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# Solutions to the Gardner equation with multi-parameters and the rational case

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## Abstract

We construct solutions to the Gardner equation in terms of trigonometric and hyperbolic functions, depending on several real parameters. Using a passage to the limit when one of these parameters goes to 0, we get, for each positive integer  $N$ , rational solutions as a quotient of polynomials in  $x$  and  $t$  depending on  $2N$  parameters. We construct explicit expressions of these rational solutions for orders  $N = 1$  until  $N = 3$ . We easily deduce solutions to the mKdV equation in terms of wronskians as well as rational solutions depending on  $2N$  real parameters.

**Key Words :** Gardner equation, wronskians, rational solutions.

**PACS numbers :**

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1 Introduction

We consider the Gardner equation in the following normalization

$$u_t - 6u(1+u)u_x + u_{xxx} = 0, \quad (1)$$

where the subscripts  $x$  and  $t$  denote partial derivatives.

The introduction of this equation is attributed to Gardner [1] in 1968. It was first considered [1] as an auxiliary mathematical tool in the derivation of the infinite set of local conservation laws of the Korteweg de Vries equation. This equation is a fundamental mathematical model for the description of weakly nonlinear dispersive waves. It can describe nonlinear wave effects in several physical contexts: for example, in plasma physics [2, 3], fluid flows [4], quantum fluid dynamics [5], in dusty plasmas [6], in ocean and atmosphere [7]. It is fundamental tool to describe large-amplitude internal waves [4, 8, 9]. These

waves are such that their vertical amplitudes underwater can exceed 170 meters as described in [10].

Many methods have been used to solve this equation, as for example the Hirota method [11], the series expansion method [12], the mapping method [13] or the method of leading-order analysis [14].

Recently published work on this equation can also be mentioned, such as [23, 24, 25, 26, 27]

Here, we used the Darboux transformation to construct different type of solutions. We give a representation of solutions in terms of a quotient of a wronskian of order  $N + 1$  by a wronskian of order  $N$ . We get what we will call  $N$ -order solutions which depend on  $2N$  real parameters in terms of trigonometric or hyperbolic functions. Then we construct rational solutions in performing a passage to the limit when one of these parameters goes to 0. We obtain rational solutions as a quotient of polynomials in  $x$  and  $t$ , depending on  $2N$  parameters. We give explicit solutions in the simplest cases  $N = 1, 2, 3$ .

We easily deduce solutions to the mKdV equation and their corresponding rational solutions depending on  $2N$  real parameters.

## 2 Nth-order solutions to the Gardner equation in terms of wronskians

### 2.1 First type of solutions

#### 2.1.1 Nth-order solutions in terms of wronskians of sine functions

We consider the Gardner equation

$$u_t - 6u(1 + u)u_x + u_{xxx} = 0.$$

In the following, we will use the wronskian of order  $N$  of the functions  $f_1, \dots, f_N$  which is the determinant denoted  $W(f_1, \dots, f_N)$ , defined by  $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$ ,  $\partial_x^i$  being the partial derivative of order  $i$  with respect to  $x$  and  $\partial_x^0 f_j$  being the function  $f_j$ .

We consider  $a_j, b_j$ , arbitrary real numbers  $1 \leq j \leq N$ . Then, we have the following statement :

**Theorem 2.1** *Let  $f_j, f$  be the functions defined by*

$$\begin{aligned} f_j(x, t) &= \sin\left(\frac{1}{2}a_j x + \frac{1}{2}a_j^3 t + b_j\right), & \text{for } 1 \leq j \leq N, \\ f(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \tag{2}$$

*then the function  $u$  defined by*

$$u(x, t) = \partial_x \ln\left(\frac{W(f_1, \dots, f_N, f)}{W(f_1, \dots, f_N)}\right) - \frac{1}{2} \tag{3}$$

is a solution to the Gardner equation (1) depending on  $2N$  real parameters  $a_j$ ,  $b_j$ ,  $1 \leq j \leq N$ .

