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Sylvain Ervedoza, Kévin Le Balc'h. Cost of observability inequalities for elliptic equations in 2-d with potentials and applications to control theory. *Communications in Partial Differential Equations*, 2023, 48 (4), pp.623–677. <10.1080/03605302.2023.2190526>. <hal-03616317v2>

**HAL Id: hal-03616317**

**<https://hal.science/hal-03616317v2>**

Submitted on 15 Nov 2023

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# Cost of observability inequalities for elliptic equations in 2-d with potentials and applications to control theory\*

Sylvain Ervedoza<sup>†</sup>      Kévin Le Balc'h<sup>‡</sup>

March 14, 2023

## Abstract

The goal of this article is to obtain observability estimates for non-homogeneous elliptic equations in the presence of a potential, posed on a smooth bounded domain  $\Omega$  in  $\mathbb{R}^2$  and observed from a non-empty open subset  $\omega \subset \Omega$ . More precisely, for  $V \in L^\infty(\Omega; \mathbb{R})$ , our main result shows that, when  $\Omega \subset \mathbb{R}^2$  has a finite number of holes, the observability constant of the elliptic operator  $-\Delta + V$ , with domain  $H^2(\Omega) \cap H_0^1(\Omega)$ , is of the form  $C \exp(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)}))$  where  $C$  is a positive constant depending only on  $\Omega$  and  $\omega$ . Our methodology of proof is crucially based on the one recently developed by Logunov, Malinnikova, Nadirashvili, and Nazarov [LMNN20], in the context of the Landis conjecture on exponential decay of solutions to homogeneous elliptic equations in the plane  $\mathbb{R}^2$ . The main difference and additional difficulty compared to [LMNN20] is that the zero set of the solutions to elliptic equations with source term can be very intricate and should be dealt with carefully. As a consequence of these new observability estimates, we obtain new results concerning control of semi-linear elliptic equations in the spirit of Fernández-Cara, Zuazua's open problem concerning small-time global null-controllability of slightly super-linear heat equations.

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\*Both authors are partially supported by the Project TRECOS ANR-20-CE40-0009 funded by the ANR (2021–2024).

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# 1 Introduction

## 1.1 Quantitative unique continuation and Landis conjecture

We first present the problem of quantitative unique continuation property for elliptic operators and its link with the Landis conjecture on exponential decay of solutions to elliptic equations.

Let  $\Omega$  be an open bounded smooth connected set of  $\mathbb{R}^N$ ,  $\omega \subset \Omega$  be a non-empty open set and  $P$  be an elliptic operator on  $\Omega$ . The unique continuation property for  $P$  is as follows

$$(Pu = 0 \text{ in } \Omega \text{ and } u = 0 \text{ in } \omega) \quad \Rightarrow \quad u \equiv 0 \text{ in } \Omega. \quad (\text{UCP})$$

When the coefficients of  $P$  are analytic in  $\Omega$ , it is a well-known fact, see [Mik78], that  $u$  is analytic on  $\Omega$  so (UCP) trivially holds. However, even with  $C^\infty$  coefficients, there is no reason to hope for more than  $C^\infty$  solutions.

In 1939, Carleman introduced in [Car39] a new method to prove unique continuation property for 2-d smooth (not necessary analytic) elliptic operators. Basically, it is based on weighted  $L^2$ -energy inequalities. This type of estimate, now referred to as Carleman estimates, were generalized and systematized by Hörmander and others for a large class of differential operators in arbitrary dimensions, see [LRL12] and references therein. For instance, when  $P$  is the Laplace operator  $-\Delta$ , with an appropriate choice of  $\varphi$ , one can prove, see for instance [F196, IP02], that there exists a constant  $C = C(\Omega, \omega) > 0$  such that for every  $s \geq 1$  and  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , we have

$$s^3 \|e^{s\varphi} u\|_{L^2(\Omega)}^2 + s \|e^{s\varphi} \nabla u\|_{L^2(\Omega)}^2 \leq C \left( \|e^{s\varphi} (-\Delta u)\|_{L^2(\Omega)}^2 + s^3 \|e^{s\varphi} u\|_{L^2(\omega)}^2 \right). \quad (\text{Carleman})$$

The estimate (Carleman) immediately gives the following quantitative unique continuation property

$$\|u\|_{L^2(\Omega)} \leq C \left( \|-\Delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right) \quad (u \in H^2(\Omega) \cap H_0^1(\Omega)), \quad (\text{QUCP})$$

for some constant  $C > 0$  that depends on  $\Omega$ ,  $\omega$ .

In this article, we focus on operators of the form  $P = -\Delta + V$  with  $V \in L^\infty(\Omega)$ . First, by applying (Carleman), one easily checks that, taking  $s \geq C(1 + \|V\|_{L^\infty(\Omega)}^{2/3})$  for some  $C = C(\Omega, \omega) > 0$  sufficiently large, we obtain

$$\|u\|_{L^2(\Omega)} \leq C \exp\left(C\|V\|_{L^\infty(\Omega)}^{2/3}\right) \left( \|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right) \quad (u \in H^2(\Omega) \cap H_0^1(\Omega)). \quad (1.1)$$

A natural question when looking at (1.1) is the optimality of the constant  $C \exp(C\|V\|_{L^\infty(\Omega)}^{2/3})$  in terms of its dependence with respect to  $\|V\|_{L^\infty(\Omega)}$ .

Note that we shall also refer to the best constant  $C_V^*$  in the estimate

$$\|u\|_{L^2(\Omega)} \leq C_V^* \left( \|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right) \quad (u \in H^2(\Omega) \cap H_0^1(\Omega)), \quad (1.2)$$

as the cost of observability, as inequalities of the form (1.2) can also be considered as observability estimates.

In fact, estimating this constant has attracted of lot of attention in the last years, and is related to the Landis conjecture on exponential decay that we present in the next paragraph.

In the late 1960's, see [KL88], Landis conjectured the following: for  $V \in L^\infty(\mathbb{R}^N)$  and  $\varepsilon > 0$ ,

$$(-\Delta u + Vu = 0 \text{ in } \mathbb{R}^N \text{ and } |u(x)| \leq \exp(-|x|^{1+\varepsilon}) \text{ in } \mathbb{R}^N) \quad \Rightarrow \quad u \equiv 0 \text{ in } \mathbb{R}^N. \quad (\text{Landis})$$

One can see (Landis) as a unique continuation property at infinity. The decay rate  $\exp(-|x|^{1+\varepsilon})$  seems to be a natural barrier, by considering the function  $u(x) = \exp(-C\sqrt{1+|x|^2})$  for a suitable constant  $C > 0$ .

Landis conjecture was first disproved by Meshkov in 1991 in the case of complex-valued potentials  $V$ . In fact, the work [Mes91] exhibits in the plane  $\mathbb{R}^2$  a counterexample to (Landis):

$$\exists V \in L^\infty(\mathbb{R}^2; \mathbb{C}) \text{ and } u \neq 0, \quad -\Delta u + Vu = 0 \text{ in } \mathbb{R}^2 \text{ and } |u(x)| \leq \exp(-|x|^{4/3}) \text{ in } \mathbb{R}^2. \quad (\text{Meshkov})$$

[Mes91] also shows that this is the right scale: for  $V \in L^\infty(\mathbb{R}^N)$  and  $\varepsilon > 0$ , we have

$$(-\Delta u + Vu = 0 \text{ in } \mathbb{R}^N \text{ and } |u(x)| \leq \exp(-|x|^{4/3+\varepsilon}) \text{ in } \mathbb{R}^N) \Rightarrow u \equiv 0 \text{ in } \mathbb{R}^N. \quad (1.3)$$

Actually, from (Meshkov), a scaling argument leads to the optimality of (1.1) for complex-valued potentials  $V \in L^\infty(\Omega; \mathbb{C})$  with  $\Omega \subset \mathbb{R}^2$ . Moreover, the proof of (1.3) directly comes from the same scaling argument applied to (1.1)<sup>1</sup>. This illustrates the strong link between the Landis conjecture and the quantification of the observability estimate for non-homogeneous elliptic equations in terms of the potential.

The case of real-valued potentials has been addressed in [BK05] and is more tricky. So far, (Landis) is still a conjecture in the case of real-valued potentials. Some versions of Landis conjecture are known under additional assumptions on  $V$ , in particular positivity conditions, see for instance [KSW15, KW15, Dav20, DW20, Ros20, SS21]. It is worth mentioning that a scaling argument would prove the Landis conjecture for real-valued potential assuming that for  $V \in L^\infty(\Omega; \mathbb{R})$ ,  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , we have for some  $C = C(\Omega, \omega) > 0$ ,

$$\|u\|_{L^2(\Omega)} \leq C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2}\right) \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)}\right). \quad (1.4)$$

Before going further, it is worth mentioning that for more regular potentials  $V \in W^{1,\infty}(\Omega)$ , even complex-valued, it is proved in [Bak12] thanks to a Carleman strategy that one can get (1.4) with  $C \exp(C\|V\|_{W^{1,\infty}(\Omega)}^{1/2})$  instead of  $C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2}\right)$ .

A breakthrough was achieved recently in the 2-d case in the work [LMNN20], which proves that for  $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ , there exists  $C > 0$  sufficiently large such that

$$(-\Delta u + Vu = 0 \text{ in } \mathbb{R}^2 \text{ and } |u(x)| \leq \exp(-C|x| \log^{1/2}(1 + |x|)) \text{ in } \mathbb{R}^2) \Rightarrow u \equiv 0 \text{ in } \mathbb{R}^2. \quad (1.5)$$

In particular, (1.5) directly implies (Landis) for  $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ .

Based on [LMNN20], the goal of this article is to discuss the cost of observability of non-homogeneous elliptic equations associated to the operator  $-\Delta + V$  with  $V \in L^\infty(\Omega; \mathbb{R})$  in the two-dimensional case. In particular, we will prove (1.4) with  $C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})\right)$  instead of  $C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2}\right)$ .

## 1.2 Main results

**Notation:** In all the following, the constants  $C > 0$  can vary from line to line, can depend on  $\Omega$  and  $\omega$  but do not depend on  $\|V\|_{L^\infty(\Omega)}$ .

The first main result of the article is the following one, whose proof will be given in Section 2.

**Theorem 1.1.** *Let  $\Omega$  be a smooth simply connected bounded domain of  $\mathbb{R}^2$ ,  $\omega$  be a non-empty open subset of  $\Omega$ .*

*Then there exists a constant  $C = C(\Omega, \omega) > 0$  such that for every  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  and real-valued potential  $V \in L^\infty(\Omega; \mathbb{R})$ ,*

$$\|u\|_{L^2(\Omega)} \leq C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})\right) \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)}\right). \quad (1.6)$$

In the sequel, we use the following notation for the constant of observability in (1.6)

$$C_V = C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})\right). \quad (1.7)$$

<sup>1</sup>Indeed, for  $u, V$  satisfying the conditions in the left hand side of (1.3), for all  $R \geq 1$ ,  $u_R(x) = u(Rx)$  and  $V_R(x) = R^2 V(Rx)$  satisfy  $-\Delta u_R + V_R u_R = 0$  in  $\mathbb{R}^N$  and  $|u_R(x)| \leq \exp(-|Rx|^{4/3+\varepsilon})$  in  $\mathbb{R}^N$ . Accordingly, applying (1.1) to  $\Omega = B(0, 2)$ ,  $\omega = B(0, 2) \setminus \overline{B(0, 1)}$  and  $\chi_R u_R$  where  $\chi_R = \chi(R \cdot)$  with  $\chi \in C_c^\infty(B(0, 2))$  and  $\chi \equiv 1$  in  $B(0, 1)$ ,

$$\|u_R\|_{L^2(B(0,1))} \leq \|\chi_R u_R\|_{L^2(B(0,2))} \leq C \exp(C\|V\|_{L^\infty}^{2/3} R^{4/3}) \left(\| -2\nabla \chi_R \cdot \nabla u_R - (\Delta \chi_R) u_R \|_{L^2(B(0,2))} + \|u_R\|_{L^2(B(0,2) \setminus \overline{B(0,1)})}\right)$$

so that, using that  $\|-\nabla \chi_R \cdot \nabla u_R\|_{L^2(B(0,2))} \leq C R^2 \|u_R\|_{L^2(B(0,2) \setminus \overline{B(0,1)})}$ , which easily comes from multiplying the equation of  $u_R$  by  $|\nabla \chi_R|^2 u_R$ , we get, for all  $R > 0$ ,

$$\|u\|_{L^2(B(0,R))} \leq C R^{N/2+2} \exp(C\|V\|_{L^\infty}^{2/3} R^{4/3}) \exp(-R^{4/3+\varepsilon}).$$

Since the right-hand side goes to zero as  $R \rightarrow \infty$ , we easily deduce that  $u$  vanishes identically in  $\mathbb{R}^N$ , hence (1.3).

Before going further, let us point out several important issues in Theorem 1.1.

The estimate (1.6) is a better estimate than (1.1), but it is restricted to the cases  $\Omega \subset \mathbb{R}^2$  and  $V$  real-valued. Recall that for complex-valued potentials  $V$  the best estimate one can hope for, in dimension 2 and higher, is (1.1). As mentioned above, this optimal result comes from (Meshkov), see [Mes91].

Our result is strictly limited to the 2-d case. In 1-d, the situation is much easier and one can easily obtain the optimal observability estimate (1.4) for  $V \in L^\infty(\Omega; \mathbb{C})$ , see Section 5.1 below. In dimension 3 and higher, for real-valued potentials, the problem is widely open, and the best estimate until today is still (1.1).

As already mentioned, Theorem 1.1 is closely related to the recent work [LMNN20]. First of all, Theorem 1.1 conjugated with a scaling argument directly leads to (1.5) but the converse is not so simple, and one cannot deduce easily Theorem 1.1 from (1.5). Moreover, [LMNN20, Corollary 2.5] also gives that for  $\Omega$  a smooth 2-d compact Riemannian manifold without boundary,  $V \in L^\infty(\Omega; \mathbb{R})$  and any solution  $u$  satisfying the homogeneous elliptic equation  $-\Delta u + Vu = 0$  in  $\Omega$ , we have  $\|u\|_{L^\infty(\Omega)} \leq C_V \|u\|_{L^\infty(\omega)}$ , where  $C_V > 0$  is as in (1.7). Therefore, our main improvements compare to [LMNN20] are the following: we are considering bounded domains, and thus boundary issues; we allow non-homogeneous source terms in the equation  $-\Delta u + Vu$ . Both issues require significant adaptations from the work [LMNN20] as the proof of [LMNN20, Corollary 2.5] strongly depends on the function  $u$  under consideration, in particular through its nodal set  $\{x \in \Omega; u(x) = 0\}$ . Indeed, estimate (1.6) is proved for any  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , for which the zero set might be extremely intricate.

There is a logarithm loss in (1.7) in comparison with the estimate (1.4). We discuss this issue in Addendum 1.

The dependence of the constant  $C$  as a function of  $\Omega$  and  $\omega$  in (1.7) is a delicate question, that we briefly comment in Remark 2.23. For further references in this direction, one can see for instance the recent article [LL21, Section 1.2].

Let us also point out the fact that since  $V$  is real-valued, considering the real part  $\Re(u)$  and imaginary part functions  $\Im(u)$  instead of  $u$  if needed, we can further assume that  $u$  is real-valued, which we will always do in the following. Therefore, the sign of the function  $u$  will be properly defined, and we will be able to use Harnack's estimates in the areas in which  $u$  does not vanish, which will turn out to be useful in the proof of Theorem 1.1.

In Theorem 1.1, we assume that  $\Omega$  is simply connected. This is mainly a technical hypothesis, and we explain in Section 3 how to extend Theorem 1.1 to connected domains  $\Omega \subset \mathbb{R}^2$  having a finite number of holes and prove the following result.

**Theorem 1.2.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  with a finite number  $N \in \mathbb{N}$  of holes  $(\mathcal{H}_i)_{i \in \{1, \dots, N\}}$ , and  $\omega$  be a non-empty open set included in  $\Omega$ .*

*Then there exists a constant  $C = C(\Omega, \omega) > 0$  such that for every  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $V \in L^\infty(\Omega; \mathbb{R})$ , estimate (1.6) holds.*

Here, the main idea to prove this result is to rely on a three spheres inequality inside  $\Omega$  (see Theorem 3.3 below), and on Theorem 1.1 to estimate  $u$  in a neighborhood of the external boundary and of the holes.

The arguments to obtain a three spheres inequality inside  $\Omega$ , developed in Theorem 3.3, are very close to the ones developed in Theorem 1.1, but need to be suitably adapted, in particular due to the fact that the nodal sets of functions  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  may intersect in an intricate manner the spheres under consideration for the three spheres argument.

### 1.3 Applications to control theory

In this part, we first present results and open questions about null-controllability of parabolic equations, mainly due to Fernández-Cara and Zuazua. Motivated by these open problems, we establish new results about control of elliptic equations that can be obtained as an application of our main results Theorem 1.1 and Theorem 1.2.

#### 1.3.1 Null-controllability of parabolic equations as a motivation.

Let  $T > 0$ ,  $\Omega$  be a smooth bounded domain  $\mathbb{R}^N$ ,  $\omega \subset \Omega$  be a non-empty open set,  $Q_T = (0, T) \times \Omega$ ,  $q_T = (0, T) \times \omega$ ,  $V = V(t, x) \in L^\infty(Q_T)$  (possibly complex valued),  $F \in L^2(Q_T)$ ,  $u_0 \in L^2(\Omega)$ , and consider the following parabolic equation

$$\begin{cases} \partial_t u - \Delta u + V(t, x)u = F, & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (1.8)$$

By a parabolic Carleman estimate, see [FCG06, Lemma 1.3], one can prove that for every  $F \in L^2(Q_T)$ ,  $u_0 \in L^2(\Omega)$ , the solution  $u$  to (1.8) satisfies the observability estimate

$$\|u(T, \cdot)\|_{L^2(\Omega)} \leq C \exp\left(C\left(\frac{1}{T} + T\|V\|_{L^\infty(Q_T)} + \|V\|_{L^\infty(Q_T)}^{2/3}\right)\right) (\|F\|_{L^2(Q_T)} + \|u_0\|_{L^2(\Omega)}). \quad (1.9)$$

By duality, see [Cor07, Theorem 2.44], (1.9) leads to a small-time null-controllability result for the controlled parabolic equation

$$\begin{cases} \partial_t y - \Delta y + V(t, x)y = h1_\omega, & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases} \quad (1.10)$$

In (1.10), at time  $t \in (0, T)$ ,  $y(t, \cdot) : \Omega \rightarrow \mathbb{C}$  is the state while  $h(t, \cdot) : \omega \rightarrow \mathbb{C}$  is the control.

More precisely, for every  $T > 0$ ,  $y_0 \in L^2(\Omega)$ , there exists  $h \in L^2(Q_T)$  such that the solution  $y$  of (1.10) satisfies  $y(T, \cdot) = 0$  and we have

$$\|y\|_{L^2(Q_T)} + \|h\|_{L^2(Q_T)} \leq C \exp\left(C\left(\frac{1}{T} + T\|V\|_{L^\infty(Q_T)} + \|V\|_{L^\infty(Q_T)}^{2/3}\right)\right) \|u_0\|_{L^2(\Omega)}. \quad (1.11)$$

Using (Meshkov), the article [DZZ08] shows the optimality in small time of (1.9) then the optimality in small time of (1.11) for  $V \in L^\infty(Q_T; \mathbb{C})$ . According to the elliptic case, it is natural to expect better cost estimates than (1.11) for  $V \in L^\infty(Q_T; \mathbb{R})$ , but this is open even in the 1-d case (except for time-independent potentials  $V \in L^\infty(\Omega; \mathbb{C})$  in 1-d according to [DZZ08, Section 8.5]).

In [FCZ00], Fernández-Cara and Zuazua considered the semi-linear parabolic equation

$$\begin{cases} \partial_t y - \Delta y + f(y) = h1_\omega, & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases} \quad (1.12)$$

where  $f \in C^1(\mathbb{R}; \mathbb{R})$  such that  $f(0) = 0$ . They prove two results for (1.12):

- (Positive result) If  $f(s) = o_{|s| \rightarrow +\infty}(|s| \log^{3/2}(|s|))$  then (1.12) is small-time globally null-controllable, i.e. for every  $T > 0$ , for every  $y_0 \in L^\infty(\Omega)$ , there exists  $h \in L^\infty(Q_T)$  such that the solution  $y$  of (1.12) satisfies  $y(T, \cdot) = 0$ . The proof of this positive result is strongly related to the cost estimate (1.9) for solutions of (1.8).

- (Negative result) For  $p > 2$ , if  $f$  is defined by  $f(s) = \int_0^{|s|} \log^p(1 + |\tilde{s}|) d\tilde{s}$ ,  $s \in \mathbb{R}$ , (1.12) is not small-time (even large-time) globally null-controllable.

