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Solutions to the Gardner equation with multi-parameters and the rational case

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Abstract

We construct solutions to the Gardner equation in terms of trigonometric and hyperbolic functions, depending on several real parameters. Using a passage to the limit when one of these parameters goes to 0, we get, for each positive integer N , rational solutions as a quotient of polynomials in x and t depending on $2N$ parameters. We construct explicit expressions of these rational solutions for orders $N = 1$ until $N = 3$. We easily deduce solutions to the mKdV equation in terms of wronskians as well as rational solutions depending on $2N$ real parameters.

Key Words : Gardner equation, wronskians, rational solutions.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We consider the Gardner equation in the following normalization

$$u_t - 6u(1+u)u_x + u_{xxx} = 0, \quad (1)$$

where the subscripts x and t denote partial derivatives.

The introduction of this equation is attributed to Gardner [1] in 1968. It was first considered [1] as an auxiliary mathematical tool in the derivation of the infinite set of local conservation laws of the Korteweg de Vries equation. This equation is a fundamental mathematical model for the description of weakly nonlinear dispersive waves. It can describe nonlinear wave effects in several physical contexts: for example, in plasma physics [2, 3], fluid flows [4], quantum fluid dynamics [5], in dusty plasmas [6], in ocean and atmosphere [7]. It is fundamental tool to describe large-amplitude internal waves [4, 8, 9]. These

waves are such that their vertical amplitudes underwater can exceed 170 meters as described in [10].

Many methods have been used to solve this equation, as for example the Hirota method [11], the series expansion method [12], the mapping method [13] or the method of leading-order analysis [14].

Recently published work on this equation can also be mentioned, such as [?, ?, ?, ?, ?]

Here, we used the Darboux transformation to construct different type of solutions. We give a representation of solutions in terms of a quotient of a wronskian of order $N + 1$ by a wronskian of order N . We get what we will call N -order solutions which depend on $2N$ real parameters in terms of trigonometric or hyperbolic functions. Then we construct rational solutions in performing a passage to the limit when one of these parameters goes to 0. We obtain rational solutions as a quotient of polynomials in x and t , depending on $2N$ parameters. We give explicit solutions in the simplest cases $N = 1, 2, 3$.

We easily deduce solutions to the mKdV equation and their corresponding rational solutions depending on $2N$ real parameters.

2 Nth-order solutions to the Gardner equation in terms of wronskians

2.1 First type of solutions

2.1.1 Nth-order solutions in terms of wronskians of sine functions

We consider the Gardner equation

$$u_t - 6u(1 + u)u_x + u_{xxx} = 0.$$

In the following, we will use the wronskian of order N of the functions f_1, \dots, f_N which is the determinant denoted $W(f_1, \dots, f_N)$, defined by $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$, ∂_x^i being the partial derivative of order i with respect to x and $\partial_x^0 f_j$ being the function f_j .

We consider a_j, b_j , arbitrary real numbers $1 \leq j \leq N$. Then, we have the following statement :

Theorem 2.1 *Let f_j, f be the functions defined by*

$$\begin{aligned} f_j(x, t) &= \sin\left(\frac{1}{2}a_j x + \frac{1}{2}a_j^3 t + b_j\right), & \text{for } 1 \leq j \leq N, \\ f(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \tag{2}$$

then the function u defined by

$$u(x, t) = \partial_x \ln\left(\frac{W(f_1, \dots, f_N, f)}{W(f_1, \dots, f_N)}\right) - \frac{1}{2} \tag{3}$$

is a solution to the Gardner equation (1) depending on $2N$ real parameters a_j , b_j , $1 \leq j \leq N$.

Remark 2.1 This result looks like to this given in [17], but with another similar normalization of the Gardner equation

$$u_t + 6u(1-u)u_x + u_{xxx} = 0. \quad (4)$$

In this paper [17], the generating functions are hyperbolic and the choice of the general solution $f(x, t)$ gives another types of solutions to the Gardner equation. Only solutions of orders 1 and 2 have been explicitly presented.

2.1.2 Some examples of solutions to the Gardner equation

To shortened the paper, we only give the solutions of order 1, 2 and 3 in the case of generating trigonometric sinus functions.

Solution of order 1

Proposition 2.1 The function u defined by

$$u(x, t) = 1/2 \frac{a_1^2}{\sin(1/2 a_1 x + 1/2 a_1^3 t + b_1)(\sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) - \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1)}.$$

is a solution to the Gardner equation (1) with a_1 , b_1 arbitrarily real parameters.

