_{0,\varepsilon}(\mathscr{C}) = h\circ\beta$. Hence $\tau_{0,2\varepsilon}(\mathscr{C}) = K^{\varepsilon}_{\varphi}(\tau_{0,\varepsilon}(\mathscr{C}))\circ\tau_{0,\varepsilon}(\mathscr{C}) = K^{\varepsilon}_{\varphi}(h)\circ K^{\varepsilon}_{\varphi}(\beta)\circ\tau_{0,\varepsilon}(\mathscr{C}) = K^{\varepsilon}_{\varphi}(h)\circ g$ vanishes, which means $\gamma(\mathscr{C},0)\le 2\varepsilon$.

(2) Since $\tau_{0,\varepsilon}(\mathscr{C}) = 0$, both morphisms f and g vanish. Since $K^{\varepsilon}_{\varphi}(\alpha)\circ\tau_{0,\varepsilon}(\mathscr{F}') = f$ vanishes, $\tau_{0,\varepsilon}(\mathscr{F}')$ factorizes as $\tau_{0,\varepsilon}(\mathscr{F}') = K^{\varepsilon}_{\varphi}(u)\circ w$ for some w.

In the same way, the vanishing of g gives a factorization $\tau_{0,\varepsilon}(\mathscr{F}) = w'\circ u$ but we cannot say that $w = w'$.

We set $v = K^{\varepsilon}_{\varphi}(w')\circ\tau_{0,\varepsilon}(\mathscr{F}')\colon \mathscr{F}'\to K^{2\varepsilon}_{\varphi}(\mathscr{F})$. In the following computation we omit abusively to write $K^{\varepsilon}_{\varphi}$ and write τ for whatever $\tau_{-,-}(-)$. A diagram chase on the diagram below gives $v\circ u = w'\circ\tau\circ u = w'\circ u\circ\tau = \tau\circ\tau = \tau$ and $u\circ v = u\circ w'\circ\tau = u\circ w'\circ u\circ w = u\circ\tau\circ w = \tau\circ u\circ w = \tau\circ\tau = \tau$, as required.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "u"] \arrow[d] & \mathscr{F}' \arrow[d] \arrow[dl, "w"'] \\
K^{\varepsilon}_{\varphi}(\mathscr{F}) \arrow[r, "u"] \arrow[d] \arrow[ur, phantom, "\circlearrowleft"] & K^{\varepsilon}_{\varphi}(\mathscr{F}') \arrow[d] \arrow[dl, "w'"] \\
K^{2\varepsilon}_{\varphi}(\mathscr{F}) \arrow[r, "u"'] \arrow[ur, phantom, "\circlearrowleft"] & K^{2\varepsilon}_{\varphi}(\mathscr{F}')
\end{tikzcd}$$

□

6.2. **Link with spectral invariants.** In this paragraph we work on $N\times\mathbb{R}$ and choose $g = \tau$. So we work with the category $D_{\tau\ge 0}(N\times\mathbb{R})$ as in §5. We prove Propositions 6.11 and 6.13 which explain the relations between the γ_τ-distance and the spectral invariants.

We first give a useful variation on the Morse theorem for sheaves. The following result is mentioned in [Nad16, §6.1].

Proposition 6.9. *Let M be a manifold and I an open interval of $\mathbb{R}$. Let $\mathscr{F}\in D(M), \mathscr{G}\in D(M\times I)$ and set $\mathscr{G}_t = \mathscr{G}|_{M\times\{t\}}$ for $t\in I$. We assume*

(1) there exists a compact set C of M such that $\mathscr{G}|_{(M\setminus C)\times I}$ is of the form $\mathscr{G}'\boxtimes k_I$ for some $\mathscr{G}'\in D(M\setminus C)$,
(2) $SS^{\bullet}(\mathscr{G})\cap(0_M\times T^*I) = \emptyset$,
(3) $(SS^{\bullet}(\mathscr{F})\times T^*I)\cap SS^{\bullet}(\mathscr{G}) = \emptyset$.

Then $\mathrm{RHom}(\mathscr{F},\mathscr{G}_t)$ *does not depend on $t\in I$.*

Proof. We let $i_t\colon M\times\{t\}\to M\times I$, $t\in I$, be the inclusion. We set $\mathscr{H} = \mathrm{R}\mathscr{H}\mathrm{om}(\mathscr{F}\boxtimes k_I,\mathscr{G})$. By [KS90, Prop. 5.4.14], condition (3) gives $SS(\mathscr{H})\subset$

$\Lambda := (SS(\mathscr{F}) \times 0_I)^a + SS(\mathscr{G})$. We can see moreover that Λ is also non-characteristic for all i_t (that is, $\Lambda \cap (0_M \times T^*I) \subset 0_{M\times I}$). Using condition (2) $i_t^{-1}\mathscr{G} \simeq i_t^!\mathscr{G}[1]$ and $i_t^{-1}\mathscr{H} \simeq i_t^!\mathscr{H}[1]$ by [KS90, Prop. 5.4.13] and we deduce $\mathrm{R}\mathscr{H}\mathrm{om}(\mathscr{F},\mathscr{G}_t) \simeq i_t^{-1}\mathscr{H}$ by the adjunction formula $i_t^!\mathrm{R}\mathscr{H}\mathrm{om}(-,-) \simeq \mathrm{R}\mathscr{H}\mathrm{om}(i_t^{-1}(-), i_t^!(-))$. Hence $\mathrm{RHom}(\mathscr{F},\mathscr{G}_t) \simeq \mathrm{R}\Gamma(M; i_t^{-1}\mathscr{H})$. By the condition (1) $\mathscr{H}|_{(M\setminus C)\times I}$ is of the form $\mathscr{H}' \boxtimes k_I$, for some $\mathscr{H}' \in D(M\setminus C)$, and we can use the base change formula and the bound for the singular support of a direct image. Hence $\mathrm{R}\Gamma(M; i_t^{-1}\mathscr{H}) \simeq (Rq_*(\mathscr{H}))_t$, where $q\colon M\times I \to I$ is the projection and it is enough to check that $Rq_*(\mathscr{H})$ is constant. Since $\Lambda \cap (0_M \times T^*I) \subset 0_{M\times I}$, we have $SS(Rq_*(\mathscr{H})) \subset 0_I$ and the result follows. □

Corollary 6.10. *Let $\mathscr{F} \in D_{\tau\geq 0}(N\times\mathbb{R})$ and t_0 satisfy* (5.3)*. We assume that $SS(\mathscr{F}) \cap SS(T_{c*}(\mathscr{F})) \subset 0_{N\times\mathbb{R}}$ for any $c \neq 0$. Then the restriction map*

$$\mathrm{RHom}(\mathscr{F}, T_{c*}(\mathscr{F})) \to \mathrm{RHom}(\mathscr{F}|_U, T_{c*}(\mathscr{F})|_U) \simeq \mathrm{RHom}(k_U, k_U) \simeq \mathrm{R}\Gamma(N; k_N)$$

is an isomorphism for any $c > 0$, where $U = N\times]c+t_0, +\infty[$. In particular, if $u\colon \mathscr{F} \to T_{c*}(\mathscr{F})$ is id *near $+\infty$, then $u = \tau_{0,c}(\mathscr{F})$.*

When $SS^\bullet(\mathscr{F})$ is the cone over a Legendrian, the hypothesis means that there is no Reeb chord. In particular the hypothesis is satisfied for the sheaves $\mathscr{F}_L$, $L \in \mathscr{L}(T^*N)$.

Proof. We are going to apply Proposition 6.9 with $M = N\times\mathbb{R}$, $I =]0,d]$ for $d > 0$ and $\mathscr{G} = \delta^{-1}(\mathscr{F})$, where $\delta\colon N\times\mathbb{R}^2 \to N\times\mathbb{R}$ is the map

$$(x, t_1, t_2) \mapsto (x, t_1 - t_2)$$

so that $\mathscr{G}|_{N\times\mathbb{R}\times\{c\}} \simeq T_{c*}(\mathscr{F})$.

For $a < b$, $\mathscr{G}$ is constant on $N\times]-\infty, a-t_0[\times[a,b]$ and $N\times]b+t_0,+\infty[\times[a,b]$. Hence $\mathscr{G}$ satisfies (1) in Proposition 6.9. It is not difficult to check (2) since its singular support is contained in the conormal of $t_1 = t_2$ that is $\tau_1 + \tau_2 = 0$. For (3) it follows from the hypothesis on $SS(\mathscr{F})$. The proposition gives $\mathrm{RHom}(\mathscr{F}, T_{c*}(\mathscr{F})) \simeq \mathrm{RHom}(\mathscr{F}, T_{d*}(\mathscr{F}))$, for any $0 < c \leq d$. □

Proposition 6.11. *Let $L_1, L_2 \in \mathscr{L}(T^*N)$. Then*

$$\gamma_\tau(\mathscr{F}_{L_1}, \mathscr{F}_{L_2}) = \begin{cases} c_+(L_1,L_2) & \text{if } c_-(L_1,L_2) > 0, \\ -c_-(L_1,L_2) & \text{if } c_+(L_1,L_2) < 0, \\ c_+(L_1,L_2) - c_-(L_1,L_2) & \text{else.} \end{cases}$$

Proof. We recall that $\mathscr{F}_{L_i} \simeq k_{N\times\mathbb{R}}$ near $+\infty$ and that $-c_-(L_1,L_2)$ is the infimum on the a such that there exists $u\colon \mathscr{F}_{L_1} \to T_{a*}(\mathscr{F}_{L_2})$ restricting to $\mathrm{id}_{k_{N\times\mathbb{R}}}$ near $+\infty$. Similarly $c_+(L_1,L_2)$ is the infimum on the b such that there exists $v\colon \mathscr{F}_{L_2} \to T_{b*}(\mathscr{F}_{L_1})$ restricting to $\mathrm{id}_{k_{N\times\mathbb{R}}}$ near $+\infty$. We

then have $T_{a*}(v)\circ u=\tau_{0,a+b}(\mathscr{F}_{L_1})$ and $T_{b*}(u)\circ v=\tau_{0,a+b}(\mathscr{F}_{L_2})$ by Corollary 6.10. Hence u, v satisfy Definition 6.2, except that here a,b may be negative. Assuming moreover $a,b\geq 0$ we find that $\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})$ is less than the right hand side of the equality in the statement.

Conversely, if $\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})<c$, there exists a,b,u,v as above with $a+b=c$, $a,b\geq 0$ but u, v only satisfy the condition on the composition. Near $+\infty$ we only know that they are $u=\lambda_u\mathrm{id}_{k_{N\times\mathbb{R}}}$, $v=\lambda_v\mathrm{id}_{k_{N\times\mathbb{R}}}$ (indeed $\mathrm{Hom}(k_{N\times\mathbb{R}},k_{N\times\mathbb{R}})\simeq k$). Since $T_{a*}(v)\circ u=\tau_{0,a+b}(\mathscr{F}_{L_1})$, we have $\lambda_u\lambda_v=1$. Hence λ_u, λ_v are not zero and, replacing u, v by a non zero multiple, we can assume that $u=v=\mathrm{id}_{k_{N\times\mathbb{R}}}$ near $+\infty$. We deduce $a\geq\max\{-c_-(L_1,L_2),0\}$, $b\geq\max\{c_+(L_1,L_2),0\}$. □

Remark 6.12. Note that in [Vit22] the metric on $\mathscr{L}(T^*N)$ is defined by $c(L_1,L_2)=|c_+(L_1,L_2)|+|c_-(L_1,L_2)|$ as . The map $L\mapsto\mathscr{F}_L$ is then a bi-Lipschitz embedding. Indeed if $c_-(L_1,L_2)>0$ then $\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})=c_+(L_1,L_2)$ and since $c_-(L_1,L_2)\leq c_+(L_1,L_2)$, we have

$$\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})\leq c(L_1,L_2)=c_+(L_1,L_2)+c_-(L_1,L_2)\leq 2\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})$$

If $c_+(L_1,L_2)<0$ the same inequality holds by exchanging L_1,L_2. Finally if $c_-(L_1,L_2)<0<c_+(L_1,L_2)$ we have $\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})=c_+(L_1,L_2)-c_-(L_1,L_2)$ hence

$$\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})=c(L_1,L_2)$$

So, in all three cases we have

$$\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})\leq c(L_1,L_2)\leq 2\gamma_\tau(\mathscr{F}_{L_1},\mathscr{F}_{L_2})$$

The proof of Proposition 6.11 works exactly the same way for Hamiltonian maps, replacing the condition "$\mathscr{F}_{L_i}\simeq k_{N\times\mathbb{R}}$ near $+\infty$" by "$\mathscr{K}_\varphi\simeq k_{\Delta_N\times[0,+\infty[}$ near $+\infty$". Since we know that $c_-(\varphi)\leq 0\leq c_+(\varphi)$ by [Vit92] we obtain the following statement:

Proposition 6.13. *Let φ_1, φ_2 be compactly supported Hamiltonian maps of T^*N. Then $\gamma_\tau(\mathscr{K}_{\varphi_1},\mathscr{K}_{\varphi_2})=\gamma(\varphi_1,\varphi_2)$.*

We end this section by the useful result that $\circledast$ is Lipschitz for the γ_τ-distance.

Lemma 6.14. *Let M, N, P be manifolds and $\mathscr{K}_1\in D_{\tau\geq 0}(M\times N\times\mathbb{R})$, $\mathscr{K}_2\in D_{\tau\geq 0}(N\times P\times\mathbb{R})$. Then, for any $a\in\mathbb{R}$, we have natural isomorphisms $T_{a*}(\mathscr{K}_1\circledast\mathscr{K}_2)\simeq T_{a*}(\mathscr{K}_1)\circledast\mathscr{K}_2\simeq\mathscr{K}_1\circledast T_{a*}(\mathscr{K}_2)$ and, through these isomorphisms, we have the equalities, for any $a\leq b$,*

$$\tau_{a,b}(\mathscr{K}_1\circledast\mathscr{K}_2)=\tau_{a,b}(\mathscr{K}_1)\circledast\mathscr{K}_2=\mathscr{K}_1\circledast\tau_{a,b}(\mathscr{K}_2)$$

Remark 6.15. The notation $\tau_{a,b}(\mathcal{K}_1)\circledast\mathcal{K}_2$ means that we apply the functor $\circledast\mathcal{K}_2$ to the morphism $\tau_{a,b}(\mathcal{K}_1)$. This means that $(\tau_{a,b}(\mathcal{K}_1)\circledast\mathcal{K}_2)$ is a morphism

$$\tau_{a,b}(\mathcal{K}_1)\circledast\mathcal{K}_2 : T_{a*}(\mathcal{K}_1)\circledast\mathcal{K}_2 \longrightarrow T_{b*}(\mathcal{K}_1)\circledast\mathcal{K}_2$$

Proof. In the proof the manifolds M,N,P play an auxiliary role and we can consider that our sheaves live on $\mathbb{R}$ to simplify the notations. Hence $\mathcal{K}_1\circledast\mathcal{K}_2 = Rs_!(\mathcal{K}_1\boxtimes\mathcal{K}_2)$ where $s\colon\mathbb{R}^2\to\mathbb{R}$ is the sum. The isomorphisms then follow from the equalities $T_a\circ s = s\circ(T_a\times\mathrm{id}_{\mathbb{R}}) = s\circ(\mathrm{id}_{\mathbb{R}}\times T_a)$. However to prove the equalities of morphisms we need to be more precise.

