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Growth order and blow up points for the parabolic Burger's equation under dynamical boundary conditions

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Abstract: We investigate the blow up points of the one–dimensional parabolic Burger's equation

$$\partial_t u = \partial_x^2 u - u\partial_x u + u^p$$

under a dissipative dynamical boundary condition $\sigma\partial_t u + \partial_\nu u = 0$ for one bump initial data. A numerical example of a solution pertaining exactly two bumps stemming from its initial data is presented. Moreover, we discuss the growth order of the L^∞–norm of the solutions when approaching the blow up time.

Keywords: Nonlinear parabolic problems, dynamical boundary conditions, blow up.

AMS Subject Classification 2000: 35B05, 35B40, 35K55, 35K57, 35K65, 35R45.

1 Introduction

This paper is devoted to the blow up phenomenon of the one–dimensional parabolic Burger's equation with a convection term changing sign under dynamical conditions on the time lateral boundary

$$\left\{\begin{array}{ll} \partial_t u = \partial_x^2 u - u\partial_x u + u^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ B_\sigma(u) = \sigma\partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) & \end{array}\right. \tag{1}$$

with the interval $\Omega = (-a, a)$ and $1 < p \in \mathbb{R}$. The results presented here improve and extend those of Section 5 from [3]. It is well known that the system defines a local flow, especially local existence results are well established, see [3] and the references therein. The basic hypotheses on σ and φ are

$$\sigma(-a) \geq 0, \quad \sigma(a) \geq 0, \tag{2}$$

$$\varphi \geq 0, \quad \varphi \in C^2([-a, a]) \ , \quad \varphi'' - \varphi\varphi' + \varphi^p \geq 0 \text{ in } [-a, a]. \tag{3}$$

Let $T = T(\sigma, \varphi)$ denote the blow up time of the maximal solution $u = u_\sigma$ of Problem (1), that is in our context the maximal existence time of u with respect to the L^∞–norm:

$$T(\sigma, \varphi) = \inf \left\{ s > 0 \,\middle|\, \lim_{t \nearrow s} \sup\{|u_\sigma(x, t)| \,|\, x \in \overline{\Omega}\} = \infty \right\}.$$

It has been shown in [3] that for $1 < p \leq 2$, the solutions of Problem (1) exist globally, while for $p > 2$ they blow up in finite time if $\varphi > 0$ in $\overline{\Omega}$. In the present paper we improve the growth order result in [3] and determine the blow up set for one bump initial data. The last section is devoted to a numerical example displaying a solution pertaining exactly two bumps stemming from its initial data.

2 Growth order

For $p > 3 + \frac{1}{3}$ the solution grow like $\frac{1}{(T-t)^{\frac{1}{p-1}}}$ when approaching the blow up time $T = T(\sigma, \varphi)$, see [3]. Here we shall show that the same holds for $p > 3$ with $4p(p-1)(p-3) > 1$.[1] Recall the following basic

Lemma 2.1 *([3]) Under (2) and (3), the solution $u_\sigma \in C^{2,1}\left(\overline{\Omega} \times [0, T(\sigma, \varphi))\right)$ of (1) satisfies $\partial_t u_\sigma \geq 0$ in $\overline{\Omega} \times [0, T(\sigma, \varphi))$. If, in addition, φ is not identically vanishing on Ω, then $\partial_t u_\sigma > 0$ in $\overline{\Omega} \times (0, T(\sigma, \varphi))$.*

First, we show that the lower growth order amounts always to $\frac{-1}{p-1}$ for any $p > 2$.

Lemma 2.2 *Under the hypotheses (2) and (3), the maximal solution u of (1) satisfies*

$$\| u_\sigma(\cdot, t) \|_\infty \geq \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \Big(T(\sigma, \varphi) - t\Big)^{\frac{-1}{p-1}} \quad \textit{for all } t \in [0, T(\sigma, \varphi)).$$

[1] Numerically this means that it is sufficient to choose

$$p > \frac{4}{3} + \frac{2\sqrt{7}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{107\sqrt{7}}{392}\right)\right) \approx 3.040302200285463617843889443 4...$$

Proof. Set $g(t) = \| u(\cdot,t) \|_\infty$ for $t \in [0,T)$. By (3) and Lemma 2.1 we have $\partial_t u \geq 0$ on $[-a,a] \times [0,T)$. Thus, g is increasing and continuous on $[0,T)$, and, thereby, g is differentiable almost everywhere in $[0,T)$. Choose $t \in [0,T)$, $0 < h < t$, x and x_h such that g is differentiable at t and

$$\| u(\cdot,t) \|_\infty = u(x,t) \quad \text{and} \quad \| u(\cdot,t-h) \|_\infty = u(x_h,t-h).$$