**Remark 2.1** This result looks like to this given in [17], but with another similar normalization of the Gardner equation

$$u_t + 6u(1-u)u_x + u_{xxx} = 0. \quad (4)$$

In this paper [17], the generating functions are hyperbolic and the choice of the general solution  $f(x, t)$  gives another types of solutions to the Gardner equation. Only solutions of orders 1 and 2 have been explicitly presented.

### 2.1.2 Some examples of solutions to the Gardner equation

To shortened the paper, we only give the solutions of order 1, 2 and 3 in the case of generating trigonometric sinus functions.

#### Solution of order 1

**Proposition 2.1** The function  $u$  defined by

$$u(x, t) = 1/2 \frac{a_1^2}{\sin(1/2 a_1 x + 1/2 a_1^3 t + b_1)(\sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) - \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1)}.$$

is a solution to the Gardner equation (1) with  $a_1$ ,  $b_1$  arbitrarily real parameters.

#### Solution of order 2

**Proposition 2.2** The function  $u$  defined by

$$u(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (5)$$

with

$$\begin{aligned} n(x, t) = & -a_2^4 - a_1^4 + a_2^4 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 + a_1^4 (\cos(1/2 a_2 x + 1/2 a_2^3 t + \\ & b_2))^2 - a_2^2 a_1^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 - a_2^2 a_1^2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 + \\ & 2 a_2^2 a_1^2 + a_1^4 \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) - \cos(1/2 a_1 x + \\ & 1/2 a_1^3 t + b_1) a_1^3 a_2^2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) - a_1^2 \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2^3 \sin(1/2 a_2 x + \\ & 1/2 a_2^3 t + b_2) + \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 a_2^4 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \end{aligned}$$

and,

$$\begin{aligned} d(x, t) = & -2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 a_2^2 + 2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 a_2^2 (\cos(1/2 a_1 x + \\ & 1/2 a_1^3 t + b_1))^2 - 2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2^3 + \\ & 2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2^3 (\cos(1/2 a_1 x + 1/2 a_1^3 t + \\ & b_1))^2 + 4 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2 \sin(1/2 a_2 x + 1/2 a_2^3 t + \\ & b_2) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 + 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_2 x + 1/2 a_2^3 t + \\ & b_2) a_2^3 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 + 2 \sin(1/2 a_2 x + \\ & 1/2 a_2^3 t + b_2) a_1^2 \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2 - 2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_1^2 \cos(1/2 a_2 x + \\ & 1/2 a_2^3 t + b_2) a_2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 - 2 a_2^2 a_1^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 + \\ & 4 a_2^2 a_1^2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 + 2 \sin(1/2 a_1 x + \end{aligned}$$

$$\begin{aligned}
& 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 a_2^2 - 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + \\
& 1/2 a_1^3 t + b_1) a_1 a_2^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 - 2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 a_1^2 + \\
& 2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 a_1^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 - 2 a_2^2 a_1^2 (\cos(1/2 a_1 x + \\
& 1/2 a_1^3 t + b_1))^2 - 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1^3 + \\
& 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1^3 (\cos(1/2 a_2 x + 1/2 a_2^3 t + \\
& b_2))^2 + 2 \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1^3 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \sin(1/2 a_1 x + \\
& 1/2 a_1^3 t + b_1) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2
\end{aligned}$$

is a solution to the Gardner equation (1) with  $a_1, a_2, b_1, b_2$  arbitrarily real parameters.

### Solution of order 3

For this third order, we only present solution in the particular case where  $a_1 = 1, a_2 = 2, a_3 = 3, b_1 = 0, b_2 = 0, b_3 = 0$  to shorten the paper.