Let (t_1,t_2,τ_1,τ_2) be the coordinates on $T^*\mathbb{R}^2$. Then the singular support of $\mathcal{K}_1\boxtimes\mathcal{K}_2$ is contained in $\{\tau_1\geq 0\}\times\{\tau_2\geq 0\}$ and, setting $g_u = u\tau_1+(1-u)\tau_2$, $\mathcal{K}_1\boxtimes\mathcal{K}_2$ belongs to $D_{g_u\geq 0}(\mathbb{R}^2)$ for any $u\in[0,1]$. Let us write $\tau^u_{a,b}$ for the morphism $\tau_{a,b}$ induced by g_u. Then $\tau^1_{a,b}$ gives $\tau_{a,b}(\mathcal{K}_1)\circledast\mathcal{K}_2$ by appyling $Rs_!$ and $\tau^0_{a,b}$ gives $\mathcal{K}_1\circledast\tau_{a,b}(\mathcal{K}_2)$. It remains to see that $\tau^u_{a,b}$ gives the same morphism for all u. We define $\mathcal{K} = \mathcal{K}_1\boxtimes\mathcal{K}_2\boxtimes k_{[0,1]}\in D(\mathbb{R}^2\times[0,1])$ and $g = u\tau_1+(1-u)\tau_2$. Then $K\in D_{g\geq 0}(\mathbb{R}^2\times[0,1])$ (here we are cheating a little bit because the singular support is not defined on a manifold with boundary – however we can replace $[0,1]$ by $]-\varepsilon,1+\varepsilon[$ and u by a smooth function with values in $[0,1]$ which is 0 on $]-\varepsilon,0]$ and 1 on $[1,1+\varepsilon[)$. We thus obtain a morphism $\tau^g_{a,b}\colon T'_{a*}(\mathcal{K})\to T'_{b*}(\mathcal{K})$ where $T'_c(t_1,t_2,u) = (t_1+uc, t_2+(1-u)c, u)$. Let $s'\colon\mathbb{R}^2\times[0,1]\to\mathbb{R}\times[0,1]$ be the sum of the first two variables. Then $s'\circ T'_c = T_c\times\mathrm{id}_{[0,1]}$ and $\tau^g_{a,b}$ induces $\tau'_{a,b}\colon(T_{a*}(\mathcal{K}_1\circledast\mathcal{K}_2))\boxtimes k_{[0,1]}\to(T_{b*}(\mathcal{K}_1\circledast\mathcal{K}_2))\boxtimes k_{[0,1]}$ which restricts to $\tau^s_{a,b}$ along $\mathbb{R}\times\{s\}$. Now restricting along $\mathbb{R}\times\{s\}$ always induces the same isomorphism $\mathrm{Hom}(\mathcal{F}\boxtimes k_{[0,1]},\mathcal{G}\boxtimes k_{[0,1]})\xrightarrow{\sim}\mathrm{Hom}(\mathcal{F},\mathcal{G})$, for all $\mathcal{F},\mathcal{G}$, namely, the inverse of the isomorphism induced by the pull-back along the projection to $[0,1]$. This concludes the proof. □

We deduce an analogue of [PS21], theorem 2.4.1, in our situation.

Lemma 6.16. *Let $\mathcal{K}_1,\mathcal{K}_2\in D_{\tau\geq 0}(M\times N\times\mathbb{R})$ and $\mathcal{F}\in D_{\tau\geq 0}(M\times\mathbb{R})$. Then $\mathcal{K}_j\circledast\mathcal{F}\in D_{\tau\geq 0}(N\times\mathbb{R})$ and we have*

$$\gamma_\tau(\mathcal{K}_1^{\circledast}(\mathcal{F}),\mathcal{K}_2^{\circledast}(\mathcal{F}))\leq\gamma_\tau(\mathcal{K}_1,\mathcal{K}_2)$$

Proof. We have $\gamma_\tau(\mathcal{K}_1,\mathcal{K}_2)<c$, for a given c, if and only if there exist $a,b\geq 0$ with $a+b=c$ and morphisms $u\colon\mathcal{K}_1\to T_{a*}(\mathcal{K}_2)$, $v\colon\mathcal{K}_2\to T_{b*}(\mathcal{K}_1)$ satisfying the conditions in Definition 6.2. The morphisms induce $u\circ\mathrm{id}_{\mathcal{F}}$, $v\circ\mathrm{id}_{\mathcal{F}}$ which also satisfies the conditions of the definition by Lemma 6.14. □

6.3. **γ_g-limits and colimits.** In this paragraph we shall prove that any sequence $(\mathscr{F}_n)_{n\in\mathbb{N}}$ which γ_g-converges to $\mathscr{F}$ can be turned into an inductive system whose homotopy colimit is $\mathscr{F}$. We recall the notion of homotopy colimit in $D(N)$ (see [BN93, Def. 2.1])

Definition 6.17. *Let $(\mathscr{F}_n, f_n)_{n\in\mathbb{N}}$, with $f_n\colon \mathscr{F}_n \to \mathscr{F}_{n+1}$, be an inductive system in $D(N)$; then* $\operatorname{hocolim}_n \mathscr{F}_n$ *is the cone of the morphism*

$$(\mathrm{id} - f)\colon \bigoplus_{n\geq 0} \mathscr{F}_n \to \bigoplus_{n\geq 0} \mathscr{F}_n$$

where $f = \bigoplus_{n\geq 0} f_n$.

We thus have a distinguished triangle

$$\bigoplus_{n\geq 0} \mathscr{F}_n \xrightarrow{\mathrm{id}-f} \bigoplus_{n\geq 0} \mathscr{F}_n \xrightarrow{p} \operatorname{hocolim}_n \mathscr{F}_n \xrightarrow{+1} \tag{6.3}$$

However, like any cone, $\operatorname{hocolim}_n \mathscr{F}_n$ is only defined up to (non unique) isomorphism and the morphism p itself is not uniquely defined. For a given triangle (6.3) and for any $n_0 \in \mathbb{N}$, we have a morphism $i_{n_0}\colon \mathscr{F}_{n_0} \to \operatorname{hocolim}_n \mathscr{F}_n$ defined as $i_{n_0} = p \circ j_{n_0}$ where j_{n_0} is the obvious morphism to the sum. For another choice of triangle, with p' instead of p, there exists an automorphism ψ of $\operatorname{hocolim}_n \mathscr{F}_n$ such that $p' = \psi \circ p$. Then $i'_{n_0} = p' \circ j_{n_0}$ is "conjugate" to i_{n_0} in the sense $i'_{n_0} = \psi \circ i_{n_0}$.

We recall the "nine diagram" from [BBD82, Prop. 1.1.11] (see [KS06, Ex. 10.6] where it is stated precisely which arrows are given and which ones are obtained), a variant of the octaedron axiom. We assume to be given a commutative square as on the top left corner of the diagram below and four distinguished triangles (two vertical and two horizontal) given by the unmarked solid arrows. The arrows marked [1] are just translations of already defined solid arrows.

The proposition claims that the dotted arrows can be chosen to make all squares commutative, except for the one with Z^2 at its top left corner, which is anticommutative, and the new line and the new column are distinguished triangles.

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d, dotted] & X^0[1] \arrow[d, "{[1]}"'] \\
Y^0 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & Y^2 \arrow[r] \arrow[d, dotted] & Y^0[1] \arrow[d, "{[1]}"'] \\
Z^0 \arrow[r, dotted] \arrow[d] & Z^1 \arrow[r, dotted] \arrow[d] & Z^2 \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\ominus"] & Z^0[1] \arrow[d, "{-[1]}"'] \\
X^0[1] \arrow[r, "{[1]}"] & X^1[1] \arrow[r, "{[1]}"] & X^2[1] \arrow[r, "{-[1]}"] & X^0[2]
\end{tikzcd}$$

Remark 6.18. Using the nine diagram built on the square

$$\begin{array}{ccc} \bigoplus_{n\geq n_0}\mathscr{F}_n & \xrightarrow{\mathrm{id}-f} & \bigoplus_{n\geq n_0}\mathscr{F}_n \\ \downarrow & & \downarrow \\ \bigoplus_{n\geq 0}\mathscr{F}_n & \xrightarrow{\mathrm{id}-f} & \bigoplus_{n\geq 0}\mathscr{F}_n \end{array}$$

we can check that $\mathrm{hocolim}_{n\geq 0}\mathscr{F}_n \xrightarrow{\sim} \mathrm{hocolim}_{n\geq n_0}\mathscr{F}_n$. With the above notations, we take $X^0 = X^1 = \bigoplus_{n\geq n_0}\mathscr{F}_n$, $Y^0 = Y^1 = \bigoplus_{n\geq 0}\mathscr{F}_n$, $Z^0 = Z^1 = \bigoplus_{0\leq n\leq n_0-1}\mathscr{F}_n$. The map from Z^0 to Z^1 is also equal to $\mathrm{id}-f$, but as is an upper triangular finite-dimensional matrix with Id on the diagonal, so it is invertible. This implies $Z^2 = 0$. As a result the vertical arrow from X^2 to Y^2 is an isomorphism (in $D(N)$).

A more general statement of the following Lemma can be found in Lemma 13.33.4 of [Stacks, Section 0A5K]. It tells us, as we might expect, that the limit of a subsequence coincides with the limit of the original sequence.

Lemma 6.19. *Let $(\mathscr{F}_n, f_n)_{n\in\mathbb{N}}$ be an inductive system in $D(N)$ and $\sigma\colon \mathbb{N}\to\mathbb{N}$ an increasing map. Then the compositions of the f_n's define an inductive system $(\mathscr{F}_{\sigma(n)}, f'_n)$ and there exists an isomorphism $j\colon \mathrm{hocolim}_n\mathscr{F}_{\sigma(n)} \xrightarrow{\sim} \mathrm{hocolim}_n\mathscr{F}_n$ such that $j\circ i'_{\sigma(n)} = i_{\sigma(n)}$, where $i'_{\sigma(n)}$ is the morphism $i_\bullet$ for $(\mathscr{F}_{\sigma(n)})$.*

A family of morphisms $u_n\colon \mathscr{F}_n\to\mathscr{G}$ such that $u_{n+1}\circ f_n = u_n$ induces a morphism $u\colon \mathrm{hocolim}_n\mathscr{F}_n\to\mathscr{G}$ such that $u\circ i_n = u_n$ for all n. Indeed we have $\bigoplus_n u_n\circ(\mathrm{id}-f) = 0$, hence $\bigoplus_n u_n$ factorizes as $\bigoplus_n u_n = u\circ p$. Again this morphism u is not uniquely defined.

Lemma 6.20. *Let $(\mathscr{F}_n, f_n)_{n\in\mathbb{N}}$, with $f_n\colon \mathscr{F}_n\to\mathscr{F}_{n+1}$, be an inductive system in $D(N)$. We let $\mathscr{C}(f_n)$ be the cone of f_n and we assume that $\sum_n \gamma_g(\mathscr{C}(f_n),0)$ is finite. Then the sequence $(\mathscr{F}_n)_{n\in\mathbb{N}}$ γ_g-converges to $\mathrm{hocolim}_k\mathscr{F}_k$. More precisely, if we let $i_n\colon \mathscr{F}_n\to\mathrm{hocolim}_k\mathscr{F}_k$ be the morphism defined after* (6.3)*, we have $\gamma_g(\mathscr{C}(i_n),0)\leq 2\sum_{k=n}^{\infty}\gamma_g(\mathscr{C}(f_k),0)$.*

Proof. Let n_0 be given. We set for short $\mathscr{S} = \bigoplus_{n\geq n_0}\mathscr{F}_n$, $\mathscr{C} = \mathrm{hocolim}_n\mathscr{F}_n$. We checked that $\mathscr{C}$ is the cone of $u = \mathrm{id}-\bigoplus_{n\geq n_0} f_n\colon \mathscr{S}\to\mathscr{S}$. We also consider the similar sum but associated to the constant sequence $\mathscr{G}_n = \mathscr{F}_{n_0}$ for all $n\geq n_0$ and $\mathscr{T} = \bigoplus_{n\geq n_0}\mathscr{G}_n$. We define $w\colon \mathscr{T}\to\mathscr{T}$ as $\mathrm{id}-s$ where $s_n\colon \mathscr{G}_n\longrightarrow\mathscr{G}_{n+1}$ is induced by $\mathrm{Id}_{\mathscr{F}_{n_0}}$. The cone of w is $\mathscr{F}_{n_0}$. We let $f_n^m\colon \mathscr{F}_n\to\mathscr{F}_m$ be the composition of the f_k's and we define $F = \bigoplus_{n\geq n_0} f_{n_0}^n\colon \mathscr{T}\to\mathscr{S}$. Then $F\circ w = u\circ F$ and the nine diagram recalled above gives the commutative (except for one anticommuting square) diagram of distinguished

triangles

$$\begin{tikzcd}
\mathscr{T} \arrow[r, "w"] \arrow[d, "F"'] & \mathscr{T} \arrow[r, "q"] \arrow[d, "F"'] & \mathscr{F}_{n_0} \arrow[r, "+1"] \arrow[d, "j_{n_0}"] & {} \\
\mathscr{S} \arrow[r, "u"] \arrow[d] & \mathscr{S} \arrow[r, "p"] \arrow[d] & \mathscr{C} \arrow[r, "+1"] \arrow[d] & {} \\
\mathscr{S}' \arrow[r] \arrow[d, "+1"] & \mathscr{S}' \arrow[r] \arrow[d, "+1"] & \mathscr{C}(j_{n_0}) \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where $\mathscr{S}' = \bigoplus_{n\geq n_0} \mathscr{C}(f_{n_0}^n)$, j_{n_0} is conjugate to i_{n_0} (unfortunately we cannot say they are equal because of the ambiguity in the definition of the cone) and $\mathscr{C}(j_{n_0})$ is the cone of j_{n_0}.

We set $r\colon \mathscr{F}_{n_0} \to \mathscr{T}$ be the inclusion of the first factor. The morphisms r and w give a splitting $\mathscr{T} \simeq \mathscr{F}_{n_0} \oplus \mathscr{T}$ and it follows that $q \circ r$ is an isomorphism. Since i_{n_0} is "conjugate" to $p \circ F \circ r$ (see after (6.3)), it is also conjugate to j_{n_0} and $\mathscr{C}(j_{n_0}) \simeq \mathscr{C}(i_{n_0})$.

Now we claim that $\gamma_g(\mathscr{C}(f_n^m), 0) \leq \sum_{k=n}^{m-1} \gamma_g(\mathscr{C}(f_k), 0)$. We prove this by induction on $m-n$. When $m = n+1$ this is clear. Since $f_n^{m+1} = f_m \circ f_n^m$, we have, using Lemma 6.4 (6), $\gamma_g(\mathscr{C}(f_n^{m+1}), 0) \leq \gamma_g(\mathscr{C}(f_n^m), 0) + \gamma_g(\mathscr{C}(f_m), 0)$.

In particular $\gamma_g(\mathscr{C}(f_{n_0}^n), 0) \leq \varepsilon_{n_0} := \sum_{k=n_0}^{\infty} \gamma_g(\mathscr{C}(f_k), 0)$, for all $n \geq n_0$, and then $\gamma_g(\mathscr{S}', 0) \leq \varepsilon_{n_0}$. By the triangle in the last row of the above diagram and Lemma 6.4 (5) we deduce $\gamma_g(\mathscr{C}(j_{n_0}), 0) \leq 2\varepsilon_{n_0}$, as required. □

Proposition 6.21. *Any Cauchy sequence in $D(N)$ with respect to γ_g is convergent.*

Remark 6.22. The reader should be careful : the distance γ_g is degenerate, so the limit is by no means unique. However we shall see that the restriction of γ_g to the set of limits of constructible sheaves (i.e. in $D_{lc}(N)$) the distance is non-degenerate (see Appendix B).

Proof. Let $(\mathscr{F}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Up to taking a subsequence we can assume that $\gamma_g(\mathscr{F}_n, \mathscr{F}_m) \leq 2^{-n}$ for any $n \leq m$. In particular, by Lemma 6.8, there exist $f_n\colon \mathscr{F}_n \to K_\varphi^{2^{-n}}(\mathscr{F}_{n+1})$ such that $\gamma_g(\mathscr{C}(f_n), 0) \leq 2^{-n+1}$. We then set $\varepsilon_n = \sum_{k\geq n} 2^{-k}$ and $\mathscr{F}'_n = K_\varphi^{-\varepsilon_n}(\mathscr{F}_n)$ so f_n induces a map $f'_n : \mathscr{F}'_n \longrightarrow \mathscr{F}'_{n+1}$ and according to Lemma 6.20, $\mathscr{F}'_n$ γ_g converges to $\operatorname{hocolim} \mathscr{F}'_n$. Since $\gamma_g(\mathscr{F}_n, \mathscr{F}'_n) \leq \varepsilon_n$ we have that $(\mathscr{F}_n)_{n\geq 1}$ has the same limit. □

We just proved that to a Cauchy sequence we can associate a limit that is a homotopy colimit. In the next Proposition we prove that conversely any γ_g-limit of a sequence $(\mathscr{F}_n)_{n\in\mathbb{N}}$ is a homotopy colimit of some

(slightly modified) subsequence. We first notice an easy consequence of Lemma 6.20.