These two results lead to the following open problem: if  $|f(s)| \sim |s| \log^p(1 + |s|)$ , as  $|s| \rightarrow +\infty$  with  $p \in [3/2, 2]$ , is (1.12) small-time/large-time globally null-controllable? Despite several works on this problem, this is still an open question in general. Note that in [AT02], the authors prove that even in the dissipative case, i.e.  $f(s) = s \log^p(1 + |s|)$  for some  $p > 2$ , small-time global null-controllability of (1.12) does not hold. Recently, the second author proves in [LB20] that if  $f$  satisfies  $|f(s)| = o_{|s| \rightarrow +\infty}(|s| \log^2(|s|))$ , and  $f$  is semi-dissipative, including in particular the case  $f(s) = \pm |s| \log^p(1 + |s|)$  with  $p < 2$ , then (1.12) is large-time globally null-controllable. Similar questions for semi-linearities depending on the gradient of the solution are also considered in [DFCGBZ02].

### 1.3.2 New control results for linear and semi-linear elliptic equations in 2-d

Motivated by the questions raised by the previous part on null-controllability of parabolic equations, we give new control results for elliptic equations in the two-dimensional case that are based on Theorem 1.1.

Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  having a finite number of holes,  $\omega$  be a non-empty open subset of  $\Omega$  and  $V \in L^\infty(\Omega; \mathbb{R})$  be a real-valued potential.

For  $F \in L^2(\Omega)$ , we consider the linear elliptic control problem

$$\begin{cases} -\Delta y + V(x)y = F + h1_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.13)$$

From Theorem 1.2 and a duality argument, we have the following result.

**Theorem 1.3.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  having a finite number of holes,  $\omega$  be a non-empty open subset of  $\Omega$  and  $V \in L^\infty(\Omega; \mathbb{R})$  be a real-valued potential.*

For every  $F \in L^2(\Omega)$ , one can find a pair  $(y, h) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\omega)$  satisfying (1.13) and

$$\|y\|_{H^2(\Omega)} + \|h\|_{L^2(\omega)} \leq C_V \|F\|_{L^2(\Omega)}, \quad (1.14)$$

where  $C_V$  is given by (1.7). Moreover, this construction can be done such that, given  $F \in L^2(\Omega)$ , the operator  $V \in L^\infty(\Omega) \mapsto (y_V, h_V) \in H^2(\Omega) \times L^2(\omega)$ , where  $(y_V, h_V)$  corresponds to the above pair associated to the potential  $V$ , satisfies the following property: if  $(V_k)_{k \in \mathbb{N}}$  converges strongly in  $L^\infty(\Omega)$  toward  $V$ , then  $(y_{V_k}, h_{V_k})$  weakly converges as  $k \rightarrow \infty$  to  $(y_V, h_V)$  in  $H^2(\Omega) \times L^2(\omega)$ .

**Remark 1.4.** Theorem 1.3 is specific to the two-dimensional case and real-valued potentials. In fact, it is some kind of dual statement of Theorem 1.1 (when  $\Omega$  is simply connected) or Theorem 1.2 (when  $\Omega$  is multiply connected) and similar remarks apply, in particular: in the multi-dimensional case when  $V \in L^\infty(\Omega; \mathbb{C})$ ,  $\Omega \subset \mathbb{R}^N$ , Theorem 1.3 holds true with  $C_V$  replaced by  $C \exp(C\|V\|_{L^\infty(\Omega)}^{2/3})$ , as a consequence of (1.1); in 1-d, for  $V \in L^\infty(\Omega; \mathbb{C})$ , Theorem 1.3 holds true with  $C_V$  replaced by  $C \exp(C\|V\|_{L^\infty(\Omega)}^{1/2})$ , by duality with Theorem 5.1 presented in Section 5.1 below.

For  $f \in C^1(\mathbb{R}; \mathbb{R})$  such that  $f(0) = 0$  and  $F \in L^2(\Omega)$ , let us deal with the semi-linear elliptic control problem

$$\begin{cases} -\Delta y + f(y) = F + h1_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.15)$$

From similar arguments as in [FCZ00] and Theorem 1.3, we prove the following result.

**Theorem 1.5.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  having a finite number of holes,  $\omega$  be a non-empty open subset of  $\Omega$ , and  $f \in C^1(\mathbb{R}; \mathbb{R})$  such that  $f(0) = 0$ .*

*We have the following dichotomy.*

- (Positive result) *If  $f(s) = o_{|s| \rightarrow +\infty}(|s| \log^p(|s|))$ ,  $p < 2$ , then for every  $F \in L^2(\Omega)$ , there exists a pair  $(y, h) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\omega)$  satisfying (1.15).*

- (Negative result) *For  $p > 2$ , setting  $f_p(s) = \int_0^{|s|} \log^p(1 + |\bar{s}|) d\bar{s}$ , for  $s \in \mathbb{R}$ , there exists  $F \in L^2(\Omega)$  such that for every  $h \in L^2(\omega)$ , the elliptic equation (1.15) has no solution  $y \in H^2(\Omega) \cap H_0^1(\Omega)$ .*

Actually, we will prove that the positive result of Theorem 1.5 holds true assuming the weaker asymptotic condition on  $f$

$$\lim_{|s| \rightarrow \infty} \frac{f(s)}{|s| \log^2(|s|)} \cdot \log \left( 1 + \left| \frac{f(s)}{s} \right| \right) = 0. \quad (1.16)$$

**Remark 1.6.** Again, the positive result of Theorem 1.5 is specific to the two-dimensional case. In  $N$ -d with  $N \geq 2$ , a similar positive result holds true assuming  $f(s) = o_{|s| \rightarrow +\infty}(|s| \log^{3/2}(|s|))$ , even when  $f$  is assumed to be complex valued, and this can be proved following the arguments in [FCZ00] based on classical Carleman estimates for the elliptic equation in dimension  $N$  (as in (Carleman)). In 1-d, a positive result can be proved assuming only that  $f(s) = o_{|s| \rightarrow +\infty}(|s| \log^2(|s|))$ , using the specific one-dimensional Carleman estimate in Theorem 5.1.

**Remark 1.7.** The negative result of Theorem 1.5 in fact holds true in any dimension, and can be proven similarly. It shows that there exists semi-linearities of the order of  $|s| \log^p(1 + |s|)$  as  $|s| \rightarrow \infty$  with  $p > 2$  such that for some specific source terms  $F \in L^2(\Omega)$ , whatever the choice of the controls  $h \in L^2(\omega)$ , there are no solution  $y \in H^2(\Omega) \cap H_0^1(\Omega)$  to the elliptic equation (1.15).

We believe that, even if the results of Theorem 1.5 do not allow to answer the open question in the work of Fernández-Cara and Zuazua [FCZ00] concerning the global null-controllability of the semi-linear heat equation, it sheds some light to this problem. It might suggest that it  $|f(s)| = o(|s| \log^p(1 + |s|))$ , as  $|s| \rightarrow +\infty$  for some  $p < 2$ , then (1.12) is globally null-controllable, at least in 1-d, but to our knowledge, this is still an open problem.

The difficulty in addressing this issue would be in obtaining the counterpart of Theorem 1.1 for parabolic 1-d equations with a real-valued time dependent potential with a cost  $C_T \exp(C_T \|V\|_{L^\infty(Q_T)}^{1/2})$  in (1.9).

Unfortunately, it is clear from the proof of Theorem 1.1 that we are strongly using quasi-conformal theory (see Section 1.4, Step 5) to “absorb” the potential, so that our strategy seems difficult to apply directly for the 1-d heat equation with potential, as the time and space variables play different roles. Still, let us mention that, when considering the 1-d heat equation with a potential which does not depend on time, the situation is much better and one can indeed prove that the observability constant in (1.9) is of the form  $C_T \exp(C_T \|V\|_{L^\infty(Q_T)}^{1/2})$ , see [DZZ08, Section 8.5].

## 1.4 Strategy of the proof of Theorem 1.1

In this part, we present the strategy of the proof of Theorem 1.1 and the main arguments of each step. This strategy actually closely follows the approach of [LMNN20], and we will explain at the end of this section where it differs from [LMNN20].

The proof of Theorem 1.1 is divided into seven main steps.

- **Step 1: Reduction to concentric balls.** Noting that here,  $\Omega$  is assumed to be simply connected, using *Riemann mapping theorem*, we first reduce Theorem 1.1 to the case where  $\omega$  and  $\Omega$  are two concentric balls centered at 0.
- **Step 2: Reduction to a  $L^\infty$  observability inequality.** By using *local elliptic regularity* and *Sobolev embeddings*, we reduce the proof of (1.6) to the proof of a similar observability inequality replacing the  $L^2$ -norms of the left hand side and second right hand side terms by  $L^\infty$ -norms, see (2.2) thereafter.
- **Step 3: Reduction to smooth functions with 0 as a regular value.** By a density argument, mainly based on *Sard's lemma* and *approximation by smooth functions* of  $H^2$ -functions, we reduce the proof of (2.2) to the proof of (2.2) for smooth functions for which 0 is a regular value and which are constant on the boundary. (For simplicity of notations, to avoid having  $\partial\Omega$  as a 0 level set of the function, we will also consider functions which are non-zero on the boundary.) Accordingly, the functions  $u$  which will be considered up to the end have a well-structured set of zeros, formed by smooth disjoint Jordan curves, which do not intersect the boundary.
- **Step 4: Construction of a positive solution  $\varphi$  of  $(-\Delta + V)\varphi = 0$  in a suitable perforated domain.** We perform a *perforation process* depending on the set of zeros of the function  $u$  such that the perforated domain has a *small Poincaré constant*, denoted by a parameter  $\varepsilon > 0$ . This enables to construct a *positive solution* to the elliptic operator  $-\Delta + V$  in this new domain. In this step, we shall first classify the zero level sets of  $u$ , consisting of *smooth disjoint Jordan curves*, into two parts: the “big curves” and the “small curves”. The “small curves” determine small domains in which the Poincaré constant is small so nice estimates can be obtained in these areas. We also deduce that the solution satisfies a uniform bound either from above or from below, in the connected components delimited by the “big curves”. The next point consists in perforating the domain  $\Omega$  using sufficiently small disks in a sufficiently large number, avoiding the “big curves”, 0 and  $x_{\max}$ , the point at which  $|u|$  is maximal. The resulting perforated domain has a small Poincaré constant so one can construct a positive solution  $\varphi$  of  $(-\Delta + V)\varphi = 0$  satisfying appropriate bounds in this new domain. In the following, we will call this solution a *multiplier*. Note that at the end of this step  $\varepsilon$  has to be chosen such that  $\varepsilon \leq C\|V\|_{L^\infty(\Omega)}^{-1/2}$ .
- **Step 5: A quasiconformal change of variable.** Thanks to the positive multiplier of the previous step, we reduce the inhomogeneous elliptic equation  $-\Delta u + Vu = f$  to an *homogeneous divergence-form elliptic equation* by a suitable change of unknowns in the perforated domain. We then apply the *theory of quasiconformal mappings* that enables to recast the divergence elliptic equation into a Laplace equation:  $\Delta h = 0$ . The last point of this step is to control how the quasiconformal change of variable transforms the perforated domain of the previous step to another perforated domain. In particular, the holes, which were disks before, will be transformed into holes which still cannot be too flattened by this quasiconformal transform. Note that at the end of this step  $\varepsilon$  has to be chosen such that  $\varepsilon \leq C\|V\|_{L^\infty(\Omega)}^{-1/2} \log^{-1/2}(\|V\|_{L^\infty(\Omega)})$ .
- **Step 6: A Carleman estimate in a perforated domain.** We now employ a *Carleman estimate* in  $\Omega$  to a cut-off version of  $h$  which vanishes in a neighborhood of the holes of the domain. The truncated terms near the holes of the perforated domain will be treated with *Harnack inequalities* taking the parameter  $s$  in the Carleman estimate such that  $s \geq C\varepsilon^{-1}$ , i.e.  $s \geq C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})$ . This combination of arguments will prove quantitative unique continuation estimates on the solution  $h$  in the perforated domain.
- **Step 7: From the Carleman estimate to the observability inequality.** We now come back to the original variable  $u$  to deduce from the observability inequality satisfied by  $h$  in the perforated domain the observability inequality for  $u$  in the whole domain. Here, we crucially use the fact that the perforation process in Step 4 avoids the point  $x_{\max}$ . Let us emphasize here, that because our goal is to get the  $L^\infty$  observability inequality of Step 2, there is no need to undo and estimate the quasiconformal change of variable of Step 4 (which depends on the multiplier  $\varphi$ ,  $\varepsilon$  and  $\|V\|_\infty$ ).

Steps 4, 5, and 6 are crucially inspired by the methodology in [LMNN20] that focuses on the case of the homogeneous elliptic equation  $-\Delta u + Vu = 0$ . Still, our strategy differs from the one in [LMNN20] in several points.

Step 3, reducing the proof of the observability estimates to smooth functions having 0 as a regular value, is new. Since we want a result valid for any  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , in principle, the zero level set of  $u$  can be extremely intricate. Here, by some easy density argument, we reduce the problem to proving the observability inequality to smooth functions  $u$  for which 0 is a regular value. This gives some structure on the zero level set of  $u$ , which is simply formed by several smooth Jordan curves which do not intersect one to another (nor themselves).

Step 4, although strongly inspired by [LMNN20], contains one new ingredient, which is the classification of the zero level set of  $u$  into two parts, small and big parts. This is new compared to [LMNN20], since when considering solutions of  $-\Delta u + Vu = 0$ , a curve of the zero level set of  $u$  cannot be included in a too small ball, except in the trivial case  $u = 0$ . In fact, in the case handled by [LMNN20], there are no small curves in the zero level set of  $u$ .

Step 5 is very similar to [LMNN20], even if we need to give an additional argument to remove the source term  $f = -\Delta u + Vu$  and obtain the equation  $\Delta h = 0$ .

Step 6 is also strongly inspired by [LMNN20], but differs from it since it relies only on a Carleman estimate and Harnack's inequality. Indeed, the approach in [LMNN20, Section 6.1] based on Carleman estimate (which gives Landis conjecture in 2-d for solutions  $u$  decaying faster than  $\exp(-C|x| \log^{3/2}(|x|))$  instead of  $\exp(-C|x| \log^{1/2}(|x|))$ , see [LMNN20]) would give (1.6) with a slightly worse bound, of the form  $C \exp(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{3/2}(\|V\|_{L^\infty(\Omega)}))$ . We thus improve the argument of [LMNN20] based on Carleman estimates to obtain (1.6) with  $C_V$  as in (1.7). We believe that this technical improvement is interesting by itself because it enables to generalize the toy problem [LMNN20, Theorem 5.1] in the multi-dimensional case without logarithm loss.

Let us finally mention that the treatment of the boundary  $\partial\Omega$  is another difficulty in the proof of Theorem 1.1 compared to [LMNN20], which we take into account by considering smooth functions  $u$  having 0 as a regular value and being a non-zero constant on the boundary, thus guaranteeing that the zero level set of  $u$  does not intersect the boundary  $\partial\Omega$ .

## 1.5 Organization of the paper

Section 2 is dedicated to the proof of the main result of the paper, i.e. the quantitative unique continuation result given in Theorem 1.1. Section 3 provides the proof of Theorem 1.2. Section 4 is devoted to the proof of the control results for elliptic equations, given in Theorem 1.3 and Theorem 1.5. We end the article in Section 5 by giving some complementary results related to observability inequalities.

**Addendum 1.** After finishing our work, Fedor Nazarov kindly indicated us in the personal communication [LMNN22] an ongoing work proving, that for  $\alpha \in [0, 1/2)$  there exist non-trivial real-valued functions  $u$  on  $\mathbb{R}^2$  such that  $|\Delta u| \leq |u|$  and  $u$  decays faster than  $\exp(-|x| \log^\alpha(1 + |x|))$  for  $|x|$  large. Accordingly, the result in [LMNN20] is sharp, and the conjecture (1.4) is disproved in the plane: the constant  $C \exp(C\|V\|_{L^\infty(\Omega)}^{1/2})$  in (1.4) has to be replaced at least by  $C \exp(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)}))$ . As a consequence, the estimate (1.6) is sharp on the constant  $C_V^*$  regarding the dependence in the  $L^\infty$ -norm of real valued potentials  $V \in L^\infty(\Omega; \mathbb{R})$ .

**Acknowledgements.** The authors are indebted to Karim Kellay for interesting discussions which led to an improvement of our results and a simplification of our arguments. We also express our gratitude to Fedor Nazarov for having pointed out the recent work [LMNN22] yielding the optimality of [LMNN20] and thus of Theorem 1.1.

## 2 Proof of the observability inequality given in Theorem 1.1

Let  $\Omega$  be a smooth simply connected bounded domain of  $\mathbb{R}^2$  and  $\omega$  be a non-empty open subset of  $\Omega$ . Let us take  $V \in L^\infty(\Omega; \mathbb{R})$ . The goal of this section is to prove Theorem 1.1, i.e. to prove the observability inequality (1.6), associated to the elliptic operator  $-\Delta + V$ .

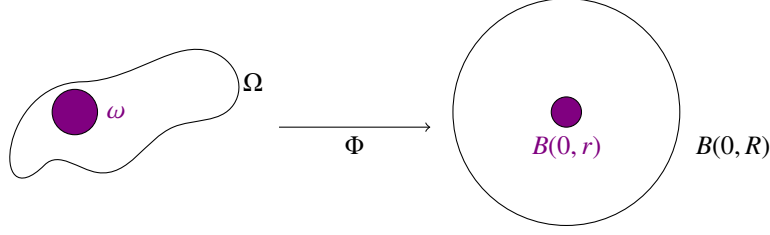


Figure 1: Image of  $\Omega$  by the bi-holomorphism  $\Phi$  of Step 1.

## 2.1 Step 1: Reduction to concentric balls

The goal of this first step is to reduce the proof of Theorem 1.1 to the case where  $\omega$  and  $\Omega$  are two concentric balls centered at 0.

**Fact 2.1.** Without loss of generality, one can assume that

$$\omega = B(0, r) \subset \Omega = B(0, R), \quad \partial\Omega = \partial B(0, R), \quad \text{for some } 0 < r < R. \quad (2.1)$$

*Proof.* Up to a translation argument, recalling that  $\omega$  is non-empty, one can assume that  $0 \in \omega$ .

By Riemann mapping theorem up to the boundary, see [Bel15, Theorem 8.2], we know that there exists  $\Phi : \bar{\Omega} \rightarrow \bar{B}(0, 1)$ , one-to-one, such that

$$\Phi \in \mathcal{H}(\Omega), \Phi \in C^\infty(\bar{\Omega}), \Phi(0) = 0, 0 < c \leq |\Phi'| \leq C \text{ in } \bar{\Omega},$$

where  $\mathcal{H}(\Omega)$  denotes the set of holomorphic functions on  $\Omega$ .

By the open mapping theorem,  $\Phi$  maps  $\omega$  to a neighborhood of 0 that contained  $B(0, r)$  for some  $r > 0$ .

For  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , by using Cauchy-Riemann's equation, setting  $\hat{u} := u \circ \Phi^{-1}$ , a straightforward computation gives

$$\Delta \hat{u}(x) = |\nabla \mathfrak{K}(\Phi^{-1})|^2 \Delta u(\Phi^{-1}(x)) \quad \forall x \in B(0, 1).$$

So setting  $\hat{V} = |\nabla \mathfrak{K}(\Phi^{-1})|^2 V \circ \Phi^{-1}$ , we obtain that  $\hat{u}$  satisfies the following elliptic equation in  $B(0, 1)$

$$-\Delta \hat{u} + \hat{V} \hat{u} = |\nabla \mathfrak{K}(\Phi^{-1})|^2 (-\Delta u(\Phi^{-1}) + V(\Phi^{-1})u(\Phi^{-1})) \text{ in } B(0, 1).$$

Moreover,  $\hat{u} = 0$  on  $\partial B(0, 1)$  as  $u = 0$  on  $\partial\Omega$  and the image of  $\partial\Omega$  through  $\Phi$  coincides with  $\partial B(0, 1)$ .

Hence the proof of the observability inequality (1.6) for  $\hat{u}$  in the unit disk  $B(0, 1)$  with the observation set  $B(0, r)$  immediately leads to (1.6) for  $u$  by coming back to the original variable.

Therefore we shall assume without loss of generality that there exist two positive constants  $0 < r < R$  such that (2.1) holds true (here  $R = 1$  in the proof).  $\square$

## 2.2 Step 2: Reduction to a $L^\infty$ observability inequality

The goal of this second step is to reduce the  $L^2$  observability inequality (1.6) to a  $L^\infty$  observability inequality.

**Fact 2.2.** Let  $\omega_0 \subset\subset \omega$ . Then the observability inequality

$$\|u\|_{L^\infty(\omega_0)} \leq C_V \left( \|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^\infty(\omega_0)} \right) \quad (u \in H^2(\Omega) \cap H_0^1(\Omega)), \quad (2.2)$$

with  $C_V$  as in (1.7), implies (1.6).

*Proof.* Recall that  $H^2(\Omega) \subset L^\infty(\Omega)$  so the two sides of the inequality (2.2) are finite because  $u \in H^2(\Omega)$ .