Solution of order 2

Proposition 2.2 The function u defined by

$$u(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (5)$$

with

$$\begin{aligned} n(x, t) = & -a_2^4 - a_1^4 + a_2^4 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 + a_1^4 (\cos(1/2 a_2 x + 1/2 a_2^3 t + \\ & b_2))^2 - a_2^2 a_1^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 - a_2^2 a_1^2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 + \\ & 2 a_2^2 a_1^2 + a_1^4 \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) - \cos(1/2 a_1 x + \\ & 1/2 a_1^3 t + b_1) a_1^3 a_2^2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) - a_1^2 \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2^3 \sin(1/2 a_2 x + \\ & 1/2 a_2^3 t + b_2) + \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 a_2^4 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \end{aligned}$$

and,

$$\begin{aligned} d(x, t) = & -2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 a_2^2 + 2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 a_2^2 (\cos(1/2 a_1 x + \\ & 1/2 a_1^3 t + b_1))^2 - 2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2^3 + \\ & 2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2^3 (\cos(1/2 a_1 x + 1/2 a_1^3 t + \\ & b_1))^2 + 4 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2 \sin(1/2 a_2 x + 1/2 a_2^3 t + \\ & b_2) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 + 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_2 x + 1/2 a_2^3 t + \\ & b_2) a_2^3 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 + 2 \sin(1/2 a_2 x + \\ & 1/2 a_2^3 t + b_2) a_1^2 \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2 - 2 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_1^2 \cos(1/2 a_2 x + \\ & 1/2 a_2^3 t + b_2) a_2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 - 2 a_2^2 a_1^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 + \\ & 4 a_2^2 a_1^2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 + 2 \sin(1/2 a_1 x + \end{aligned}$$

$$\begin{aligned}
& 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1 a_2^2 - 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + \\
& 1/2 a_1^3 t + b_1) a_1 a_2^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 - 2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 a_1^2 + \\
& 2 (\cos(1/2 a_1 x + 1/2 a_1^3 t + b_1))^2 a_1^2 (\cos(1/2 a_2 x + 1/2 a_2^3 t + b_2))^2 - 2 a_2^2 a_1^2 (\cos(1/2 a_1 x + \\
& 1/2 a_1^3 t + b_1))^2 - 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1^3 + \\
& 2 \sin(1/2 a_1 x + 1/2 a_1^3 t + b_1) \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1^3 (\cos(1/2 a_2 x + 1/2 a_2^3 t + \\
& b_2))^2 + 2 \cos(1/2 a_1 x + 1/2 a_1^3 t + b_1) a_1^3 \sin(1/2 a_2 x + 1/2 a_2^3 t + b_2) \sin(1/2 a_1 x + \\
& 1/2 a_1^3 t + b_1) \cos(1/2 a_2 x + 1/2 a_2^3 t + b_2) a_2
\end{aligned}$$

is a solution to the Gardner equation (1) with a_1, a_2, b_1, b_2 arbitrarily real parameters.

Solution of order 3

For this third order, we only present solution in the particular case where $a_1 = 1, a_2 = 2, a_3 = 3, b_1 = 0, b_2 = 0, b_3 = 0$ to shorten the paper.

Proposition 2.3 *The function u defined by*

$$u(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (6)$$

with

$$\begin{aligned}
n(x, t) = & -30 \sin(1/2 x + 1/2 t) \cos(1/2 x + 1/2 t) - 54 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) + \\
& 96 \sin(x + 4t) \cos(x + 4t) + 192 (\cos(x + 4t))^2 + 60 (\cos(1/2 x + 1/2 t))^2 + 108 \sin(1/2 x + \\
& 1/2 t) \cos(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \sin(3/2 x + \frac{27}{2} t) - 252 \sin(1/2 x + 1/2 t) \cos(x + \\
& 4t) \cos(1/2 x + 1/2 t) \sin(x + 4t) - 108 \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) \sin(x + \\
& 4t) + 84 (\cos(1/2 x + 1/2 t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - 102 (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - \\
& 234 (\cos(1/2 x + 1/2 t))^2 (\cos(x + 4t))^2 - 108 \sin(x + 4t) \cos(x + 4t) (\cos(1/2 x + 1/2 t))^2 - \\
& 24 \sin(1/2 x + 1/2 t) \cos(1/2 x + 1/2 t) (\cos(3/2 x + \frac{27}{2} t))^2 + 54 \sin(1/2 x + 1/2 t) \cos(1/2 x + \\
& 1/2 t) (\cos(x + 4t))^2 - 18 (\cos(x + 4t))^2 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) + 12 (\cos(3/2 x + \\
& \frac{27}{2} t))^2 \sin(x + 4t) \cos(x + 4t) + 72 (\cos(1/2 x + 1/2 t))^2 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) \\
& \text{and,}
\end{aligned}$$