Lemma 6.23. *Let $\mathscr{F} \in D(N)$ and let $\varepsilon_n > 0$ be a sequence decreasing to* 0. *We have an inductive system $(K_\varphi^{-\varepsilon_n}(\mathscr{F}_n), \tau_{-\varepsilon_n,-\varepsilon_{n+1}}(\mathscr{F}))$. Then the maps $\tau_{-\varepsilon_n,0}(\mathscr{F})$ induce an isomorphism* $\operatorname{hocolim} K_\varphi^{-\varepsilon_n}(\mathscr{F}_n) \xrightarrow{\sim} \mathscr{F}$.

Proof. Let $\tau\colon \operatorname{hocolim} K_\varphi^{-\varepsilon_n}(\mathscr{F}_n) \to \mathscr{F}$ be the morphism induced by the maps $\tau_{-\varepsilon_n,0}(\mathscr{F})$. With the notation i_n of Lemma 6.20 we have $\tau \circ i_n = \tau_{-\varepsilon_n,0}(\mathscr{F})$. By the octaedron axiom there exists a distinguished triangle relating the cones of these maps and Lemma 6.4 implies that $\gamma_g(0, \mathscr{C}(\tau)) \leq \gamma_g(0, \mathscr{C}(i_n)) + \gamma_g(0, \mathscr{C}(\tau_{-\varepsilon_n,0}(\mathscr{F})))$. By Lemma 6.8 $\gamma_g(0, \mathscr{C}(\tau_{-\varepsilon_n,-\varepsilon_{n+1}}(\mathscr{F}))) \leq 2(\varepsilon_n - \varepsilon_{n+1})$ and we can apply Lemma 6.20. Hence $\gamma_g(0, \mathscr{C}(i_n))$ goes to 0 and $\gamma_g(0, \mathscr{C}(\tau))$ is as small as desired. So $\mathscr{C}(\tau) \simeq 0$ and τ is an isomorphism. □

Proposition 6.24. *Let $(\mathscr{F}_n)_{n\in\mathbb{N}}$ be a sequence in $D(N)$ which γ_g-converges to $\mathscr{F}$. Then, up to taking a subsequence, there exist a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to* 0 *and morphisms $f_n\colon K_\varphi^{-\varepsilon_n}(\mathscr{F}_n) \to K_\varphi^{-\varepsilon_{n+1}}(\mathscr{F}_{n+1})$, $u_n\colon K_\varphi^{-\varepsilon_n}(\mathscr{F}_n) \to \mathscr{F}$ such that $u_{n+1} \circ f_n = u_n$, for all n, and the morphism* $\operatorname{hocolim} K_\varphi^{-\varepsilon_n}(\mathscr{F}_n) \to \mathscr{F}$ *induced by the u_n's (see after* (6.3)*) is an isomorphism.*

Proof. We set $\alpha_n = 2^{-n}$. Up to taking a subsequence we can assume that $\gamma_g(\mathscr{F}_n, \mathscr{F}) \leq \alpha_{n+1}$ for each n. This implies the existence of $0 \leq \beta_n \leq \alpha_{n+1}$ and morphisms $v_n\colon \mathscr{F} \to K_\varphi^{\beta_n}(\mathscr{F}_n)$, $w_n\colon K_\varphi^{\beta_n}(\mathscr{F}_{\sigma(n)}) \to K_\varphi^{\alpha_{n+1}}(\mathscr{F})$ such that $w_n \circ v_n = \tau_{0,\alpha_{n+1}}(\mathscr{F})$. Translating these morphisms by $K_\varphi^\bullet(-)$ and alternating the $\mathscr{F}_n$ and the $K_\varphi^{-\alpha_n}(\mathscr{F})$ we obtain an inductive system

$$\begin{aligned}\cdots \to K_\varphi^{-\alpha_n}(\mathscr{F}) &\xrightarrow{K_\varphi^{-\alpha_n}(v_n)} K_\varphi^{-\varepsilon_n}(\mathscr{F}_n) \xrightarrow{K_\varphi^{-\alpha_n}(w_n)} K_\varphi^{-\alpha_{n+1}}(\mathscr{F}) \\ &\xrightarrow{K_\varphi^{-\alpha_{n+1}}(v_{n+1})} K_\varphi^{-\varepsilon_{n+1}}(\mathscr{F}_{n+1}) \xrightarrow{K_\varphi^{-\alpha_{n+1}}(w_{n+1})} K_\varphi^{-\alpha_{n+2}}(\mathscr{F}) \to \cdots\end{aligned}$$

where $\varepsilon_n = \alpha_n - \beta_n$. We define $f_n = K_\varphi^{-\alpha_{n+1}}(v_{n+1}) \circ K_\varphi^{-\alpha_n}(w_n)$ and $u_n = \tau_{-\alpha_{n+1},0}(\mathscr{F}) \circ K_\varphi^{-\alpha_n}(w_n)$. The result follows from Lemmas 6.19 and 6.23. □

7. Proof of Theorem 1.2

Let $\mathscr{F} \in D(N)$. We want to prove that around a point $z_0 \in SS(\mathscr{F})$ there is no sequence $(\varphi_j)_{j\geq 1}$ in $\mathfrak{D}\mathfrak{H}\mathfrak{am}_c(B(z_0,\varepsilon))$ such that $\gamma - \lim_j \varphi_j = \mathrm{Id}$ and $\varphi_j(SS(\mathscr{F})) \cap B(z_0,\eta) = \emptyset$. We argue by contradiction and assume the existence of such a sequence. Let $\mathscr{K}_{\varphi_j}$ be the sheaf in $D^b(N \times N \times \mathbb{R})$ inducing φ_j as recalled in §4. We denote by $T^*_{\tau>0}(N \times \mathbb{R})$ the subset of

$T^*(N\times\mathbb{R})$ given by $\tau>0$ and we set $\rho\colon T^*_{\tau>0}(N\times\mathbb{R})\to T^*N$, $(x,t,\xi,\tau)\mapsto(x,\xi/\tau)$. Then $\rho(SS(\mathcal{K}^{\circledast}_{\varphi_j}(\mathcal{F}))\cap T^*_{\tau>0}(N\times\mathbb{R}))=\varphi_j(SS(\mathcal{F}))$ does not intersect $B(z_0,\eta)$. We want to prove that this implies that $SS(\mathcal{F})\cap B(z_0,\eta)=\emptyset$.

We will use the following propositions.

Proposition 7.1. *Let $(\varphi_j)_{j\geq1}$ be a sequence in $\mathfrak{D}\mathfrak{H}\mathfrak{am}_c(T^*N)$ such that $\gamma-\lim\varphi_j=\mathrm{Id}$ and $\mathcal{K}^{\circledast}_{\varphi_j}$ its quantization in $D^b(N\times N\times\mathbb{R})$. Then for all $\mathcal{F}\in D^b(N)$ we have*

$$\gamma_g-\lim\mathcal{K}^{\circledast}_{\varphi_j}(\mathcal{F})=\mathcal{K}^{\circledast}_{\mathrm{Id}}(\mathcal{F})=\mathcal{F}\boxtimes k_{[0,+\infty[}\in D^b(N\times\mathbb{R})$$

Proof. This follows from Lemma 6.16 and Proposition 6.13. □

Remark 7.2. Note that $\varphi_j(SS(\mathcal{F}))$ is not a homogeneous manifold, and $SS(\mathcal{K}^{\circledast}_{\varphi_j}(\mathcal{F}))\subset T^*(N\times\mathbb{R})$ represents the homogenization of $\varphi_j(SS(\mathcal{F}))$.

Now for a sequence $(X_j)_{j\geq0}$ of subsets of a topological space Z, its topological upper and lower limits are defined as

$$\begin{aligned}\limsup_j X_j&=\bigcap_n\overline{\bigcup_{j\geq n}X_j}\\&=\left\{x\in Z\mid\exists(x_j)_{j\geq1},\ x_j\in X_j\text{ for infinitely many }j,\ \lim_j x_j=x\right\}\\\liminf_j X_j&=\left\{x\in Z\mid\exists(x_j)_{j\geq1},x_j\in X_j,\lim_j x_j=x\right\}\end{aligned}$$

Obviously $\liminf_j X_j\subset\limsup_j X_j$. We have

Proposition 7.3. *Let M be a manifold and let $g\colon T^*M\setminus0_M\to\mathbb{R}$ be a 1-homogeneous autonomous Hamiltonian function. If $(\mathcal{F}_j)_{j\geq1}$ is a sequence such that $\gamma_g-\lim\mathcal{F}_j=\mathcal{F}$ then we have*

$$SS(\mathcal{F})\subset\liminf_j SS(\mathcal{F}_j)$$

Proof. By Proposition 6.24, up to taking a subsequence and replacing $\mathcal{F}_j$ by $K^{\varepsilon_j}_{\varphi_g}(F_j)$, for some sequence (ε_j) converging to 0, we can assume that there exist morphisms $f_j\colon\mathcal{F}_j\to\mathcal{F}_{j+1}$ such that $\operatorname{hocolim}\mathcal{F}_j\xrightarrow{\sim}\mathcal{F}$. The homotopy colimit is defined by a distinguished triangle where the other two terms are $\bigoplus_{n\geq n_0}\mathcal{F}_n$ (we can indeed restrict to $\bigoplus_{n\geq n_0}$ by Remark 6.18). By the triangle inequality for singular supports and Proposition 3.4, this implies that $SS(\mathcal{F})\subset\limsup_j SS(\mathcal{F}_j)$. Now let $p=(x,\xi)\notin\liminf_j SS(\mathcal{F}_j)$, we can find a subsequence $\sigma(j)$, such that $p\notin\limsup_j SS(\mathcal{F}_{\sigma(j)})$; but $\mathcal{F}_{\sigma(j)}$ has limit $\mathcal{F}$, so $p\notin SS(\mathcal{F})$. As a result we have $SS(\mathcal{F})\subset\liminf_j SS(\mathcal{F}_j)$. □

We may now conclude the proof of the Theorem.

As indicated above, we argue by contradiction and assume $SS(\mathscr{F})$ is not γ-coisotropic at $z_0 = (x_0, p_0) \in SS(\mathscr{F})$. Then there is a sequence $(\varphi_j)_{j\geq 1}$ of Hamiltonian maps and an open ball $B(z_0, \eta)$ such that

(1) $(\varphi_j)_{j\geq 1}$ γ-converges to id,
(2) $\varphi_j(SS(\mathscr{F})) \cap B(z_0, \eta) = \emptyset$.

The first statement together with Proposition 7.1 implies that

$$\mathscr{K}^{\circledast}_{\varphi_j}(\mathscr{F}) \xrightarrow{\gamma_g} \mathscr{K}^{\circledast}_{\mathrm{id}}(\mathscr{F}) = \mathscr{F} \boxtimes k_{[0,+\infty[}$$

and Proposition 7.3, applied with $M = N \times \mathbb{R}$ and $g = \tau$, then gives

$$SS(\mathscr{F}) \times (\{0\} \times \mathbb{R}) = SS(\mathscr{K}^{\circledast}_{\mathrm{id}}(\mathscr{F})) \subset \liminf_j SS(\mathscr{K}^{\circledast}_{\varphi_j}(\mathscr{F}))$$

Using $\rho(SS(\mathscr{K}^{\circledast}_{\varphi_j}(\mathscr{F})) \cap T^*_{\tau>0}(N \times \mathbb{R})) = \varphi_j(SS(\mathscr{F}))$ we deduce $SS(\mathscr{F}) \subset \liminf_j \varphi_j(SS(\mathscr{F}))$. In particular $SS(\mathscr{F}) \cap B(z_0, \eta) = \emptyset$ by the hypothesis (2), which gives a contradiction.

8. γ-COISOTROPIC VS CONE-COISOTROPIC

8.1. **A γ-coisotropic set is cone-coisotropic.** Our goal in this section is to prove Proposition 1.3.

Following Bouligand (see [Bou32]), one may define for a point x in a subset V in a smooth manifold, two cones :

Definition 8.1. *The* **paratingent cone** *of a set V at x is*

$$C^+(x, V) = \left\{\lim_n c_n(x_n - y_n) \mid x_n, y_n \in V, c_n \in \mathbb{R} \lim_n x_n = \lim_n y_n = x, \lim_n c_n = +\infty\right\}$$

The **contingent cone** *of a set V at x is*

$$C^-(x, V) = \left\{\lim_n c_n(x_n - x) \mid x_n \in V, c_n \in \mathbb{R}, \lim_n x_n = x, \lim_n c_n = +\infty\right\}$$

Clearly $C^-(x, V) \subset C^+(x, V)$. Note that $C^+(x, V)$ is invariant by $v \mapsto -v$, while it is not necessarily the case for $C^-(x, V)$. We then have the following definition, which appears under the name "involutivity" in the book of Kashiwara and Schapira

Definition 8.2 (Cone-coisotropic, see [KS90], definition 6.5.1 p. 271)**.** *We shall say that V is* **cone-coisotropic** *if whenever a hyperplane H is such that $C^+(x, V) \subset H$ then the symplectic orthogonal of H, H^ω is contained in $C^-(x, V)$.*

Proof of Proposition 1.3. We want to prove that if V is not cone-coisotropic at $z \in V$ then it is not γ-coisotropic at z. For H a hyperplane we set n_H the normal to H (for the usual scalar product). Let us assume that $z \in V$ is

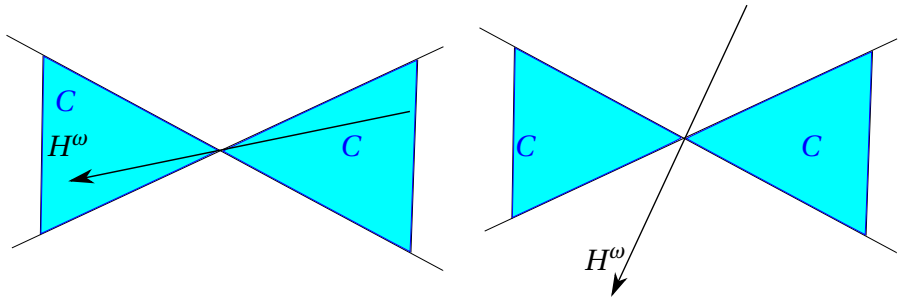

FIGURE 2. Coisotropic cone on the left, non-coisotropic on the right

such that $C^+(z,V) \subset H$ and $H^\omega \notin C^-(z,V)$. This will follow from the next two lemmata. □

Lemma 8.3. *Let V be a closed set, $z \in V$ and H a hyperplane such that $C^+(z,V) \subset H$. Then there exists a continuous function $f_{H,V}$ defined in a neighbourhood of z in H such that $f_{H,V}(z) = 0$ and $d_G f_{H,V}(z) = 0$, where d_G is the Gâteaux derivative, and such that for any neighbourhood $C^+_\varepsilon(z,V)$ of $C^+(z,V)$ we have for δ small enough*

$$V \cap B(z,\delta) \subset \left\{z + x + f_V(x) n_H \mid x \in C^+_\varepsilon(z,V)\right\}$$

Proof. We can assume for simplicity that $z = 0$. For $x \in H$ let $V_x = V \cap (x + \mathbb{R} n_H)$. We claim that for x close enough to 0, V_x contains at most one point. Indeed, if this was not the case, we would have sequences $(y_n)_{n\geq 1}, (w_n)_{n\geq 1}$ converging to 0 such that $y_n - w_n = t_n n_H$ with $t_n \neq 0$. But then by definition of $C^+(0,V)$, we would have $n_H \in C^+(0,V)$ a contradiction.

As a result, on the closed set $\Gamma \subset H$ defined as the set of x such that V_x is non empty, we have a function f such that

$$x \in \Gamma \Longleftrightarrow (x, f(x)) \in V$$

Let us prove f is continuous on Γ, at least in a neighbourhood of 0. If this was not the case, there would be a sequence $(z^n)_{n\geq 1}$ converging to 0 and for each n two sequences $(x^n_k)_{k\geq 1}$, $(y^n_k)_{k\geq 1}$ such that $\lim_k x^n_k = \lim y^n_k = z^n$ and $\lim_k f(x^n_k) \neq \lim_k f(y^n_k)$. Setting $u^n_k = (x^n_k, f(x^n_k))$, $v^n_k = (y^n_k, f(y^n_k))$ we have

$$u^n_k - v^n_k = (x^n_k - y^n_k, f(x^n_k) - f(y^n_k))$$

which can be normalized so that, for any n, it has a limit $(0,1)$ when $k \to +\infty$. As a result for $k = k(n)$ large enough, setting $u_n = u^n_{k(n)}, v_n = v^n_{k(n)}$, there is a τ_n so that

$$\|\tau_n(u_n - v_n) - (0,1)\| < 1/3$$

so we can find a converging subsequence to a vector w which does not belong to H. Since $w \in C^+(0,V)$ we have a contradiction. From now on we replace Γ by $\Gamma \cap B(0,r)$ with r small enough so that f is now continuous on Γ.