Then

$$g(t) - g(t-h) = u(x,t) - u(x_h,t-h) \leq u(x,t) - u(x,t-h),$$

and there exists $\theta \in (0,1)$ such that

$$u(x,t) = u(x,t-h) + h\partial_t u(x,t-\theta h).$$

We conclude

$$\frac{g(t) - g(t-h)}{h} \leq \partial_t u(x,t-\theta h).$$

and, letting h tend to 0,

$$g'(t) \leq \partial_t u(x,t).$$

By hypothesis on u and definition of g,

$$\begin{array}{rcl} g'(t) & \leq & \partial_x^2 u(x,t) - u(x,t)\partial_x u(x,t) + u(x,t)^p \\ & \leq & u(x,t)^p = g(t)^p. \end{array}$$

Integration between $t > 0$ and $t < \tau < T$ yields

$$g(t)^{1-p} - g(\tau)^{1-p} \leq (p-1)(\tau - t).$$

As τ tends to T, the function $g(\tau)$ tends to ∞. This shows

$$g(t)^{1-p} \leq (p-1)(T-t).$$

■

The growth order bound is optimal as the example of constant initial data $\varphi > 0$ under Neumann boundary conditions displays well.

Theorem 2.3 *Suppose conditions* (2) *and* (3) *to hold and that*

$$p > 3, \quad p(p-1)(p-3) > \frac{1}{4}. \tag{4}$$

Then there is a constant $C > 0$ *such that the maximal solution* $u_\sigma \in \mathcal{C}^{2,1}([-a,a] \times [0,T(\sigma,\varphi))$ *of* (1) *satisfies*

$$\| u_\sigma(\cdot,t) \| \leq \frac{C}{(T(\sigma,\varphi)-t)^{\frac{1}{p-1}}} \quad \textit{for all } t \in [0,T(\sigma,\varphi)).$$

Proof. We can adopt and improve the proofs by Friedman & McLeod [4] and von Below & Pincet [3]. For the reader's convenience we repeat some similar arguments here. Choose $\alpha > 1$ such that

$$p(p-1)(p-3) \geq \frac{\alpha^2}{4} > \frac{1}{4}.$$

We choose $M > 1$ et $M > \frac{1}{6}(\alpha - 1)^{\frac{-2}{p-3}}$. Suppose $\xi \in]0, \frac{t_0}{2}]$ with $t_0 = \frac{\|\varphi\|_\infty^{1-p}}{p-1}$. We shall show that there exists $\delta > 0$ such that

$$\partial_t u \geq \delta e^{-Mt}(u^p + u^3) \text{ in } [-a, a] \times [\xi, T(\sigma, \varphi)). \tag{5}$$

In the same way as in Corollary 2.7 from [3], the strong minimum principle from [1] yields that there exists a positive constant c such that

$$y := \partial_t u \geq c > 0 \text{ in } [-a, a] \times [\xi, T(\sigma, \varphi)). \tag{6}$$

Now, set $d(t) = \exp(-Mt)$ with $M > 1$ sufficiently large to be determined later and $k(u) = u^p + u^3$ and introduce

$$J = \partial_t u - \delta d(t) k(u),$$

where $\delta > 0$ is sufficiently small such that

$$J(\cdot, \xi) \geq 0 \text{ in } [-a, a] \tag{7}$$

in view to (6). By classical regularity results, $J \in C^{2,1}([-a, a] \times [\xi, T(\sigma, \varphi)))$, see e.g.[5], and J satisfies the boundary condition

$$B_\sigma(J) = B_\sigma(y) - \delta d k'(u) B_\sigma(u) - \sigma \delta d' k(u) = \sigma \delta \exp(-Mt) k(u) \geq 0.$$

Moreover, J fulfills

$$\partial_t J - \Delta J + u \partial_x J - (p u^{p-1} - \partial_x u) J = \delta d H(u) \text{ in } [-a, a] \times [\xi, T(\sigma, \varphi)),$$

where

$$H(u) := p u^{p-1} k(u) - k'(u) u^p + k''(u)(\partial_x u)^2 - \frac{d'}{d} k(u) - k(u) \partial_x u.$$

In order to show that $H(u) \geq 0$, we prove the inequality

$$\begin{aligned} |\partial_x u|(u^p + u^3) &\leq M(u^p + u^3) + (p-3)u^{p+2} \\ &+ \left(p(p-1)u^{p-2} + 6u\right)(\partial_x u)^2. \end{aligned} \tag{8}$$

It is obvious when $|\partial_x u| \leq M$. Now, we suppose that $|\partial_x u| > M$. In the case where $u^2 \leq 6|\partial_x u|$, (8) is fulfilled because $u^3 |\partial_x u| \leq 6u(\partial_x u)^2$ and $u^p |\partial_x u| \leq p(p-$

$1)u^{p-2}(\partial_x u)^2$ since $p > 3$. If $u^2 > 6|\partial_x u|$, then we have $u > \sqrt{6M} = (\alpha - 1)^{\frac{-1}{p-3}}$, and

$$u^p + u^3 \leq \alpha u^p.$$

(8) is satisfied if

$$\alpha|\partial_x u|u^p \leq (p-3)u^{p+2} + p(p-1)u^{p-2}(\partial_x u)^2$$

which is equivalent to

$$\begin{aligned} \alpha|\partial_x u|u^2 &\leq (p-3)u^4 + p(p-1)(\partial_x u)^2 \\ &\leq \left(\sqrt{p-3}u^2 - \sqrt{p(p-1)}|\partial_x u|\right)^2 + 2\sqrt{p(p-1)(p-3)}u^2|\partial_x u|. \end{aligned}$$

In other words, we just need

$$\alpha \leq 2\sqrt{p(p-1)(p-3)},$$

which is guaranteed by the definition of α and p. Thus, (5) is shown.