Sobolev embedding and local elliptic regularity give that for some positive constant  $C > 0$ , for all  $u \in H^2(\Omega)$ ,

$$\begin{aligned} \|u\|_{L^\infty(\omega_0)} &\leq C \|u\|_{H^2(\omega_0)} \leq C \left( \|-\Delta u\|_{L^2(\omega)} + \|u\|_{L^2(\omega)} \right) \\ &\leq C \left( \|-\Delta u + Vu\|_{L^2(\omega)} + (1 + \|V\|_{L^\infty(\Omega)}) \|u\|_{L^2(\omega)} \right) \leq C_V \left( \|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right). \end{aligned} \quad (2.3)$$

So by gathering (2.2) and (2.3), this directly yields (1.6).  $\square$

In the following, we will assume

$$\omega_0 = B(0, r_0), \quad \text{for some } r_0 \in (0, r), \quad (2.4)$$

where  $r$  is the radius of  $\omega$  (recall (2.1)).

### 2.3 Step 3: Reduction to smooth functions with 0 as a regular value

For simplicity, we will deal only with smooth functions  $u$  such that 0 is a regular value of  $u$  (meaning that for all  $x \in \overline{\Omega}$  with  $u(x) = 0$ ,  $\nabla u(x) \neq 0$ ) and such that  $u$  equals to a non-zero constant on  $\partial\Omega$ . Let us call  $\mathcal{U}$  this set:

$$\mathcal{U} = \{u \in C^\infty(\overline{\Omega}; \mathbb{R}) ; 0 \text{ is a regular value of } u \text{ and } u \text{ is a non-zero constant on } \partial\Omega\}. \quad (2.5)$$

**Lemma 2.3.** *For any  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , there exists a sequence  $(u_n)_{n \in \mathbb{N}}$  of elements of  $\mathcal{U}$  which converges to  $u$  in  $H^2(\Omega)$ .*

*Proof.* Equivalently to the statement of Lemma 2.3, we will show that for  $u \in H^2(\Omega) \cap H_0^1(\Omega; \mathbb{R})$  and  $\varepsilon > 0$ , we can find  $u_\varepsilon \in \mathcal{U}$  such that  $\|u - u_\varepsilon\|_{H^2(\Omega)} \leq \varepsilon$ .

By [Eva10, Section 5.3.3, Theorem 3], the closure of  $C^\infty(\overline{\Omega})$  for the  $H^2(\Omega)$  topology is  $H^2(\Omega)$  so there exists  $u_1 \in C^\infty(\overline{\Omega})$  such that

$$\|u - u_1\|_{H^2(\Omega)} \leq \varepsilon.$$

We then recall that there exists a lifting operator  $\mathcal{R} : H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \rightarrow H^2(\Omega)$  such that for all  $(g_0, g_1) \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ ,  $\mathcal{R}(g_0, g_1) \in H^2(\Omega)$  and  $(\mathcal{R}(g_0, g_1)|_{\partial\Omega}, \partial_n \mathcal{R}(g_0, g_1)|_{\partial\Omega}) = (g_0, g_1)$ , and for all  $(g_0, g_1) \in C^\infty(\partial\Omega)^2$ ,  $\mathcal{R}(g_0, g_1) \in C^\infty(\overline{\Omega})$  (see e.g. [LM72, Theorem 8.3 and Remark 3.3]).

Accordingly,  $u_1 - \mathcal{R}(u_1|_{\partial\Omega}, \partial_n u_1|_{\partial\Omega})$  belongs to  $H_0^2(\Omega) \cap C^\infty(\overline{\Omega})$ . Now, since by definition the closure of  $C_c^\infty(\Omega)$  for the  $H^2(\Omega)$  topology is  $H_0^2(\Omega)$ , there exists  $u_2 \in C_c^\infty(\Omega)$  such that

$$\|u_2 - (u_1 - \mathcal{R}(u_1|_{\partial\Omega}, \partial_n u_1|_{\partial\Omega}))\|_{H^2(\Omega)} \leq \varepsilon.$$

It follows that  $u_3 = u_2 + \mathcal{R}(0, \partial_n u_1|_{\partial\Omega})$  satisfies

$$\begin{aligned} \|u_3 - u\|_{H^2(\Omega)} &\leq \|u_2 - (u_1 - \mathcal{R}(u_1|_{\partial\Omega}, \partial_n u_1|_{\partial\Omega}))\|_{H^2(\Omega)} + \|\mathcal{R}(u_1|_{\partial\Omega}, 0)\|_{H^2(\Omega)} + \|u - u_1\|_{H^2(\Omega)} \\ &\leq \varepsilon + C\|u - u_1\|_{H^2(\Omega)} \leq C\varepsilon, \end{aligned}$$

since, using the continuity of the lifting operator  $\mathcal{R}$  from  $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$  to  $H^2(\Omega)$  and  $u \in H_0^1(\Omega)$ ,

$$\|\mathcal{R}(u_1|_{\partial\Omega}, 0)\|_{H^2(\Omega)} \leq C\|u_1\|_{H^{3/2}(\partial\Omega)} \leq C\|u_1 - u\|_{H^{3/2}(\partial\Omega)} \leq C\|u_1 - u\|_{H^2(\Omega)}.$$

Let us further note that  $u_3 \in C^\infty(\overline{\Omega})$  and that  $u_3|_{\partial\Omega} = 0$ .

We now use Sard's lemma to get the existence of some positive parameters  $\gamma \in (0, \varepsilon)$  such that 0 is not a critical value of the function  $u_4 = u_3 + \gamma$ .

By construction,  $\|u_4 - u\|_{H^2(\Omega)} \leq C\varepsilon$ , and  $u_4$  belongs to the set  $\mathcal{U}$  defined in (2.5):  $u_4$  belongs to  $C^\infty(\overline{\Omega})$ , is a non-zero constant on  $\partial\Omega$ , and has 0 as regular value. This proves the result since  $\varepsilon > 0$  is arbitrary, by replacing  $\varepsilon$  by  $\varepsilon/C$ .  $\square$

From Lemma 2.3, since the topology of  $H^2(\Omega)$  allows to pass the limit in each terms of (2.2) (recall that in dimension 2,  $H^2(\Omega) \subset L^\infty(\Omega)$ ), we directly obtain the following result.

**Fact 2.4.** The observability inequality

$$\|u\|_{L^\infty(\Omega)} \leq C_V \left( \|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^\infty(\omega_0)} \right) \quad (u \in \mathcal{U}), \quad (2.6)$$

with  $C_V$  as in (1.7), implies (2.2).

### 2.4 Step 4: Construction of a positive multiplier in a suitable perforated domain

We now begin the proof of (2.6). From now on, we fix  $u \in \mathcal{U}$ , and we set

$$f := -\Delta u + Vu \text{ in } \Omega,$$

which belongs to  $L^2(\Omega)$  since  $u \in H^2(\Omega)$  and  $V \in L^\infty(\Omega)$ .

This step consists in constructing a suitable perforation of the domain  $\Omega$  in order to construct a domain in which the Poincaré constant is small, and such that the perforations we add do not meet some particular nodal lines of  $u$ . Then in this perforated domain, because the Poincaré constant is small, one can construct a positive solution  $\varphi$  satisfying  $-\Delta\varphi + V\varphi = 0$ , which we will call a multiplier hereafter.

From now and in all the following, the constants  $C > 0$  and  $c > 0$  are independent of  $u$  and  $V$ .

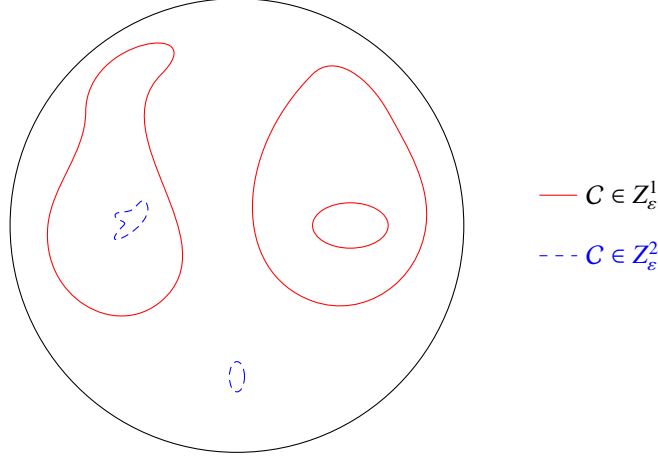


Figure 2: An example of nodal set of  $u$ .

### 2.4.1 Decomposition of the nodal sets of $u$

Let us denote by  $Z$  the nodal set of  $u$ :

$$Z := \{x \in \Omega ; u(x) = 0\}.$$

The first important property of this set is that, since  $u \in \mathcal{U}$ , recalling the definition (2.5),  $Z$  is the union of smooth compact connected curves contained in  $\Omega$  which do not intersect each other. In other words, we have that for some set of index  $J$ ,

$$Z = \cup_{j \in J} C_j, \quad C_j \text{ are disjoint smooth Jordan curves that do not intersect } \partial\Omega.$$

By classical differential geometry arguments, we have the following result.

**Fact 2.5.** For every  $j \in J$ ,  $C_j$  satisfies the following properties:

- (*Jordan curve*)  $C_j$  separates  $\Omega$  into exactly two connected components denoted by  $O_{C_j}$ , the interior of  $C_j$ , strictly contained in  $\Omega$ , and  $\mathcal{E}_{C_j}$ , the exterior of  $C_j$ ,
- (*Separation and Sign property*) there exists an open connected neighbourhood  $V_j$  of  $C_j$  contained in  $\Omega$  such that:  $V_j \cap Z = C_j$ ; for every  $x_0, x_1 \in V_j \cap \mathcal{E}_{C_j}$ ,  $u(x_0)u(x_1) > 0$ ; for every  $x_0, x_1 \in V_j \cap O_{C_j}$ ,  $u(x_0)u(x_1) < 0$ .

We then separate the nodal sets  $Z$  of  $u$  into two parts, depending on a parameter  $\varepsilon > 0$ , a small parameter that will be fixed later and that will always be assumed to be smaller than 1.

Namely, we consider the following property: for  $j \in J$ ,  $C_j$  satisfies the property **(P- $\varepsilon$ )** if

$$\forall x_0 \in C_j, \forall r \in (0, \varepsilon], \partial B(x_0, r) \cap C_j \neq \emptyset. \quad (\text{P-}\varepsilon)$$

Accordingly, we write

$$J_\varepsilon^1 = \{j \in J, C_j \text{ satisfies } (\text{P-}\varepsilon)\}, \quad J_\varepsilon^2 = \{j \in J, C_j \text{ does not satisfy } (\text{P-}\varepsilon)\}, \quad (2.7)$$

which of course forms a partition of  $J$ :

$$J = J_\varepsilon^1 \cup J_\varepsilon^2.$$

We then decompose  $Z$  accordingly:

$$Z = Z_\varepsilon^1 \cup Z_\varepsilon^2, \quad \text{with } Z_\varepsilon^1 := \cup_{j \in J_\varepsilon^1} C_j \text{ and } Z_\varepsilon^2 := \cup_{j \in J_\varepsilon^2} C_j. \quad (2.8)$$

In Figure 2, we have represented the nodal set of  $u$  by distinguishing the curves that belong to  $Z_\varepsilon^1$  and  $Z_\varepsilon^2$ . Roughly speaking,  $Z_\varepsilon^1$  corresponds to the large components of the nodal set of  $u$ , while  $Z_\varepsilon^2$  corresponds to the small ones.

Then, by construction and a connectedness argument, we have that for every  $j \in J_\varepsilon^2$ , there exists  $x_j \in C_j$  such that  $C_j \subset B(x_j, \varepsilon)$ . Accordingly, the Poincaré constant in  $O_{C_j}$  should be of the order of  $\varepsilon$ . In particular, if  $\varepsilon^2 \|V\|_{L^\infty}$  is small enough, from the fact that  $u$  satisfies  $-\Delta u + Vu = f$  in  $O_{C_j}$  and vanishes on  $C_j$ , Lax-Milgram lemma gives  $\|\nabla u\|_{L^2(O_{C_j})} \leq C \|f\|_{L^2(\Omega)}$ .

The next proposition makes this idea precise, and also states that each connected component of  $O$  of  $\Omega \setminus Z_\varepsilon^1$  then corresponds to open sets in which the solution enjoys a uniform bound, either from above, or from below.

**Proposition 2.6.** *There exist two constants  $C > 0$  and  $c > 0$  (independent of  $u$  and  $V$ ) such that for every  $\varepsilon > 0$  satisfying*

$$\varepsilon^2 \|V\|_{L^\infty(\Omega)} \leq c, \quad (2.9)$$

the following holds.

- For each connected component  $C$  of  $Z_\varepsilon^2$ ,

$$\|u\|_{H_0^1(O_C)} + \|u\|_{L^\infty(O_C)} \leq C \|f\|_{L^2(O_C)}. \quad (2.10)$$

- For each connected component  $O$  of  $\Omega \setminus Z_\varepsilon^1$ , we have

$$\left( \forall x \in O, \quad u(x) \geq -C \|f\|_{L^2(\Omega)} \right) \quad \text{or} \quad \left( \forall x \in O, \quad u(x) \leq C \|f\|_{L^2(\Omega)} \right). \quad (2.11)$$

**Remark 2.7.** In the case  $f = 0$ , we obtain that  $u \equiv 0$  in  $O_C$  when  $C$  is a connected component of  $Z_\varepsilon^2$ , which is in apparent contradiction with the fact that 0 is a regular value of  $u$ . This is due to the fact that if  $f = 0$ , the set  $Z_\varepsilon^2$  is empty, in agreement with [LMNN20, Lemma 6.14].

**Remark 2.8.** Actually, from a scaling argument, one can obtain a more precise estimate than (2.10), namely, under the condition (2.9), for each connected component  $C$  of  $Z_\varepsilon^2$ ,

$$\|u\|_{L^2(O_C)} + \varepsilon \|u\|_{L^\infty(O_C)} + \varepsilon \|\nabla u\|_{L^2(O_C)} \leq C \varepsilon^2 \|f\|_{L^2(\Omega)}.$$

Before going into the proof of Proposition 2.6, let us first define the Poincaré constant.

**Definition 2.9.** Let  $O$  be a smooth bounded domain of  $\mathbb{R}^2$ . The Poincaré constant  $C_P(O)$  is the smallest constant for which we have the Poincaré inequality

$$\forall v \in H_0^1(O), \quad \|v\|_{L^2(O)} \leq C_P(O) \|\nabla v\|_{L^2(O)}.$$

For later use, let us recall that if  $O_1 \subset O_2$  then  $C_P(O_1) \leq C_P(O_2)$  and that  $C_P(B(R)) = C_{0,P}R$  for an universal constant  $C_{0,P} > 0$ .

*Proof of Proposition 2.6.* In the following proof, the constants  $C > 0$  can vary from line to line, but they are always independent of  $\varepsilon > 0$ ,  $u$  and  $V$ .

Take  $C \subset Z_\varepsilon^2$ . We know from (P- $\varepsilon$ ), (2.8) and a connectedness argument that there exists  $x_0 \in C$  such that  $O_C \subset B(x_0, \varepsilon)$  so in particular we have  $C_P(O_C) \leq C_P(B(x_0, \varepsilon)) \leq C_{0,P}\varepsilon$ .

We have that  $u \in H_0^1(O_C)$  satisfies  $-\Delta u + Vu = f$  in  $O_C$ . By multiplying by  $u$ , and integrating by parts, we obtain the following energy identity

$$\int_{O_C} |\nabla u|^2 dx + \int_{O_C} Vu^2 dx = \int_{O_C} fu dx.$$

We then deduce by using Young's inequality and Poincaré's inequality that

$$\|\nabla u\|_{L^2(O_C)}^2 \leq \|V\|_{L^\infty(\Omega)} \|u\|_{L^2(O_C)}^2 + \|f\|_{L^2(\Omega)} \|u\|_{L^2(O_C)} \leq \left( C_{0,P}^2 \varepsilon^2 \|V\|_{L^\infty(\Omega)} + \frac{1}{2} \right) \|\nabla u\|_{L^2(O_C)}^2 + \frac{C_{0,P}^2 \varepsilon^2}{2} \|f\|_{L^2(\Omega)}^2.$$

Taking  $C_{0,P}^2 \varepsilon^2 \|V\|_{L^\infty(\Omega)} \leq 1/4$ , we obtain  $\|\nabla u\|_{L^2(O_C)}^2 \leq 2C_{0,P}^2 \varepsilon^2 \|f\|_{L^2(\Omega)}^2$ , which leads to

$$\|u\|_{L^2(O_C)} \leq C \varepsilon^2 \|f\|_{L^2(\Omega)}, \quad \|\nabla u\|_{L^2(O_C)} \leq C \varepsilon \|f\|_{L^2(\Omega)}, \quad (2.12)$$

for some constant  $C > 0$ , and this of course implies the first part of (2.10).

To prove the  $L^\infty$ -estimate in (2.10), we write  $-\Delta u = -Vu + f =: g$  in  $O_C$ . From (2.12), we have that

$$\|g\|_{L^2(O_C)} \leq \|Vu\|_{L^2(O_C)} + \|f\|_{L^2(O_C)} \leq C \varepsilon^2 \|V\|_{L^\infty(O_C)} \|f\|_{L^2(O_C)} + \|f\|_{L^2(O_C)} \leq C \|f\|_{L^2(O_C)}.$$

Then, we use de Giorgi's method in the spirit of [GT83, Section 8.1, Theorem 8.1], see Lemma A.1, to obtain that

$$\|u\|_{L^\infty(O_C)} \leq C \|f\|_{L^2(\Omega)}. \quad (2.13)$$

Here, we underline that the constant  $C$  does not depend on the measure of  $\mathcal{O}_C$ . Since this point is important for our argument, we provide the proof of Lemma A.1 in Appendix. Also note that the scaling argument sketched in Remark 2.8 would provide the better estimate  $\|u\|_{L^\infty(\mathcal{O}_C)} \leq C\varepsilon \|f\|_{L^2(\Omega)}$ .

To prove property (2.11), we proceed as follows. First, let us fix a connected component  $\mathcal{O}$  of  $\Omega \setminus Z_\varepsilon^1$ .

It is obvious that  $u$  cannot change sign in connected components of  $\mathcal{O} \setminus \overline{\cup_{C \in Z_\varepsilon^2} \mathcal{O}_C}$ , since there are no  $x \in \mathcal{O} \setminus \overline{\cup_{C \in Z_\varepsilon^2} \mathcal{O}_C}$  for which  $u(x) = 0$ . Indeed, such  $x$  would necessarily belong to  $Z_\varepsilon^2$  and thus to some Jordan curve  $C_0 \in Z_\varepsilon^2$ . Thus, we should have  $x \in \overline{\mathcal{O}_{C_0}}$ , which would contradict  $x \in \mathcal{O} \setminus \overline{\cup_{C \in Z_\varepsilon^2} \mathcal{O}_C}$ .

Let us now proceed by contradiction and take  $x_0$  and  $x_1$  in  $\mathcal{O}$  such that  $u(x_0) > C \|f\|_{L^2(\Omega)}$  and  $u(x_1) < -C \|f\|_{L^2(\Omega)}$ , where  $C$  is as in (2.10) and set

$$\mathcal{O}_+ = \{x \in \mathcal{O} \setminus \overline{\cup_{C \in Z_\varepsilon^2} \mathcal{O}_C}, \text{ with } u(x) > 0\}, \quad \mathcal{O}_- = \{x \in \mathcal{O} \setminus \overline{\cup_{C \in Z_\varepsilon^2} \mathcal{O}_C}, \text{ with } u(x) < 0\}.$$

Since for all  $x \in \cup_{C \in Z_\varepsilon^2} \mathcal{O}_C$ , from (2.13), we necessarily have  $|u(x)| \leq C \|f\|_{L^2(\Omega)}$ , it is clear that  $x_0$  and  $x_1$  belong to  $\mathcal{O} \setminus \overline{\cup_{C \in Z_\varepsilon^2} \mathcal{O}_C}$ , and thus  $x_0 \in \mathcal{O}_+$ , and  $x_1 \in \mathcal{O}_-$ .

By using the separation property of Fact 2.5, it is easy to see that  $\Omega \setminus Z_\varepsilon^1$  is locally connected so its connected components are open hence they are path-connected.

Take  $\gamma : [0, 1] \rightarrow \mathcal{O}$  a continuous path such that  $\gamma(0) = x_0$  and  $\gamma(1) = x_1$ . Then set

$$t_0 = \sup\{t \in [0, 1] ; \gamma(t) \in \mathcal{O}_+\}.$$

Since  $x_1 \in \mathcal{O}_-$ , we necessarily have  $t_0 < 1$ , and, since  $x_0 \in \mathcal{O}_+$ ,  $t_0 > 0$ . Besides, by construction  $u(\gamma(t_0)) = 0$ , so that  $\gamma(t_0) \in C_0$  for some  $C_0 \in Z_\varepsilon^2$ , and for all  $t > t_0$ ,  $\gamma(t) \in \mathcal{O}_- \cup \overline{\cup_{C \in Z_\varepsilon^2} \mathcal{O}_C}$ . Now, since  $C_0$  is a Jordan curve,  $\mathcal{O}_+$  is neighborhood of  $C_0$  in  $\mathcal{E}_{C_0}$ .