Let us now prove that $\lim\limits_{\substack{|x|\to 0\\ x\in\Gamma}} \dfrac{f(x)}{|x|} = 0$. Indeed, if we had a sequence x_k in Γ converging to 0 and such that $\lim\limits_k \dfrac{f(x_k)}{|x_k|} \neq 0$, the sequence of vectors $\frac{1}{|x_k|}(x_k, f(x_k))$ would have a limit which does not belong to H, so $C^+(0,V) \not\subset H$, a contradiction.

We now will extend f to $f_{H,V}$ such that

(1) $f_{H,V}(x) = f(x)$ for $x \in \Gamma$
(2) $f_{H,V}$ is continuous
(3) $d_G(f_{H,V}(0)) = 0$

For this we can assume for simplicity that $r = 1$ and write $x = \rho \cdot \theta$ with $0 \le \rho \le 1, \theta \in S^{n-1}$. We write $g(s,\theta) = f(e^s \cdot \theta)$ with $s \in]-\infty, 0]$ and identify Γ with its image in $]-\infty,0] \times S^{n-1}$. Now let $g_n(s,\theta)$ be a continuous function defined on $C_n =]-n-1, -n+1[\times S^{n-1}$ for $n \in \mathbb{N}$, coinciding with g on $\Gamma \cap C_n$ and having the same bound, i.e.

$$\sup\left\{g_n(s,\theta) \mid (s,\theta) \in C_n\right\} = \sup\left\{g(s,\theta) \mid (s,\theta) \in \Gamma \cap C_n\right\}$$

$$\inf\left\{g_n(s,\theta) \mid (s,\theta) \in C_n\right\} = \inf\left\{g(s,\theta) \mid (s,\theta) \in \Gamma \cap C_n\right\}$$

Such a function exists by Tietze's extension theorem.

Now let χ_n be a partition of unity subordinated to the covering of $]-\infty,0] \times S^{n-1}$ by the C_n. We set

$$G(s,\theta) = \sum_{n\ge 0} \chi_n(s) g_n(s,\theta)$$

Note that this is well defined since whenever $g_n(s,\theta)$ is not defined, we have $\chi_n(s,\theta) = 0$, so we set $\chi_n(s)g_n(s,\theta) = 0$ outside of C_n. And since whenever $g_n(s,\theta)$ is defined and $(s,\theta) \in \Gamma$ we have $g_n(s,\theta) = g(s,\theta)$, we see that F is an extension of G. We now set

$$f_{H,V}(e^s \cdot \theta) = G(s,\theta)$$

Properties (1) and (2) are then obvious. As for the third one, we notice that

$$\sup\left\{f_{H,V}(x) \mid |x| \le e^{-n}\right\} = \sup\{G(s,\theta) \mid (s,\theta) \mid s \le -n\} \le$$
$$\sup_{k\ge n}\sup\left\{g_k(s,\theta) \mid (s,\theta) \in C_k\right\} \le \sup\left\{g(s,\theta) \mid (s,\theta) \in \Gamma,\, s \le -n\right\} =$$
$$\sup\left\{f(x) \mid x \in \Gamma,\, |x| \le e^{-n}\right\}$$

We see that since $\lim\limits_{\substack{|x|\longrightarrow 0\\ x\in\Gamma}} \dfrac{f(x)}{|x|} = 0$ we have $\lim\limits_{x\longrightarrow 0} \dfrac{f_{H,V}(x)}{|x|} = 0$.

As a result

$$\lim_{t\longrightarrow 0} \frac{1}{t}(f_{H,V}(th) - f_{H,V}(0)) = 0$$

and we may conclude that $d_G f_{H,V}(z) = 0$. □

Lemma 8.4. *Let f be a continuous function on the hyperplane H such that $d_G f(0) = 0$ and C a cone in H not containing H^ω. Set*

$$\widetilde{C} = \left\{x + f(x)n_H \mid x \in C\right\}$$

Then for any positive ε, there is a Hamiltonian map φ such that $\gamma(\varphi) \le \varepsilon$ and

$$\varphi(\widetilde{C} \cap B(0,1)) \cap B(0,1) = \emptyset$$

Proof. We shall assume w.l.o.g. that $H = \{p_n = 0\}$ and $z = 0$, so that H^ω is given by the direction q_n. First let us mention that the case $f = 0$ is trivial. Just take a Hamiltonian $H(p_n)$ such that $H(0) = 0$, $H'(0) = A$ and H is C^0 small, A being chosen so that in $B(0,1)$ we have

$$C \cap B(0,1) \cap (A\frac{\partial}{\partial q_n} + C) = \emptyset$$

This extends to the case where f is smooth. Indeed, let us set $K(q_n, f(q_n,\bar{q},\bar{p}),\bar{q},\bar{p}) = 0$ and $\frac{\partial K}{\partial p_n}(q_n, f(q_n,\bar{q},\bar{p}),\bar{q},\bar{p}) = A$. We have, for any $(q_n,p_n,\bar{q},\bar{p}) \in \widetilde{C}$, $\varphi_H^t(q_n,p_n,\bar{q},\bar{p}) = (q_n + At, p_n(t), \bar{q}(t), \bar{p}(t))$ hence

$$\varphi_K^t(\widetilde{C}) \cap B(0,1) = \emptyset$$

Of course we have to truncate the flow, so that it is unchanged on the trajectory of points in $\varphi_K^t(\widetilde{C}) \cap B(0,1)$, but this is not difficult.

Note that the same holds if we replace $\widetilde{C}$ by the region

$$\widetilde{C}_{g,g'} = \left\{(q_n,p_n,\bar{q},\bar{p}) \mid (q_n,\bar{q},\bar{p}) \in C, g(q_n\bar{q},\bar{p}) \le p_n \le g'(q_n\bar{q},\bar{p})\right\}$$

where $g \le g'$ are smooth functions with $dg(0) = dg'(0) = 0$, except that now we cannot have $\gamma(\varphi_K)$ arbitrarily small because K has oscillation bounded from below by $A\|g - g'\|_{C^0}$. Indeed we must have $\frac{\partial K}{\partial p_n} = A$ on

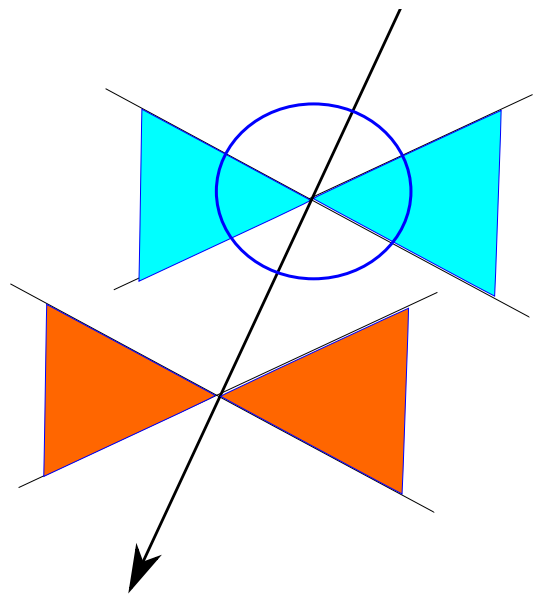

FIGURE 3. The cone C and its translation by $A\frac{\partial}{\partial q_n}$

a segment in the p_n direction of length $g'(q_n\bar{q},\bar{p}) - g'(q_n\bar{q},\bar{p})$. However we can assume $\gamma(\varphi_K) \le A\|g - g'\|_{C^0} + \varepsilon$ whith $\varepsilon > 0$ arbitrarily small.

The general case is obtained as follows. We take a sequence $g_k \le f \le g'_k$ such that $\|g_k - g'_k\|_{C^0(B(0,r))}$ goes to zero with k. We can then displace $\widetilde{C}_{g_k,g'_k}$ hence $\widetilde{C}$ by a map with $\gamma(\varphi_k) \le A\|g_k - g'_k\|_{C^0(B(0,r))} + \varepsilon_k$. As k goes to infinity, we may let ε_k go to zero. This concludes our proof. □

Remark 8.5. The proof actually shows that V is locally rigid in the sense of Usher (see [Ush19]). So we have that γ-coisotropic implies locally rigid which implies cone-coisotropic.

8.2. **A cone-coisotropic set that is not γ-coisotropic.** Let K_a be the central Cantor set of ratio a. In other words we set

$$F_1 = [0,a] \cup [1-a,1]$$

$$F_2 = [0,a^2] \cup [a(1-a),a] \cup [1-a,1-a+a^2] \cup [1-a^2,1]$$

and $K_a = \bigcap_n F_n$.

Proposition 8.6. *For $a < 2^{-2n}$, the set $X = K_a^{2n}$ is cone-coisotropic but not γ-coisotropic.*

Proof. We claim that $C^+(z,X)$ is not contained in any hyperplane. Indeed for $x \in K = K_a$ the cone $C^+(x,K)$ is never empty since any point in

K is a cluster point of K. In fact $C^-(x,K)$ is either $\mathbb{R}_-, \mathbb{R}_+$ or $\mathbb{R}$ depending whether x is the limit of points of K less than x, the limit of points of K greater than x or both, and $C^+(x,K) = \mathbb{R}$. Even though the paratingent cone of a product does not necessarily contain the product of the paratingent cones of the factors, in the present case $C^+(z,X)$ contains the rays $\{0^{j-1}\} \times C^+(x_j,K) \times \{0^{n-j}\}$ where $z = (x_1,..,x_{2n})$, and $0^k = (0,...,0) \in \mathbb{R}^k$. Therefore $C^+(z,X)$ is not contained in any hyperplane and X is trivially cone-coisotropic.

We now claim X is not γ-coisotropic. Pick $z \in X$ and balls $B(z,\varepsilon)$, $B(z,\eta)$, $0 < \eta < \varepsilon$. Let k big enough (to be made precise later). Since K is contained in F_k which consists of 2^k intervals of length a^k, our X is contained in a family $\mathscr{I}$ of cubes of edge length a^k with $|\mathscr{I}| = 2^{2nk}$. The projection $(q,p) \mapsto (q_2,\dots,q_n,p_1,\dots,p_n)$ maps $\mathscr{I}$ to a family $\mathscr{I}'$ of cubes of $\mathbb{R}^{2n-1}$ (again these cubes have edge length a^k and $|\mathscr{I}'| = 2^{(2n-1)k}$). We choose disjoints neighbourhoods U_i, $i \in \mathscr{I}'$, of the cubes in $\mathscr{I}'$. For a given $i_0 \in \mathscr{I}'$, let C_j, $j = 1,\dots,N$, be the cubes of $\mathscr{I}$ contained in $B(z,\varepsilon) \cap (\mathbb{R} \times U_{i_0})$, ordered according to the q_1 variable. Let N_1 be the maximal index j such that C_j meets $B(z,\eta)$. We push all cubes $C_2,\dots,C_{N_1}$ to the left, close to C_1, in the space between $B(z,\varepsilon)$ and $B(z,\eta)$ (if k is big enough), as follows. We move C_2 by a translation T_2 with direction $-\partial_{q_1}$, so that $T_2(C_2)$ is close to C_1. Then we move C_3 by a similar translation T_3, so that $T_3(C_3)$ is close to $T_2(C_2)$. We go on until C_{N_1}. We have thus accumulated at most 2^k cubes of edge length a^k near C_1. Since $a < 1/2$, we have room to put all these cubes in $B(z,\varepsilon) \setminus B(z,\eta)$ for k big enough.

Now the translation T_j can be realized by a Hamiltonian function of the form $h_j = h(p_1)\rho_j(q,p)$, where $h' = 1$ and ρ_j is a bump function vanishing outside $B(z,\varepsilon) \cap (\mathbb{R} \times U_{i_0})$ and over $C_1 \cup \dots \cup T_{j-1}(C_{j-1}) \cup C_{j+1} \cup \dots \cup C_{N_1}$ and equals to 1 over the path swept by C_j. We have $||h|| \sim a^k$ and we apply the isotopy for a length of time less than $\varepsilon < 1$. We have to do this less than $|\mathscr{I}|$ times. Since we chose $a < 2^{-2n}$, the composition of all these isotopies has norm as small as required.

□

Remark 8.7. We can weaken the assumption $a < 2^{-2n}$ to $a < 1/4$ at the expense of a more complicated proof.

Question 8.8. Find an example where the image of a smooth non-coisotropic submanifold C by a symplectic homeomorphism, φ, is such that $\varphi(C)$ is cone-coisotropic (but will not be γ-coisotropic by [Vit22]).

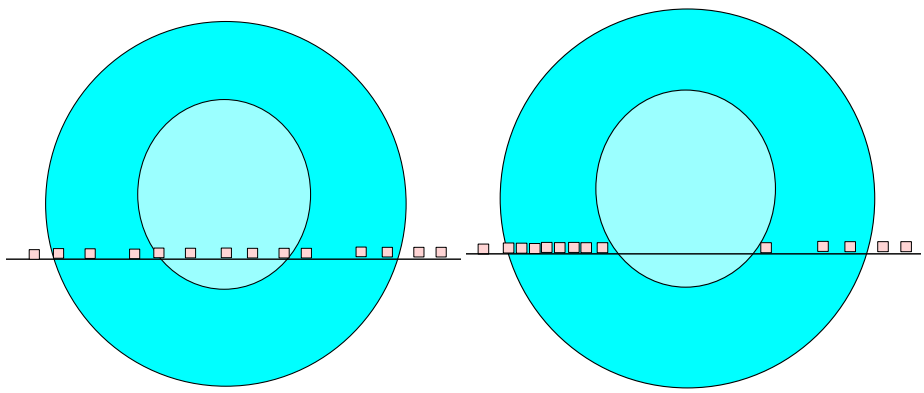

FIGURE 4. Moving the cubes..

9. QUESTIONS AND COMMENTS

We gather in this section a number of questions that sprang up naturally while writing this paper.

Question 9.1. What is the γ_g-distance of k_X and k_Y for two closed sets X, Y. If X, Y do not have the same cohomology, the distance is infinite. So we can focus on the case when $Y = X_\varepsilon$ is an ε-neighbourhood of X. When is it true that $\gamma_g(k_X, k_{X_\varepsilon}) \leq C\varepsilon$? Or when do we have $\gamma - \lim k_{X_\varepsilon} = k_X$?

For the next four questions we choose on N a real analytic structure. We let $D_{lc}(N)$ be the category of limits of constructible sheaves considered in Appendix B.

Question 9.2. In Appendix B we describe the objects of $D_{lc}(\mathbb{R})$ with microsupport contained in $\{\tau \geq 0\}$. Is there a similar description in the general case ?

Question 9.3. Using Lemma B.3 and the proof of Lemma B.1 it is probably not hard to see that $D_{lc}(N)$ is a triangulated category. If we have a map $f\colon N \to M$, can we find conditions so that $Rf_*(D_{lc}(N)) \subset D_{lc}(M)$, $f^{-1}(D_{lc}(M)) \subset D_{lc}(N)$?

For example this is true for a map of the type $f \times \mathrm{id}_{\mathbb{R}}\colon N \times \mathbb{R} \to M \times \mathbb{R}$, if we consider the γ_τ-distance, because Rf_* and f^{-1} are Lipschitz for this distance. In general f^{-1} is not Lipschitz: take $f\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, 0)$, and D_ε the closed disc with center $(0,1)$ and radius $1-\varepsilon$; then $\gamma_g(k_{D_0}, k_{D_\varepsilon}) = \varepsilon$ but $\gamma_g(f^{-1}(k_{D_0}), f^{-1}(k_{D_\varepsilon})) = +\infty$ (here $g(q,p) = ||p||$). Hence it is not clear that $f^{-1}(D_{lc}(M)) \subset D_{lc}(N)$.