**Proposition 2.3** *The function  $u$  defined by*

$$u(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (6)$$

with

$$\begin{aligned}
n(x, t) = & -30 \sin(1/2 x + 1/2 t) \cos(1/2 x + 1/2 t) - 54 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) + \\
& 96 \sin(x + 4t) \cos(x + 4t) + 192 (\cos(x + 4t))^2 + 60 (\cos(1/2 x + 1/2 t))^2 + 108 \sin(1/2 x + \\
& 1/2 t) \cos(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \sin(3/2 x + \frac{27}{2} t) - 252 \sin(1/2 x + 1/2 t) \cos(x + \\
& 4t) \cos(1/2 x + 1/2 t) \sin(x + 4t) - 108 \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) \sin(x + \\
& 4t) + 84 (\cos(1/2 x + 1/2 t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - 102 (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - \\
& 234 (\cos(1/2 x + 1/2 t))^2 (\cos(x + 4t))^2 - 108 \sin(x + 4t) \cos(x + 4t) (\cos(1/2 x + 1/2 t))^2 - \\
& 24 \sin(1/2 x + 1/2 t) \cos(1/2 x + 1/2 t) (\cos(3/2 x + \frac{27}{2} t))^2 + 54 \sin(1/2 x + 1/2 t) \cos(1/2 x + \\
& 1/2 t) (\cos(x + 4t))^2 - 18 (\cos(x + 4t))^2 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) + 12 (\cos(3/2 x + \\
& \frac{27}{2} t))^2 \sin(x + 4t) \cos(x + 4t) + 72 (\cos(1/2 x + 1/2 t))^2 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) \\
& \text{and,}
\end{aligned}$$

$$\begin{aligned}
d(x, t) = & 128 (\cos(x + 4t))^2 + 81 (\cos(3/2 x + \frac{27}{2} t))^2 + 5 (\cos(1/2 x + 1/2 t))^2 + 276 \sin(1/2 x + \\
& 1/2 t) \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \cos(3/2 x + \frac{27}{2} t) \sin(x + 4t) + \\
& 54 \sin(1/2 x + 1/2 t) \cos(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \sin(3/2 x + \frac{27}{2} t) - 56 \sin(1/2 x + \\
& 1/2 t) \cos(x + 4t) \cos(1/2 x + 1/2 t) \sin(x + 4t) - 216 \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \\
& \frac{27}{2} t) \sin(x + 4t) - 54 \sin(1/2 x + 1/2 t) (\cos(x + 4t))^2 \cos(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \sin(3/2 x + \\
& \frac{27}{2} t) + 56 \sin(1/2 x + 1/2 t) \cos(x + 4t) (\cos(3/2 x + \frac{27}{2} t))^2 \cos(1/2 x + 1/2 t) \sin(x + \\
& 4t) + 216 (\cos(1/2 x + 1/2 t))^2 \cos(3/2 x + \frac{27}{2} t) \sin(x + 4t) \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) - \\
& 86 (\cos(1/2 x + 1/2 t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - 209 (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - \\
& 84 \sin(x + 4t) \cos(x + 4t) (\cos(1/2 x + 1/2 t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 + 156 \sin(3/2 x + \\
& \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) (\cos(1/2 x + 1/2 t))^2 (\cos(x + 4t))^2 + 204 \sin(1/2 x + 1/2 t) \cos(1/2 x + \\
& 1/2 t) (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 + 214 (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 (\cos(1/2 x + \\
& 1/2 t))^2 - 133 (\cos(1/2 x + 1/2 t))^2 (\cos(x + 4t))^2 + 30 \sin(x + 4t) \cos(x + 4t) (\cos(1/2 x + \\
& 1/2 t))^2 - 108 \sin(1/2 x + 1/2 t) \cos(1/2 x + 1/2 t) (\cos(3/2 x + \frac{27}{2} t))^2 - 96 \sin(1/2 x + \\
& 1/2 t) \cos(1/2 x + 1/2 t) (\cos(x + 4t))^2 - 96 (\cos(x + 4t))^2 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \\
& \frac{27}{2} t) + 54 (\cos(3/2 x + \frac{27}{2} t))^2 \sin(x + 4t) \cos(x + 4t) - 60 (\cos(1/2 x + 1/2 t))^2 \sin(3/2 x + \\
& \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t)
\end{aligned}$$

is a solution to the Gardner equation (1).