We then set

$$t_1 = \sup\{t \in (t_0, 1] ; \gamma(t) \in \overline{\mathcal{O}_{C_0}}\}.$$

It is clear that  $t_1 < 1$ , due to the fact that  $\gamma(1) = x_1 \notin \overline{\mathcal{O}_{C_0}}$ . Besides, by construction, we have that  $\gamma(t_1) \in C_0$  and thus there exists  $\alpha > 0$  such that for all  $t \in (t_1, t_1 + \alpha]$ ,  $\gamma(t) \in \mathcal{O}_+$ . This contradicts the definition of  $t_0$ .  $\square$

## 2.4.2 A perforation process

The next step is to construct a suitable perforation of the domain  $\Omega$  which avoids the nodal set  $Z_\varepsilon^1$ , and which, for convenience, will also avoid the point in which the function  $u$  achieves its maximum on  $\overline{\Omega}$ .

Let us take  $x_{\max} \in \overline{\Omega}$  such that

$$|u(x_{\max})| = \sup_{\overline{\Omega}} |u|. \quad (2.14)$$

We then get the following lemma.

**Lemma 2.10.** *For all  $C_0 \geq 5$ , for every  $\varepsilon > 0$ , there exist finitely many  $C_0\varepsilon$ -separated closed disks of radius  $\varepsilon$ , whose union is denoted by  $F_\varepsilon$ , satisfying the following properties:*

- *these disks are  $C_0\varepsilon$ -separated from each other, from  $Z_\varepsilon^1$ , from  $\partial\Omega$ , from  $x_{\max}$  and from 0,*
- *the set  $Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\Omega$  is a  $6C_0\varepsilon$ -net in  $\Omega$ , meaning that for all  $x \in \Omega$ ,  $B(x, 6C_0\varepsilon) \cap (Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\Omega) \neq \emptyset$ .*
- *the set*

$$\Omega_\varepsilon := \Omega \setminus (Z_\varepsilon^1 \cup F_\varepsilon) \quad (2.15)$$

*satisfies  $C_P(\Omega_\varepsilon) \leq C\varepsilon$  for some constant  $C > 0$  depending on  $C_0$  but independent of  $\varepsilon$ , ( $u$  and  $V$ ).*

In Figure 3, we have represented the perforated domain.

*Proof.* Let  $C_0 \geq 5$ . Among the union of separated closed disks of radius  $(C_0 + 1)\varepsilon$  in  $\Omega \setminus Z_\varepsilon^1$ , choose a maximal family for the inclusion, and denote by  $\tilde{F}_\varepsilon$  the union of disks of radius  $\varepsilon$  with the same centers as the one of the previous family. By construction, the disks of  $\tilde{F}_\varepsilon$  are  $C_0\varepsilon$  separated one from another, from  $Z_\varepsilon^1$  and from  $\partial\Omega$ . Besides, for all  $x \in \Omega$ ,  $B(x, (2C_0 + 1)\varepsilon) \cap (Z_\varepsilon^1 \cup \tilde{F}_\varepsilon \cup \partial\Omega) \neq \emptyset$ , otherwise one could add  $B(x, (C_0 + 1)\varepsilon)$  to the maximal family of separated closed disks of radius  $(C_0 + 1)\varepsilon$  considered above.

We then set

$$F_\varepsilon = \tilde{F}_\varepsilon \setminus \{\text{disks of } \tilde{F}_\varepsilon \text{ which intersects } B(0, C_0\varepsilon) \text{ or } B(x_{\max}, C_0\varepsilon)\}.$$

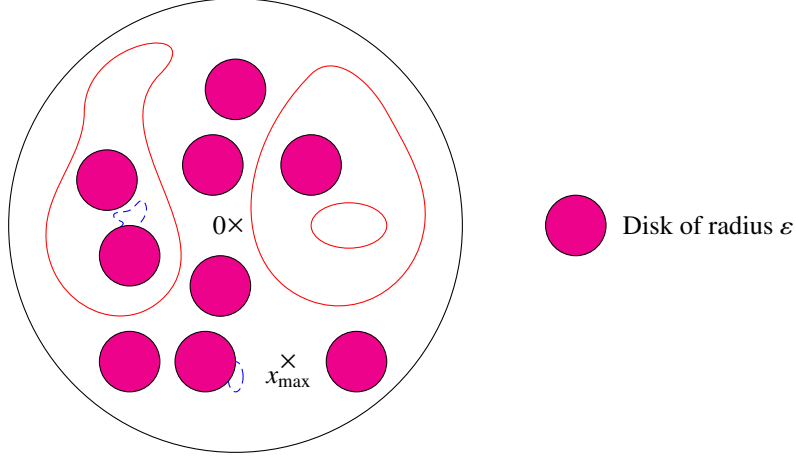


Figure 3: The perforation process.

It is then clear that the disks of  $F_\varepsilon$  are  $C_0\varepsilon$ -separated from each other, from  $Z_\varepsilon^1$ , from  $\partial\Omega$ , from  $x_{\max}$  and from 0, i.e. that the first item of Lemma 2.10 holds.

Let us prove the second item of Lemma 2.10.

For  $x \in \Omega$ , if  $B(x, (3C_0 + 3)\varepsilon) \cap \{0, x_{\max}\} = \emptyset$ , then we necessarily have that  $B(x, (2C_0 + 1)\varepsilon) \cap (Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\Omega) = B(x, (2C_0 + 1)\varepsilon) \cap (Z_\varepsilon^1 \cup \tilde{F}_\varepsilon \cup \partial\Omega)$  since  $F_\varepsilon$  and  $\tilde{F}_\varepsilon$  differ only in  $B(0, (C_0 + 2)\varepsilon) \cup B(x_{\max}, (C_0 + 2)\varepsilon)$  and  $B(x, (2C_0 + 1)\varepsilon) \cap (B(0, (C_0 + 2)\varepsilon) \cup B(x_{\max}, (C_0 + 2)\varepsilon)) = \emptyset$ . Therefore, if  $B(x, (3C_0 + 3)\varepsilon) \cap \{0, x_{\max}\} = \emptyset$ , we necessarily have  $B(x, (2C_0 + 1)\varepsilon) \cap (Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\Omega) \neq \emptyset$ .

For  $x \in \Omega$  such that  $B(x, (3C_0 + 3)\varepsilon) \cap \{0, x_{\max}\} \neq \emptyset$ , we necessarily have that  $B(x, (3C_0 + 4)\varepsilon) \setminus (B(0, (3C_0 + 3)\varepsilon) \cup B(x_{\max}, (3C_0 + 3)\varepsilon)) \neq \emptyset$ . Accordingly, taking  $x_0 \in B(x, (3C_0 + 4)\varepsilon) \setminus (B(0, (3C_0 + 3)\varepsilon) \cup B(x_{\max}, (3C_0 + 3)\varepsilon))$ , we necessarily have that  $B(x_0, (3C_0 + 3)\varepsilon) \cap \{0, x_{\max}\} = \emptyset$ , so that the previous argument applies and  $B(x_0, (2C_0 + 1)\varepsilon) \cap (Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\Omega) \neq \emptyset$ . Since  $B(x, (5C_0 + 5)\varepsilon)$  contains  $B(x_0, (2C_0 + 1)\varepsilon)$ , we thus get  $B(x, (5C_0 + 5)\varepsilon) \cap (Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\Omega) \neq \emptyset$ .

The second item of Lemma 2.10 follows from the above analysis since  $6C_0 \geq \max\{2C_0 + 1, 5C_0 + 5\}$  for  $C_0 \geq 5$ .

The third item of Lemma 2.10 is an immediate corollary of Lemma 3.1 in [LMNN20], due to the fact that the set  $Z_\varepsilon^1$  satisfies the key property (P- $\varepsilon$ ).  $\square$

In the sequel, it will be useful to choose a very large  $C_0$ . For simplicity, from now on, we set  $C_0 = 18 \cdot 32^2$  (this choice will be made clearer later).

### 2.4.3 Construction of the positive multiplier in the perforated domain

Based on the fact that the Poincaré constant  $C_P(\Omega_\varepsilon)$  can be made arbitrarily small by Lemma 2.10, applying Lemma 3.2 in [LMNN20], we derive the existence of a positive multiplier  $\varphi$  in  $\Omega_\varepsilon$ .

**Lemma 2.11.** *There exist two positive constant  $C > 0$  and  $c > 0$  (independent of  $u$  and  $V$ ) such that for every  $\varepsilon > 0$  satisfying (2.9), there exists a solution  $\varphi$  of*

$$\begin{cases} -\Delta\varphi + V\varphi = 0 & \text{in } \Omega_\varepsilon, \\ \varphi = 1 & \text{on } \partial\Omega_\varepsilon, \end{cases} \quad (2.16)$$

and  $\tilde{\varphi} = \varphi - 1$  satisfies

$$\tilde{\varphi} \in H_0^1(\Omega_\varepsilon) \text{ and } \|\tilde{\varphi}\|_\infty \leq C\varepsilon^2 \|V\|_{L^\infty(\Omega)}. \quad (2.17)$$

Note that  $\varepsilon > 0$  still is a free parameter, but it will always be taken such that Proposition 2.6 and Lemma 2.11 hold. Furthermore, we will always assume in the following that (2.9) holds true with  $c > 0$  small enough so that

$$\|\varphi - 1\|_\infty \leq \frac{1}{2}. \quad (2.18)$$

Moreover, since  $u$  is smooth,  $Z_\varepsilon^1$  is made of smooth curves, and  $\varphi$  is thus smooth on any connected component of  $\Omega_\varepsilon$ , taking value one on the boundary of each connected component. We can thus extend  $\varphi$  by 1 in  $\overline{\Omega} \setminus \Omega_\varepsilon$ , which yields a continuous function  $\varphi$  in  $\overline{\Omega}$ .

## 2.5 Step 5: A quasiconformal change of variable

The next step is to use a quasiconformal change of variable to recast the elliptic problem  $-\Delta u + Vu = f$  in  $\Omega$  into an harmonic equation. This part closely follows [LMNN20, Section 4], except for the introduction of  $\psi$  in Lemma 2.13 to remove the source term.

### 2.5.1 An homogeneous elliptic divergence equation

The first step is to rewrite the elliptic problem  $-\Delta u + Vu = f$  in  $\Omega$  into an equation of divergence form without source term.

Unfortunately, we are not able to do it in the whole set  $\Omega$  directly, but only in the set

$$\Omega'_\varepsilon = \Omega \setminus F_\varepsilon, \quad (2.19)$$

i.e. a set which is slightly larger than the set  $\Omega_\varepsilon = \Omega \setminus (Z_\varepsilon^1 \cup F_\varepsilon)$  defined in (2.15).

Using the equation of  $\varphi$  in (2.16), it is clear that, setting  $v = u/\varphi$  in  $\Omega_\varepsilon$ , we have  $\nabla \cdot (\varphi^2 \nabla v) = f\varphi$  in  $\Omega_\varepsilon$ . In fact, since  $\Omega'_\varepsilon = \Omega_\varepsilon \cup Z_\varepsilon^1$ , and  $u$  vanishes on  $Z_\varepsilon^1$ , a straightforward adaptation of [LMNN20, Lemma 4.1] yields that the equation  $\nabla \cdot (\varphi^2 \nabla v) = f\varphi$  also holds in  $\Omega'_\varepsilon$ . To be more precise, we get the following result.

**Lemma 2.12.** *The function  $v$  defined in  $\Omega'_\varepsilon$  by*

$$v := \frac{u}{\varphi} \text{ in } \Omega'_\varepsilon, \quad (2.20)$$

*belongs to  $H^1(\Omega'_\varepsilon)$  and satisfies in the weak sense*

$$\nabla \cdot (\varphi^2 \nabla v) = f\varphi \text{ in } \Omega'_\varepsilon. \quad (2.21)$$

Next, we remove the source term  $f\varphi$  of (2.21) as follows.

**Lemma 2.13.** *There exists a unique weak solution  $\psi \in H_0^1(\Omega'_\varepsilon)$  of*

$$\begin{cases} \nabla \cdot (\varphi^2 \nabla \psi) = f\varphi & \text{in } \Omega'_\varepsilon, \\ \psi = 0 & \text{on } \partial\Omega'_\varepsilon. \end{cases} \quad (2.22)$$

*Besides, there is a positive constant  $C > 0$  (independent of  $\varepsilon$ ,  $u$  and  $V$ ) such that*

$$\|\psi\|_{L^\infty(\Omega'_\varepsilon)} \leq C \|f\|_{L^2(\Omega)}. \quad (2.23)$$

*Proof.* First, the solvability of the divergence equation (2.22) directly comes from [GT83, Section 8.2, Theorem 8.3] and from the bounds (2.17).

For the  $L^\infty$ -bound (2.23), we recall first that from (2.18)  $\varphi$  takes value in  $[1/2, 3/2]$ . We then use de Giorgi's method as in the proof of [GT83, Section 8.1, Theorem 8.1], see Lemma A.1. The important point is that the constant  $C > 0$  can be made independent of  $\varepsilon$ , which is the reason why we recall for the sake of completeness the proof of Lemma A.1 in Appendix.  $\square$

We then simply set

$$\hat{v} = v - \psi \text{ in } \Omega'_\varepsilon, \quad (2.24)$$

where  $v$  is defined in (2.20) and  $\psi$  is defined in Lemma 2.13. From Lemma 2.12, Lemma 2.13. and the second point of Proposition 2.6, i.e. (2.11), we deduce the following result.

**Corollary 2.14.** *The function  $\hat{v}$  belongs to  $H^1(\Omega'_\varepsilon)$  and satisfies in the weak sense*

$$\nabla \cdot (\varphi^2 \nabla \hat{v}) = 0 \text{ in } \Omega'_\varepsilon. \quad (2.25)$$

*Moreover, there exists a positive constant  $C > 0$  (independent of  $\varepsilon$ ,  $u$  and  $V$ ) such that for every disk  $D \subset F_\varepsilon$ ,*

$$\left( \forall x \in \Omega'_\varepsilon \setminus D, \text{ with } \text{dist}(x, D) < C_0\varepsilon, \hat{v}(x) \geq -C \|f\|_{L^2(\Omega)} \right) \text{ or} \\ \left( \forall x \in \Omega'_\varepsilon \setminus D, \text{ with } \text{dist}(x, D) < C_0\varepsilon, \hat{v}(x) < C \|f\|_{L^2(\Omega)} \right),$$

*where  $F_\varepsilon$  and  $C_0$  are defined in Lemma 2.10.*

## 2.5.2 A quasiconformal change of variable

We then use the theory of quasiconformal mappings, which roughly speaking guarantees that solutions to homogeneous elliptic divergence equations behave as harmonic functions, see e.g. [AIM09].

**Lemma 2.15.** *There exists an homeomorphic mapping  $L$  of  $\overline{\Omega} = \overline{B(0, R)}$  into itself such that*

- $L$  is a  $K$ -quasiconformal mapping of  $\Omega$  into itself, with  $K$  satisfying

$$1 \leq K \leq 1 + C\varepsilon^2 \|V\|_{L^\infty(\Omega)}, \quad (2.26)$$

- $L(0) = 0$ ,
- $h = \hat{v} \circ L^{-1}$  satisfies  $\Delta h = 0$  in  $L(\Omega'_\varepsilon)$ , is constant on  $\partial\Omega$  and belongs to  $H^2_{loc}(L(\Omega'_\varepsilon))$  and to  $H^2$  in a neighborhood of  $\partial\Omega$ .

*Proof.* Let us first consider a real-valued function  $\hat{v} \in H^1_{loc}(B)$  for some ball  $B$  of  $\mathbb{R}^2$  and such that  $\nabla \cdot (\varphi^2 \nabla \hat{v}) = 0$  in  $B$  for some real-valued  $\varphi$  satisfying  $0 < c < \varphi(x) < C < +\infty$  in  $B$ .

By Poincaré lemma, see for instance [LMNN20, Section 6.5], one can then find a function  $\tilde{v} \in H^1_{loc}(B)$  such that  $\varphi^2 \hat{v}_x = \tilde{v}_y$  and  $\varphi^2 \hat{v}_y = -\tilde{v}_x$ . Setting  $w = \hat{v} + i\tilde{v}$ , we easily check that  $w$  is a solution to the Beltrami equation

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z} \text{ in } B,$$

with the Beltrami coefficient

$$\mu = \frac{1 - \varphi^2}{1 + \varphi^2} \cdot \frac{\hat{v}_x + i\hat{v}_y}{\hat{v}_x - i\hat{v}_y}, \quad (2.27)$$

where we set  $\mu = 0$  when  $\nabla \hat{v} = 0$ .

Note that, since  $\Omega_\varepsilon$  is not simply connected,  $w$  and  $\tilde{v}$  cannot be a priori defined in the whole set  $\Omega_\varepsilon$ . However, since  $\hat{v}$  is well-defined in  $\Omega_\varepsilon$ , we can safely define the Beltrami coefficient  $\mu$  by (2.27) in  $\Omega'_\varepsilon$ , and we further have, by Lemma 2.11,

$$\|\mu\|_{L^\infty(\Omega_\varepsilon)} \leq \left\| \frac{1 - \varphi^2}{1 + \varphi^2} \right\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^2 \|V\|_{L^\infty(\Omega)}. \quad (2.28)$$

We then extend  $\mu$  by zero outside  $\Omega'_\varepsilon$  to the whole complex plane, and remark that  $\mu$  has compact support.

We then use [AIM09, Theorem 5.3.2] to obtain the existence of a  $K$ -quasiconformal homeomorphism  $\Psi$  of the complex plane such that  $\Psi \in H^1_{loc}(\mathbb{C})$ ,  $\Psi$  satisfies the Beltrami equation

$$\frac{\partial \Psi}{\partial \bar{z}} = \mu \frac{\partial \Psi}{\partial z} \text{ in } \mathbb{C},$$

and  $K = \frac{1 + \sup|\mu|}{1 - \sup|\mu|}$ . In our case, according to (2.28), we have (2.26).

The function  $\hat{v} \circ \Psi^{-1}$  is harmonic in  $\Psi(\Omega'_\varepsilon)$ . Indeed, for any ball  $B$  included in  $\Omega'_\varepsilon$ , one can define  $w \in H^1_{loc}(B)$  such that  $\hat{v} = \Re(w)$  and  $w, \Psi$  solve the same Beltrami equation. Stoilow factorization theorem [AIM09, Theorem 5.5.1] then implies that there is a holomorphic function  $W$  such that  $w = W \circ \Psi$  in  $B$ , and therefore the harmonic function  $\Re(W)$  satisfies  $\hat{v} = \Re(W) \circ \Psi$  so we have  $\hat{v} \circ \Psi^{-1} = \Re(W)$  is a harmonic function in  $\Psi(B)$ . Since  $B$  is arbitrary in  $\Omega'_\varepsilon$ ,  $\hat{v} \circ \Psi^{-1}$  is harmonic in  $\Psi(\Omega'_\varepsilon)$ .

Since  $\Psi(\Omega)$  is a simply connected domain which does not fill the whole plane, by Riemann mapping theorem, see [Bel15, Theorem 8.2], there exists  $\alpha : \overline{\Psi(\Omega)} \rightarrow \overline{\Omega}$ , one-to-one, such that  $\alpha$  is holomorphic in  $\Psi(\Omega)$  and  $\alpha(\Psi(0)) = 0$ . The mapping  $L = \alpha \circ \Psi$  from  $\overline{\Omega}$  onto itself is a  $K$ -quasiconformal mapping from  $\Omega$  onto itself with  $L(0) = 0$ . Then the function  $h = \hat{v} \circ L^{-1} = \hat{v} \circ \Psi^{-1} \circ \alpha^{-1}$  is a harmonic function in  $L(\Omega'_\varepsilon)$ , as a composition of a harmonic function with a holomorphic function.

Besides, following the construction of  $h$ , recalling that  $u$  is constant on  $\partial\Omega$ , it is clear that  $h$  is constant on  $L(\partial\Omega) = \partial\Omega$ . Thus, since  $h$  is a harmonic function in  $L(\Omega'_\varepsilon)$ , it belongs to  $H^2$  of any compact subset of  $L(\Omega'_\varepsilon)$  and in a neighborhood of  $\partial\Omega$  hence  $h \in H^2_{loc}(L(\Omega'_\varepsilon))$  and  $h$  is  $H^2$  in a neighborhood of  $\partial\Omega$ .  $\square$

**Remark 2.16.** Note that we have used Riemann mapping theorem for the proof of Lemma 2.15, this is the main reason explaining why we present Theorem 1.1 in the case of a simply connected domain  $\Omega$  independently.