$$\begin{aligned}
d(x, t) = & 128 (\cos(x + 4t))^2 + 81 (\cos(3/2 x + \frac{27}{2} t))^2 + 5 (\cos(1/2 x + 1/2 t))^2 + 276 \sin(1/2 x + \\
& 1/2 t) \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \cos(3/2 x + \frac{27}{2} t) \sin(x + 4t) + \\
& 54 \sin(1/2 x + 1/2 t) \cos(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \sin(3/2 x + \frac{27}{2} t) - 56 \sin(1/2 x + \\
& 1/2 t) \cos(x + 4t) \cos(1/2 x + 1/2 t) \sin(x + 4t) - 216 \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \\
& \frac{27}{2} t) \sin(x + 4t) - 54 \sin(1/2 x + 1/2 t) (\cos(x + 4t))^2 \cos(3/2 x + \frac{27}{2} t) \cos(1/2 x + 1/2 t) \sin(3/2 x + \\
& \frac{27}{2} t) + 56 \sin(1/2 x + 1/2 t) \cos(x + 4t) (\cos(3/2 x + \frac{27}{2} t))^2 \cos(1/2 x + 1/2 t) \sin(x + \\
& 4t) + 216 (\cos(1/2 x + 1/2 t))^2 \cos(3/2 x + \frac{27}{2} t) \sin(x + 4t) \cos(x + 4t) \sin(3/2 x + \frac{27}{2} t) - \\
& 86 (\cos(1/2 x + 1/2 t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - 209 (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 - \\
& 84 \sin(x + 4t) \cos(x + 4t) (\cos(1/2 x + 1/2 t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 + 156 \sin(3/2 x + \\
& \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t) (\cos(1/2 x + 1/2 t))^2 (\cos(x + 4t))^2 + 204 \sin(1/2 x + 1/2 t) \cos(1/2 x + \\
& 1/2 t) (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 + 214 (\cos(x + 4t))^2 (\cos(3/2 x + \frac{27}{2} t))^2 (\cos(1/2 x + \\
& 1/2 t))^2 - 133 (\cos(1/2 x + 1/2 t))^2 (\cos(x + 4t))^2 + 30 \sin(x + 4t) \cos(x + 4t) (\cos(1/2 x + \\
& 1/2 t))^2 - 108 \sin(1/2 x + 1/2 t) \cos(1/2 x + 1/2 t) (\cos(3/2 x + \frac{27}{2} t))^2 - 96 \sin(1/2 x + \\
& 1/2 t) \cos(1/2 x + 1/2 t) (\cos(x + 4t))^2 - 96 (\cos(x + 4t))^2 \sin(3/2 x + \frac{27}{2} t) \cos(3/2 x + \\
& \frac{27}{2} t) + 54 (\cos(3/2 x + \frac{27}{2} t))^2 \sin(x + 4t) \cos(x + 4t) - 60 (\cos(1/2 x + 1/2 t))^2 \sin(3/2 x + \\
& \frac{27}{2} t) \cos(3/2 x + \frac{27}{2} t)
\end{aligned}$$

is a solution to the Gardner equation (1).

2.2 Other types of solutions

We obtain similar results with other types of generating functions whose proofs are identical.

2.2.1 Solutions with cosine generating functions

Theorem 2.2 Let g_j, g be the following functions

$$\begin{aligned} g_j(x, t) &= \cos\left(\frac{1}{2}a_jx + \frac{1}{2}a_j^3t + b_j\right), \quad \text{for } 1 \leq i \leq N, \\ g(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (7)$$

then the function u defined by

$$u(x, t) = \partial_x \ln\left(\frac{W(g_1, \dots, g_N, g)}{W(g_1, \dots, g_N)}\right) - \frac{1}{2} \quad (8)$$

is a solution to the Gardner equation (1) with a_j, b_j $1 \leq j \leq N$ arbitrarily real parameters.