Finally, for the growth order, it suffices to estimate $\|u(\cdot,t)\|_\infty$ for $t \in [\frac{t_0}{2}, T(\sigma,\varphi))$. For each $x \in [-a,a]$ the integral

$$\int_t^S \frac{\partial_t u(x,s)}{u^p(x,s)}\,ds = \int_{u(x,t)}^{u(x,S)} \frac{1}{\eta^p}\,d\eta$$

converges as $S \nearrow T(\sigma,\varphi)$. We conclude

$$\frac{u^{1-p}(x,T(\sigma,\varphi)) - u^{1-p}(x,t)}{1-p} \geq \delta\left(T(\sigma,\varphi) - t\right),$$

and

$$u(x,t) \leq (p-1)^{\frac{1}{1-p}} \max\left\{\delta^{\frac{1}{1-p}}, \left(\frac{2T(\sigma,\varphi)}{t_0} - 1\right)^{\frac{1}{p-1}}\right\} \frac{1}{(T(\sigma,\varphi) - t)^{1/p-1}},$$

which holds especially at the blow up points. ∎

3 Blow up points

Let $p > 2$ and suppose that σ is a constant satisfying

$$\sigma > 0. \tag{9}$$

For the initial values, we require in addition to condition (3)

$$\varphi(x) \geq \varphi(-x) \text{ in } [0,a], \tag{10}$$

and the one bump condition

$$\begin{cases} \varphi'(x) > 0 & \text{in } [-a, x_0), \\ \varphi'(x) < 0 & \text{in } (x_0, a] \end{cases} \tag{11}$$

with some $x_0 \in (-a, a)$. Set $T = T(\sigma, \varphi)$. Following the proof of Lemma 5.2 of [2], and using an argument by Friedman & McLeod, see Lemma 5.2 from [4], in the case where $h(u) = u^p$, we can state

Lemma 3.1 *There exists a continuous function $x = s(t)$ such that for $0 < t < T$*

$$\begin{aligned} &\partial_x u(x,t) > 0 \quad \text{if } -a \le x < s(t), \\ &\partial_x u(x,t) < 0 \quad \text{if } s(t) < x \le a. \end{aligned}$$

Proof. By Lemma 2.1, $\partial_t u > 0$ on $\overline{\Omega} \times [0, T)$. The boundary condition $\mathcal{B}_\sigma(u) = 0$ with $\sigma > 0$ shows $\partial_\nu u < 0$ on $\partial\Omega \times (0, T)$, i.e.

$$\partial_x u > 0 \text{ on } \{-a\} \times (0,T) \text{ and } \partial_x u < 0 \text{ on } \{a\} \times (0,T). \tag{12}$$

Moreover, (11) implies

$$\partial_x u > 0 \text{ on } [-a, x_0) \times \{0\} \text{ and } \partial_x u < 0 \text{ on } (x_0, a] \times \{0\}.$$

First, by continuity of the solution u, there are two relatively open subsets A_+ and A_- of $\overline{\Omega} \times [0, T)$ such that

1. $A_+ \subseteq \{\partial_x u > 0\}$ and $A_- \subseteq \{\partial_x u < 0\}$.
2. A_+ and A_- are connected.
3. the parabolic boundary $\overline{\Omega} \times \{0\} \cup \partial\Omega \times (0, T)$ belongs to $\partial A_+ \cup \partial A_- \cup \{x_0\}$.
4. A_+ and A_- are maximal with respect to the properties 1. - 3.

We shall show that $A_- = \{\partial_x u < 0\}$ and $A_+ = \{\partial_x u > 0\}$. Recall that $u \in \mathcal{C}^{3,1}(\overline{\Omega} \times [0, T))$, see [5]. Thus, $y := \partial_x u \in \mathcal{C}^{2,1}(\overline{\Omega} \times [0, T))$ and

$$\begin{cases} \partial_t y = \Delta y - y^2 - u\partial_x y + pu^{p-1} y & \text{in } \overline{\Omega} \times [0, T), \\ \mathcal{B}_\sigma(y) = 0 & \text{on } \partial\Omega \times [0, T), \\ y(\cdot, 0) = \varphi' & \text{in } \overline{\Omega}. \end{cases}$$

For $0 < \tau < T$, set

$$\Omega_\tau := \overline{\Omega} \times [0, \tau).$$

Let Q^- denote a connected component of $\{\partial_x u < 0\} \cap \Omega_\tau$ and prove that $Q^- \subseteq A^-$. We first show that ∂Q^- and the parabolic boundary of Ω_τ meet at least in one point different from $\{x_0\}$. For that purpose, suppose that

$$\partial Q^- \cap \Big(\overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, \tau) \Big) \subseteq \{x_0\}.$$