Question 9.4. Let $D_{dom}(N)$ be the subcategory of $D(N)$ generated in the triangulated sense by the k_U, where U runs over the domains with smooth boundary. It should not be difficult to prove that $D_{dom}(N) \subset D_{lc}(N)$ by approximating domains with smooth boundary by domains with analytic boundary. Is it true that, for any subanalytic open subset V of X and any $\varepsilon > 0$, there exists a domain U with smooth boundary such that $U \subset V$ and $K^{\varepsilon}_{\varphi_g}(k_U) \simeq k_{U_\varepsilon}$ with U_ε open and $V \subset U_\varepsilon$? In this case we can see that $D_{lc}(N)$ coincides with the γ-closure of $D_{dom}(N)$, which would imply that $D_{lc}(N)$ only depends on the smooth structure of N.

Question 9.5. Let $D_{tri}(N)$ be the set of sheaves which are constructible with respect to some (non-fixed !) triangulation of N. Do the γ-closures of $D_{dom}(N)$ and $D_{tri}(N)$ coincide ? Are they equal to $D_{lc}(N)$?

We remind the reader of the following (see [KS18], theorem 2.11, and [PS21], corollary 2.4.2)

Proposition 9.6. *We have for all* 1*-Lipschitz maps* $f : M \longrightarrow N$ *the inequality*

$$\gamma_g((Rf)_*\mathscr{F}, (Rf)_*\mathscr{G}) \leq \gamma_g(\mathscr{F}, \mathscr{G})$$

Question 9.7. Can we give conditions on $\mathscr{F}$,$\mathscr{G}$ (for example $\gamma_g(\mathscr{F}, \mathscr{G}) < \infty$) such that the opposite inequality

$$\gamma_g(\mathscr{F}, \mathscr{G}) \leq C \sup_{f \in Lip_1(X,\mathbb{R})} \gamma_{|\tau|}(Rf_*(\mathscr{F}), Rf_*(\mathscr{G}))$$

holds for some constant C?

Question 9.8. Let f be a continuous function on N. Then $\mathscr{F}_f = k_{\{(x,t)|f(x)\leq t\}}$ is a sheaf on $N \times \mathbb{R}$, and Vichery defined a subdifferential as

$$\partial f = SS(\mathscr{F}_f) \cap \{\tau = 1\}/(t) \subset T^*N$$

Its intersection with $T^*_x N$ is $\partial f(x)$. On the other hand $df \in \widehat{\mathfrak{L}}(T^*N)$, so we may ask whether

$$SS(\mathscr{F}_f) \cap \{\tau = 1\}/(t) = \partial f = \gamma - \mathrm{supp}(df)$$

This leads to more general questions. First notice the following

Proposition 9.9. *There is a functor* $Q : \widehat{\mathscr{L}}(T^*N) \longrightarrow D_{lc}(N \times \mathbb{R})$ *extending the usual quantization from [Gui12; Vit19]. Moreover* Q *is bi-Lipschitz of ratio* 2 *from* $(\widehat{\mathscr{L}}(T^*N), \gamma)$ *to* $(D_{lc}(N \times \mathbb{R}), \gamma_\tau)$.

Proof. Let L_n be a Cauchy sequence in $\mathscr{L}(T^*N)$. Then $\mathscr{F}_{L_n}$ is a Cauchy sequence in $D_{lc}(N \times \mathbb{R})$. The map $L \longrightarrow \mathscr{F}_L$ is a bi-Lipschitz map from the metric c to the (pseudo-) metric γ_τ. As we saw in Remark 6.12, the metric γ_τ is bounded between $1/2c$ and c where c is the metric defined on

$\mathscr{L}(T^*N)$ in [Vit22] by $c(\widetilde{L}_1,\widetilde{L}_2) = |c_+(L_1,L_2)|+|c_-(L_1,L_2)|$. Since according to Proposition B.7 the metric γ_τ is a bona fide metric in $D_{lc}(N\times\mathbb{R})$, limits are unique and the limit of $\mathscr{F}_{L_n}$ is well-defined. □

Question 9.10. Set $\rho(x,t;\xi,\tau) = (x;\xi/\tau)$. Given $\Lambda \in \widehat{\mathscr{L}}(T^*N)$, do we have $\gamma-\mathrm{supp}(\Lambda) = \rho(SS^\bullet(Q(\Lambda)))$?

A positive answer would also imply a positive answer to Question 9.8, i.e. $\partial f = \gamma-\mathrm{supp}(df)$ by taking $\Lambda_n = df_n$ for some sequence of differentiable functions converging to f.

Question 9.11. Can one characterize the sheaves $\mathscr{F}$ such that there exists an element $\widehat{SS}(\mathscr{F})$ in $\widehat{\mathscr{L}}(T^*N)$ so that $SS(\mathscr{F}) = \gamma-\mathrm{supp}(\widehat{SS}(\mathscr{F}))$? In the previous question we considered the $\mathscr{F}$ which are γ-limits of sheaves $\mathscr{F}_j$ quantizing some exact Lagrangian.

Let us give an example of such a situation : let M,N be closed manifolds. For a map $f\in C^\infty(M,N)$ set Λ_f be the correspondence in $T^*M\times T^*N$ defined by

$$\Lambda_f = \left\{(x,p_x,y,p_y) \mid y = f(x), p_y\circ df(x) = -p_x\right\}$$

Now let f_n be a sequence of smooth maps C^0-converging to $f\in C^0(M,N)$. For $\mathscr{F}\in D^b(M)$ the sequence $(R(f_n)_*\mathscr{F})_{n\geq 1}$ γ-converges to $Rf_*(\mathscr{F})$ and $SS(R(f_n)_*\mathscr{F})\subset \Lambda_{f_n}\circ SS(\mathscr{F})$. Thus $SS(R(f_n)_*\mathscr{F})$) should γ-converge to a subset of $\Lambda_f\circ SS(\mathscr{F})$. Of course since f is only C^0, Λ_f is not defined as a set, but it is well defined in $\widehat{\mathscr{L}}(T^*N\times T^*M)$, hence $\Lambda_f\circ SS(\mathscr{F})\in\widehat{\mathscr{L}}(T^*N)$. Thus we expect $\widehat{SS}(Rf_*(\mathscr{F})) = \Lambda_f\circ SS(\mathscr{F})$ and $SS(Rf_*\mathscr{F})\subset\gamma-\mathrm{supp}(\Lambda_f\circ SS(\mathscr{F}))$.

Question 9.12. Let $\mathscr{F}_n$ be a sequence in $D(N)$ γ_g-converging to $\mathscr{F}_\infty$. We set $\Lambda_n = SS(\mathscr{F}_n)$ and we assume that Λ_n Hausdorff converges to Λ_∞. We know that $SS(\mathscr{F}_\infty)\subset\Lambda_\infty$ but when is it true that $\Lambda_\infty = SS(\mathscr{F}_\infty)$? We can prove it if Λ_∞ is a smooth Lagrangian or more generally minimally γ-coisotropic. Indeed Proposition 7.3 shows that $SS(\mathscr{F}_\infty)\subset\Lambda_\infty$ and the involutivity of the singular support gives the equality (by an argument similar to the one in the proof of Proposition 9.13). Assuming $N = M\times\mathbb{R}$ and the $\mathscr{F}_n$ are quantizations of exact Lagrangian, Proposition 9.13 shows that this still hold when Λ_∞ is the conification of some $\varphi(L_0)$ for L_0 smooth Lagrangian, φ a symplectic homeomorphism.

Note however that the answer cannot be positive in general: Let $f(x)$ be a smooth bounded function on $\mathbb{R}$, such that $\max f' = 1$, $\min f' = -1$. Let $f_n(x) = f(nx)/n$ and $\mathscr{F}_n = k_{\{t\geq f_n(x)\}}$. Then $\mathscr{F}_n$ converges to $\mathscr{F}_\infty = k_{\mathbb{R}\times[0,\infty[}$ but $SS(\mathscr{F}_n)$ converges to a set Λ_∞ bigger than $SS(F_\infty)$. In fact Λ is the cone over a "cross" C in $T^*\mathbb{R}$ with $C = 0_\mathbb{R}\cup(\{0\}\times[-1,1])$.

Proposition 9.13. *Let $L = \varphi(L_0)$ where L_0 is smooth Lagrangian and $\varphi \in \mathscr{H}_\gamma(M,\omega) = \widehat{\mathfrak{DHam}}(M,\omega) \cap Homeo(M)$. Then any closed proper subset of L is not γ-coisotropic.*

Proof. Let $L' \subset L$. Since being γ-coisotropic is invariant by $\widehat{\mathfrak{DHam}}(M,\omega) \cap Homeo(M)$ it is enough to prove the Proposition for L_0 i.e. in the smooth case. Now there is a ball $B(z,r)$ in L such that $\overline{B(z,r)} \cap L' = \{z'\}$. Moreover we may choose r to be arbitrarily small. Let us consider a flow on L such that $\rho^t(B(z,r) \subset B(z',r)$. In a chart containing $B(z,2r)$ we may assume that ρ^t is a translation. Then $\rho^1(L') \cap B(z',r) = \emptyset$. Since if X_t is the vector field generating ρ^t, the Hamiltonian written in action-angle coordinates (i.e. L is given locally by $p = 0$) can be chosen of the form $\chi(|p|)\langle p, X_t(x)\rangle$ for some bump function χ, and this can be made arbitrarily small. As a result, L' is not γ-coisotropic.

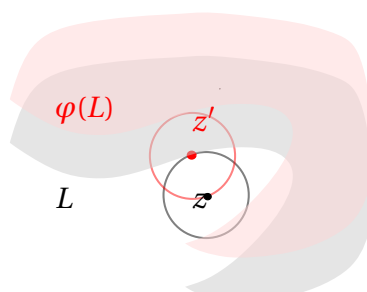

FIGURE 5. Proof of Proposition 9.13

□

Question 9.14. Let V be γ-coisotropic of dimension n. Then for $z \in V$ and ε small enough we have $V \setminus B(z,\varepsilon)$ is not coisotropic.

APPENDIX A. PERSISTENCE MODULES AND BARCODES

For this section we refer also to [Zha20]

We have already noticed after Definition 5.3 that the spectral invariants of $L_1, L_2 \in \mathscr{L}(T^*N)$ are encoded in the sheaf over $\mathbb{R}$ given by $\mathscr{H} = Rt_* \mathscr{H}\mathrm{om}^*(\mathscr{F},\mathscr{G}))$, where $t\colon N \times \mathbb{R} \to \mathbb{R}$ is the projection, because of the isomorphism

$$\mathrm{R}\Gamma_{N\times\{t\}}(N \times \mathbb{R}; \mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1}, \mathscr{F}_{L_2}))) \simeq \mathrm{R}\Gamma_{\{t\}}(\mathbb{R}; \mathscr{H})$$

We recall that $\mathscr{H}$ belongs to the category $D_{\tau\geq 0}(\mathbb{R})$ of sheaves with singular support in $\{\tau \geq 0\}$.

We remark that $D_{\tau\geq 0}(\mathbb{R})$ contains the derived category of *persistence modules*, where by a persistence module, we mean a constructible sheaf on $(\mathbb{R},\leq)$. We refer to [Bar94; Cha+09; ELZ02; KS18; ZC05] for the theory and applications. Such a persistence module is uniquely defined by the finite-dimensional vector spaces $V_t = \mathscr{F}(]-\infty,t[)$ and the linear maps $r_{s,t}: V_t \longrightarrow V_s$ defined for $s\leq t$, such that

(1) for $s<t<u$ we have $r_{u,t}\circ r_{s,t} = r_{s,u}$,
(2) $\lim_{t>s} V_t = V_s$ where the limit is that of the directed system given by the $r_{s,t}$,
(3) $r_{t,t} = \mathrm{Id}$.

It is a well-known result that persistence modules have a decomposition as sum of barcodes : this is Gabriel's theorem ([Gab72] on quivers in the finite case and Crawley-Boevey's theorem ([Cra15]) in the locally finite case. More precisely, using Gabriel's theorem and the fact that any complex of objects in an Abelian category of homological dimension 1 is the sum of its cohomology, we find: any $\mathscr{F}\in D_{\tau\geq 0}(\mathbb{R})$ which is constructible for a finite stratification can be written $\mathscr{F}\simeq \bigoplus_{j\in\mathscr{I}} k_{[a_j,b_j[}[-n_j]$, where $\mathscr{I}$ is a finite family, and this decomposition is unique up to isomorphism. (The hypothesis that the singular support is in $\{\tau\geq 0\}$ is useless here.) We remark that the stratification consists only of a finite set of points in $\mathbb{R}$ and the connected components of its complement, where $\mathscr{F}$ is constant. This finite set is thus the projection to $\mathbb{R}$ of $SS^\bullet(\mathscr{F})$. If $L\in\mathscr{L}(T^*N)$ is generic in the sense that L meets 0_N transversally, then $Rt_*(\mathscr{F}_L)$ is of the above type, where the stratification is given by the set of actions of the points in $L\cap 0_N$.

In [KS18] it is explained how to go from the case of a finite stratification to the case of a locally finite one (then $\mathscr{I}$ is at most countable). By [Cra15] the same result holds in fact (at least in $D_{\tau\geq 0}(\mathbb{R})$) under the more general hypothesis that $\mathscr{F}_x$ is finite dimensional for all $x\in\mathbb{R}$. However the sheaves we encounter in this paper are not necessarily of this type: if f is the function $f(x) = x^2\sin(1/x)$ and $\mathscr{F} = Rt_*(k_{\{t\geq f(x)\}})$, then $\mathscr{F}_0$ is infinite dimensional. But, by a small perturbation of L, we see that all sheaves of the form $Rt_*(\mathscr{F}_L)$ are in the closure of the set of constructible sheaves. Using Corollary B.11 we deduce:

Proposition A.1. *Let $L_1, L_2\in\mathscr{L}(T^*N)$. Then there exists an at most countable family $\mathscr{I}$ (which is finite in the case of transverse intersection) such that*

$$(Rt)_*(\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_2})) = \bigoplus_{j\in\mathscr{I}} k_{[a_j,b_j[}[-n_j],$$

where for each $j\in\mathscr{I}$ we have $a_j\in\mathbb{R}\cup\{-\infty\}$, $b_j\in\mathbb{R}$, $n_j\in\mathbb{Z}$. Moreover there is a unique j_- such that $a_{j_-} = -\infty$ and $n_{j_-} = -1$ and then $b_{j_-} =$

$c_-(L_1,L_2)$ and a unique j_+ such that $a_{j_+}=+\infty$ and $n_{j_+}=n-1$ and then $b_{j_+}=c_+(L_1,L_2)$.

More generally, with the notations of the proposition we let $\mathscr{I}_\infty$ be the set of j such that $a_j=-\infty$. Then $H^*_{t_0}((Rt)_*(\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_2})))\simeq\bigoplus_{j\in\mathscr{I}_\infty}k[-n_j]$, for $t_0\ll 0$, and we deduce that the spectral invariants $c(\alpha,L_1,L_2)$ are the b_j's with $j\in\mathscr{I}_\infty$.

Note that the filtered Floer cohomology, $FH^*(L_1,L_2;-\infty,t)$ defines a persistence module equal to the one obtained from $\mathscr{H}\mathrm{om}^*(\mathscr{F}_{L_1},\mathscr{F}_{L_2})$ as follows from Theorem 5.1.

It will be useful to remind the reader that

Lemma A.2 (see [KS18], (1.10))**.**

$$\mathrm{RHom}(k_{[a,b[},k_{[c,d[})\simeq\begin{cases}k & \textit{for } a\le c<b\le d,\\ k[-1] & \textit{for } c<a\le d<b,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. The formula $k_{[c,d[}\simeq \mathrm{R}\mathscr{H}\mathrm{om}(k_{]d,c]},k_{\mathbb{R}})$ together with the adjunction between $\otimes$ and $\mathrm{R}\mathscr{H}\mathrm{om}$ give $\mathrm{RHom}(k_{[a,b[},k_{[c,d[})\simeq\mathrm{RHom}(k_I,k_{\mathbb{R}})$, where $I=[a,b[\cap]c,d]$. If I is half closed or empty, the result is 0. The two other cases correspond to I open or I closed. □

Appendix B. Decomposition in the completion of $D_{lc}(\mathbb{R})$

B.1. **Limits of constructible sheaves.** We assume in this section that N is endowed with a real analytic structure and a real analytic metric g. We recall that we work with coefficients in a field k. We denote by $D_c(N)$ the subcategory of $D(N)$ of constructible sheaves. We let $D_{lc}(N)$ be the set of objects which are limits of constructible sheaves with respect to the γ_g-topology.