2.2.2 Solutions with hyperbolic generating functions

Theorem 2.3 Let h_j, h be the following functions

$$\begin{aligned} h_j(x, t) &= \sinh\left(\frac{1}{2}a_jx - \frac{1}{2}a_j^3t + b_j\right), \quad \text{for } 1 \leq i \leq N, \\ h(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (9)$$

then the function u defined by

$$u(x, t) = \partial_x \ln\left(\frac{W(h_1, \dots, h_N, h)}{W(h_1, \dots, h_N)}\right) - \frac{1}{2} \quad (10)$$

is a solution to the Gardner equation (1) with a_j, b_j $1 \leq j \leq N$ arbitrarily real parameters.

Theorem 2.4 Let k_j, k be the following functions

$$\begin{aligned} k_j(x, t) &= \cosh\left(\frac{1}{2}a_jx - \frac{1}{2}a_j^3t + b_j\right), \quad \text{for } 1 \leq i \leq N, \\ k(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (11)$$

then the function u defined by

$$u(x, t) = \partial_x \ln \left(\frac{W(k_1, \dots, k_N, k)}{W(k_1, \dots, k_N)} \right) - \frac{1}{2} \quad (12)$$

is a solution to the Gardner equation (1) with a_j, b_j $1 \leq j \leq N$ arbitrarily real parameters.

3 Rational solutions to the Gardner equation

3.1 Rational solutions to the Gardner equation as a limit

To obtain rational solutions to the Gardner equation (1), we are going to perform a limit when a parameter e tends to 0. For this we replace all parameters a_j and b_j , $1 \leq j \leq N$ by $\hat{a}_j = \sum_{k=1}^N a_k(je)^{2k-1}$ and $\hat{b}_j = \sum_{k=1}^N b_k(je)^{2k-1}$ with e an arbitrary real parameter.

We get the following result :

Theorem 3.1 *Let ψ_j, ψ be the functions*

$$\psi_j(x, t, e) = \sin \left(\frac{1}{2} \sum_{k=1}^N a_k(je)^{2k-1} x + \frac{1}{2} \left(\sum_{k=1}^N a_k(je)^{2k-1} \right)^3 t + \sum_{k=1}^N b_k(je)^{2k-1} \right),$$

for $1 \leq j \leq N$,

$$\psi(x, t) = \exp \left(\frac{1}{2} (x - t) \right)$$

then the function u defined by

$$u(x, t) = \lim_{e \rightarrow 0} \partial_x \ln \left(\frac{W(\psi_1, \dots, \psi_N, \psi)}{W(\psi_1, \dots, \psi_N)} \right) - \frac{1}{2} \quad (13)$$

is a rational solution to the Gardner equation (1).

We have similar results with generating cosine or hyperbolic functions.

3.2 Degenerate rational solutions to the Gardner equation

We can give the expression of the rational solutions of the Gardner equation avoiding the presence of a limit. For this we consider another type of functions. We get the following result :

Theorem 3.2 *Let ψ, φ_j, φ be the functions*

$$\psi(x, t, e) = \sin \left(\frac{1}{2} \left(\sum_{k=1}^N a_k e^{2k-1} \right) x + \frac{1}{2} \left(\sum_{k=1}^N a_k e^{2k-1} \right)^3 t + \sum_{k=1}^N b_k e^{2k-1} \right),$$

$$\varphi_j(x, t) = \frac{\partial^{2j-1} \psi(x, t, 0)}{\partial_{2j-1} e}, \text{ for } 1 \leq j \leq N,$$

$\varphi(x, t) = \exp\left(\frac{1}{2}(x - t)\right)$
then the function v defined by

$$v(x, t) = \partial_x \ln \left(\frac{W(\varphi_1, \dots, \varphi_N, \varphi)}{W(\varphi_1, \dots, \varphi_N)} \right) - \frac{1}{2} \quad (14)$$

is a rational solution to the Gardner equation (1) depending on $2N$ parameters $a_j, b_j, 1 \leq j \leq N$.

So we obtain an infinite hierarchy of rational solutions to the Gardner equation depending on the integer N .

In the following we give some examples of rational solutions.

These results are consequences of the previous result.

But, it is also possible to prove it directly in replacing the expressions of each of the solutions given in the corresponding equation and check that the relation is verified.