It yields $y \equiv 0$ on $\partial Q^- \cap \Omega_\tau$. Moreover, by definition of Q^- we have

$$\begin{cases} \partial_t y \leq \Delta y - u\partial_x y & \text{in } Q^-, \\ y \leq 0 & \text{on } \partial Q^-. \end{cases}$$

Since y attains its maximum 0 on ∂Q^-, the strong maximum principle from [7] implies $y \equiv 0$ in Q^-, which is impossible. Clearly, $Q^- \cap A_- \neq \emptyset$, which implies $A_- \cup Q^- = A_-$ by maximality of A_-. Now, for a connected component Q^+ of $\{\partial_x u > 0\} \cap \Omega_\tau$, suppose that

$$\partial Q^+ \cap \left(\overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, \tau)\right) \subseteq \{x_0\}.$$

We still have $y \equiv 0$ on $\partial Q^+ \cap \Omega_\tau$, which implies that the interior of the set $P^+ := Q^+ \setminus \{pu^{p-1} - y < 0\}$ is not empty. Then y satisfies

$$\begin{cases} \partial_t y \geq \Delta y - u\partial_x y + y(pu^{p-1} - y) \geq \Delta y - u\partial_x y & \text{in } P^+, \\ y \geq 0 & \text{on } \partial P^+. \end{cases}$$

Since y attains its minimum 0 on $\partial Q^+ \subseteq \partial P^+$, the strong minimum principle from [7] again, leads to $y \equiv 0$ in P^+, in contradiction with the definition of P^+. Thus, we can conclude that $\{\partial_x u > 0\} = A_+$.

Then, since $\{\partial_x u > 0\}$ and $\{\partial_x u < 0\}$ are connected, for $(x^+, t) \in \{\partial_x u > 0\}$ and $(x^-, t) \in \{\partial_x u < 0\}$ with $0 < t < T$, we have $x^+ < x^-$. Furthermore, we can prove that for all $\tilde{x}$ such that $(\tilde{x}, t) \in \{\partial_x u = 0\}$, $x^+ < \tilde{x} < x^-$. Indeed, for $x_1 < x_2 < x_3$ and $s \in (0, T)$ with (x_1, s) and (x_3, s) belonging to $\{\partial_x u > 0\}$ and $(x_2, t) \in \{\partial_x u = 0\}$, we can choose x_1, x_3, s and $r \in (0, s)$ such that $[x_1, x_3] \times [r, s) \subseteq \{\partial_x u > 0\}$. For x_1 and x_3 sufficiently close to x_2 and r sufficiently close to s, we have $pu^{p-1} - y \geq 0$, and we obtain

$$\begin{cases} \partial_t y \geq \Delta y - u\partial_x y & \text{in } [x_1, x_3] \times [r, s], \\ y \geq 0 & \text{on } \{x_1, x_3\} \times [r, s] \cup [x_1, x_3] \times \{r\}. \end{cases}$$

Thus, y attains its minimum in (x_2, s) and the strong minimum principle leads to $y \equiv 0$ in $[x_1, x_3] \times [r, s]$, which is impossible. In the same way, points of the form (x_1, s) and (x_3, s) with $x_1 < x_2 < x_3$ and $s \in (0, T)$ with (x_1, s) and (x_3, s) belonging to $\{\partial_x u < 0\}$ and $(x_2, t) \in \{\partial_x u = 0\}$ can be excluded.

Finally, the analytic non linearity implies the analyticity of the function $u(\cdot, t)$, and, if we consider $x_1, x_2 \in \overline{\Omega}$ and $s \in [0, T)$ such that $\partial_x u(\cdot, s) = 0$ in $[x_1, x_2]$, we obtain $\partial_x u(\cdot, s) = 0$ in $\overline{\Omega}$, in contradiction with (12). ∎

Now, we can state the following

Theorem 3.2 *Under Conditions* (9)–(11)*, for each* $(x, t) \in [0, a] \times [0, T)$ *it holds*

$$u(x, t) \geq u(-x, t).$$

Proof. Set $v(x,t) := u(-x,t)$ in $[0,a]\times[0,T)$ and introduce the function y defined by $y(x,t) := u(x,t) - v(x,t)$. Then y is a solution belonging to $\mathcal{C}([0,a]\times[0,T))\cap \mathcal{C}^{2,1}([0,a]\times](0,T))$ of the problem

$$\begin{cases} \partial_t y = \partial_x^2 y + v\partial_x y + \delta y - (u+v)\partial_x u & \text{in } [0,a]\times(0,T), \\ y(x,0) \geq 0 & \text{in } [0,a], \\ \mathcal{B}_\sigma(y) = 0 & \text{on } \{a\}\times(0,T), \\ y = 0 & \text{on } \{0\}\times(0,T), \end{cases}$$

with

$$\delta(x,t) = \frac{u^p(x,t) - u^p(-x,t)}{u(x,t) - u(-x,t)}.$$

Now, suppose that there exists a point $(\hat{x},\tilde{t}) \in [0,a]\times(0,T)$ with

$$y(\hat{x},\tilde{t}) < 0, \tag{13}$$

then $y(\cdot,\tilde{t})$ attains a negative minimum at $(\tilde{x},\tilde{t}) \in [0,a]\times(0,T)$ such that

$$\begin{cases} u(\tilde{x},\tilde{t}) < v(\tilde{x},\tilde{t}) \\ \partial_x u(\tilde{x},\tilde{t}) = \partial_x v(\tilde{x},\tilde{t}). \end{cases} \tag{14}$$