Lemma B.1. *Let $\mathscr{F},\mathscr{G}\in D_{lc}(N)$ with compact supports and $a<b$. Then the morphism $\mathrm{Hom}(\mathscr{F},K^a_\varphi(\mathscr{G}))\to\mathrm{Hom}(\mathscr{F},K^b_\varphi(\mathscr{G}))$ induced by $\tau_{a,b}(\mathscr{G})$ has finite dimensional image.*

Proof. We set $c=(b-a)/2$. Applying K^{-c}_φ we identify $\mathrm{Hom}(\mathscr{F},K^b_\varphi(\mathscr{G}))$ with $\mathrm{Hom}(K^{-c}_\varphi(\mathscr{F}),K^{b-c}_\varphi(\mathscr{G}))$ and the morphism of the lemma gets identified with

$$\begin{aligned} f\colon \mathrm{Hom}(\mathscr{F},K^a_\varphi(\mathscr{G}))&\to\mathrm{Hom}(K^{-c}_\varphi(\mathscr{F}),K^{b-c}_\varphi(\mathscr{G}))\\ u&\mapsto\tau_{a,b-c}(\mathscr{G})\circ u\circ\tau_{-c,0}(\mathscr{F})\end{aligned}$$

Since $\mathscr{F},\mathscr{G}\in D_{lc}(N)$, there exist constructible sheaves $\mathscr{F}'$, $\mathscr{G}'$ and morphisms $K^{-c}_\varphi\mathscr{F}\xrightarrow{u_1}\mathscr{F}'\xrightarrow{u_2}\mathscr{F}$, $K^a_\varphi(\mathscr{G})\xrightarrow{v_1}\mathscr{G}'\xrightarrow{v_2}K^{b-c}_\varphi(\mathscr{G})$ such that $u_2\circ$

$u_1 = \tau_{-c,0}(\mathscr{F})$, $v_2 \circ v_1 = \tau_{a,b-c}(\mathscr{G})$. We choose a compact set Z containing the supports of $\mathscr{F}$ and $K_\varphi^{-c}\mathscr{F}$. Applying the functor $(-)_Z$ we obtain $K_\varphi^{-c}\mathscr{F} \xrightarrow{(u_1)_Z} \mathscr{F}'_Z \xrightarrow{(u_2)_Z} \mathscr{F}$ and we still have $(u_2)_Z \circ (u_1)_Z = (u_2 \circ u_1)_Z = \tau_{-c,0}(\mathscr{F})$. So we may assume from the beginning that $\mathscr{F}'$ has a compact support. Now f factorizes through $\mathrm{Hom}(\mathscr{F}', \mathscr{G}')$ which is finite dimensional. □

As a consequence, for a given $\varepsilon > 0$, the images of $\mathrm{Hom}(\mathscr{F}, K_\varphi^\delta(\mathscr{G}))$ in $\mathrm{Hom}(\mathscr{F}, K_\varphi^\varepsilon(\mathscr{G}))$, for $0 < \delta \le \varepsilon$, stabilize when $\delta \to 0$. In other words the projective system $(\mathrm{Hom}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G})))_{n\in\mathbb{N}}$, with morphisms induced by $\tau_{1/(n+1),1/n}(\mathscr{G})$, satisfies the Mittag-Leffler criterion. Let us introduce the notations, for $\varepsilon > 0$, $\mathrm{Hom}_\varepsilon(\mathscr{F}, \mathscr{G}) = \mathrm{Im}(\mathrm{Hom}(\mathscr{F}, K_\varphi^{-\varepsilon}(\mathscr{G})) \to \mathrm{Hom}(\mathscr{F}, \mathscr{G}))$ and $\mathrm{Hom}_{(\varepsilon)}(\mathscr{F}, \mathscr{G}) = \bigcap_{\delta<\varepsilon} \mathrm{Hom}_\delta(\mathscr{F}, \mathscr{G})$. A consequence of Lemma B.3 below is that $\mathrm{Hom}_\varepsilon(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{(\varepsilon)}(\mathscr{F}, \mathscr{G})$. We will mainly consider

$$\mathrm{Hom}_{(\varepsilon)}(\mathscr{F}, K_\varphi^\varepsilon(\mathscr{G})) = \bigcap_{0<\delta\le\varepsilon} \mathrm{Im}\big(\mathrm{Hom}(\mathscr{F}, K_\varphi^\delta(\mathscr{G})) \to \mathrm{Hom}(\mathscr{F}, K_\varphi^\varepsilon(\mathscr{G}))\big)$$

By Lemma B.1 this space has finite dimension and, by construction, the morphisms $\mathrm{Hom}_{(\varepsilon)}(\mathscr{F}, K_\varphi^\varepsilon(\mathscr{G})) \to \mathrm{Hom}_{(\varepsilon')}(\mathscr{F}, K_\varphi^{\varepsilon'}(\mathscr{G}))$, for $0 < \varepsilon \le \varepsilon'$, are surjective.

In the next lemma Ab denotes the category of Abelian groups and $D(\mathrm{Ab})$ its derived category. The statement is a well known consequence of the Mittag-Leffler condition; it is for example a variation on [KS90, Prop. 1.12.4].

Lemma B.2. *Let (A_n, a_n), $a_n\colon A_n \to A_{n-1}$, be a projective system in $D(\mathrm{Ab})$ and define $\mathrm{holim}_n A_n$ by the distinguished triangle $\mathrm{holim}_n A_n \to \prod_n A_n \xrightarrow{\alpha} \prod_n A_n \xrightarrow{+1}$ where $\alpha = \mathrm{id} - \prod_n a_n$. We assume that $H^i A_n$ satisfies the Mittag-Leffler criterion, for some i. Then $H^{i+1}(\mathrm{holim}_n A_n) \simeq \varprojlim_n H^{i+1} A_n$.*

Proof. In Ab the product $\prod$ is exact, hence H^i and $\prod$ commute and we have a long exact sequence $\prod_n H^i A_n \xrightarrow{H^i\alpha} \prod_n H^i A_n \to H^{i+1}(\mathrm{holim}_n A_n) \to \prod_n H^{i+1} A_n \to \prod_n H^{i+1} A_n$. Hence the statement is equivalent to the surjectivity of $H^i\alpha$. We write for short $B_n = H^i A_n$, $b_n = H^i a_n$ and $B_n^\infty = \mathrm{im}(B_m \to B_n)$ for $m \gg n$. We let b_n^∞, $\bar{b}_n$ be the maps induced by b_n on B_n^∞, B_n/B_n^∞. We have $H^i\alpha = \mathrm{id} - \prod b_n$ and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod B_n^\infty \arrow[r] \arrow[d, "\mathrm{id}-\prod b_n^\infty"] & \prod B_n \arrow[r] \arrow[d, "\mathrm{id}-\prod b_n"] & \prod B_n/B_n^\infty \arrow[r] \arrow[d, "\mathrm{id}-\prod \bar{b}_n"] & 0 \\
0 \arrow[r] & \prod B_n^\infty \arrow[r] & \prod B_n \arrow[r] & \prod B_n/B_n^\infty \arrow[r] & 0
\end{tikzcd}$$

We see that $\mathrm{id} - \prod b_n^\infty$ is surjective because the b_n^∞'s are surjective. Let us write $t = \prod \bar{b}_n$. By the Mittag-Leffler criterion, for each n, there exists $m \geq n$ such that $\bar{b}_{n+1} \circ \bar{b}_{n+2} \circ \cdots \circ \bar{b}_m$ is the zero map. It follows that $t' = \sum_{i=0}^\infty t^i$ is well-defined. Hence $\mathrm{id} - t$ is an isomorphism, with inverse t'. We deduce that $\mathrm{id} - \prod b_n$ is also surjective, as required. □

Lemma B.3. *Let $\mathscr{F}, \mathscr{G} \in D_{lc}(N)$ with compact supports. We have natural isomorphisms*

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_{(1/n)}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G})) \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G}))$$

and, for any n, the map $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{(1/n)}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G}))$, $u \mapsto [u]_n := \tau_{0,1/n}(\mathscr{G}) \circ u$, *is surjective.*

Proof. By Lemma 6.23 we have $\mathrm{hocolim}\, K_\varphi^{-1/n}(\mathscr{F}) \xrightarrow{\sim} \mathscr{F}$ and we deduce $\mathrm{RHom}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{holim}\, \mathrm{RHom}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G}))$. Since the projective system of vector spaces $H^{-1}\mathrm{RHom}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G})) \simeq \mathrm{Hom}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G})[-1])$ satisfies Mittag-Leffler, Lemma B.2 gives $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G}))$. The second isomorphism of the lemma and the last assertion are clear by the definition of $\mathrm{Hom}_{(1/n)}$. □

Lemma B.4. *For $\mathscr{F}, \mathscr{G}, \mathscr{H} \in D_{lc}(N)$ with compact supports, the composition induces a well-defined map, for any n,*

$$\circ_n \colon \mathrm{Hom}_{(1/n)}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{G})) \times \mathrm{Hom}_{(1/n)}(\mathscr{G}, K_\varphi^{1/n}(\mathscr{H})) \to \mathrm{Hom}_{(1/n)}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{H}))$$
$$(\bar{u}, \bar{v}) \mapsto \bar{v} \circ_n \bar{u} := [v \circ u]_n,$$

where u, v in $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$, $\mathrm{Hom}(\mathscr{G}, \mathscr{H})$ *satisfy $[u]_n = \bar{u}$, $[v]_n = \bar{v}$, and we use the notation $[-]_n$ of Lemma B.3. This turns* $\mathrm{Hom}_{(1/n)}(\mathscr{F}, K_\varphi^{1/n}(\mathscr{F}))$ *into an algebra which is a finite dimensional quotient of* $\mathrm{Hom}(\mathscr{F}, \mathscr{F})$ *and whose unit is $\tau_{0,1/n}(\mathscr{F})$ (assuming $\mathscr{F} \neq 0$ and n is big enough so that $\tau_{0,1/n}(\mathscr{F}) \neq 0$ – see Lemma 6.4).*

Proof. It is enough to see that if $[u]_n = 0$ or $[v]_n = 0$, then $[v \circ u]_n = 0$. For u this follows from the commutative diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "u"] \arrow[dr, "{[u]_n}"'] & \mathscr{G} \arrow[r, "v"] \arrow[d, "\tau_{0,1/n}(\mathscr{G})"] & \mathscr{H} \arrow[d, "\tau_{0,1/n}(\mathscr{H})"] \\
& K_\varphi^{1/n}(\mathscr{G}) \arrow[r, "K_\varphi^{1/n}(v)"'] & K_\varphi^{1/n}(\mathscr{H}))
\end{tikzcd}$$

which shows that $[v \circ u]_n = K_\varphi^{1/n}(v) \circ [u]_n$. The case $[v]_n = 0$ is similar. □

Lemma B.5. *Let $\mathscr{F},\mathscr{G}\in D_{lc}(N)$ with compact supports such that $\gamma_g(\mathscr{F},\mathscr{G})=0$. We set $A_n=\mathrm{Hom}_{(1/n)}(\mathscr{F},K_\varphi^{1/n}(\mathscr{F}))$ and let $A_n^\times$ be its subset of invertible elements. We also define, using the notation $\circ_n$ of Lemma B.4,*

$$B_n=\big\{(u,v)\in\mathrm{Hom}_{(1/n)}(\mathscr{F},K_\varphi^{1/n}(\mathscr{G}))\times\mathrm{Hom}_{(1/n)}(\mathscr{G},K_\varphi^{1/n}(\mathscr{F}));\\ v\circ_n u=\tau_{0,1/n}(\mathscr{F}),\ u\circ_n v=\tau_{0,1/n}(\mathscr{G})\big\}$$

Then

1. *for any n the set B_n is non empty,*
2. *for any n and any $(u,v)\in B_n$, the maps $\alpha_{u,v}\colon A_n^\times\to B_n$, $a\mapsto(u\circ_n a,a^{-1}\circ_n v)$ and $\beta_{u,v}\colon B_n\to A_n^\times$, $(u',v')\mapsto v\circ_n u'$, are mutually inverse bijections,*
3. *for $m\geq n$ the natural map $B_m\to B_n$ is surjective.*

Proof. (1) By Lemma B.1 there exists $m\geq n$ such that $\mathrm{Hom}_{(1/n)}(\mathscr{F},K_\varphi^{1/n}(\mathscr{G}))$ is the image of $\mathrm{Hom}(\mathscr{F},K_\varphi^{1/m}(\mathscr{G}))$ in $\mathrm{Hom}(\mathscr{F},K_\varphi^{1/n}(\mathscr{G}))$ (and the same with $\mathscr{F},\mathscr{G}$ switched). Since $\gamma_g(\mathscr{F},\mathscr{G})=0$, we can find $u_1\colon\mathscr{F}\to K_\varphi^{1/2m}(\mathscr{G})$ and $v_1\colon\mathscr{G}\to K_\varphi^{1/2m}(\mathscr{F})$ such that $u_1\circ K_\varphi^{1/2m}(v_1)$ and $v_1\circ K_\varphi^{1/2m}(u_1)$ are the morphisms τ. Letting u be the image of u_1 in $\mathrm{Hom}_{(1/n)}(\mathscr{F},K_\varphi^{1/n}(\mathscr{G}))$ and defining v in the same way, we then have $(u,v)\in B_n$.

(2) We first remark that $\beta_{u,v}(u',v')$ belongs to A'_n: indeed the inverse of $v\circ_n u'$ is $v'\circ_n u$. Now $\beta_{u,v}(\alpha_{u,v}(a))=v\circ_n u\circ_n a=a$ and $\alpha_{u,v}(\beta_{u,v}(u',v'))=(u\circ_n v\circ_n u',v'\circ_n u\circ_n v)=(u',v')$.

(3) We pick $(u,v)\in B_m$ and let $(\bar u,\bar v)$ be its image by the natural map $B_m\to B_n$. We obtain a commutative diagram

$$\begin{array}{ccc} A'_m & \xrightarrow{\alpha_{u,v}} & B_m \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ A'_n & \xrightarrow{\alpha_{\bar u,\bar v}} & B_n \end{array}$$

where the horizontal maps are bijections and the vertical map q is surjective by Lemma B.6. The result follows. □

Lemma B.6. *Let A, B be finite dimensional algebras over k and let $f\colon A\to B$ be a surjective algebra morphism. Then the map induced on the sets of invertible elements $f^\times\colon A^\times\to B^\times$ is also surjective.*

Proof. We pick $x\in A$ and assume that $f(x)$ is invertible. We have to prove that there exists $z\in\ker(f)$ such that $x+z$ is invertible. We let $A'=\langle x\rangle$ be the subalgebra of A generated by x and define similarly $B'=\langle f(x)\rangle$. The maps $u\colon k[X]\to A$, $X\mapsto x$, and $f\circ u$ identify A' and B' with quotients

of $k[X]$. Since $k[X]$ is principal, we can write $A' = k[X]/\langle PQ\rangle$, with $x = [X]_{A'}$, and $B' = k[X]/\langle Q\rangle$, with $f(x) = [X]_{B'}$.

Since $f(x)$ is invertible in B, it is not a zero divisor and we have $X \nmid Q$. Let us write $P = X^n P'$ with $X \nmid P'$. We set $R = X + P'Q$. Then $f([R]_{A'}) = f(x)$. We claim that $[R]_{A'}$ is invertible, which concludes the proof. We have to check that $\langle R, PQ\rangle = k[X]$. Let D be a generator of $\langle R, PQ\rangle$. Then $D \mid X^n R - PQ = X^{n+1}$. Either $X \mid D$, but then $X \mid R$ and finally $X \mid P'Q$ which is false, or $D \in k^\times$ and we are done. □

Proposition B.7. *Let $\mathscr{F}, \mathscr{G}$ be elements in $D_{lc}(N)$ with coefficients in a field k and having compact supports. Assume that that $\gamma_g(\mathscr{F}, \mathscr{G}) = 0$. Then $\mathscr{F} \simeq \mathscr{G}$.*

Question B.8. Is the Proposition true if we replace the field k by $\mathbb{Z}$ or more generally a ring ?