### 2.5.3 Image of the perforated domain by the quasiconformal mapping

We conclude this part by the analysis of the distortion of distances through the quasiconformal mapping  $L$ , which is precisely given by Mori's theorem, see [Ahl66, Chapter III, Section C]: for a  $K$ -quasiconformal mapping  $L$  of  $B(0, R)$  into itself, for all  $z_1, z_2 \in B(0, R)$ ,

$$\frac{1}{16} \left| \frac{z_1 - z_2}{R} \right|^K \leq \frac{|L(z_1) - L(z_2)|}{R} \leq 16 \left| \frac{z_1 - z_2}{R} \right|^{1/K}. \quad (2.29)$$

Based on this result, it is not difficult to prove that the balls of  $F_\varepsilon$  are not too much distorted by the map  $L$ , see the lemma afterwards.

**Lemma 2.17.** *Let  $L$  be the mapping as in Lemma 2.15.*

*There exist a positive constant  $c > 0$  (independent of  $u$  and  $V$ ) such that for every  $\varepsilon > 0$  satisfying*

$$\varepsilon^2 \|V\|_{L^\infty(\Omega)} \left( \log \left( \frac{R}{\varepsilon} \right) + \log \left( \frac{R}{r_0} \right) \right) \leq c, \quad (2.30)$$

- $L(\omega_0)$  contains  $B(0, r_0/32)$ , recall that  $\omega$  is defined in (2.4),
- the images of the disks of  $F_\varepsilon$  (recall its definition in Lemma 2.10) are contained in disks  $(D'_j)_{j \in J}$  of size  $\varepsilon' = 32\varepsilon$ , indexed by  $j \in J$ , that are  $(C_0\varepsilon/32 - 64\varepsilon)$ -separated from each other, from  $L(Z_\varepsilon^1)$ , from  $\partial\Omega (= L(\partial\Omega))$ , and from  $L(x_{\max})$ .

*Proof.* To prove that  $L(\omega_0)$  contains  $B(0, r_0/32)$  when condition (2.30) holds, we simply use the first inequality of (2.29): if  $|z_1| \geq r_0$ , then  $|L(z_1)| \geq |z_1|^K R^{-K+1}/16 \geq r_0(r_0/R)^{K-1}/16$ . Therefore, if  $(r_0/R)^{K-1} \geq 1/2$ ,  $L(\omega_0)$  contains  $B(0, r_0/32)$ . It thus remains to check that, using the estimate (2.26), the condition  $(r_0/R)^{K-1} \geq 1/2$  simply follows from the condition

$$\varepsilon^2 \|V\|_{L^\infty(\Omega)} \log \left( \frac{R}{r_0} \right) \text{ small enough.}$$

To prove that disks of  $F_\varepsilon$  are contained in disks of size  $32\varepsilon$ , it is sufficient to prove that

$$|z_a - z_b| \leq \varepsilon \Rightarrow |L(z_a) - L(z_b)| \leq 32\varepsilon.$$

For  $z_a$  and  $z_b$  in  $\Omega$  with  $|z_a - z_b| \leq \varepsilon$ , we have by the second inequality of (2.29),

$$|L(z_a) - L(z_b)| \leq 16|\varepsilon|^{1/K} R^{1-1/K},$$

which is thus bounded by  $32\varepsilon$  if  $(R/\varepsilon)^{1-1/K} \leq 2$ , that is if

$$(K-1) \log \left( \frac{R}{\varepsilon} \right) \leq K \log(2) \leq 2 \log(2).$$

if we assume that  $K \leq 2$  (which can be done without loss of generality by taking  $\varepsilon^2 \|V\|_{L^\infty(\Omega)}$  smaller if necessary). Therefore, using the estimate (2.26), the image of the disks  $(B(x_j, \varepsilon))_{j \in J}$  of  $F_\varepsilon$  are contained in disks of size  $32\varepsilon$  of the form  $(B(L(x_j), 32\varepsilon))_{j \in J}$  if

$$\varepsilon^2 \|V\|_{L^\infty(\Omega)} \log \left( \frac{R}{\varepsilon} \right) \text{ is small enough.}$$

To estimate the separation between two disks of the form  $(B(L(x_j), 32\varepsilon))_{j \in J}$ , using that the  $(x_j)_{j \in J}$  are  $C_0\varepsilon$  separated and the same argument as above, the centers  $(L(x_j))_{j \in J}$  are  $C_0\varepsilon/32$  separated provided

$$\varepsilon^2 \|V\|_{L^\infty(\Omega)} \log \left( \frac{R}{\varepsilon} \right) \text{ is small enough.}$$

The balls  $(B(L(x_j), 32\varepsilon))_{j \in J}$  are thus  $(C_0/32 - 64)\varepsilon$  separated.

Using similar arguments, we can also prove that the disks  $B(L(x_j), 32\varepsilon)_{j \in J}$  are  $(C_0\varepsilon/32 - 32\varepsilon)$  separated from  $L(Z_\varepsilon^1)$ ,  $\partial\Omega$ , and from  $L(x_{\max})$ .  $\square$

**Remark 2.18.** Note that we have used Mori's theorem in a disk to quantify precisely the distortion induced by the map  $L$ . This is the second main reason for dealing with a simply connected domain  $\Omega$  in Theorem 1.1.

Before ending this step of the proof, let us set  $\omega'_0 = B(0, r_0/32)$ ,  $r'_0 = r_0/32$ ,  $\varepsilon' = 32\varepsilon$  and remark that by construction, and recalling the choice  $C_0 = 18 \cdot 32^2$ , for which we have  $C_0/32 - 64 = 16 \cdot 32$ , the disks  $(D'_i)_{i \in I}$  given by Lemma 2.17 are  $16\varepsilon'$ -separated from each other, from  $L(Z_\varepsilon^1)$ , from  $\partial\Omega$ , and from  $L(x_{\max})$ .

Then, as a straightforward application of Corollary 2.14, Lemma 2.15 and Lemma 2.17, (2.23), we prove the following result.

**Corollary 2.19.** *Under the assumptions of Lemma 2.17, there exists a constant  $C > 0$  independent of  $\varepsilon$  such that for every disk  $D'$  as in Lemma 2.17, we have*

$$\begin{aligned} & (\forall x \in L(\Omega'_\varepsilon) \setminus D' \text{ with } \text{dist}(x, D') < 16\varepsilon', \quad h(x) \geq -C \|f\|_{L^2(\Omega)}) \\ & \text{or } (\forall x \in L(\Omega'_\varepsilon) \setminus D' \text{ with } \text{dist}(x, D') < 16\varepsilon', \quad h(x) \leq C \|f\|_{L^2(\Omega)}). \end{aligned} \quad (2.31)$$

From now on, we assume that  $\varepsilon > 0$  always satisfies (2.9), (2.30) and also

$$\varepsilon \leq r_0/32^3. \quad (2.32)$$

Note that the smallness condition (2.32) does not appear previously but it will be used in the next part.

## 2.6 Step 6: A Carleman estimate for the Laplace equation in a perforated domain

The next step is to use a Carleman estimate for the Laplace operator in the ball  $\Omega = B(0, R)$ , and to apply it to a suitably cut-off version of  $h$ , absorbing the terms arising from the cut-off using Harnack's inequality.

### 2.6.1 An elliptic Carleman estimate in a ball

To start with, we begin by stating a Carleman estimate for the Laplace operator in the ball  $\Omega = B(0, R)$ .

Let us take

$$\beta(x) = \sqrt{R^2 + 1} - \sqrt{|x|^2 + 1}, \quad (x \in B(0, R)),$$

and, for  $\lambda \geq 1$ , set

$$\alpha(x) = e^{\lambda\beta(x)}, \quad (x \in B(0, R)). \quad (2.33)$$

We recall the following classical Carleman estimate, see Appendix A.3 for a proof that mainly comes from [FI96].

**Proposition 2.20.** *There exist constants  $C > 0$ ,  $\lambda \geq 1$ , and  $s_0 \geq 1$ , depending on  $r'_0$ ,  $R$ , such that for every  $s \geq s_0$  and every  $y \in H^2(B(0, R))$ , constant on  $\partial B(0, R)$ ,  $\tilde{y} = ye^{s\alpha}$  satisfies*

$$\begin{aligned} & \int_{\Omega} |-\Delta \tilde{y} - s^2 |\nabla \alpha|^2 \tilde{y}|^2 dx + \int_{\Omega} |s \nabla \alpha \cdot \nabla \tilde{y}|^2 dx + s^3 \int_{\Omega} |\tilde{y}|^2 dx + s \int_{\Omega} |\nabla \tilde{y}|^2 dx + s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma \\ & \leq C \left( \|(\Delta y) e^{s\alpha}\|_{L^2(\Omega)}^2 + s^3 \int_{\omega'_0} |\tilde{y}|^2 dx \right). \end{aligned} \quad (2.34)$$

The first two left hand side terms of the previous Carleman estimate (2.34) are usually not mentioned although they appear naturally in the classical computations for deriving Carleman estimates. Here, as we will use them afterwards, we shall keep them.

### 2.6.2 A Carleman estimate in a perforated domain conjugated with Harnack inequality

We would like to apply the previous Carleman estimate to  $h$ , but  $h$  is defined only in  $L(\Omega'_\varepsilon)$  and is only  $H^2_{loc}$  in  $L(\Omega'_\varepsilon)$  and near the boundary  $\partial\Omega$ . We therefore introduce a smooth cut-off function  $\chi$  taking value 0 on  $B(0, 3)$  and value 1 on  $\mathbb{R}^2 \setminus B(0, 4)$ , and set

$$y(x) = \begin{cases} h(x) \prod_{j \in J} \chi\left(\frac{x - x'_j}{\varepsilon'}\right) & \text{for } x \in L(\Omega'_\varepsilon), \\ 0 & \text{for } x \in \Omega \setminus L(\Omega'_\varepsilon), \end{cases} \quad (2.35)$$

where the family  $(x'_j)_{j \in J}$  denotes the centers of the disks  $(D'_j)_{j \in J}$  introduced in Lemma 2.17.

From Lemma 2.15, we have that  $y \in H^2(\Omega)$  and is constant on  $\partial\Omega$ . We are then in position to apply the Carleman estimate (2.34) to  $y$ . Starting from there, and based on Harnack's inequality to deal with the cut-off terms, recalled in Theorem A.2, we prove the following result.

**Proposition 2.21.** *There exist  $a > 0$  and  $C > 0$  (independent of  $\varepsilon$ ,  $u$  and  $V$ ) such that for all  $s \geq s_0$  satisfying*

$$s\varepsilon' \geq a, \quad (2.36)$$

(recall that  $\varepsilon' = \varepsilon/32$ ), we have

$$\int_{\Omega} |y|^2 dx + \int_{\Omega} |\nabla y|^2 dx + \int_{\partial\Omega} |y|^2 d\sigma \leq C^2 e^{2Cs} \|h\|_{L^\infty(L(\Omega'_\varepsilon) \cap \omega'_0)}^2 + C^2 e^{2Cs} \|f\|_{L^2(\Omega)}^2. \quad (2.37)$$

**Remark 2.22.** It is worth mentioning that (2.36) gives that  $s \geq C\varepsilon^{-1}$  and  $\varepsilon' \leq \|V\|_{L^\infty(\Omega)}^{-1/2} \log^{-1/2}(\|V\|_{L^\infty(\Omega)})$  by (2.30) so  $s \geq C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})$ . Then, the quantitative observability estimate (2.37) on the perforated domain will lead to the expected observability estimate (2.6) on the whole domain with  $C_V$  of the form  $C \exp(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)}))$ . Note that the Carleman approach in [LMNN20], only leads to a weaker observability estimate (2.6) with an observability constant of the form  $C \exp(C\|V\|_{L^\infty(\Omega)}^{1/2} \log^{3/2}(\|V\|_{L^\infty(\Omega)}))$ . The new ingredient here is to use the second term in the left hand side of the Carleman estimate (2.34). Recall that this improvement in comparison with [LMNN20, Section 6.1] leads to the generalization of [LMNN20, Theorem 5.1] in the multi-dimensional case without logarithm loss.

*Proof.* In the following proof, the constants  $C > 0$  are allowed to vary from line to line but they do not depend on  $\varepsilon$  and  $s$ .

Setting  $\tilde{y} = ye^{s\alpha}$ , we get from Proposition 2.20 that

$$\int_{\Omega} |s\nabla\alpha \cdot \nabla\tilde{y}|^2 dx + s^3 \int_{\Omega} |\tilde{y}|^2 dx + s \int_{\Omega} |\nabla\tilde{y}|^2 dx + s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma \leq C \left\| \Delta \left( \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) h \right) e^{s\alpha} \right\|_{L^2(\Omega)}^2 + Cs^3 \int_{\omega'_0} |\tilde{y}|^2 dx.$$

By using Leibniz formula, the harmonicity of  $h$  in  $L^2(\Omega'_\varepsilon)$  and the fact that the gradient of the cut-off functions  $(\chi((\cdot - x'_j)/\varepsilon'))_{j \in J}$  are supported in disjoint separated sets since the balls  $D'_j$  are  $16\varepsilon'$ -separated sets, we then develop as follows

$$\begin{aligned} \Delta \left( \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) h \right) &= \Delta \left( \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) \right) h + 2\nabla \left( \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) \right) \cdot \nabla h + \left( \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) \right) \Delta h \\ &= \Delta \left( \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) \right) h + 2\nabla \left( \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) \right) \cdot \nabla h = \sum_{j \in J} \Delta \left( \chi \left( \frac{x - x'_j}{\varepsilon'} \right) \right) h + 2 \sum_{j \in J} \nabla \left( \chi \left( \frac{x - x'_j}{\varepsilon'} \right) \right) \cdot \nabla h. \end{aligned} \quad (2.38)$$

Thus, for all  $s \geq s_0$ ,

$$\begin{aligned} \int_{\Omega} |s\nabla\alpha \cdot \nabla\tilde{y}|^2 dx + s^3 \int_{\Omega} |\tilde{y}|^2 dx + s \int_{\Omega} |\nabla\tilde{y}|^2 dx + s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma &\leq Cs^3 \int_{\omega'_0} |\tilde{y}|^2 dx \\ &+ C \sum_{j \in J} \left( \frac{1}{|\varepsilon'|^4} \|he^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \|\nabla h e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 \right). \end{aligned} \quad (2.39)$$

We thus look locally around each ball  $D'_j$ , and first derive from (2.39) nice estimates on  $h$  in annulus of the form  $B(x'_j, 8\varepsilon') \setminus B(x'_j, \varepsilon')$  for  $j \in J$ . Since we will deal differently with these terms depending if  $B(x'_j, 8\varepsilon') \cap B(r'_0/2)$  is empty or not, we define  $J_1$  as the set of indexes  $j \in J$  such that  $B(x'_j, 8\varepsilon') \cap B(r'_0/2) = \emptyset$ , and we set  $J_2 = J \setminus J_1$ . We will also use the notation  $C(x'_j, \varepsilon') = B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon')$  in the sequel of the proof.

We first focus on the case  $B(x'_j, 8\varepsilon') \cap B(r'_0/2) = \emptyset$ , i.e.  $j \in J_1$ , and we point out that the weight function  $\alpha$  satisfies the following properties:  $\alpha$  is radial,  $\alpha$  is decreasing, and there exists  $c > 0$  such that  $\inf_{\Omega \setminus B(0, r'_0/2)} |\nabla\alpha| \geq c$ .

Accordingly, using that  $\tilde{y}$  vanishes on  $B(x'_j, 2\varepsilon')$ , and the properties of the weight function, when  $B(x'_j, 8\varepsilon') \cap B(r'_0/2) = \emptyset$ , setting  $x''_j = x'_j - 6\varepsilon' x'_j/|x'_j|$ , Poincaré's estimate in the one-dimensional case (integrating along a radial line) yields the existence of a constant  $C > 0$  independent of  $\varepsilon'$  such that for all  $j \in J_1$ ,

$$\int_{B(x''_j, \varepsilon')} |\tilde{y}|^2 dx \leq C(\varepsilon')^2 \int_{B(x'_j, 8\varepsilon')} |\nabla\alpha \cdot \nabla\tilde{y}|^2 dx.$$

Using then that  $\tilde{y} = he^{s\alpha}$  in  $B(x'_j, \varepsilon')$ , it follows that

$$\begin{aligned} \left(\frac{s}{\varepsilon'}\right)^2 e^{2s \inf_{B(x'_j, \varepsilon')} \alpha} |B(x'_j, \varepsilon')| \inf_{B(x'_j, \varepsilon')} |h|^2 &\leq C \left(\frac{s}{\varepsilon'}\right)^2 \int_{B(x'_j, \varepsilon')} |h|^2 e^{2s\alpha} dx \\ &\leq C \left(\frac{s}{\varepsilon'}\right)^2 \int_{B(x'_j, \varepsilon')} |\tilde{y}|^2 dx \leq C s^2 \int_{B(x'_j, 8\varepsilon')} |\nabla \alpha \cdot \nabla \tilde{y}|^2 dx. \end{aligned} \quad (2.40)$$

We now set  $C(x'_j, \varepsilon') = B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon')$ . With our choice of cut-off functions, we have, for all  $j \in J_1$ ,

$$\begin{aligned} \int_{C(x'_j, \varepsilon')} \left( \frac{1}{(\varepsilon')^4} |h|^2 + \frac{1}{(\varepsilon')^2} |\nabla h|^2 \right) e^{2s\alpha} dx \\ \leq C e^{2s \sup_{C(x'_j, \varepsilon')} \alpha} |C(x'_j, \varepsilon')| \left( \frac{1}{(\varepsilon')^4} \sup_{C(x'_j, \varepsilon')} |h|^2 + \frac{1}{(\varepsilon')^2} \sup_{C(x'_j, \varepsilon')} |\nabla h|^2 \right). \end{aligned} \quad (2.41)$$

Using (2.40), (2.41), we get from (2.39) that for all  $s \geq s_0$ ,

$$\begin{aligned} s^3 \int_{\Omega} |y|^2 e^{2s\alpha} dx + s \int_{\Omega} |\nabla y|^2 e^{2s\alpha} dx + s^3 \int_{\partial\Omega} |y|^2 e^{2s\alpha} d\sigma + \sum_{j \in J_1} \left(\frac{s}{\varepsilon'}\right)^2 e^{2s \inf_{B(x'_j, \varepsilon')} \alpha} |B(x'_j, \varepsilon')| \inf_{B(x'_j, \varepsilon')} |h|^2 \\ \leq C s^3 \int_{\omega'_0} |y|^2 e^{2s\alpha} dx + C \sum_{j \in J_1} e^{2s \sup_{C(x'_j, \varepsilon')} \alpha} |C(x'_j, \varepsilon')| \left( \frac{1}{(\varepsilon')^4} \sup_{C(x'_j, \varepsilon')} |h|^2 + \frac{1}{(\varepsilon')^2} \sup_{C(x'_j, \varepsilon')} |\nabla h|^2 \right) \\ + C \sum_{j \in J_2} \left( \frac{1}{|\varepsilon'|^4} \|he^{s\alpha}\|_{L^2(C(x'_j, \varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \|\nabla h e^{s\alpha}\|_{L^2(C(x'_j, \varepsilon'))}^2 \right). \end{aligned} \quad (2.42)$$

Now, using  $\Delta h = 0$  in the annulus  $B(x'_j, 8\varepsilon') \setminus B(x'_j, \varepsilon')$  (for any  $j \in J$  and in particular for  $j \in J_1$ ) and that  $h$  has a strict sign in this set up to a constant  $C \|f\|_{L^2(\Omega)}$  by (2.31), the Harnack's inequality, see Theorem A.2, implies the existence of a constant  $C > 0$  such that

$$\sup_{B(x'_j, 7\varepsilon') \setminus B(x'_j, 2\varepsilon')} |h| + \sup_{B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon')} \varepsilon' |\nabla h| \leq C \left( \inf_{B(x'_j, 7\varepsilon') \setminus B(x'_j, 2\varepsilon')} |h| + \|f\|_{L^2(\Omega)} \right). \quad (2.43)$$

We now remark that:

- the ratio of the measures of the sets  $B(x'_j, \varepsilon')$  and  $C(x'_j, \varepsilon') = B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon')$  is bounded from below and from above independently from  $\varepsilon'$ ;
- since the sets  $(C(x'_j, \varepsilon'))_{j \in J}$  are disjoint,

$$\sum_{j \in J_1} e^{2s \sup_{C(x'_j, \varepsilon')} \alpha} |C(x'_j, \varepsilon')| \leq |\Omega| e^{2s \|\alpha\|_{L^\infty(\Omega)}};$$

- there exists  $c > 0$  independent from  $\varepsilon'$  such that, for all  $j \in J_1$

$$\sup_{C(x'_j, \varepsilon')} \alpha \leq \inf_{B(x'_j, \varepsilon')} \alpha - c\varepsilon'.$$

Accordingly, for any constant  $C$ , we can choose  $s$  big enough to get that, for all  $j \in J_1$ ,

$$\left(\frac{s}{\varepsilon'}\right)^2 e^{2s \inf_{B(x'_j, \varepsilon')} \alpha} \geq C \left(\frac{1}{\varepsilon'}\right)^4 e^{2s \sup_{C(x'_j, \varepsilon')} \alpha},$$

which is the case if  $(s\varepsilon')^2 e^{2cs\varepsilon'} \geq C$ , that is  $s\varepsilon'$  large enough, as in (2.36). We then derive from (2.42) and (2.43) that there exists  $a > 0$  large enough so that for all  $s \geq s_0$  and  $s\varepsilon' \geq a$ ,

$$\begin{aligned} s^3 \int_{\Omega} |y|^2 e^{2s\alpha} dx + \int_{\Omega} |\nabla y|^2 e^{2s\alpha} dx + s^3 \int_{\partial\Omega} |y|^2 e^{2s\alpha} d\sigma \\ \leq C s^3 \int_{\omega'_0} |y|^2 e^{2s\alpha} dx + C e^{2s \|\alpha\|_{L^\infty}} \|f\|_{L^2(\Omega)}^2 + C \sum_{j \in J_2} \left( \frac{1}{|\varepsilon'|^4} \|he^{s\alpha}\|_{L^2(C(x'_j, \varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \|\nabla h e^{s\alpha}\|_{L^2(C(x'_j, \varepsilon'))}^2 \right). \end{aligned} \quad (2.44)$$

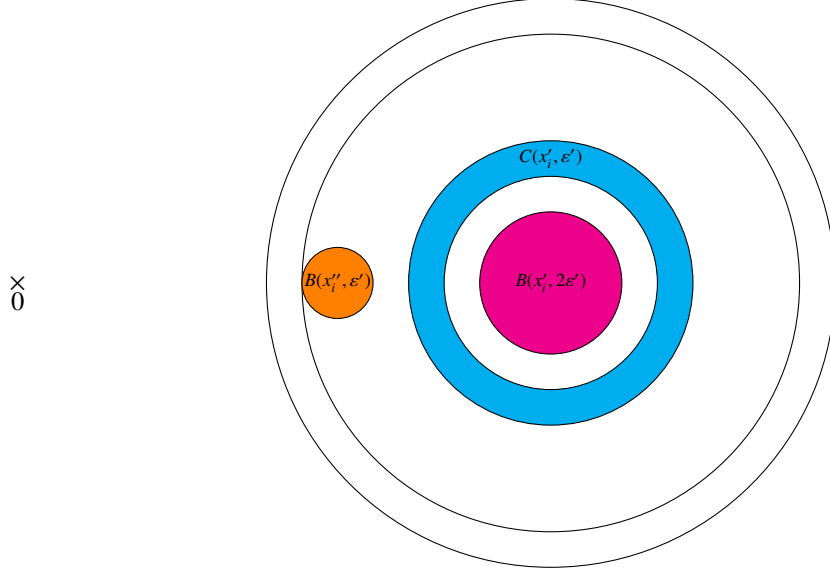


Figure 4: The neighborhood of a disk  $D'_i$  of the perforated domain.