If $\tilde{x} > s(\tilde{t})$, then, by Lemma 3.1, $\partial_x u(\tilde{x},\tilde{t}) < 0$. In a neighbourhood V of $(\tilde{x},\tilde{t})$ in which $\partial_x u < 0$, we have

$$\partial_t y - \partial_x^2 y - v\partial_x y - \delta y \geq 0.$$

The comparison principle in [1] leads to $y \geq 0$, in contradiction with (14). On the other hand, if $\tilde{x} = s(\tilde{t})$ we have

$$y(\tilde{x},\tilde{t}) = u(\tilde{x},\tilde{t}) - u(-\tilde{x},\tilde{t}) \geq 0,$$

in contradiction with (13). Finally, for $0 \leq \tilde{x} < s(\tilde{t})$, we have $\partial_x u(\tilde{x},\tilde{t}) > 0$ and $\partial_x u(-\tilde{x},\tilde{t}) > 0$ since $-s(\tilde{t}) < -\tilde{x} \leq 0 < s(\tilde{t})$. Then we conclude that

$$\partial_x y(\tilde{x},\tilde{t}) = \partial_x u(\tilde{x},\tilde{t}) + \partial_x u(-\tilde{x},\tilde{t}) > 0,$$

again, in contradiction with (14). ∎

The last two results immediately yield the following.

Corollary 3.3 *Under the aforementioned hypotheses,* $s(t) \geq 0$ *and*

$$\partial_x u(x,t) > 0 \quad \textit{in } [-a,0)\times(0,T).$$

Definition 3.4 $s^- = \liminf_{t\to T} s(t)$ and $s^+ = \limsup_{t\to T} s(t)$

The two following theorems will show that blow up can only occur in the interval $[s^-, s^+]$.

Theorem 3.5 *Under the aforementioned assumptions, no point in $[-a, s^-)$ is a blow up point.*

Proof. By Corollary 3.3, $s^- \geq 0$. Suppose $\beta \in [-a, s^-)$ and choose $T_0 < T$ sufficiently close to T such that $s(t) > \beta$ for $T_0 \leq t < T$. Introduce

$$R = [-a, \beta] \times (T_0, T),$$

and

$$J = \partial_x u - \varepsilon dk$$

with

$$d(x) = (x-\beta)^n, \ n \in \mathbb{N}^* \text{ even},$$
$$k(u) = u^q \quad , \quad q = p - 1,$$

and $\varepsilon > 0$ sufficiently small such that

$$J \geq 0 \text{ in } [-a, \beta] \times \{T_0\}, \tag{15}$$

and

$$u^q + \frac{n(n-1)}{(x-\beta)^2} + \frac{n}{|x-\beta|} u \geq \varepsilon u^{q-1} \left[(x-\beta)^n u + 2nq|x-\beta|^{n-1} \right]. \tag{16}$$

Condition (15) is due to Lemma 3.1, where as (16) is a consequence of the assumptions on p, q and n. Moreover $J(\beta, \cdot) > 0$ in $[T_0, T)$ by Lemma 3.1, since $d(\beta) = 0$. Set $f(u) := u^p$, (16) implies that H defined by

$$\begin{aligned} H(u) \ &:= \ f'(u)k(u) - f(u)k'(u) - \varepsilon \left(dk^2(u) - 2d'k(u)k'(u) \right) \\ &- \ \frac{d'}{d} k(u)u + \frac{d''}{d} k(u) + k''(u)(\partial_x u)^2 + \frac{1}{\varepsilon d}(\partial_x u)^2 \end{aligned}$$

satisfies $H(u) \geq k''(u)(\partial_x u)^2 + \frac{1}{\varepsilon d}(\partial_x u)^2 \geq 0$ since u is strictly positive in R, as well as k'' for $q > 1$. Recall that $u > 0$ for $t > 0$ by the comparison principle of [1] and by Lemma 2.1. Moreover, J satisfies on $\{-a\} \times (T_0, T)$,

$$\mathcal{B}_\sigma(J) = \mathcal{B}_\sigma(\partial_x u) - \varepsilon dk'(u)\mathcal{B}_\sigma(u) - \varepsilon d'k(u) = -\varepsilon n(-a-\beta)^{n-1} u^q(-a, t)$$