Proof. If $\mathscr{F} = 0$, then $\gamma_g(0, \mathscr{G}) = 0$ and we obtain $\mathscr{G} = 0$ by Lemma 6.4. We assume $\mathscr{F} \neq 0$, hence $\tau_{0,1/n}(\mathscr{F}) \neq 0$ for n big enough. We use the notation B_n of Lemma B.5. By this lemma the inductive system $\cdots \to B_n \to B_{n-1} \to \cdots$ is made of surjective maps between non empty sets for $n \gg 0$. Hence its limit is non empty and we deduce a pair $(u, v) \in \mathrm{Hom}(\mathscr{F}, \mathscr{G}) \times \mathrm{Hom}(\mathscr{F}, \mathscr{G})$ such that $([u]_n, [v]_n) \in B_n$ for all n, that is, $[v \circ u]_n = \tau_{0,1/n}(\mathscr{F})$ and $[u \circ v]_n = \tau_{0,1/n}(\mathscr{G})$. Let $\mathscr{C}$ be the cone of u. For any n we have the following morphism of triangles given by the morphisms $\tau_{0,1/n}(-)$:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "u"] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] \arrow[dl, "{[v]_n}"'] & \mathscr{C} \arrow[r] \arrow[d] & \mathscr{F}[1] \arrow[d] \\
K_\varphi^{1/n}(\mathscr{F}) \arrow[r] & K_\varphi^{1/n}(\mathscr{G}) \arrow[r] & K_\varphi^{1/n}(\mathscr{C}) \arrow[r] & K_\varphi^{1/n}(\mathscr{F})[1],
\end{tikzcd}$$

where both maps $\tau_{0,1/n}(\mathscr{F})$ and $\tau_{0,1/n}(\mathscr{G})$ factorize through $[v]_n$. By the same argument as in the proof of Lemma 6.8-(1) we deduce that $\tau_{0,2/n}(\mathscr{C}) = 0$. Hence $\gamma_g(0, \mathscr{C}) \leq 2/n$, for any n. Thus the cone of u vanishes and u is an isomorphism. □

B.2. **Decomposition in $D_{lc}(\mathbb{R})$.** Let $\mathrm{Mod}(\mathbb{R})$ be the category of sheaves of k-vector spaces on $\mathbb{R}$. We consider it as usual as the full subcategory of $D(\mathbb{R})$ of complexes concentrated in degree 0. A sheaf $\mathscr{F} \in \mathrm{Mod}(\mathbb{R})$ which is constructible with respect to a finite stratification of $\mathbb{R}$ is described by a quiver representation of type A_n. Hence Gabriel theorem implies that $\mathscr{F}$ is decomposed as $\mathscr{F} \simeq \bigoplus_{I \in \mathscr{I}} k_I$ where $\mathscr{I}$ is a finite family of intervals. In this section we extend this result to γ_g-limits of constructible sheaves under the assumption that the microsupport is contained in $\{\tau \geq 0\}$.

Let Int^+ be the set of intervals of the form $[a,b[$, $a\in\mathbb{R}$, $b\in\mathbb{R}\cup\{+\infty\}$. We endow Int^+ with the order

$$[a,b[\le[a',b'[\qquad \text{if and only if} \qquad a\le a' \text{ and } b\le b' \tag{B.1}$$

By Lemma A.2, for $I,I'\in Int^+$ we have $\mathrm{Hom}(k_I,k_{I'})\simeq k$ canonically if and only if $I\cap I'\neq\emptyset$ and $I\le I'$ and in this case we denote by $e_I^{I'}$ its generator. Otherwise $\mathrm{Hom}(k_I,k_{I'})\simeq 0$ and we set $e_I^{I'}=0$. For $I\le I'\le I''$ we have $e_{I'}^{I''}\circ e_I^{I'}=e_I^{I''}$. For two finite families $\mathscr{I}$, $\mathscr{I}'\subset Int^+$ and $\mathscr{G}=\bigoplus_{I\in\mathscr{I}}k_I$, $\mathscr{G}'=\bigoplus_{J\in\mathscr{I}'}k_J$, we obtain a natural morphism

$$E=E_{\mathscr{I}}^{\mathscr{I}'}\colon \mathrm{Mat}_{lt}(\mathscr{I}'\times\mathscr{I})\to\mathrm{Hom}(\mathscr{G},\mathscr{G}'),\quad (a_{I',I})_{I'\in\mathscr{I}',I\in\mathscr{I}}\mapsto\sum a_{I',I}e_I^{I'},$$

where Mat_{lt} denotes the space of lower triangular matrices ($a_{I',I}\neq 0$ implies $I\le I'$). The map E is surjective and, for three families $\mathscr{I}$, $\mathscr{I}'$, $\mathscr{I}''$, we have $E_{\mathscr{I}'}^{\mathscr{I}''}(B)\circ E_{\mathscr{I}}^{\mathscr{I}'}(A)=E_{\mathscr{I}}^{\mathscr{I}''}(BA)$.

Lemma B.9. *Let $\mathscr{I}$, $\mathscr{I}'\subset Int^+$ be two finite families and $\mathscr{G}=\bigoplus_{I\in\mathscr{I}}k_I$, $\mathscr{G}'=\bigoplus_{J\in\mathscr{I}'}k_J$. Let $\varepsilon>0$ be given and let $u\colon\mathscr{G}\to\mathscr{G}'$, $v\colon\mathscr{G}'\to T_{\varepsilon*}(\mathscr{G})$ be such that $v\circ u=\tau_{0,\varepsilon}(\mathscr{G})$. We assume that each $I\in\mathscr{I}$ is of length $>\varepsilon$. Then there exist an isomorphism $\phi\colon\mathscr{G}'\xrightarrow{\sim}\mathscr{G}'$ and an injective map $\sigma\colon\mathscr{I}\to\mathscr{I}'$ such that $I\le\sigma(I)$ for all $I\in\mathscr{I}$ (for the order $\le$ defined in* (B.1)*) and $\phi\circ u\colon\mathscr{G}\to\mathscr{G}'$ is the sum of the natural morphisms $k_I\to k_{\sigma(I)}$.*

Proof. We set $\mathscr{I}''=\{T_\varepsilon(I);\ I\in\mathscr{I}\}$. In view of the discussion after (B.1) we can find lower triangular matrices $A\in\mathrm{Mat}_{lt}(\mathscr{I}'\times\mathscr{I})$, $B\in\mathrm{Mat}_{lt}(\mathscr{I}''\times\mathscr{I}')$ representing u, v. Then $BA\in\mathrm{Mat}_{lt}(\mathscr{I}''\times\mathscr{I})$ is lower triangular. We identify $\mathscr{I}$ and $\mathscr{I}''$ through T_ε and we claim that $BA\in\mathrm{Mat}_{lt}(\mathscr{I}\times\mathscr{I})$ is still lower triangular. We take $I_1,I_2\in\mathscr{I}$ and assume that $0\neq(BA)_{I_1,I_2}=\sum_{J\in\mathscr{I}'}B_{T_\varepsilon(I_1),J}A_{J,I_2}$. There exists J such that $B_{T_\varepsilon(I_1),J},A_{J,I_2}\neq 0$, hence $I_2\le J\le T_\varepsilon(I_1)$. If we assume moreover that $I_2\not\le I_1$ we must have $T_\varepsilon(I_1)\cap I_2\neq\emptyset$ because I_1 and I_2 are of length $>\varepsilon$. Hence $e_{I_2}^{T_\varepsilon(I_1)}\neq 0$ and we deduce that the decomposition of $v\circ u$ on the basis of the non zero $e_I^{I''}$'s has a non zero coefficient on $e_{I_2}^{T_\varepsilon(I_1)}$. Since $v\circ u=\tau_{0,\varepsilon}(\mathscr{G})$ this means that $I_1=I_2$, contradiciting $I_2\not\le I_1$. Hence $I_2\le I_1$, as claimed.

Finally BA is lower triangular (in $\mathrm{Mat}_{lt}(\mathscr{I}\times\mathscr{I})$) and has entries 1 along the diagonal since it represents $\tau_{0,\varepsilon}(\mathscr{G})$. Hence BA is invertible. In particular $|\mathscr{I}|\le|\mathscr{I}'|$, A has rank $|\mathscr{I}|$, we can decompose $\mathscr{I}'$ as $\mathscr{I}'=\mathscr{I}\sqcup\mathscr{I}_1'$ and find an invertible matrix $A'\in\mathrm{Mat}_{lt}(\mathscr{I}'\times\mathscr{I}')$ such that the block $\mathscr{I}\times\mathscr{I}$ in $A'A$ is the identity matrix. Now we let σ be the map given by the partition $\mathscr{I}'=\mathscr{I}\sqcup\mathscr{I}_1'$ and ϕ be the morphism induced by A'. □

Proposition B.10. *Let $\mathscr{F}\in\mathrm{Mod}(\mathbb{R})$ such that $SS(\mathscr{F})\subset\{(t;\tau)\in T^*\mathbb{R};\ \tau\ge 0\}$. We assume that $\mathscr{F}$ is a γ_g-limit of constructible sheaves $\mathscr{F}_n\in\mathrm{Mod}(\mathbb{R})$ such*

that $SS(\mathscr{F}_n) \subset \{(t;\tau) \in T^*\mathbb{R};\ \tau \geq 0\}$. *Then there exists an at most countable set of intervals* $\mathscr{I}$ *such that* $\mathscr{F} \simeq \bigoplus_{I\in\mathscr{I}} k_I$.

We notice that the singular support hypothesis in the proposition implies that our intervals are of the type $[a,b[$, $[a,\infty[$ or $]-\infty,b[$.

Proof. (i) Up to taking a subsequence we can assume $\gamma_g(\mathscr{F},\mathscr{F}_n) \leq \varepsilon_n = 2^{-n}$. For each n there exists a finite family $\mathscr{I}_n \subset Int^+$ such that $\mathscr{F}_n := \bigoplus_{I\in\mathscr{I}_n} k_I$. By Proposition 6.24 we can assume, up to translating $\mathscr{F}_n$ by less than ε_n, that there exist compatible morphisms $f_n\colon \mathscr{F}_n \to \mathscr{F}_{n+1}$ and that $\operatorname{hocolim}\mathscr{F}_n \xrightarrow{\sim} \mathscr{F}$. Since $\mathscr{F}_n \in \operatorname{Mod}(\mathbb{R})$, we have $\operatorname{hocolim}\mathscr{F}_n \simeq \varinjlim \mathscr{F}_n$. We can also assume (maybe changing ε_n by $4\varepsilon_n$) that there exist morphisms $g_n\colon \mathscr{F}_{n+1} \to T_{\varepsilon_n *}(\mathscr{F}_n)$ such that $g_n \circ f_n = \tau_{0,\varepsilon_n}(\mathscr{F}_n)$.

(ii) It remains to check that $\varinjlim \mathscr{F}_n$ is decomposed. For this we modify the f_n's using Lemma B.9. We let $\mathscr{J}_n \subset \mathscr{I}_n$ be the set of intervals of length $> 2\varepsilon_n$ and we set $\widetilde{\mathscr{F}}_n = \bigoplus_{I\in\mathscr{J}_n} k_I$. We first put the f_n's in "diagonal form", $\widehat{f}_n$ (see the diagram (B.2)). Lemma B.9 applied with $f_0|_{\widetilde{\mathscr{F}}_0}\colon \widetilde{\mathscr{F}}_0 \to \mathscr{F}_1$ gives an injective map $\sigma_0\colon \mathscr{J}_0 \to \mathscr{I}_1$ and an isomorphism $\phi_1\colon \mathscr{F}_1 \xrightarrow{\sim} \mathscr{F}_1$ such that $\phi_1 \circ f_0|_{\widetilde{\mathscr{F}}_0}$ has a "diagonal form" (that is, it is the sum of the natural morphisms $k_I \to k_{\sigma_0(I)}$). We set $\widehat{f}_0 = \phi_1 \circ f_0$ and $f_1' = f_1 \circ \phi_1^{-1}$. Now we apply the lemma with $f_1'|_{\widetilde{\mathscr{F}}_1}\colon \mathscr{F}_1 \to \mathscr{F}_2$ and obtain $\phi_2\colon \mathscr{F}_2 \xrightarrow{\sim} \mathscr{F}_2$ and $\sigma_1\colon \mathscr{I}_1 \to \mathscr{I}_2$. We set $\widehat{f}_1 = \phi_2 \circ f_1'$ and $f_2' = f_2 \circ \phi_2^{-1}$. We go on inductively and end up with a sequence of injective maps $\sigma_n\colon \mathscr{J}_n \to \mathscr{I}_{n+1}$ and morphisms $\widehat{f}_n\colon \mathscr{F}_n \to \mathscr{F}_{n+1}$ such that $\widehat{f}_n$ restricted to $\widetilde{\mathscr{F}}_n$ has a "diagonal form".

(B.2)
$$\begin{tikzcd}
\widetilde{\mathscr{F}}_0 \arrow[r, equal] \arrow[d, "\widetilde{f}_0"'] & \bigoplus_{I_0\in\mathscr{J}_0} k_{I_0} \arrow[r] \arrow[dr] & \bigoplus_{I_0\in\mathscr{I}_0} k_{I_0} \arrow[r, equal] \arrow[d, "\widehat{f}_0"] & \mathscr{F}_0 \arrow[d, "f_0"] \\
\widetilde{\mathscr{F}}_1 \arrow[r, equal] \arrow[d, "\widetilde{f}_1"'] & \bigoplus_{I_1\in\mathscr{J}_1} k_{I_1} \arrow[r] \arrow[dr] & \bigoplus_{I_1\in\mathscr{I}_1} k_{I_1} \arrow[r, "\phi_1", "\sim"'] \arrow[d, "\widehat{f}_1"] & \mathscr{F}_1 \arrow[d, "f_1"] \\
\widetilde{\mathscr{F}}_2 \arrow[r, equal] \arrow[d, "\widetilde{f}_2"'] & \bigoplus_{I_2\in\mathscr{J}_2} k_{I_2} \arrow[r] \arrow[dr] & \bigoplus_{I_2\in\mathscr{I}_2} k_{I_2} \arrow[r, "\phi_2", "\sim"'] \arrow[d, "\widehat{f}_2"] & \mathscr{F}_2 \arrow[d, "f_2"] \\
\widetilde{\mathscr{F}}_3 \arrow[r, equal] & \bigoplus_{I_3\in\mathscr{J}_3} k_{I_3} \arrow[r] & \bigoplus_{I_3\in\mathscr{I}_3} k_{I_3} \arrow[r, "\phi_3", "\sim"'] & \mathscr{F}_3
\end{tikzcd}$$

(iii) With our definition of $\mathscr{J}_n$ we have moreover $\sigma_n(\mathscr{J}_n) \subset \mathscr{J}_{n+1}$. Indeed, modifying g_n as $\widehat{g}_n = T_{\varepsilon_n *}(\phi_n) \circ g_n \circ \phi_{n+1}^{-1}$, we see that $\widehat{g}_n \circ \widehat{f}_n = \tau_{0,\varepsilon_n}(\mathscr{F}_n)$. It follows that, for any $[a,b[\in \mathscr{J}_n$ and $[a',b'[= \sigma_n([a,b[)$, we must have

$a \le a' \le a+\varepsilon_n$ and $b \le b' \le b+\varepsilon_n$. Hence $b'-a' \ge b-a-\varepsilon_n \ge 2\varepsilon_n - \varepsilon_n = 2\varepsilon_{n+1}$, which shows that $[a', b'[\in \mathscr{J}_{n+1}$.