We then focus on the case  $B(x'_j, 8\varepsilon') \cap B(r'_0/2) \neq \emptyset$ , i.e.  $j \in J_2$ . For these terms, we use some Cacciopoli's inequality. More precisely, we introduce a cut-off function  $\eta_j$  such that  $\eta_j = 1$  in  $C(x'_j, \varepsilon')$  and  $\eta_j = 0$  outside  $B(x'_j, 5\varepsilon') \setminus B(x'_j, 2\varepsilon')$ , with  $\varepsilon \|\nabla \eta_j\|_{L^\infty} + \varepsilon^2 \|\Delta \eta_j\|_{L^\infty} \leq C$ , and we multiply the equation  $\Delta h = 0$  in  $B(x'_j, 5\varepsilon') \setminus B(x'_j, 2\varepsilon')$  by  $e^{2s\alpha} \eta_j / h$ . By integrating by parts, we obtain

$$\int_{B(x'_j, 5\varepsilon') \setminus B(x'_j, 2\varepsilon')} \eta_j |\nabla h|^2 e^{2s\alpha} dx = \frac{1}{2} \int_{B(x'_j, 5\varepsilon') \setminus B(x'_j, 2\varepsilon')} |h|^2 \Delta(\eta_j e^{2s\alpha}) dx.$$

This entails

$$\|\nabla h|e^{s\alpha}\|_{L^2(C(x'_j, \varepsilon'))}^2 \leq C \left( s^2 + \frac{1}{\varepsilon'^2} \right) \int_{B(x'_j, 5\varepsilon') \setminus B(x'_j, 2\varepsilon')} |h|^2 e^{2s\alpha} dx.$$

So we get, using  $B(x'_j, 5\varepsilon') \setminus B(x'_j, 2\varepsilon') \subset B(x'_j, 8\varepsilon') \subset \omega'_0$  because  $\varepsilon$  satisfies (2.32),

$$\sum_{j \in J_2} \left( \frac{1}{|\varepsilon'|^4} \|h e^{s\alpha}\|_{L^2(C(x'_j, \varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \|\nabla h|e^{s\alpha}\|_{L^2(C(x'_j, \varepsilon'))}^2 \right) \leq C \left( \frac{s^2}{|\varepsilon'|^2} + \frac{1}{|\varepsilon'|^4} \right) \int_{L(\Omega'_\varepsilon) \cap \omega'_0} |h|^2 e^{2s\alpha} dx, \quad (2.45)$$

which is less than  $C s^4 \|h\|_{L^\infty(L(\Omega'_\varepsilon) \cap \omega'_0)}^2 e^{2s|\alpha|_\infty}$  for  $s$  satisfying (2.36).

Combining (2.44) and (2.45), and bounding the weight function  $e^{s\alpha}$  from below by 1 and from above by  $e^{s|\alpha|_{L^\infty(\Omega)}}$  leads to (2.37), hence to the conclusion of the proof of Proposition 2.21.  $\square$

## 2.7 Step 7: From the Carleman estimate to the observability inequality

We now prove (2.6) from Proposition 2.21, this will end the proof of Theorem 1.1.

*Proof of (2.6).* We choose  $\varepsilon$  small enough such that (2.9), (2.30) and (2.32) hold, and then  $s \geq a/(32\varepsilon)$  so that (2.36) holds. These choices are done by choosing  $1/\varepsilon$  and  $s$  of the order  $\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})$ .

We will distinguish two cases, depending if  $x'_{\max} := L(x_{\max})$ , where  $x_{\max}$  is defined in (2.14) and  $L$  is the quasiconformal map given by Lemma 2.15, is close to the boundary  $\partial\Omega$  or not.

Let us first consider the case  $d(x'_{\max}, \partial\Omega) \geq \varepsilon$ . Based on Lemma 2.17, we necessarily have  $B(x'_{\max}, \varepsilon) \subset L(\Omega'_\varepsilon)$ . Accordingly by the mean value property applied to the harmonic function  $h$ , we have

$$|h(x'_{\max})| \leq \pi^{-1} \varepsilon^{-2} \int_{B(x'_{\max}, \varepsilon)} |h(z)| dz \leq \pi^{-1/2} \varepsilon^{-1} \left( \int_{B(x'_{\max}, \varepsilon)} |h(z)|^2 dz \right)^{1/2}.$$

Then using the fact that  $y = h$  on  $B(x'_{\max}, \varepsilon)$  by (2.35) (recall that the balls  $(B(x'_j, \varepsilon'))_{j \in J}$  are  $16\varepsilon'$  separated from  $x'_{\max}$  by Lemma 2.17), Proposition 2.21 gives

$$|h(x'_{\max})| \leq C e^{C_s} \|h\|_{L^\infty(L(\Omega'_\varepsilon) \cap \omega'_0)} + C e^{C_s} \|f\|_{L^2(\Omega)}.$$

We now come back to the variable  $\hat{v} = h \circ L$  thanks to Lemma 2.15, this leads to

$$|\hat{v}(x_{\max})| \leq C e^{C_s} \|\hat{v}\|_{L^\infty(L^{-1}(L(\Omega'_\varepsilon) \cap \omega'_0))} + C e^{C_s} \|f\|_{L^2(\Omega)}.$$

Then, using again Lemma 2.17 that tells us that  $L^{-1}(L(\Omega'_\varepsilon) \cap \omega'_0) \subset \omega_0$ , we deduce

$$|\hat{v}(x_{\max})| \leq C e^{C_s} \|\hat{v}\|_{L^\infty(\omega_0)} + C e^{C_s} \|f\|_{L^2(\Omega)}.$$

By using the definition of  $\hat{v} := v - \psi$  and Lemma 2.13, the bound (2.23), we deduce the following estimate on  $v$

$$|v(x_{\max})| \leq C e^{C_s} \|v\|_{L^\infty(\omega_0)} + C e^{C_s} \|f\|_{L^2(\Omega)}.$$

By Lemma 2.12, we recall that  $u = v\varphi$  and using the pointwise estimates on  $\varphi$  from (2.17) of Lemma 2.11, we deduce

$$\|u\|_{L^\infty(\Omega)} \leq C e^{C_s} \|u\|_{L^\infty(\omega_0)} + C e^{C_s} \|f\|_{L^2(\Omega)}.$$

Recalling that  $s$  is of the order  $\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})$ , the previous estimate coincides with (2.6).

We next consider the case  $d(x'_{\max}, \partial\Omega) \leq \varepsilon$ . Recalling Lemma 2.17, the disks  $(D'_j)_{j \in J}$  are  $16\varepsilon'$ -separated from  $\partial\Omega$ . In particular, using the definition of  $y$  in (2.35), we have  $y \equiv h$  in  $V_{2\varepsilon'} = \{x \in \Omega; \text{dist}(x, \partial\Omega) \leq 2\varepsilon'\}$ . We then deduce from (2.37) that

$$\int_{V_{2\varepsilon'}} |h|^2 dx + \int_{V_{2\varepsilon'}} |\nabla h|^2 dx + \int_{\partial\Omega} |h|^2 d\sigma \leq C^2 e^{2C_s} \|h\|_{L^\infty(L(\Omega'_\varepsilon) \cap \omega'_0)}^2 + C^2 e^{2C_s} \|f\|_{L^2(\Omega)}^2. \quad (2.46)$$

Now we recall that  $h$  is equal to some constant  $\delta$  on  $\partial\Omega$  by Lemma 2.15, so (2.46) immediately gives

$$\int_{V_{2\varepsilon'}} |h|^2 dx + \int_{V_{2\varepsilon'}} |\nabla h|^2 dx + \delta^2 \leq C^2 e^{2C_s} \|h\|_{L^\infty(L(\Omega'_\varepsilon) \cap \omega'_0)}^2 + C^2 e^{2C_s} \|f\|_{L^2(\Omega)}^2. \quad (2.47)$$

Choosing now a cut-off function  $\eta$  such that  $\eta \equiv 1$  in  $\mathbb{R}^2 \setminus B(0, R(1 - \varepsilon'))$  and  $\eta \equiv 0$  in  $\overline{B(0, R(1 - 2\varepsilon'))}$  with  $\varepsilon \|\nabla \eta\|_{L^\infty} + \varepsilon^2 \|\Delta \eta\|_{L^\infty} \leq C$  for some constant  $C$  independent of  $\varepsilon$ , we observe that  $\tilde{h} = \eta h$  satisfies

$$\begin{cases} -\Delta \tilde{h} = -2\nabla \eta \cdot \nabla h - (\Delta \eta)h & \text{in } V_{2\varepsilon'}, \\ \tilde{h} = \delta & \text{on } \partial\Omega, \\ \tilde{h} = 0 & \text{on } \partial B(0, R(1 - 2\varepsilon')). \end{cases}$$

The maximum principle, see for instance [GT83, Section 3.3, Theorem 3.7], leads to

$$\|\tilde{h}\|_{L^\infty(V_{2\varepsilon'})} \leq C \left( |\delta| + \|-2\nabla \eta \cdot \nabla h - (\Delta \eta)h\|_{L^2(V_{2\varepsilon'})} \right).$$

The constant  $C$  in the above estimate does not depend on  $\varepsilon$  (this can be proved as in Lemma A.1 for the distributed term). Using now the property of  $\eta$  and the fact that  $d(x'_{\max}, \partial\Omega) \leq \varepsilon (\leq \varepsilon')$ , we obtain that

$$|h(x'_{\max})| \leq C \left( |\delta| + \varepsilon^{-1} \|\nabla h\|_{L^2(V_{2\varepsilon'})} + \varepsilon^{-2} \|h\|_{L^2(V_{2\varepsilon'})} \right).$$

With (2.47) and the fact that  $s$  and  $1/\varepsilon$  are of the same order  $\|V\|_{L^\infty(\Omega)}^{1/2} \log^{1/2}(\|V\|_{L^\infty(\Omega)})$ , we obtain

$$|h(x'_{\max})| \leq C e^{C_s} \|h\|_{L^\infty(L(\Omega'_\varepsilon) \cap \omega'_0)} + C e^{C_s} \|f\|_{L^2(\Omega)}.$$

We now conclude exactly as in the first case.  $\square$

**Remark 2.23.** It is clear that the constant  $C = C(\Omega, \omega)$  in the statement of Theorem 1.1 depends on the smooth simply connected bounded domain  $\Omega$  and the non-empty open subset  $\omega$  of  $\Omega$ . In the above proof, this dependence appears only within the first step, as we reduce ourselves to the case of concentric balls. In particular, the constant  $C(\Omega, \omega)$  can be estimated from the knowledge of a conformal mapping  $\Phi : \overline{\Omega} \rightarrow \overline{B(0, 1)}$  such that  $0 \in \Phi(\omega)$ , a parameter  $r \in (0, 1)$  such that  $B(0, r) \subset \Phi(\omega)$ , and the constant  $C(r)$  defined as the constant given by Theorem 1.1 corresponding to the case  $\Omega^* = B(0, 1)$  and  $\omega^* = B(0, r)$ .

### 3 Extension to multiply connected domains

In this part, we present a strategy to extend Theorem 1.1 to the case of multiply connected domains having a finite number of holes.

#### 3.1 The case of a finite number of holes each one circled by $\omega$

We first begin with the case of one hole circled by  $\omega$ .

**Theorem 3.1.** *Let  $\Omega$  be a smooth bounded doubly connected domain of  $\mathbb{R}^2$ , assume that  $\omega$  is a non-empty open subset of  $\Omega$  which circles the hole (that is,  $\omega$  contains a Jordan curve  $\gamma$  such that the hole of  $\Omega$  is in the interior of  $\gamma$ ).*

*Then there exists a constant  $C = C(\Omega, \omega) > 0$  such that for every  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $V \in L^\infty(\Omega; \mathbb{R})$ , estimate (1.6) holds.*

*Proof.* Since  $\Omega$  is a doubly connected domain, i.e. a connected domain with a single hole called  $\mathcal{H}$ , which, without loss of generality, can be assumed to contain a neighborhood of 0, such that  $\omega$  circles  $\mathcal{H}$ , i.e. containing a Jordan curve  $\gamma$  such that  $\mathcal{H}$  is in the interior  $\mathcal{O}_\gamma$  of  $\gamma$ , we can construct a tubular neighborhood  $\omega_0$  of  $\gamma$  such that  $\omega_0 \subset\subset \omega$ ,  $\omega_0$  circles  $\mathcal{H}$ , and a smooth cut-off function  $\eta$  such that  $\eta = 1$  in  $\mathcal{O}_\gamma \setminus \omega_0$ , and  $\eta$  vanishes outside  $\mathcal{O}_\gamma$ .

Let  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , and write  $u = \eta u + (1 - \eta)u$ . We will analyze each of the functions in the right hand-side of this identity and show that they both satisfy the estimate (1.6).

Setting  $\Omega_e = \Omega \cup \mathcal{O}_\gamma$ , we easily check that the function  $u_e = (1 - \eta)u$ , extended by 0 in  $\mathcal{H}$ , is well-defined on the simply connected domain  $\Omega_e$ , and satisfies  $u_e \in H^2(\Omega_e) \cap H_0^1(\Omega_e)$  and  $-\Delta u_e + V u_e = [\Delta, \eta]u + (1 - \eta)(-\Delta u + V u)$  in  $\Omega_e$ , where  $V$  has been extended by 0 in  $\mathcal{H}$ . We can then apply Theorem 1.1 to  $u_e$ , and we get

$$\|u_e\|_{L^2(\Omega)} \leq \|u_e\|_{L^2(\Omega_e)} \leq C_V \left( \|-\Delta u_e + V u_e\|_{L^2(\Omega_e)} + \|u_e\|_{L^2(\omega_0)} \right) \leq C_V \left( \|-\Delta u + V u\|_{L^2(\Omega)} + \|u\|_{H^1(\omega_0)} \right),$$

with  $C_V$  of the form (1.7).

Since  $\omega_0 \subset\subset \omega$ , we then simply use basic local elliptic regularity to get

$$\|u\|_{H^1(\omega_0)} \leq C(1 + \|V\|_\infty) \left( \|-\Delta u + V u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right),$$

which yields, for a constant  $C_V$  still of the form (1.7):

$$\|u_e\|_{L^2(\Omega)} \leq C_V \left( \|-\Delta u + V u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right). \quad (3.1)$$

Estimating  $u_i = \eta u$  is slightly more delicate. The first point is to check that  $u$  can be extended to all the exterior of  $\Omega$  by setting  $u_i = 0$  outside  $\Omega$ . Thus,  $u_i$  now belongs to  $H^2(\Omega_i) \cap H_0^1(\Omega_i)$  where  $\Omega_i$  denotes the exterior domain  $\Omega_i = \mathbb{R}^2 \setminus \mathcal{H}$ ; besides, since  $\Omega$  is bounded,  $u_i$  is compactly supported in some ball  $|x| \leq R$ . We then set  $V_i = 1_\Omega V$  in  $\Omega_i$ , and we check that  $-\Delta u_i + V_i u_i = 1_\Omega \eta(-\Delta u + V u) - [\Delta, \eta]u$  in  $\Omega_i$ . Of course  $\Omega_i$  is not simply connected, so we cannot use directly Theorem 1.1. However, if we define  $\Phi : z \mapsto 1/z$  from  $\mathbb{C}^*$  to  $\mathbb{C}^*$ ,  $\tilde{\Omega}_i = \Phi(\Omega_i)$  is simply connected and bounded,  $\tilde{u}_i = u_i \circ \Phi$  satisfies  $-\Delta \tilde{u}_i + \tilde{V}_i \tilde{u}_i = |\nabla \mathfrak{R}(\Phi)|^2 (-\Delta u_i(\Phi) + V_i(\Phi) u_i(\Phi))$  in  $\tilde{\Omega}_i$ , where  $\tilde{V}_i = V_i \circ \Phi$ , and  $\tilde{u}_i \in H^2(\tilde{\Omega}_i) \cap H_0^1(\tilde{\Omega}_i)$  since  $u_i$  vanishes in a neighborhood of infinity and thus  $\tilde{u}_i$  vanishes in a neighborhood of 0. Therefore, we can apply Theorem 1.1 to  $\tilde{u}_i$  in  $\tilde{\Omega}_i$  with observation in  $\tilde{\omega}_0 = \Phi(\omega_0)$ :

$$\|\tilde{u}_i\|_{L^2(\tilde{\Omega}_i)} \leq C \exp(C \|\tilde{V}_i\|_{L^\infty(\tilde{\Omega}_i)}^{1/2} \log^{1/2}(\|\tilde{V}_i\|_{L^\infty(\tilde{\Omega}_i)})) \left( \|-\Delta \tilde{u}_i + \tilde{V}_i \tilde{u}_i\|_{L^2(\tilde{\Omega}_i)} + \|\tilde{u}_i\|_{L^2(\tilde{\omega}_0)} \right).$$

Since  $\tilde{u}_i$  vanishes in a neighborhood of 0, and  $\tilde{V}_i$  also vanishes in a neighborhood of 0, we easily deduce, for some constant  $C_V$  of the form (1.7),

$$\|u_i\|_{L^2(\Omega_i)} \leq C_V \left( \|-\Delta u_i + V_i u_i\|_{L^2(\Omega_i)} + \|u_i\|_{L^2(\omega_0)} \right).$$

Due to the form of  $u_i$  and of  $-\Delta u_i + V_i u_i$ , similarly as for  $u_e$ , we easily deduce, for some constant  $C_V$  of the form (1.7) that

$$\|u_i\|_{L^2(\Omega)} \leq C_V \left( \|-\Delta u + V u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right). \quad (3.2)$$

Combining estimates (3.1)–(3.2), we immediately get estimate (1.6).  $\square$

In fact, an easy adaptation of the above proof left to the reader (which can be done by recurrence on the number of holes for instance) gives the following result:

**Theorem 3.2.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  with a finite number  $N \in \mathbb{N}$  of holes  $(\mathcal{H}_i)_{i \in \{1, \dots, N\}}$ , and assume that  $\omega$  is a non-empty open subset of  $\Omega$  such that for each hole  $\mathcal{H}_i$  of  $\Omega$  ( $i \in \{1, \dots, N\}$ ),  $\omega$  contains a Jordan curve  $\gamma_i$  such that the interior of  $\gamma_i$  contains  $\mathcal{H}_i$  and no other holes of  $\Omega$ .*

*Then there exists a constant  $C = C(\Omega, \omega) > 0$  such that for every  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $V \in L^\infty(\Omega; \mathbb{R})$ , estimate (1.6) holds.*

### 3.2 A three-spheres type inequality

The goal of this section is to prove the following result:

**Theorem 3.3.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  and  $r > 0$ .*

*Then there exists a constant  $\delta \in (0, 1)$  such that for all  $x \in \Omega$  with  $B(x, 2^{10}r) \subset \Omega$ , for every  $u \in H^2(\Omega)$  and real-valued potentials  $V \in L^\infty(\Omega; \mathbb{R})$ ,*

$$\|u\|_{L^2(B(x, 2r))} \leq C_V \left( \|u\|_{L^2(B(x, r))} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^\delta \left( \|u\|_{L^2(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta}, \quad (3.3)$$

and

$$\|u\|_{L^\infty(B(x, 2r))} \leq C_V \left( \|u\|_{L^\infty(B(x, r))} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^\delta \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta}, \quad (3.4)$$

where  $C_V$  is of the form (1.7).