## 2.2 Other types of solutions

We obtain similar results with other types of generating functions whose proofs are identical.

### 2.2.1 Solutions with cosine generating functions

**Theorem 2.2** Let  $g_j, g$  be the following functions

$$\begin{aligned} g_j(x, t) &= \cos\left(\frac{1}{2}a_jx + \frac{1}{2}a_j^3t + b_j\right), & \text{for } 1 \leq i \leq N, \\ g(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (7)$$

then the function  $u$  defined by

$$u(x, t) = \partial_x \ln\left(\frac{W(g_1, \dots, g_N, g)}{W(g_1, \dots, g_N)}\right) - \frac{1}{2} \quad (8)$$

is a solution to the Gardner equation (1) with  $a_j, b_j$   $1 \leq j \leq N$  arbitrarily real parameters.

### 2.2.2 Solutions with hyperbolic generating functions

**Theorem 2.3** Let  $h_j, h$  be the following functions

$$\begin{aligned} h_j(x, t) &= \sinh\left(\frac{1}{2}a_jx - \frac{1}{2}a_j^3t + b_j\right), & \text{for } 1 \leq i \leq N, \\ h(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (9)$$

then the function  $u$  defined by

$$u(x, t) = \partial_x \ln\left(\frac{W(h_1, \dots, h_N, h)}{W(h_1, \dots, h_N)}\right) - \frac{1}{2} \quad (10)$$

is a solution to the Gardner equation (1) with  $a_j, b_j$   $1 \leq j \leq N$  arbitrarily real parameters.

**Theorem 2.4** Let  $k_j, k$  be the following functions

$$\begin{aligned} k_j(x, t) &= \cosh\left(\frac{1}{2}a_jx - \frac{1}{2}a_j^3t + b_j\right), & \text{for } 1 \leq i \leq N, \\ k(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (11)$$

then the function  $u$  defined by

$$u(x, t) = \partial_x \ln \left( \frac{W(k_1, \dots, k_N, k)}{W(k_1, \dots, k_N)} \right) - \frac{1}{2} \quad (12)$$

is a solution to the Gardner equation (1) with  $a_j, b_j$   $1 \leq j \leq N$  arbitrarily real parameters.

### 3 Rational solutions to the Gardner equation

#### 3.1 Rational solutions to the Gardner equation as a limit

To obtain rational solutions to the Gardner equation (1), we are going to perform a limit when a parameter  $e$  tends to 0. For this we replace all parameters  $a_j$  and  $b_j$ ,  $1 \leq j \leq N$  by  $\hat{a}_j = \sum_{k=1}^N a_k (je)^{2k-1}$  and  $\hat{b}_j = \sum_{k=1}^N b_k (je)^{2k-1}$  with  $e$  an arbitrary real parameter.

We get the following result :

**Theorem 3.1** *Let  $\psi_j, \psi$  be the functions*

$$\psi_j(x, t, e) = \sin \left( \frac{1}{2} \sum_{k=1}^N a_k (je)^{2k-1} x + \frac{1}{2} \left( \sum_{k=1}^N a_k (je)^{2k-1} \right)^3 t + \sum_{k=1}^N b_k (je)^{2k-1} \right),$$

for  $1 \leq j \leq N$ ,

$$\psi(x, t) = \exp \left( \frac{1}{2} (x - t) \right)$$

then the function  $u$  defined by

$$u(x, t) = \lim_{e \rightarrow 0} \partial_x \ln \left( \frac{W(\psi_1, \dots, \psi_N, \psi)}{W(\psi_1, \dots, \psi_N)} \right) - \frac{1}{2} \quad (13)$$

is a rational solution to the Gardner equation (1).

We have similar results with generating cosine or hyperbolic functions.