We can then define $\widetilde{f}_n = \widehat{f}_n|_{\widetilde{\mathscr{F}}_n} : \widetilde{\mathscr{F}}_n \to \widetilde{\mathscr{F}}_{n+1}$ and $\widetilde{f}_n$ has a "diagonal form" with respect to the inclusion $\sigma_n|_{\mathscr{J}_n} : \mathscr{J}_n \hookrightarrow \mathscr{J}_{n+1}$. For a given $I \in \mathscr{J}_n$ we write $\sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_n(I) = [a_k, b_k[$; the sequences (a_k), (b_k) are non decreasing and have limits, say a_∞, b_∞. Then $\varinjlim k_{[a_k,b_k[} \simeq k_{[a_\infty,b_\infty[}$. We set $I_\infty = [a_\infty, b_\infty[$. We let $\mathscr{J}$ be the increasing union of the $\mathscr{J}_n$'s. Then each element of $\mathscr{J}$ corresponds to an interval I_∞ as above and we have proved $\varinjlim \widetilde{\mathscr{F}}_n \simeq \bigoplus_{I_\infty \in \mathscr{J}} k_{I_\infty}$. By Lemma 6.20 the maps $\widetilde{\mathscr{F}}_k \to \varinjlim \widetilde{\mathscr{F}}_n$ and $\mathscr{F}_k \to \varinjlim \mathscr{F}_n$ have cones which go to 0 when $k \to \infty$. It follows by Lemma 6.4 that the natural map $\varinjlim \widetilde{\mathscr{F}}_n \to \varinjlim \mathscr{F}_n$ has a cone which is arbitrarily close to 0, hence it is an isomorphism. □

Corollary B.11. *We recall that k is a field. Let $\mathscr{F} \in D_{lc}(\mathbb{R}) \cap D_{\tau\ge 0}(\mathbb{R})$ be a limit of constructible objects of $D_{\tau\ge 0}(\mathbb{R})$. We assume that $\mathscr{F}$ is constant outside $[-A, A]$ for some A. Then there exists an at most countable set of intervals $\mathscr{I}$ and integers d_I, $I \in \mathscr{I}$, such that $\mathscr{F} \simeq \bigoplus_{I\in\mathscr{I}} k_I[d_I]$.*

Proof. If we prove the result for $\mathscr{F} \otimes k_{[-A-1,A+1[}$ we deduce the result for $\mathscr{F}$ by gluing the decomposition with the constant sheaves $\mathscr{F} \otimes k_{]-\infty,-A[}$ and $\mathscr{F} \otimes k_{]A,\infty[}$. Hence we can assume $\mathscr{F}$ has compact support.

As in the case of constructible sheaves the result follows from Proposition B.10 and a decomposition of a complex as sum of its cohomology. However we didn't prove that limits of constructible sheaves have no Ext^2 so we have to be careful to decompose the cohomology before we take the limit, as follows.

There exists a sequence of constructible objects $\mathscr{F}_n \in D_{\tau\ge 0}(\mathbb{R})$ which γ_g-converges to F. Since $\mathrm{Ext}^2(\mathscr{G}, \mathscr{G}') = 0$ for any two constructible $\mathscr{G}, \mathscr{G}' \in \mathrm{Mod}(\mathbb{R})$, we have $\mathscr{F}_n \simeq \bigoplus_{i\in\mathbb{Z}} H^i\mathscr{F}_n[-i]$ (see for example [KS06, Cor. 13.1.20]).

In $D_{\tau\ge 0}(\mathbb{R})$ we have $K^a_\varphi(-) = T_{a*}(-)$, where T_a is as usual the translation by a, and this functor commutes with the cohomology, that is, $H^i(T_{a*}(\mathscr{G})) \simeq T_{a*}(H^i(\mathscr{G}))$. It follows that $\gamma_g(H^i\mathscr{G}, H^i\mathscr{G}') \le \gamma_g(\mathscr{G}, \mathscr{G}')$, for any $\mathscr{G}, \mathscr{G}' \in D_{\tau\ge 0}(\mathbb{R})$. Hence $H^i\mathscr{F}_n$ γ_g-converges to $H^i\mathscr{F}$.

It follows that $\mathscr{F}_n \simeq \bigoplus_i H^i\mathscr{F}_n[-i]$ γ_g-converges to $\bigoplus_i H^i\mathscr{F}[-i]$. By unicity of the limit (Proposition B.7) we have $\mathscr{F} \simeq \bigoplus_i H^i\mathscr{F}[-i]$. Now the result follows from Proposition B.10. □

Appendix C. Comparing different interleaving metrics on $D(N \times \mathbb{R})$.

To prove Theorem 1.2 we have used the distance γ_τ on sheaves over $N \times \mathbb{R}$ and the fact that its restriction to sheaves associated with Hamiltonian maps coincides with the spectral distance (see Proposition 6.13).

We can also consider the distance γ_g on sheaves, where g is some Riemannian metric on N. For the sake of completeness we describe the distance γ_g from the point of view of spectral distance and we prove that the distances γ_τ and γ_g induce equivalent topologies (when restricted to sheaves associated with Hamiltonian maps).

In the following we let $H\colon T^*N \to \mathbb{R}$ be a function which coincides with $(q,p) \mapsto ||p||_g$ outside a compact neighbourhood of 0_N and $\varphi = \varphi_H$ denotes its Hamiltonian flow. We will use spectral invariants and sheaves for φ^s but in §4 and §5 we only defined them for compactly supported isotopies. However we can extend them to a positive Hamiltonian H as follows. Let H_r be a truncation of H to the disc bundle of T^*N of radius r, so that H_r is an increasing sequence of Hamiltonians converging to H. Then the homogeneous lift of $s \mapsto \varphi_{H_r}^{-s} \circ \varphi_{H_{r+1}}^{s}$ to $T^*(N \times \mathbb{R}) \setminus 0_{N\times\mathbb{R}}$ is a non negative Hamiltonian isotopy. Hence we have a natural map $\mathcal{K}_{\mathrm{id}} \to \mathcal{K}_{\varphi_{H_r}^{-s}} \circ \mathcal{K}_{\varphi_{H_{r+1}}^{s}}$, or equivalently, $\mathcal{K}_{\varphi_{H_r}^{s}} \to \mathcal{K}_{\varphi_{H_{r+1}}^{s}}$. We define $\mathcal{K}_{\varphi_s}$ as the homotopy colimit of this inductive system (say r runs over the integers). In the same way $c_+(\varphi_r^{-s}\psi)$ is a decreasing sequence, and we say that $c_+(\varphi^{-s}\psi) = 0$ if $\lim_r c_+(\varphi_r^{-s}\psi) = 0$. We shall prove that the sequence is in fact stationary, so this means $c_+(\varphi_r^{-s}\psi) = 0$ for r large enough.

Now Lemma 5.7 extends as follows: there are morphisms $\mathcal{K}_\psi \longrightarrow \mathcal{K}_{\varphi^s} \longrightarrow \mathcal{K}_{\varphi^{2s}\circ\psi}$ if and only if $c_-(\varphi^s\psi) = 0$ and $c_-(\varphi^s\psi^{-1}) = 0$. Using Corollary 6.10 we obtain

Lemma C.1. *If $c_-(\varphi^s\psi) = 0$ and $c_-(\varphi^s\psi^{-1}) = 0$, the composition of the map of the map $\mathcal{K}_\psi \longrightarrow \mathcal{K}_{\varphi^s}$ and $\mathcal{K}_{\varphi^s} \longrightarrow \mathcal{K}_{\varphi^{2s}\circ\psi}$ coincides with the canonical map $\mathcal{K}_\psi \longrightarrow \mathcal{K}_{\varphi^{2s}\psi}$.*

Now we may define the distance γ^g on $\mathrm{Ham}(T^*N)$ as follows

Definition C.2. *We set*

$$c_-^g(\psi) = -\inf\{s \geq 0 \mid c_-(\varphi^s\psi) = 0\}$$

and

$$c_+^g(\psi) = -c_-^g(\psi^{-1}) = \inf\{s \geq 0 \mid c_+(\varphi^{-s}\psi) = 0\}$$

Finally we set

$$\gamma^g(\psi) = c_+^g(\psi) - c_-^g(\psi)$$

Proposition C.3. *We have*

(1) The function γ^g defines a metric by $\gamma^g(\psi_1,\psi_2) = \gamma^g(\psi_1\psi_2^{-1})$.
(2) The topologies defined by γ^g and γ coincide.

Proof. (1) Apply section 3 of [PS21] to the Kernels $\mathcal{K}_{\varphi^s}$ (or the symplectic flow φ^s).

(2) We shall first prove that $c_+(\psi) \leq C_W c_+^g(\psi)$ and $\gamma(\psi) \leq C_W \gamma^g(\psi)$ for all ψ such that $\mathrm{supp}(\psi) \subset K$. We consider $c_+(\varphi^{-s}\psi)$ and remember that φ^s is generated by a positive homogeneous Hamiltonian H. By a classical argument (see e.g. [Vit92], lemma 4.7 p. 699) the map $s \mapsto c_+(\varphi^{-s}\psi)$ is piecewise C^1 and where it is C^1 we have

$$\frac{d}{ds} c_+(\varphi^{-s}\psi)_{|s=s_0} = -K_s(z)$$

where K_s is the Hamiltonian associated to the map $\varphi^{-s}\psi$ and z is some fixed point of $\varphi^{-s}\psi$ having non-negative action. Indeed, if S_s is a Generating function Quadratic at infinity for $\varphi^{-s}\psi$ we know that $c_+(\varphi^{-s}\psi)$ is a critical value of S_s and is non-negative. It thus corresponds to fixed points of $\varphi^{-s}\psi$ with non-negative action and its derivative is $\frac{d}{ds} S_s(\xi_z)$ where ξ_z corresponds to z, one of the fixed points such that $S_s(\xi_z) = c_+(\varphi^{-s}\psi)$.

In our case, if z is outside the support of ψ, the fixed point is a fixed point of φ^{-s} and these have action 0 since H is 1-homogeneous (as on an orbit $p\dot{q} - H(q,p) = p\frac{\partial H}{\partial p} - H(q,p) = 0$). So we only need to estimate H in the support of ψ and we set W to be a neighbourhood of this support. Then if $C_W = \sup\{H(x,p) \mid (x,p) \in W\}$ we have

$$\frac{d}{ds} c_+(\varphi^{-s}\psi)_{|s=s_0} \geq -C_W$$

so that

$$c_+(\varphi^{-s}\psi) - c_+(\psi) \geq -C_W s$$

and if $c_+(\varphi^{-s}\psi) = 0$ we have

$$C_W s \geq c_+(\psi)$$

hence

$$c_+(\psi) \leq C_W c^g(\psi)$$

Applying the same to ψ^{-1}, we get

$$\gamma(\psi) \leq C_W \gamma^g(\psi)$$

Note that we also proved that $\gamma(\varphi_r^{-s}\psi)$ is stationary for r such that $B(0,r)$ contains the support of ψ.

Conversely, let us write $\varphi^s = \xi_r^s \circ \eta_r^s$ where ξ_r^s (resp. η_r^s) is the flow of $f_r(H)$ (resp. $(1-f_r)(H)$) where f_r is a function given by

(1) $f_r(t) = t$ for $t \geq 2r$
(2) $f_r = \frac{3r}{2}$ for $t \leq r$
(3) f_r is non-decreasing

Note that the flows ξ_r^s, η_r^s are autonomous, commute, and $f_r(H)$ is bounded from below by $\frac{3r}{2}$ and η_r^s is supported in a ball of radius $K \cdot r$.

Then

$$c_+(\varphi^{-s}\psi) = c_+(\xi^{-s}\eta_r^{-s}\psi) \leq c_+(\eta_r^{-s}\psi) - \frac{3r}{2}s$$

as long as $c_+(\varphi^{-s}\psi) > 0$ so we have

$$\frac{3r}{2}c_+^g(\psi) \leq c_+(\eta_r^{-s}\psi) \leq \pi K^2 \cdot r^2 + c_+(\psi)$$

As a result if $c_+(\psi_j)$ converges to zero, let $r_j = \sqrt{c_+(\psi_j)}$. Then

$$c_+^g(\psi_j) \leq \frac{2}{3}\left(\pi K \cdot r_j + \frac{c_+(\psi_j)}{r_j}\right) \leq \frac{2}{3}(\pi K + 1)\sqrt{c_+(\psi_j)}$$

and we deduce that $\lim_j c_+^g(\psi_j) = 0$. □

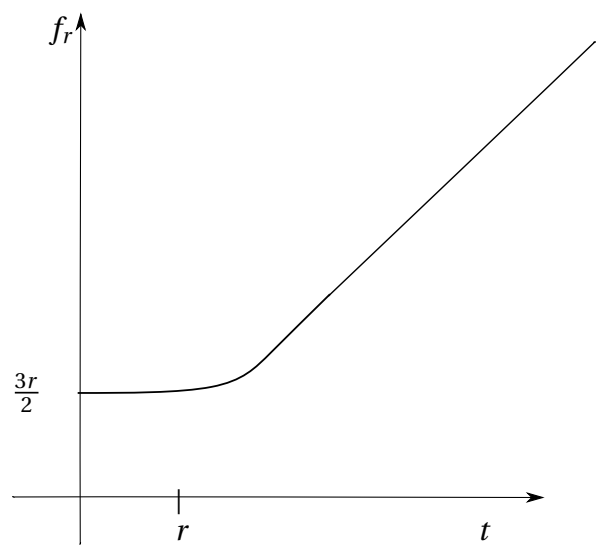

FIGURE 6. The function f_r

Remark C.4. (1) We do not know if the metrics γ and γ^g are equivalent, only that they define the same topology.

(2) Here is an analogy for the difference between γ and γ^g. Consider the following norm on $C_b^0(X,\mathbb{R})$: we first define

$$\max_f(g) = \inf\{s \geq 0 \mid \max(g - sf) \leq 0\}$$

and

$$\min_f(g) - \max_f(-g) = \inf\{s \geq 0 \mid \min(g + sf) \geq 0\}$$

Then we set $\|g\|_f = \max_f(g) - \min_f(g)$. Provided

$$0 < \inf_{x\in X} f(x) \leq \sup_{x\in X} f(x) < +\infty$$

it is obvious that $\|\bullet\|_f$ and $\|\bullet\|$ are equivalent.

APPENDIX D. LOCAL FLOER COHOMOLOGY

Our goal is to prove the following statement, which was used in Subsection 5.2.

Let φ be a Hamiltonian map supported in a Darboux chart, V. Since $\Gamma(\varphi)$ coincides with the diagonal, $\Gamma(\mathrm{Id})$ outside $W = V \times V$, we can consider two types of Floer cohomology :

(1) $FH^*(\varphi, W)$ obtained by using an almost complex structure making ∂W pseudoconvex and building the boundary operator by considering only the Floer trajectories contained in W
(2) The full Floer cohomology $FH^*(\varphi) = FH^*(\Gamma(\varphi, \Delta))$

Note that we may also restrict the action to some interval $[a, b]$.

Proposition D.1. *The above two Floer cohomologies coincide. The same holds when restricting the action to* $[a, b]$.

This is a special case of the following. Let L_t be an exact Lagrangian isotopy supported in a pseudoconvex domain W (i.e. L_t is constant outside W). We can consider $FH^*_{[a,b]}(L_1, L_0; W)$ and $FH^*_{[a,b]}(L_1, L_0)$.

Proposition D.2. *We have* $FH^*_{[a,b]}(L_1, L_0; W) = FH^*_{[a,b]}(L_1, L_0)$.

Proof. We may assume that W is contained in a Weinstein neighbourhood of L_0. We shall then use the structure of the cotangent bundle to write in these coordinates $s \cdot (q, p) = (q, sp)$ and $s \cdot L_1$ is the image of L_1 by the action of s. We have that $FH^*_{[sa,sb]}(s \cdot L_1, L_0; W)$ does not depend on s, since the intersection points $s \cdot L_1 \cap L_0$ are constant, their action is multiplied by s. We choose an almost complex structure making ∂W pseudoconvex and such that $L_0 \cap \partial W$ is a Legendrian, say Λ_0, and for all $x \in \Lambda_0$ we have $JT_x\Lambda_0 \subset T_x\partial W$.

Then the holomorphic strips cannot exit from W by pseudoconvexity, since an interior point of such a curve cannot be tangent to ∂W and if the boundary of the strip in L_j was tangent to ∂W then the holomorphic curve would be tangent to ∂W from inside, which is again impossible by pseudoconvexity. As a result the coboundary map defining $FH^*_{[sa,sb]}(s \cdot L_1, L_0; W)$ is also constant. Of course the same holds for $FH^*_{[a,b]}(L_1, L_0)$, as in that case we do not even have to worry about the location of the holomorphic strips. Now for s small enough all holomorphic strips must have area less than any given ε; but there is a positive

ε_0 such that no holomorphic strip of area less than ε_0 can exit from ∂W (by a monotonicity argument). As a result for s small enough we have $FH^*_{[sa,sb]}(s \cdot L_1, L_0; W) = FH^*_{[sa,sb]}(s \cdot L_1, L_0)$ and this implies the Proposition. □

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