**Remark 3.4.** By following the proof of Theorem 3.3, one can track the dependence of  $\delta$ . This constant only depends on  $r$  but its precise value is unknown. For a more precise discussion, one can also see [ARRV09, Theorem 1.10 and Section 1.8, Remark (4)].

*Proof.* Without loss of generality, we can assume that

$$x = 0 \quad \text{and} \quad \Omega = B(0, 2^{10}r).$$

The proof of Theorem 3.3 follows the one of Theorem 1.1, and we are thus going to point out how it should be modified here, some very similar details being left to the reader.

**Step 1.** Mimicking Step 2 of the proof of Theorem 1.1, we check that, for (3.3), it is sufficient to prove for  $r_0 = r/2$ ,  $\omega_0 = B(0, r_0)$ ,  $V \in L^\infty(\Omega; \mathbb{R})$  and  $u \in H^2(\Omega)$ ,

$$\|u\|_{L^\infty(B(0, 2r))} \leq C_V \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^\delta \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta}. \quad (3.5)$$

Note that (3.5) would also imply (3.4) immediately, and we thus focus on the proof of (3.5) from now on.

In fact, we will prove that there exist positive constants  $A, B, C$  and  $s_0$  such that for  $V \in L^\infty(\Omega; \mathbb{R})$ , for all  $s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))$ , and  $u \in H^2(\Omega)$ ,

$$\|u\|_{L^\infty(B(0, 2r))} \leq C e^{As} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right) + C e^{-Bs} \|u\|_{L^\infty(\Omega)}. \quad (3.6)$$

Indeed, optimizing in  $s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))$ , we easily find that, if

$$\frac{1}{(A+B)} \log \left( \frac{\|u\|_{L^\infty(\Omega)}}{\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^\infty(\omega_0)}} \right) \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty)),$$

we obtain

$$\begin{aligned} \inf_{s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))} \left\{ C e^{As} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right) + C e^{-Bs} \|u\|_{L^\infty(\Omega)} \right\} \\ = C \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^\delta \|u\|_{L^\infty(\Omega)}^{1-\delta}, \quad \text{with } \delta = \frac{B}{A+B}, \end{aligned}$$

which implies (3.5).

If

$$\frac{1}{(A+B)} \log \left( \frac{\|u\|_{L^\infty(\Omega)}}{\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^\infty(\omega_0)}} \right) \leq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty)),$$

then

$$\inf_{s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))} \left\{ C e^{As} \left( \|u\|_{L^\infty(\omega_0)}^2 + \|-\Delta u + Vu\|_{L^2(\Omega)}^2 \right) + C e^{-Bs} \|u\|_{L^\infty(\Omega)} \right\} \\ \leq C e^{As_0} e^{As_0 \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty)} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right),$$

which of course also implies (3.5).

**Step 2.** Similarly as in Step 3 of the proof of Theorem 1.1, we can also check that it is sufficient to check (3.6) for functions  $u \in C^\infty(\hat{\Omega}; \mathbb{R})$ , where  $\hat{\Omega} = B(0, 2^{11}r)$ , which are equal to a non-zero constant on  $\partial\hat{\Omega}$ , and for which 0 is a regular value.

From now on, we consider  $u \in C^\infty(\bar{\Omega}; \mathbb{R})$  which is equal to some non-zero constant on  $\partial\hat{\Omega}$ , and for which 0 is a regular value, and we set

$$f := -\Delta u + Vu \quad \text{in } \Omega.$$

**Step 3.** We then consider the nodal set  $Z$  of  $u$ , i.e.  $Z = \{x \in \hat{\Omega} ; u = 0\}$ , which, as in Step 4 of the proof of Theorem 1.1, consists in the union of disjoint smooth Jordan curves indexed by  $J$ .

For  $\varepsilon > 0$  to be fixed later, we introduce the property (P- $\varepsilon$ ) and partition  $Z$  and  $J$  as in (2.7)–(2.8) under the form  $Z_\varepsilon^1 \cup Z_\varepsilon^2$ . We have the following variant of Proposition 2.6, whose proof is left to the reader as it is similar to the one of Proposition 2.6:

**Proposition 3.5.** *There exist two constants  $C > 0$  and  $c > 0$  (independent of  $u$  and  $V$ ) such that for every  $\varepsilon \in (0, r)$  satisfying (2.9), the following holds:*

- For each connected component  $C$  of  $Z_\varepsilon^2$  such that  $O_C \subset \Omega$ , estimate (2.10) holds.
- For each connected component  $O$  of  $B(0, 2^9r) \setminus Z_\varepsilon^1$ , (2.11) holds.

Note that, compared to Proposition 2.6, the difference is that the level set  $\{u = 0\}$  may cross  $\Omega = B(0, 2^{10}r)$ . Therefore, the first property holds only for connected component  $C$  of  $Z_\varepsilon^2$  such that  $O_C \subset \Omega$ , which guarantees that the source term  $f = -\Delta u + Vu$  is known in  $O_C$ . Furthermore, we cannot deduce the second property for connected components of  $B(0, 2^{10}r) \setminus Z_\varepsilon^1$ , and we should slightly reduce the ball we are looking at, by considering for instance the connected components  $O$  of  $B(0, 2^9r) \setminus Z_\varepsilon^1$  as above (in fact it would work too for the connected components of  $B(0, 2^{10}r - 2\varepsilon) \setminus Z_\varepsilon^1$ ).

We then construct a perforation process similar to the one of Lemma 2.10 with  $x_{\max}$  chosen such that

$$|u(x_{\max})| = \sup_{B(0, 2r)} |u|.$$

More precisely, we use the following version of Lemma 2.10, which is in fact used in [LMNN20, Section 3]:

**Lemma 3.6.** *For all  $C_0 \geq 5$ , for every  $\varepsilon > 0$ , there exist  $C_0\varepsilon$ -separated closed disks of radius  $\varepsilon$ , whose union is denoted by  $F_\varepsilon$ , satisfying the following properties:*

- these disks are  $C_0\varepsilon$ -separated from each other, from  $Z_\varepsilon^1$ , from  $\partial\hat{\Omega}$ , from  $x_{\max}$  and from 0,
- the set  $Z_\varepsilon^1 \cup F_\varepsilon$  is a  $6C_0\varepsilon$ -net in  $\hat{\Omega}$ , meaning that for all  $x \in \hat{\Omega}$ ,  $B(x, 6C_0\varepsilon) \cap (Z_\varepsilon^1 \cup F_\varepsilon \cup \partial\hat{\Omega}) \neq \emptyset$ .
- the set

$$\hat{\Omega}_\varepsilon := \hat{\Omega} \setminus (Z_\varepsilon^1 \cup F_\varepsilon)$$

satisfies  $C_P(\hat{\Omega}_\varepsilon) \leq C\varepsilon$  for some constant  $C > 0$  depending on  $C_0$  but independent of  $\varepsilon$ , ( $u$  and  $V$ ).

As in the proof of Theorem 1.1, we then set  $C_0 = 18 \cdot 32^2$ .

According to the last item of Lemma 3.6, for  $\varepsilon$  small enough as in (2.9), the following variant of Lemma 2.11 applies (again, its proof is left to the reader as it is based on the same ingredients as the one of Lemma 2.11):

**Lemma 3.7.** *There exist two positive constants  $C > 0$  and  $c > 0$  (independent of  $u$  and  $V$ ) such that for every  $\varepsilon > 0$ , satisfying (2.9), there exists a solution  $\varphi$  of*

$$\begin{cases} -\Delta\varphi + 1_\Omega V\varphi = 0 & \text{in } \hat{\Omega}_\varepsilon, \\ \varphi = 1 & \text{on } \partial\hat{\Omega}_\varepsilon, \end{cases}$$

and  $\tilde{\varphi} = \varphi - 1$  satisfies (2.17).

**Step 4.** Mimicking Step 5 of the proof of Theorem 1.1, we introduce

$$v = \frac{u}{\varphi} \quad \text{in the set} \quad \Omega'_\varepsilon = B(0, 2^9 r) \setminus F_\varepsilon,$$

similarly as in (2.19), and  $v$  satisfies (2.21). We now introduce  $\psi$  solution of

$$\begin{cases} \nabla \cdot (\varphi^2 \nabla \psi) = f \varphi 1_\Omega & \text{in } \hat{\Omega}'_\varepsilon = \hat{\Omega} \setminus F_\varepsilon \\ \psi = 0 & \text{on } \partial \hat{\Omega}'_\varepsilon. \end{cases}$$

Using the same argument as in Lemma 2.13, there exists a positive constant  $C > 0$  (independent of  $\varepsilon$ ) such that

$$\|\psi\|_{L^\infty(\hat{\Omega}'_\varepsilon)} \leq C \|f\|_{L^2(\Omega)}. \quad (3.7)$$

We then set  $\hat{v} = v - \psi$  in  $\Omega'_\varepsilon$ , and Corollary 2.14 applies:  $\hat{v}$  satisfies  $\nabla \cdot (\varphi^2 \nabla \hat{v}) = 0$  in  $\Omega'_\varepsilon$  and we get the following property: there exists a positive constant  $C > 0$  such that for every disk  $D \subset F_\varepsilon$  intersecting  $B(0, 2^9 r)$ ,

$$\begin{aligned} (\forall x \in \Omega'_\varepsilon \setminus D, \text{ with } \text{dist}(x, D) < C_0 \varepsilon, \hat{v}(x) \geq -C \|f\|_{L^2(\Omega)}) \text{ or} \\ (\forall x \in \Omega'_\varepsilon \setminus D, \text{ with } \text{dist}(x, D) < C_0 \varepsilon, \hat{v}(x) \leq C \|f\|_{L^2(\Omega)}), \end{aligned}$$

where  $F_\varepsilon$  is the collection of disks used in the construction of the perforated domain (based on Lemma 3.6).

We can then construct a quasiconformal mapping  $L$  of  $B(0, 2^9 r)$  into itself satisfying the properties of Lemma 2.15, i.e. such that

$$h = \hat{v} \circ L^{-1},$$

defined in  $L(\Omega'_\varepsilon)$ , is harmonic in  $L(\Omega'_\varepsilon)$  and thus belongs to  $H_{loc}^2(L(\Omega'_\varepsilon))$ .

Besides, provided  $\varepsilon > 0$  satisfies (2.30), the distortion estimates of Lemma 2.17 apply. In particular,  $L(B(0, r_0))$  contains  $\omega'_0 = B(0, r'_0)$  with  $r'_0 = r_0/32 (= r/64)$ , and the images of the disks of  $F_\varepsilon$  that intersect  $B(0, 2^9 r)$  are contained in disks  $(D'_j)_{j \in J}$  of size  $\varepsilon' = 32\varepsilon$ , indexed by  $j \in J$  (which can intersect the boundary of  $B(0, 2^9 r)$ ), that are  $16\varepsilon'$ -separated from each other, from  $L(Z_\varepsilon^1 \cap \overline{B(0, 2^9 r)})$ , and from  $L(x_{\max})$ . Similarly, one can prove, under the condition (2.30), that

$$L(B(0, 4r)) \subset B(0, 2^7 r).$$

Estimate (2.31) also applies and yields the existence of a constant  $C > 0$  independent of  $\varepsilon$  such that for every disk  $D'$  as in Lemma 2.17,

$$\begin{aligned} (\forall x \in L(\Omega'_\varepsilon) \setminus D' \text{ with } \text{dist}(x, D') < 16\varepsilon', \quad h(x) \geq -C \|f\|_{L^2(\Omega)}) \\ \text{or } (\forall x \in L(\Omega'_\varepsilon) \setminus D' \text{ with } \text{dist}(x, D') < 16\varepsilon', \quad h(x) \leq C \|f\|_{L^2(\Omega)}). \end{aligned}$$

From now on,  $\varepsilon$  is always assumed to satisfy estimates (2.9), (2.30) and also (2.32).

**Step 5.** Following Step 6 of the proof of Theorem 1.1, we apply the Carleman estimate (2.39) (with  $\Omega = B(0, 2^9 r)$ ) to  $y$  defined by

$$y(x) = \begin{cases} \eta(x) h(x) \prod_{j \in J} \chi\left(\frac{x - x'_j}{\varepsilon'}\right) & \text{for } x \in L(\Omega'_\varepsilon), \\ 0 & \text{for } x \in B(0, 2^9 r) \setminus L(\Omega'_\varepsilon), \end{cases}$$

where the family  $(x'_j)_{j \in J}$  denotes the centers of the disks  $(D'_j)_{j \in J}$  introduced above,  $\chi$  denotes a smooth cut-off function  $\chi$  taking value 0 on  $B(0, 3)$  and value 1 on  $\mathbb{R}^2 \setminus B(0, 4)$ , and  $\eta$  is a smooth cut-off radial function taking value 1 in  $B(0, 2^8 r + 16\varepsilon')$ , vanishing identically outside  $B(0, 2^9 r - 16\varepsilon')$ .

For this to be possible, from now on, we need to check that  $2^9 r - 16\varepsilon' > 2^8 r + 16\varepsilon'$ . This last condition is guaranteed by (2.32).

Our goal is to prove the following result.

**Proposition 3.8.** *For  $\varepsilon$  satisfying (2.9), (2.30), (2.32) and  $s$  as in (2.36) for  $a > 0$  large enough (independent of  $\varepsilon$ ,  $u$  and  $V$ ), the function  $h$  of the previous step satisfies:*

$$s^3 \int_{L(B(0, 2^7 r)) \setminus (\cup_{j \in J} B(x'_j, 4\varepsilon'))} e^{2s\alpha} |h|^2 dx \leq C s^4 e^{2s\|\alpha\|_{L^\infty(B(0, 2^9 r), B(0, 2^8 r))}} \|h\|_{L^2(L(\Omega'_\varepsilon))}^2 + C s^4 \int_{\omega'_0} |h|^2 e^{2s\alpha} dx. \quad (3.8)$$

*Proof.* Setting  $\tilde{y} = ye^{s\alpha}$  with  $\alpha$  as in (2.33), we get by the Carleman estimate (2.39)

$$\begin{aligned} & \int_{B(0,2^9r)} |s\nabla\alpha \cdot \nabla\tilde{y}|^2 dx + s^3 \int_{B(0,2^9r)} |\tilde{y}|^2 dx + s \int_{B(0,2^9r)} |\nabla\tilde{y}|^2 dx \\ & \leq C \left\| \left( \Delta \left( \eta \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} h \right) \right) \right) e^{s\alpha} \right\|_{L^2(B(0,2^9r))}^2 + Cs^3 \int_{\omega'_0} |\tilde{y}|^2 dx. \quad (3.9) \end{aligned}$$

Using similar computations as in (2.38), we then bound the first term in the right-hand side of (3.9) as follows:

$$\begin{aligned} & \left\| \left( \Delta \left( \eta \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} h \right) \right) \right) e^{s\alpha} \right\|_{L^2(B(0,2^9r))}^2 \\ & \leq C \sum_{j \in J} \left( \frac{1}{|\varepsilon'|^4} \|\eta h e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \left( \|\eta |\nabla h| e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 + \|\nabla \eta\| h e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 \right) \right) \\ & + C \|\Delta \eta\| h \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) e^{s\alpha}\|_{L^2(B(0,2^9r))}^2 + C \|\nabla \eta\| |\nabla h| \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) e^{s\alpha}\|_{L^2(B(0,2^9r))}^2. \quad (3.10) \end{aligned}$$

The first term in the right hand side can be handled as in the proof of Proposition 2.21, depending where the ball  $B(x'_j, 8\varepsilon')$  is located. Namely, we distinguish the three following cases:

- $B(x'_j, 8\varepsilon') \cap B(0, r'_0/2) = \emptyset$  and  $B(x'_j, 8\varepsilon') \cap B(0, 2^8r) \neq \emptyset$ , and we call  $J_1$  the set of such  $j \in J$ ;
- $B(x'_j, 8\varepsilon') \cap B(0, r'_0/2) \neq \emptyset$  and we call  $J_2$  the set of such  $j \in J$ ;
- $B(x'_j, 8\varepsilon') \cap B(0, 2^8r) = \emptyset$ , and we call  $J_3$  the set of such  $j \in J$ .

We then discuss below how to handle the right hand side of (3.10) in each case:

- For  $j \in J_1$ , i.e. for  $j$  such that  $B(x'_j, 8\varepsilon') \cap B(0, r'_0/2) = \emptyset$  and  $B(x'_j, 8\varepsilon') \cap B(0, 2^8r) \neq \emptyset$ , using that  $\eta = 1$  in  $B(x'_j, 8\varepsilon')$  in this case, we can argue as in Proposition 2.21 and absorb the terms

$$\sum_{j \in J_1} \left( \frac{1}{|\varepsilon'|^4} \|\eta h e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \|\eta |\nabla h| e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 \right)$$

by the left hand side of (3.9) for  $s$  such that  $s\varepsilon'$  is large enough.

- For  $j \in J_2$ , i.e. for  $j$  such that  $B(x'_j, 8\varepsilon') \cap B(0, r'_0/2) \neq \emptyset$  since we also have  $\eta = 1$  in  $B(x'_j, 8\varepsilon')$  in this case, similarly as in (2.45), using Cacciopoli's inequality we obtain

$$\begin{aligned} & \sum_{j \in J_2} \left( \frac{1}{|\varepsilon'|^4} \|\eta h e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \|\eta |\nabla h| e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 \right) \\ & \leq C \left( \frac{s^2}{|\varepsilon'|^2} + \frac{1}{|\varepsilon'|^4} \right) \int_{L(\Omega'_\varepsilon) \cap \omega'_0} |h|^2 e^{2s\alpha} dx. \quad (3.11) \end{aligned}$$

- For  $j \in J_3$ , i.e. for  $j$  such that  $B(x'_j, 8\varepsilon') \cap B(0, 2^8r) = \emptyset$ , using again Cacciopoli's inequality, it is straightforward to get

$$\begin{aligned} & \sum_{j \in J_3} \left( \frac{1}{|\varepsilon'|^4} \|\eta h e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 + \frac{1}{|\varepsilon'|^2} \left( \|\eta |\nabla h| e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 + \|\nabla \eta\| h e^{s\alpha}\|_{L^2(B(x'_j, 4\varepsilon') \setminus B(x'_j, 3\varepsilon'))}^2 \right) \right) \\ & \leq C \left( \frac{s^2}{|\varepsilon'|^2} + \frac{1}{|\varepsilon'|^4} \right) e^{2s\|\alpha\|_{L^\infty(B(0,2^9r) \setminus B(0,2^8r))}} \|h\|_{L^2(L(\Omega'_\varepsilon))}^2. \quad (3.12) \end{aligned}$$

To estimate the last terms in the right hand side of (3.10), we simply use that  $\nabla\eta$  and  $\Delta\eta$  are localized in  $B(0, 2^9 r) \setminus B(0, 2^8 r)$ :

$$\begin{aligned} & \| \Delta\eta h \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) e^{s\alpha} \|_{L^2(B(0, 2^9 r))}^2 + \| |\nabla\eta| |\nabla h| \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) e^{s\alpha} \|_{L^2(B(0, 2^9 r))}^2 \\ & \leq C e^{2s\|\alpha\|_{L^\infty(B(0, 2^9 r) \setminus B(0, 2^8 r))}} \|h\|_{H^1(\Omega \setminus \cup_{j \in J} B(x'_j, 3\varepsilon'))}^2. \end{aligned}$$

Using again Cacciopoli's inequality,

$$\begin{aligned} & \| \Delta\eta h \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) e^{s\alpha} \|_{L^2(B(0, 2^9 r))}^2 + \| |\nabla\eta| |\nabla h| \prod_{j \in J} \chi \left( \frac{x - x'_j}{\varepsilon'} \right) e^{s\alpha} \|_{L^2(B(0, 2^9 r))}^2 \\ & \leq C \left( s^2 + \frac{1}{|\varepsilon'|^2} \right) e^{2s\|\alpha\|_{L^\infty(B(0, 2^9 r) \setminus B(0, 2^8 r))}} \|h\|_{L^2(L(\Omega'_\varepsilon))}^2. \quad (3.13) \end{aligned}$$

Accordingly, we deduce from (3.9), (3.10), (3.11), (3.12) and (3.13) that there exists  $a > 0$  such that for all  $s \geq 1$  satisfying (2.36), with  $\varepsilon = a/s$ ,

$$\begin{aligned} & \int_{B(0, 2^9 r)} |s \nabla \alpha \cdot \nabla \tilde{y}|^2 dx + s^3 \int_{B(0, 2^9 r)} |\tilde{y}|^2 dx + s \int_{B(0, 2^9 r)} |\nabla \tilde{y}|^2 dx \\ & \leq C s^4 e^{2s\|\alpha\|_{L^\infty(B(0, 2^9 r) \setminus B(0, 2^8 r))}} \|h\|_{L^2(L(\Omega'_\varepsilon))}^2 + C s^4 \int_{\omega'_0} |h|^2 e^{2s\alpha} dx. \quad (3.14) \end{aligned}$$

The estimate (3.8), i.e. Proposition 3.8 easily follows from (3.14).  $\square$

**Step 6.** To conclude (3.6), we follow the step 7 of the proof of Theorem 1.1. We first choose  $\varepsilon$  satisfying estimates (2.9), (2.30), (2.32), i.e.  $\varepsilon = \varepsilon_0 / (1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))$  for some  $\varepsilon_0 \in (0, 1)$  small enough independent of  $u$  and  $V$ , and  $s$  is any number larger than  $a/(32\varepsilon)$ , with  $a > 0$  as in Proposition 3.8. All these choices lead to  $s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))$ , for some  $s_0 \geq 1$  independent of  $u$  and  $V$ .