#### 3.2 Degenerate rational solutions to the Gardner equation

We can give the expression of the rational solutions of the Gardner equation avoiding the presence of a limit. For this we consider another type of functions. We get the following result :

**Theorem 3.2** *Let  $\psi, \varphi_j, \varphi$  be the functions*

$$\psi(x, t, e) = \sin \left( \frac{1}{2} \left( \sum_{k=1}^N a_k e^{2k-1} \right) x + \frac{1}{2} \left( \sum_{k=1}^N a_k e^{2k-1} \right)^3 t + \sum_{k=1}^N b_k e^{2k-1} \right),$$

$$\varphi_j(x, t) = \frac{\partial^{2j-1} \psi(x, t, 0)}{\partial_{2j-1} e}, \text{ for } 1 \leq j \leq N,$$

$\varphi(x, t) = \exp\left(\frac{1}{2}(x - t)\right)$   
then the function  $v$  defined by

$$v(x, t) = \partial_x \ln \left( \frac{W(\varphi_1, \dots, \varphi_N, \varphi)}{W(\varphi_1, \dots, \varphi_N)} \right) - \frac{1}{2} \quad (14)$$

is a rational solution to the Gardner equation (1) depending on  $2N$  parameters  $a_j, b_j, 1 \leq j \leq N$ .

So we obtain an infinite hierarchy of rational solutions to the Gardner equation depending on the integer  $N$ .

In the following we give some examples of rational solutions.

These results are consequences of the previous result.

But, it is also possible to prove it directly in replacing the expressions of each of the solutions given in the corresponding equation and check that the relation is verified.

### 3.3 First order rational solutions

We have the following result at order  $N = 1$  :

**Proposition 3.1** *The function  $v$  defined by*

$$v(x, t) = \frac{2a_1^2}{(a_1x + 2b_1)(-2a_1 + 2b_1 + a_1x)}, \quad (15)$$

is a rational solution to the Gardner equation (1) with  $a_1, b_1$ , arbitrarily real parameters.

### 3.4 Second order rational solutions

**Proposition 3.2** *The function  $v$  defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (16)$$

with

$$n(x, t) = -6a_1a_2(-a_1^5a_2 - a_2^5a_1 + 2a_2^3a_1^3)x^4 - 6a_1a_2(4a_2^5a_1 + 4a_1^5a_2 - 8a_2^3a_1^3)x^3 - 6a_1a_2(-48a_2^3ta_1^3 - 48b_2a_1^3 + 48b_2a_1a_2^2 + 24a_1^5ta_2 + 48b_1a_2a_1^2 - 48b_1a_2^3 + 24a_2^5ta_1)x^2 - 6a_1a_2(-48b_2a_1a_2^2 - 48b_1a_2a_1^2 + 48a_2^3ta_1^3 + 48b_2a_1^3 - 24a_1^5ta_2 + 48b_1a_2^3 - 24a_2^5ta_1),$$

and,

$$d(x, t) = (a_1^3a_2 - a_1a_2^3)^2x^6 + (a_1^3a_2 - a_1a_2^3)(-6a_1^3a_2 + 6a_1a_2^3)x^5 + (a_1^3a_2 - a_1a_2^3)(12a_1^3a_2 - 12a_1a_2^3)x^4 + 2(-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)(a_1^3a_2 - a_1a_2^3)x^3 + (-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)(-6a_1^3a_2 + 6a_1a_2^3)x^2 + (-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)(12a_1^3a_2 - 12a_1a_2^3)x + (-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)^2$$

is a rational solution to the Gardner equation (1) dependant on 4 real parameters  $a_1, a_2, b_1, b_2$ .



### 3.5 Rational solutions of order three

The explicit solution depending on 6 real parameters being too long, we give only the rational solution without parameters. We get the following rational solutions given by :

**Proposition 3.3** *The function  $v$  defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (17)$$

with

$$n(x, t) = 12x^{10} - 120x^9 + 360x^8 + 8640tx^5 + 64800t^2x^4 - 259200t^2x^3 + 518400t^2x^2 + 518400t^3x - 518400t^3$$

and,

$$d(x, t) = x^{12} - 12x^{11} + 60x^{10} + (120t - 120)x^9 - 1080tx^8 + 4320tx^7 + (-1440t^2 - 60t(120 - 60t) - 1440t)x^6 - 12960t^2x^5 + (720t^2(120 - 60t) - 60t(1440t + 720t^2))x^3 + 259200t^3x^2 - 518400t^3x + 720t^2(1440t + 720t^2)$$

is a rational solution to the Gardner equation (1).