3.3 First order rational solutions

We have the following result at order $N = 1$:

Proposition 3.1 *The function v defined by*

$$v(x, t) = \frac{2a_1^2}{(a_1x + 2b_1)(-2a_1 + 2b_1 + a_1x)}, \quad (15)$$

is a rational solution to the Gardner equation (1) with a_1, b_1 , arbitrarily real parameters.

3.4 Second order rational solutions

Proposition 3.2 *The function v defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (16)$$

with

$$n(x, t) = -6a_1a_2(-a_1^5a_2 - a_2^5a_1 + 2a_2^3a_1^3)x^4 - 6a_1a_2(4a_2^5a_1 + 4a_1^5a_2 - 8a_2^3a_1^3)x^3 - 6a_1a_2(-48a_2^3ta_1^3 - 48b_2a_1^3 + 48b_2a_1a_2^2 + 24a_1^5ta_2 + 48b_1a_2a_1^2 - 48b_1a_2^3 + 24a_2^5ta_1)x^2 - 6a_1a_2(-48b_2a_1a_2^2 - 48b_1a_2a_1^2 + 48a_2^3ta_1^3 + 48b_2a_1^3 - 24a_1^5ta_2 + 48b_1a_2^3 - 24a_2^5ta_1),$$

and,

$$d(x, t) = (a_1^3a_2 - a_1a_2^3)^2x^6 + (a_1^3a_2 - a_1a_2^3)(-6a_1^3a_2 + 6a_1a_2^3)x^5 + (a_1^3a_2 - a_1a_2^3)(12a_1^3a_2 - 12a_1a_2^3)x^4 + 2(-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)(a_1^3a_2 - a_1a_2^3)x^3 + (-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)(-6a_1^3a_2 + 6a_1a_2^3)x^2 + (-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)(12a_1^3a_2 - 12a_1a_2^3)x + (-12a_2^3ta_1 - 24b_2a_1 + 12a_1^3ta_2 + 24b_1a_2)^2$$

is a rational solution to the Gardner equation (1) dependant on 4 real parameters a_1, a_2, b_1, b_2 .

3.5 Rational solutions of order three

The explicit solution depending on 6 real parameters being too long, we give only the rational solution without parameters. We get the following rational solutions given by :

Proposition 3.3 *The function v defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (17)$$

with

$$n(x, t) = 12x^{10} - 120x^9 + 360x^8 + 8640tx^5 + 64800t^2x^4 - 259200t^2x^3 + 518400t^2x^2 + 518400t^3x - 518400t^3$$

and,

$$d(x, t) = x^{12} - 12x^{11} + 60x^{10} + (120t - 120)x^9 - 1080tx^8 + 4320tx^7 + (-1440t^2 - 60t(120 - 60t) - 1440t)x^6 - 12960t^2x^5 + (720t^2(120 - 60t) - 60t(1440t + 720t^2))x^3 + 259200t^3x^2 - 518400t^3x + 720t^2(1440t + 720t^2)$$

is a rational solution to the Gardner equation (1).

4 Case of the mKdV equation

4.1 Solutions in terms of wronskians

As a consequence of the previous study, we easily deduce solutions to the modified Korteweg-de Vries (mKdV) equation in the following normalization (??)

$$w_t - 6w^2w_x + w_{xxx} = 0.$$

We have the following result :

Theorem 4.1 *Let f_j, f be the following functions*

$$\begin{aligned} f_j(x, t) &= \sin\left(\frac{1}{2}a_jx + \frac{1}{2}a_j^3t + b_j\right), \quad \text{for } 1 \leq i \leq N, \\ f(x, t) &= \exp\left(\frac{1}{2}(x - t)\right) \end{aligned} \quad (18)$$

then the function w defined by

$$w(x, t) = \partial_x \ln\left(\frac{W(f_1, \dots, f_N, f)}{W(f_1, \dots, f_N)}\right) \quad (19)$$

is a solution to the mKdV equation (??) depending on $2N$ real parameters $a_j, b_j, 1 \leq j \leq N$.