since σ is a constant. As Ω is an interval of the real line, classical interior regularity results, see e.g. [5], imply that $u \in \mathcal{C}^{3,2}([-a, \beta] \times [T_0, T))$. Thus, J belongs to $\mathcal{C}^{2,2}([-a, \beta] \times [T_0, T))$ and is a solution of

$$\begin{cases} \partial_t J - \Delta J + u\partial_x J + K(u)J = \varepsilon dH(u) \geq 0 & \text{in } [-a, \beta) \times (T_0, T), \\ \mathcal{B}_\sigma(J) \geq 0 & \text{on } \{-a\} \times (T_0, T), \\ J(\cdot, T_0) \geq 0 & \text{in } [-a, \beta], \\ J \geq 0 & \text{on } \{\beta\} \times (T_0, T), \end{cases}$$

where $K(u) = J - f'(u) + 2\varepsilon dk(u) - 2\varepsilon d'k'(u)$. By the comparison principle from [1] we conclude that $J \geq 0$ in $[-a, \beta] \times [T_0, T)$, in other words

$$\frac{\partial_x u}{k(u)} \geq \varepsilon d.$$

Integration between x and y with $-a \leq x < y < \beta$ and the parity of n yield

$$u^{1-q}(x,t) - u^{1-q}(y,t) \geq \varepsilon \frac{q-1}{n+1}[(y-\beta)^{n+1} - (x-\beta)^{n+1}] > 0.$$

Finally, suppose that $x \in [-a, \beta)$ is a blow up point. Using $1 - q < 0$, we conclude that $u^{1-q}(y,t) < 0$ for $y \in (x, \beta)$, and t sufficiently close to T, which is impossible. ∎

Theorem 3.6 *Under the aforementioned assumptions, no point of* $(s^+, a]$ *is a blow up point.*

Proof. We can follow the idea of the proof of Theorem 3.5. Suppose that $s^+ < a$ and $\gamma \in (s^+, a]$. Choose $T_0 < T$ sufficiently close to T such that $s(t) < \gamma$ for $T_0 \leq t < T$. Set $R = [\gamma, a] \times (T_0, T)$ and

$$J = \partial_x u + \varepsilon dk$$

with

$$d(x) = e^{\frac{1}{\gamma - x}},$$
$$k(u) = u^q, \ 2 < q < p,$$

and $\varepsilon > 0$ sufficiently small such that

$$J(\cdot, T_0) \leq 0 \text{ in } [\gamma, a]. \tag{17}$$

Condition (17) is possible by Lemma 3.1. Using $f(u) := u^p$, set

$$\begin{aligned} H(u) &= f'(u)k(u) - f(u)k'(u) + \varepsilon dk^2(u) - 2\varepsilon d'k(u)k'(u) - \frac{d'}{d}k(u)u \\ &\quad + \frac{d''}{d}k(u) + (\partial_x u)^2 k''(u) \\ &= (p-q)u^{p+q-1} + \frac{u^{q+1}}{(\gamma - x)^2} + \left(\frac{1}{(\gamma - x)^4} - \frac{2}{(\gamma - x)^3}\right) u^q \\ &\quad + \varepsilon e^{\frac{-1}{\gamma - x}} \left(\frac{2qu^{2q-1}}{(\gamma - x)^2} + u^{2q}\right) + q(q-1)(\partial_x u)^2 u^{q-2}. \end{aligned}$$

Then $H(u) \geq 0$ since $p > q > 2$. Moreover, J satisfies on $\{a\} \times (T_0, T)$ the dynamical boundary condition,

$$\mathcal{B}_\sigma(J) = \mathcal{B}_\sigma(\partial_x u) + \varepsilon dk'(u)\mathcal{B}_\sigma(u) + \varepsilon d'k(u) = \frac{-\varepsilon}{(\gamma - a)^2} e^{\frac{1}{\gamma - a}} u^q(a,t)$$

since σ is a constant. Thus $J \in \mathcal{C}^{2,2}([\gamma, a] \times [T_0, T))$ is a solution of

$$\begin{cases} \partial_t J - \Delta J + u\partial_x J + K(u)J = -\varepsilon dH(u) \leq 0 & \text{in } (\gamma, a] \times (T_0, T), \\ \mathcal{B}_\sigma(J) \leq 0 & \text{on } \{a\} \times (T_0, T). \\ J(\cdot, T_0) \leq 0 & \text{on } [\gamma, a], \\ J(\gamma, \cdot) \leq 0 & \text{on } (T_0, T), \end{cases}$$

where $K(u) := J - f'(u) - 2\varepsilon dk(u) + 2\varepsilon d'k'(u)$. By the comparison principle of [1], $J \leq 0$ in $[\gamma, a] \times [T_0, T)$. By integrating as above, we obtain $u^{1-q}(y,t) - u^{1-q}(x,t) > 0$ for $\gamma < x < y \leq a$. Thus if $y \in (\gamma, a]$ is a blow up point, $u^{1-q}(x,t) < 0$ for $x \in (\gamma, y)$ and t sufficiently close to T, in contradiction to the positivity of u. Thus, u remains bounded in R. ∎

In fact, it will be shown that $s^- = s^+$. In other words, blow up can only occur at a singleton.