## 4 Case of the mKdV equation

### 4.1 Solutions in terms of wronskians

As a consequence of the previous study, we easily deduce solutions to the modified Korteweg-de Vries (mKdV) equation in the following normalization (??)

$$w_t - 6w^2w_x + w_{xxx} = 0.$$

We have the following result :

**Theorem 4.1** *Let  $f_j, f$  be the following functions*

$$\begin{aligned} f_j(x, t) &= \sin\left(\frac{1}{2}a_jx + \frac{1}{2}a_j^3t + b_j\right), \quad \text{for } 1 \leq i \leq N, \\ f(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (18)$$

then the function  $w$  defined by

$$w(x, t) = \partial_x \ln\left(\frac{W(f_1, \dots, f_N, f)}{W(f_1, \dots, f_N)}\right) \quad (19)$$

is a solution to the mKdV equation (??) depending on  $2N$  real parameters  $a_j, b_j, 1 \leq j \leq N$ .

## 4.2 Rational solutions

We get the following statement :

**Theorem 4.2** *Let  $\psi_j, \psi$  be the functions*

$$\psi_j(x, t, e) = \sin \left( \frac{1}{2} \sum_{k=1}^N a_k (je)^{2k-1} x + \frac{1}{2} \left( \sum_{k=1}^N a_k (je)^{2k-1} \right)^3 t + \sum_{k=1}^N b_k (je)^{2k-1} \right),$$

for  $1 \leq j \leq N$ ,

$$\psi(x, t) = \exp \left( \frac{1}{2} (x - t) \right)$$

then the function  $u$  defined by

$$u(x, t) = \lim_{e \rightarrow 0} \partial_x \ln \left( \frac{W(\psi_1, \dots, \psi_N, \psi)}{W(\psi_1, \dots, \psi_N)} \right) \quad (20)$$

is a rational solution to the mKdV equation (??).

We can also give the expression of the rational solutions of the mKdV equation without the presence of a limit. We get the following result : We get the following result :

**Theorem 4.3** *Let  $\psi, \varphi_j, \varphi$  be the functions*

$$\psi(x, t, e) = \sin \left( \frac{1}{2} \left( \sum_{k=1}^N a_k e^{2k-1} \right) x + \frac{1}{2} \left( \sum_{k=1}^N a_k e^{2k-1} \right)^3 t + \sum_{k=1}^N b_k e^{2k-1} \right),$$

$$\varphi_j(x, t) = \frac{\partial^{2j-1} \psi(x, t, 0)}{\partial_{2j-1} e}, \text{ for } 1 \leq j \leq N,$$

$$\varphi(x, t) = \exp \left( \frac{1}{2} (x - t) \right)$$

then the function  $v$  defined by

$$v(x, t) = \partial_x \ln \left( \frac{W(\varphi_1, \dots, \varphi_N, \varphi)}{W(\varphi_1, \dots, \varphi_N)} \right) \quad (21)$$

is a rational solution to the mKdV equation (??) depending on  $2N$  parameters  $a_j, b_j, 1 \leq j \leq N$ .

## 5 Conclusion

We have given two types of representations of solutions to the Gardner equation. First, solutions as a quotient of a wronskian of order  $N + 1$  by a wronskian of order  $N$  depending on  $2N$  real parameters have been constructed. Then performing a passage to the limit when one parameter goes to 0 we get rational solutions to the Gardner equation depending on  $2N$  real parameters.

So we obtain an infinite hierarchy of multiparametric families of rational solutions to the Gardner equation as a quotient of a polynomials in  $x$  and  $t$  depending on  $2N$  real parameters.

As a byproduct, we easily deduce solutions to the mKdV equation in terms of wronskians and rational solutions, depending on  $2N$  real parameters.

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