4.2 Rational solutions

We get the following statement :

Theorem 4.2 *Let ψ_j, ψ be the functions*

$$\psi_j(x, t, e) = \sin \left(\frac{1}{2} \sum_{k=1}^N a_k (je)^{2k-1} x + \frac{1}{2} \left(\sum_{k=1}^N a_k (je)^{2k-1} \right)^3 t + \sum_{k=1}^N b_k (je)^{2k-1} \right),$$

for $1 \leq j \leq N$,

$$\psi(x, t) = \exp \left(\frac{1}{2} (x - t) \right)$$

then the function u defined by

$$u(x, t) = \lim_{e \rightarrow 0} \partial_x \ln \left(\frac{W(\psi_1, \dots, \psi_N, \psi)}{W(\psi_1, \dots, \psi_N)} \right) \quad (20)$$

is a rational solution to the mKdV equation (??).

We can also give the expression of the rational solutions of the mKdV equation without the presence of a limit. We get the following result : We get the following result :

Theorem 4.3 *Let ψ, φ_j, φ be the functions*

$$\psi(x, t, e) = \sin \left(\frac{1}{2} \left(\sum_{k=1}^N a_k e^{2k-1} \right) x + \frac{1}{2} \left(\sum_{k=1}^N a_k e^{2k-1} \right)^3 t + \sum_{k=1}^N b_k e^{2k-1} \right),$$

$$\varphi_j(x, t) = \frac{\partial^{2j-1} \psi(x, t, 0)}{\partial_{2j-1} e}, \text{ for } 1 \leq j \leq N,$$

$$\varphi(x, t) = \exp \left(\frac{1}{2} (x - t) \right)$$

then the function v defined by

$$v(x, t) = \partial_x \ln \left(\frac{W(\varphi_1, \dots, \varphi_N, \varphi)}{W(\varphi_1, \dots, \varphi_N)} \right) \quad (21)$$

is a rational solution to the mKdV equation (??) depending on $2N$ parameters $a_j, b_j, 1 \leq j \leq N$.

5 Conclusion

We have given two types of representations of solutions to the Gardner equation. First, solutions as a quotient of a wronskian of order $N + 1$ by a wronskian of order N depending on $2N$ real parameters have been constructed. Then performing a passage to the limit when one parameter goes to 0 we get rational solutions to the Gardner equation depending on $2N$ real parameters.

So we obtain an infinite hierarchy of multiparametric families of rational solutions to the Gardner equation as a quotient of a polynomials in x and t depending on $2N$ real parameters.

As a byproduct, we easily deduce solutions to the mKdV equation in terms of wronskians and rational solutions, depending on $2N$ real parameters.

References

- [1] R.M. Miura, C.S. Gardner and M.D. Kruskal, Korteweg-deVries equation and generalizations. II. Existence of conservation laws and constants of motion., *J. Math. Phys.*, V. 9, 1204-1209, 1968
- [2] S. Watanabe, Ion Acoustic Soliton in Plasma with Negative Ion, *J. Phys. Soc. Japan*, V. 53, 950-956, 1984
- [3] M.S. Ruderman, T. Talipova, and E. Pelinovsky, Dynamics of modulationally unstable ion-acoustic wavepackets in plasmas with negative ions, *J. Plasma Phys.*, V. 74, 639-656, 2008
- [4] R. Grimshaw, *Environmental Stratified Flows*, *Topics in Environmental Fluid Mechanics*, Kluwer, 2002
- [5] E. Demler and A. Maltsev, Semiclassical solitons in strongly correlated systems of ultracold bosonic atoms in optical lattices, *Ann. Phys.*, V. 326, 1775-1805, 2011
- [6] A.H. Khater, A.A. Abdallah, O.H. El-Kalaawy, D.K. Callebaut, Bäcklund transformations, a simple transformation and exact solutions for dust-acoustic solitary waves in dusty plasma consisting of cold dust particles and two-temperature isothermal ions, *Phys. Plasmas*, V. 6, 4542-4547, 1999
- [7] R. Grimshaw, D. Pelinovsky, E. Pelinovsky, T. Talipova, *Wave Group Dynamics in Weakly Nonlinear Long-Wave Models*, *Phys. D*, V. 159, 35-37, 2001
- [8] K.R. Helfrich and W.K. Melville, Long nonlinear internal waves, *Annu. Rev. Fluid Mech.*, V. 38, 395-425, 2006
- [9] J.R. Apel, L.A. Ostrovsky, Y.A. Stepanyants, and J.F. Lynch, Internal solitons in the ocean and their effect on underwater sound, *J. Acoust. Soc. Am.*, V. 121, 695-722, 2007
- [10] M.H. Alford, T. Peacock, T.Y. Tang, The formation and fate of internal waves in the South China Sea, *Nature*, V 521, 65-69, 2015
- [11] M. Wadati, Wave propagation in nonlinear lattice III, *J. Phys. Soc. Jpn.*, V. 38, 681-686, 1975
- [12] M. Coffey On series expansions giving closed-form solutions of Korteweg-de Vries-like equations, *J. Appl. Math.*, V. 50, N. 6, 1580-1592, 1990
- [13] S.Y. Lou, L.L. Chen, Solitary wave solutions and cnoidal wave solutions to the combined KdV and mKdV equation, *Math Meth. Appl. Sci.*, V. 17, 339-347, 1994