Theorem 3.7 *Under Conditions* (3) (9) (10) (11)*, the blow up set of Problem* (1) *consists at most of a single point.*

Proof. Theorem 3.5 and Theorem 3.6 imply that blow up can only occur in $[s^-, s^+]$. It remains to show that $s^- = s^+$. Suppose that $s^+ > s^-$ and that u blows up at $z_1 \in [s^-, s^+]$. Set

$$y_1 = s^- - \varepsilon,$$

with

$$0 < \varepsilon < \frac{s^+ - s^-}{2}. \tag{18}$$

By Theorem 3.5, there exists a constant $C_0 > 0$ such that

$$u(y_1, t) \leq C_0 \text{ in } [0, T). \tag{19}$$

As u explodes at z_1, there is τ_1 sufficiently close to T such that $u(z_1, t) > C_0$ for any $\tau_1 < t < T$. Moreover, by Lemma 3.1,

$$u(z_1, t) \leq \max_{\overline{\Omega}} u(\cdot, t) = u(s(t), t) \quad \text{for all} \quad 0 < t < T.$$

Thus, by Definition 3.4 and Lemma 3.1, there exists $\tau \in (0, T)$ with $T - \tau$ sufficiently small such that for $y_2 = s(\tau) > s^+ - \varepsilon$ and for $\alpha = \frac{y_1 + y_2}{2}$ we have

$$\begin{aligned} &u(y_2, \tau) > C_0, \\ &u(x, \tau) \leq u(2\alpha - x, \tau) \quad \text{for } x \in [y_1, \alpha], \\ &u(x, \tau) < u(2\alpha - x, \tau) \quad \text{for } x \in [y_1, \alpha). \end{aligned}$$

Then (19) and Lemma 2.1 yield

$$u(y_1,t) < u(y_2,t) \text{ for } \tau \le t < T.$$

Set $v(x,t) = u(2\alpha - x, t)$ in $[y_1,\alpha]\times[\tau,T)$ and

$$w(x,t) = v(x,t) - u(x,t).$$

Then $w \in \mathcal{C}^{2,1}([y_1,\alpha]\times[\tau,T))$ is a solution of

$$\begin{cases} \partial_t w = \Delta w + v\partial_x w + \delta w + (u+v)\partial_x u & \text{in } (y_1,\alpha)\times(\tau,T), \\ w(\cdot,\tau) \ge 0 & \text{in } [y_1,\alpha], \\ w > 0 & \text{on } \{y_1\}\times(\tau,T), \\ w = 0 & \text{on } \{\alpha\}\times(\tau,T), \end{cases}$$

where we have set

$$\delta(x,t) = \frac{v^p(x,t) - u^p(x,t)}{v(x,t) - u(x,t)}.$$

Suppose that

$$w(\hat{x},\tilde{t}) < 0 \tag{20}$$

at some $(\hat{x},\tilde{t}) \in (y_1,\alpha)\times(\tau,T)$. Then there exists a point $(\tilde{x},\tilde{t}) \in (y_1,\alpha)\times(\tau,T)$ at which $w(\cdot,\tilde{t})$ takes a negative minimum. Thus,

$$\begin{cases} v(\tilde{x},\tilde{t}) < u(\tilde{x},\tilde{t}), \\ \partial_x v(\tilde{x},\tilde{t}) = \partial_x u(\tilde{x},\tilde{t}). \end{cases} \tag{21}$$

If $\alpha > \tilde{x} > s(\tilde{t})$ then $\partial_x u(\tilde{x},\tilde{t}) < 0$ and $\partial_x u(2\alpha - \tilde{x},\tilde{t}) < 0$ since $2\alpha - \tilde{x} > \alpha > \tilde{x} > s(\tilde{t})$. Thus,

$$\partial_x w = -\partial_x u(2\alpha - \tilde{x},\tilde{t}) - \partial_x u(\tilde{x},\tilde{t}) > 0,$$

in contradiction with (21). Next, recall that $\tilde{x}$ is the unique zero of $\partial_x u(\cdot,\tilde{t})$, by Lemma 3.1. Thus, $\alpha > \tilde{x} = s(\tilde{t})$ would lead to another zero since $2\alpha - \tilde{x} \ne \tilde{x}$. Finally, suppose $y_1 \le \tilde{x} < s(\tilde{t})$, then $\partial_x u(\tilde{x},\tilde{t}) > 0$. Let V a neighborhood of $(\tilde{x},\tilde{t})$ in which $\partial_x u > 0$. In V, we have

$$\partial_t w - \Delta w - v\partial_x w - \delta w = (u+v)\partial_x u \ge 0,$$

and comparison principle of [1] leads to $w \ge 0$, in contradiction to (20).