- [14] J Zhang, New Solitary Wave Solution of the Combined KdV and mKdV Equation, *Int. Jour. Of Theo. Phys.*, V. 37, 1541-1546, 1998
- [15] V.B. Matveev and M.A. Salle, Darboux transformations and solitons, *Series in Nonlinear Dynamics*, Springer-Verlag, Berlin, 1991
- [16] M.Yu. Kulikov and G.M. Fraiman, Darboux transformations and solitons, Preprint of the Institute of Applied Physics, Russian Academy of Sciences, N. 404, Nizhni Novgorod, 1996
- [17] A.V. Slyunyaev and E.N. Pelinovski, Dynamics of large amplitude solitons, *Jour. Of Exp. And Theo. Phys.*, V. 89, N.1, 173-181, 1999
- [18] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. **161**, 2002
- [19] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. **68**, 77-90, 2004
- [20] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. **89**, 1-12, 2009
- [21] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. **42**, 404409-1-16, 2009
- [22] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. **185**, 247-258, 2010
- [23] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 435204-1-15, 2011
- [24] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. **13**, N. 2, 71-153, 2012
- [25] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. **54**, 013504-1-32, 2013
- [26] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. **4**, N. 4, 246-266, 2013
- [27] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. **2013**, ID 645752, 1-10, 2013
- [28] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. **7**, N. **45**, 1-6, 2013

- [29] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. **88**, 042903-1-9, 2013
- [30] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. **2013**, 1-111, 2013
- [31] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. **54**, 073519-1-22, 2013
- [32] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. **61**, 365-369, 2014
- [33] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. **89**, 015004-1-7, 2014
- [34] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. **482**, 012016-1-7, 2014
- [35] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. **26**, N. 2, 1550016-1-14, 2014
- [36] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. **5**, 093506-1-12, 2014
- [37] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. **574**, 012031-1-5, 2015
- [38] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. **355**, 293-298, 2015
- [39] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. **379**, 1309-1313, 2015
- [40] P. Gaillard, Other $2N-2$ parameters solutions to the NLS equation and $2N+1$ highest amplitude of the modulus of the N -th order AP breather, *J. Phys. A: Math. Theor.*, V. **48**, 145203-1-23, 2015
- [41] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. **4**, N. **5**, 346-364, 2015
- [42] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. **633**, 012106-1-6, 2016

- [43] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. **10**,1-7, 2016
- [44] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. **2**, 136-144, 2016
- [45] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. **367**, 1-5, 2016
- [46] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. **10**, 83-89, 2016
- [47] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016
- [48] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [49] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 3, 130-139, 2017
- [50] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N^2$ parameters, *Int. Jour. of Appl. Sci. And Math.*, V. **4**, N. 3, 60-70, 2017
- [51] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. 1, N. 1, 26-31, 2017
- [52] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 2, 92-98, 2017
- [53] P. Gaillard, N -Order rational solutions to the Johnson equation depending on $2N - 2$ parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 9, 19-37, 2017
- [54] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 11, 24-30, 2017
- [55] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. **6**, N. 1, 14-19, 2018

- [56] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. **196**, N. 2, 1174-1199, 2018
- [57] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. **2018**, 1-18, 2018
- [58] P. Gaillard, Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters, *NHIMP*, V. **2**, N. 4, 53-61, 2018
- [59] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. **1141**, 012102-1-10, 2018
- [60] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. , 109-112, 2019
- [61] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. **2**, N. 4, 1-4, 2019
- [62] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. **3**, N. 2, 44-52, 2020
- [63] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. **100**, 102667-1-9, 2021
- [64] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. **161**, 104059-1-12, 2021
- [65] P. Gaillard, 2N parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. **1**, N. **1**, 1-6, 2021
- [66] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. **3**, N. 3, 14-21, 2021