Thus, we have shown that $w \ge 0$. For any $t \in (\tau,T)$, the maximum of $u(\cdot,t)$ in $[y_1,y_2]$ can not be attained for $x \in [y_1,\alpha)$. As $\partial_x u$ vanishes at $s(t)$ by Lemma 3.1, $\alpha \le s(t)$ for $\tau < t < T$. Thus, by Definition 3.4

$$s^- = \liminf_{t\to T} \; s(t) \ge \alpha = \frac{y_1+y_2}{2} \ge \frac{s^- + s^+}{2} - \varepsilon$$

and $s^- \ge s^+ - 2\varepsilon$, in contradiction with (18). This permits to conclude that $s^- = s^+$. ∎

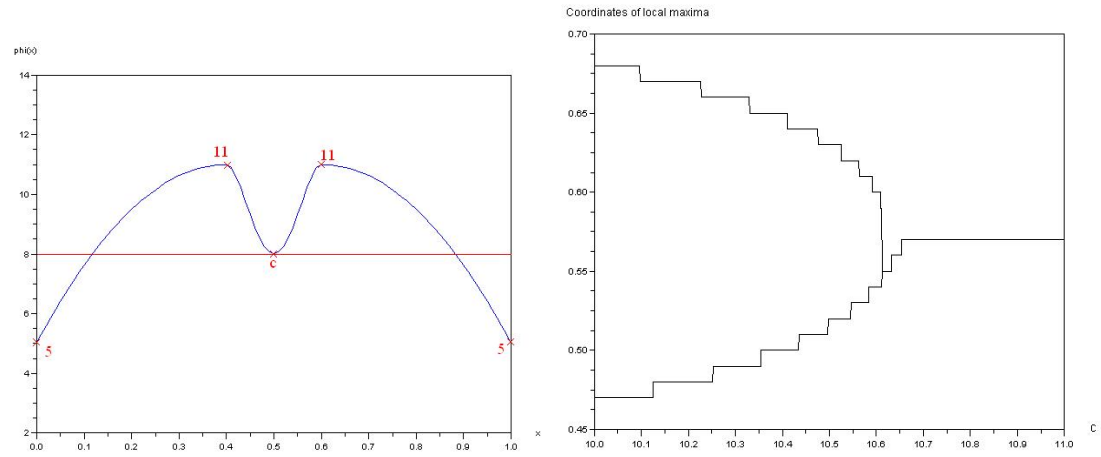

Figure 1: Initial data φ and the bifurcation of local maxima

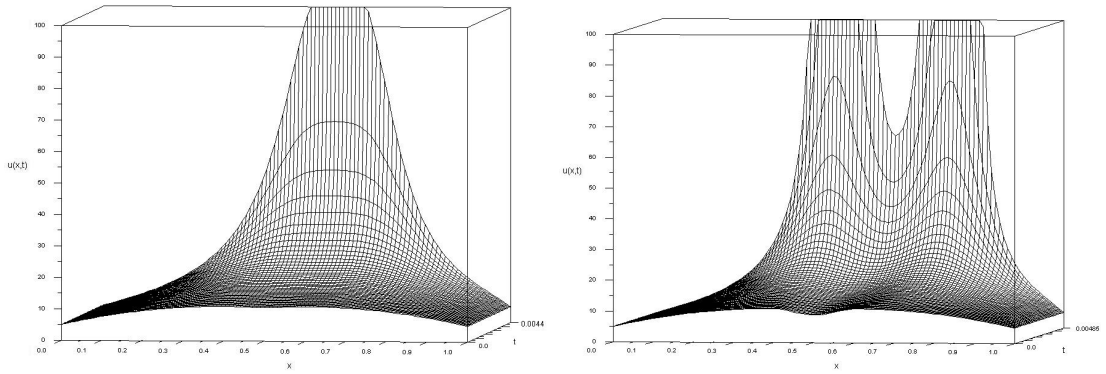

Figure 2: Convergence to one bump for $c = 10.7$ and conservation of two bumps for $c = 9$

4 A numerical example

If the initial condition displays two bumps or more, in general, the bumps can coalesce or not, as in the pure reaction–diffusion case. The following example exhibits numerically conservation of exactly two initial bumps until the blow up time. We use a consistent finite difference scheme for $p = 3$, $\sigma = 100$, with spatial mesh width $k = 1/100$ and temporal mesh width $h = \frac{k^2}{4}$ and initial data

$$\varphi(x) = \begin{cases} 5 + 6f(x) & \text{for } x \in [0, \frac{2}{5}] \\ 11 - 10000(11 - c)(x - \frac{2}{5})^2(x - \frac{3}{5})^2 & \text{for } x \in [\frac{2}{5}, \frac{3}{5}] \\ 5 + 6f(1 - x) & \text{for } x \in [\frac{3}{5}, 1] \end{cases}$$

with $f(x) = 5x - \frac{25}{4}x^2$. Note that the parameter c denotes just the value of the local minimum in the middle, see Fig. 1 for the case $c = 8$. The figure on the r.h.s. of Fig. 1 displays the maxima coordinates in $[0, 1]$ before blowing up. The left part of Fig. 2 exhibits coalescence into one bump for $c = 10.7$ when approaching the blow up time, while the right part displays conservation of the two bumps for $c = 9$.

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