We use that  $B(L(x_{\max}, \varepsilon')) \subset L(B(0, 2^7 r)) \setminus (\cup_{j \in J} B(x'_j, 4\varepsilon'))$ , so that, by the mean value property,

$$|h(L(x_{\max}))|^2 \leq \frac{C}{\varepsilon'^2} \int_{B(L(x_{\max}, \varepsilon'))} |h(z)|^2 dz.$$

Accordingly, coming from Proposition 3.8, for  $s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))$ , we get

$$\begin{aligned} e^{2s \inf_{B(0, 2^7 r)} \alpha} |h(L(x_{\max}))|^2 & \leq \frac{C}{\varepsilon'^2} \int_{B(L(x_{\max}, \varepsilon'))} |h(z)|^2 e^{2s\alpha} dz \leq \frac{C}{\varepsilon'^2} \int_{L(B(0, 2^7 r)) \setminus (\cup_{j \in J} B(x'_j, 4\varepsilon'))} |h(z)|^2 e^{2s\alpha} dz \\ & \leq C s^6 e^{2s\|\alpha\|_{L^\infty(B(0, 2^9 r) \setminus B(0, 2^8 r))}} \|h\|_{L^2(L(\Omega'_\varepsilon))}^2 + C s^6 e^{2s\|\alpha\|_{L^\infty(B(0, 2^9 r))}} \int_{\omega'_0} |h|^2 dx. \end{aligned}$$

Using now  $\hat{v} = h \circ L$ , the bound (3.7) on  $\psi$ , and the bounds on  $\varphi$ , we deduce that for all  $s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))$ ,

$$\|u\|_{L^\infty(B(0, 2r))}^2 \leq C s^6 e^{-2sB_0} \|u\|_{L^\infty(\Omega)}^2 + C s^6 e^{2sA_0} \left( \|u\|_{L^\infty(\omega_0)}^2 + \|f\|_{L^2(\Omega)}^2 \right), \quad (3.15)$$

where

$$A_0 = \|\alpha\|_{L^\infty(B(0, 2^9 r))} - \inf_{B(0, 2^7 r)} \alpha, \quad \text{and} \quad B_0 = \inf_{B(0, 2^7 r)} \alpha - \|\alpha\|_{L^\infty(B(0, 2^9 r) \setminus B(0, 2^8 r))}.$$

Since  $A_0 > 0$  and  $B_0 > 0$ , choosing  $A > A_0$  and  $B \in (0, B_0)$ , we obtain (3.6) from (3.15) for any  $s \geq s_0(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty))$ .  $\square$

### 3.3 Smooth bounded domains with a finite number of holes: Proof of Theorem 1.2

*Proof of Theorem 1.2.* Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  having a finite number of holes  $(\mathcal{H}_i)_{i \in \{1, \dots, N\}}$ , and  $\omega$  be a non-empty open subset of  $\Omega$ .

For all  $i \in \{1, \dots, N\}$ , there exists a smooth Jordan curve  $\gamma_i \subset \Omega$  such that  $\gamma_i \cap \omega \neq \emptyset$ , and the interior of  $\gamma_i$  contains  $\mathcal{H}_i$  and no other holes of  $\Omega$ . Since  $\gamma_i$  is compact, there exists  $R_i > 0$  such that the tubular neighborhood  $\gamma_{i,R_i} = \{x, d(x, \gamma_i) < R_i\} \subset \Omega$  and there exists  $x_i \in \gamma_i$  such that  $B(x_i, R_i) \subset \omega$ .

We then set  $r = 2^{-10} \inf_{i \in \{1, \dots, N\}} R_i$ ,  $\omega_{0,i} = B(x_i, r)$  for  $i \in \{1, \dots, N\}$ ,  $\omega_0 = \cup_{i \in \{1, \dots, N\}} \omega_{0,i}$ , and for  $k \in \mathbb{N}$ , we set

$$\omega_{k+1,i} = \omega_{k,i} \cup \{B(x, 2r) \text{ with } x \in \gamma_i \text{ such that } B(x, r) \subset \omega_{k,i}\}, \quad \text{and} \quad \omega_{k+1} = \cup_{i \in \{1, \dots, N\}} \omega_{k+1,i}.$$

Theorem 3.3 then gives us the existence of a positive constant  $\delta \in (0, 1)$  such that (3.4) holds for all  $x \in \Omega$  such that  $B(x, 2^{10}r) \subset \Omega$ , which can be easily transformed into: for all  $u \in H^2 \cap H_0^1(\Omega)$  and  $V \in L^\infty(\Omega; \mathbb{R})$ , if  $B(x, 2^{10}r) \subset \Omega$ , we get

$$\begin{aligned} \|u\|_{L^\infty(B(x, 2r))} + \|-\Delta u + Vu\|_{L^2(\Omega)} \\ \leq C_V \left( \|u\|_{L^\infty(B(x, r))} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^\delta \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta}, \end{aligned} \quad (3.16)$$

with  $C_V$  as in (1.7).

We claim that for all  $k \in \mathbb{N}$ ,

$$\|u\|_{L^\infty(\omega_k)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \leq C_V^{\delta^k} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{\delta^k} \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta^k}.$$

Indeed, this can be proved by induction: it is straightforward when  $k = 0$ ; to show the induction property, it suffices to check that, applying (3.16), we get

$$\begin{aligned} \|u\|_{L^\infty(\omega_{k+1})} + \|-\Delta u + Vu\|_{L^2(\Omega)} &\leq C_V \left( \|u\|_{L^\infty(\omega_k)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^\delta \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta} \\ &\leq C_V^{\delta^{k+1}} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{\delta^{k+1}} \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta^{k+1}}. \end{aligned}$$

Since all the curves  $(\gamma_i)_{i \in \{1, \dots, N\}}$  are compact, there exists  $K \in \mathbb{N}$ , depending only on  $(\gamma_i)_{i \in \{1, \dots, N\}}$ ,  $\Omega$ , and  $r$ , such that for all  $k \geq K$ ,  $\omega_k = \omega_K = \cup_{i \in \{1, \dots, N\}} \gamma_{i, 2r}$ .

By construction, this set  $\omega_K$  satisfies the assumption of Theorem 3.2, and accordingly, for  $C_V$  of the form (1.7), for every  $u \in H^2(\Omega) \times H_0^1(\Omega)$  and  $V \in L^\infty(\Omega; \mathbb{R})$ ,

$$\|u\|_{L^2(\Omega)} \leq C_V \left( \|u\|_{L^2(\omega_K)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right).$$

Accordingly, using  $bd \leq b^p/p + d^q/q$  for  $b, d \geq 0$  and  $p, q \in [1, \infty]$  with  $1/p + 1/q = 1$ , for every  $u \in H^2(\Omega) \times H_0^1(\Omega)$ ,  $V \in L^\infty(\Omega; \mathbb{R})$ , and  $a > 0$

$$\begin{aligned} \|u\|_{L^2(\Omega)} &\leq C_V^{\delta^K+1} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{\delta^K} \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right)^{1-\delta^K} \\ &\leq \delta^K a C_V^{1+\delta^{-K}} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right) + (1-\delta^K) \frac{1}{a^{\delta^K/(1-\delta^K)}} \left( \|u\|_{L^\infty(\Omega)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right). \end{aligned}$$

Next, there exists  $C > 0$  independent of  $V$  such that for  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ ,

$$\|u\|_{L^\infty(\Omega)} \leq C \|-\Delta u\|_{L^2(\Omega)} \leq C \|-\Delta u + Vu\|_{L^2(\Omega)} + C \|V\|_\infty \|u\|_{L^2(\Omega)}.$$

Choosing then  $a > 0$  such that

$$\frac{C \|V\|_\infty}{a^{\delta^K/(1-\delta^K)}} \leq \frac{1}{2},$$

for instance with  $a = (2 + 2C \|V\|_\infty)^{1/\delta^K}$  as a power of  $\|V\|_\infty$ , we get

$$\|u\|_{L^2(\Omega)} \leq 2\delta^K (2 + 2C \|V\|_\infty)^{1/\delta^K} C_V^{1+\delta^{-K}} \left( \|u\|_{L^\infty(\omega_0)} + \|-\Delta u + Vu\|_{L^2(\Omega)} \right) + C \|-\Delta u + Vu\|_{L^2(\Omega)}.$$

Of course, this yields estimate (1.6) with  $C_V$  of the form (1.7).  $\square$

## 4 Control of elliptic equations

### 4.1 Linear elliptic equations

The goal of this part is to prove Theorem 1.3. First, from Theorem 1.2, we deduce the following stronger observability inequality.

**Corollary 4.1.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^2$  having a finite number of holes, and  $\omega$  be a non-empty open subset of  $\Omega$ .*

*For every  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $V \in L^\infty(\Omega; \mathbb{R})$ , we have*

$$\|u\|_{H^2(\Omega)} \leq C_V \left( \|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right), \quad (4.1)$$

where  $C_V$  is as in (1.7).

*Proof.* Take  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ . From elliptic regularity in  $L^2$  for the Dirichlet-Laplacian, we have for some constant  $C > 0$ ,  $\|u\|_{H^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)}$ . We then deduce

$$\|u\|_{H^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} \leq C(1 + \|V\|_\infty) \left( \|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right),$$

hence applying Theorem 1.2, estimate (1.6) yields (4.1).  $\square$

From Corollary 4.1 and a duality argument, we now move to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* In the following proof,  $C_V$  are positive constants that can vary from line to line but are always of the form (1.7).

Take  $F \in L^2(\Omega)$ , and introduce the functional

$$J(w) = \int_{\Omega} |-\Delta w + Vw|^2 dx + \int_{\omega} |w|^2 dx - 2 \int_{\Omega} Fw dx, \quad (w \in H^2(\Omega) \cap H_0^1(\Omega)).$$

By the observability inequality (4.1) from Corollary 4.1, we deduce that for all  $w \in H^2(\Omega) \cap H_0^1(\Omega)$ ,

$$\begin{aligned} J(w) &\geq \int_{\Omega} |-\Delta w + Vw|^2 dx + \int_{\omega} |w|^2 dx - \frac{1}{2C_V^2} \int_{\Omega} |w|^2 dx - 2C_V^2 \|F\|_{L^2(\Omega)}^2 \\ &\geq \frac{1}{2} \left( \int_{\Omega} |-\Delta w + Vw|^2 dx + \int_{\omega} |w|^2 dx \right) - 2C_V^2 \|F\|_{L^2(\Omega)}^2 \end{aligned} \quad (4.2)$$

$$\geq \frac{1}{2C_V^2} \|w\|_{H^2(\Omega)}^2 - 2C_V^2 \|F\|_{L^2(\Omega)}^2. \quad (4.3)$$

So the functional  $J$  is coercive on  $H^2(\Omega) \cap H_0^1(\Omega)$ . Since  $J$  is obviously of class  $C^1$  and strictly convex on  $H^2(\Omega) \cap H_0^1(\Omega)$ ,  $J$  admits a unique minimizer at  $w^* \in H^2(\Omega) \cap H_0^1(\Omega)$ , satisfying the Euler-Lagrange equation

$$\int_{\Omega} (-\Delta w^* + Vw^*)(-\Delta w + Vw) dx + \int_{\omega} w^* w dx = \int_{\Omega} Fw dx, \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega). \quad (4.4)$$

We set

$$y := -\Delta w^* + Vw^*, \quad (x \in \Omega), \quad h := -w^* 1_{\omega}, \quad (x \in \omega).$$

From (4.4), we then deduce that

$$\forall w \in H^2(\Omega) \cap H_0^1(\Omega), \quad \int_{\Omega} y(-\Delta w + Vw) dx = \int_{\Omega} Fw dx + \int_{\omega} hw,$$

that is to say  $y \in L^2(\Omega)$  is a very weak solution to (1.13) in the sense of [QS07, Definition 3.1].

Using (4.2) and the fact that  $J(w^*) \leq J(0)$ , we get

$$\|y\|_{L^2(\Omega)} + \|h\|_{L^2(\omega)} \leq C_V \|F\|_{L^2(\Omega)}. \quad (4.5)$$

Now, let us define  $\tilde{y} \in H_0^1(\Omega)$  as the unique weak solution to  $-\Delta \tilde{y} = -Vy + F + h1_{\omega} \in L^2(\Omega)$ . In particular,  $y$  and  $\tilde{y}$  are two very weak solutions to  $-\Delta Y = -Vy + F + h1_{\omega}$  in  $\Omega$  with homogeneous Dirichlet boundary data. By

using [QS07, Appendix C, Theorem 49.1], we deduce that  $y = \tilde{y}$ . Hence,  $y \in H_0^1(\Omega)$  satisfies (1.13). The estimate (1.14) comes from (4.5) and a standard elliptic regularity result, since the source term  $-Vy + F + h1_\omega$  belongs to  $L^2(\Omega)$ .

It remains to study the sequential continuity of the application  $V \in L^\infty(\Omega) \mapsto (y, h) \in H^2(\Omega) \times L^2(\omega)$  for a given  $F \in L^2(\Omega)$ , when  $H^2(\Omega) \times L^2(\omega)$  is endowed with its weak topology. To make it easier to follow, given  $V \in L^\infty(\Omega)$ , we denote by  $(y_V, h_V)$  the controlled trajectory and the control constructed by the above process, and by  $w_V^*$  the minimizer of the functional  $J_V$  on  $H^2(\Omega) \cap H_0^1(\Omega)$ .

Let  $V \in L^\infty(\Omega)$  and  $(V_k)_{k \in \mathbb{N}}$  be a sequence in  $L^\infty(\Omega)$  strongly converging to  $V$ . Then the sequence  $(V_k)_{k \in \mathbb{N}}$  is uniformly bounded in  $L^\infty(\Omega)$ . According to (4.3), we deduce that the sequence  $(w_{V_k}^*)_{k \in \mathbb{N}}$  is uniformly bounded in  $H^2(\Omega) \cap H_0^1(\Omega)$ , so up to a subsequence,  $(w_{V_k}^*)_{k \in \mathbb{N}}$  weakly converges to some  $w_\infty$  in  $H^2(\Omega) \cap H_0^1(\Omega)$ . Accordingly, the controlled solutions  $(y_{V_{k_n}} = -\Delta w_{V_{k_n}}^* + V_{k_n} w_{V_{k_n}}^*)_{n \in \mathbb{N}}$  and the corresponding controls  $(h_{V_{k_n}} = -w_{V_{k_n}}^* 1_\omega)_{n \in \mathbb{N}}$  weakly converge to  $y_\infty = -\Delta w_\infty + V w_\infty$ , respectively  $h_\infty = -w_\infty 1_\omega$ , in  $L^2(\Omega)$ , respectively  $L^2(\omega)$ . Besides, according to the bound (1.14), the family  $(y_{V_{k_n}})_{n \in \mathbb{N}}$  is uniformly bounded in  $H^2(\Omega) \cap H_0^1(\Omega)$ , so it weakly converges to  $y_\infty$  in  $H^2(\Omega) \cap H_0^1(\Omega)$ . These convergences allow to pass to the limit in the equation  $-\Delta y_{V_{k_n}} + V_{k_n} y_{V_{k_n}} = F + h_{V_{k_n}} 1_\omega$  in  $\Omega$ , and to obtain that  $y_\infty \in H^2(\Omega) \cap H_0^1(\Omega)$  is such that

$$-\Delta y_\infty + V y_\infty = F + h_\infty \quad \text{in } \Omega.$$

It remains to check that  $(y_V, h_V)$  in fact coincides with  $(y_\infty, h_\infty)$ . In order to check this point, we multiply the above equation by  $w \in H^2(\Omega) \cap H_0^1(\Omega)$ , and recalling  $y_\infty = -\Delta w_\infty + V w_\infty$  and  $h_\infty = -w_\infty 1_\omega$ , we find that for all  $w \in H^2(\Omega) \cap H_0^1(\Omega)$ ,

$$\int_{\Omega} (-\Delta w_\infty + V w_\infty)(-\Delta w + V w) \, dx + \int_{\omega} w_\infty w \, dx = \int_{\Omega} F w \, dx.$$

This implies that  $w_\infty$  is a critical point for  $J_V$ . But  $J_V$  is strictly convex and therefore has only one critical point; we conclude that necessarily  $w_\infty = w_V$ , and the controlled solution  $y_\infty$  and the control  $h_\infty$  necessarily coincide with  $y_V$  and  $h_V$ . Therefore, the sequences  $(y_{V_k})_{k \in \mathbb{N}}$  and  $(h_{V_k})_{k \in \mathbb{N}}$  have only one weak accumulation point, so they converge globally to  $y_V$ , respectively  $h_V$ , for the weak topology of  $H^2(\Omega)$ , respectively  $L^2(\omega)$ .  $\square$

**Remark 4.2.** One can in fact prove the strong continuity of the application  $V \in L^\infty(\Omega) \mapsto (y_V, h_V) \in H^2(\Omega) \times L^2(\omega)$ , based on the fact that the controlled solution  $y_V$  and the control  $h_V$  in fact minimizes the norm of  $\|y\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\omega)}^2$  among all functions  $(y, h) \in H_0^1(\Omega) \times L^2(\omega)$  such that  $-\Delta y + Vy = F + h1_\omega$  in  $\Omega$ . In particular, based on this fact, it is not difficult to check that if  $(V_k)_{k \in \mathbb{N}}$  converges to  $V$  in  $L^\infty(\Omega)$ , the sequence  $(\|y_{V_k}\|_{L^2(\Omega)}^2 + \|h_{V_k}\|_{L^2(\omega)}^2)_{k \in \mathbb{N}}$  converges to  $\|y_V\|_{L^2(\Omega)}^2 + \|h_V\|_{L^2(\omega)}^2$  so that the sequence  $(y_{V_k}, h_{V_k})_{k \in \mathbb{N}}$  strongly converges in  $L^2(\Omega) \times L^2(\omega)$  to  $(y_V, h_V)$ . The strong convergence of the sequence  $(y_{V_k})_{k \in \mathbb{N}}$  to  $y_V$  in  $H^2(\Omega)$  then easily follows from the equation. We do not provide any further detail as we will not use this property in the following.

## 4.2 Semi-linear elliptic equations

### 4.2.1 Proof of the positive result of Theorem 1.5

The goal of this part is to prove the positive result of Theorem 1.5. The proof will be crucially inspired by the fixed-point argument developed in [FCZ00, Theorem 1.2] and the control result in the linear case, i.e. Theorem 1.3.

As we have said, we are going to prove this result under the weaker assumption (1.16) on the semi-linearity  $f$ .

*Proof of the positive result of Theorem 1.5.* We introduce the function  $K(s) = |s|^{1/2} \log^{1/2}(1 + |s|)$ ,  $\forall s \in \mathbb{R}$ . Note that  $K(\|V\|_{L^\infty(\Omega)})$  behaves as  $\log(C_V)$  in the limit  $\|V\|_{L^\infty(\Omega)} \rightarrow \infty$ , where  $C_V$  is defined in (1.7).

Let us define  $g \in C(\mathbb{R}; \mathbb{R})$  as follows  $g(s) = f(s)/s$ ,  $s \neq 0$  and  $g(0) = f'(0)$ . First we remark that from the assumption (1.16) on the semi-linearity  $f$ , we have that for every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that

$$K(g(s)) \leq \varepsilon \log(2 + |s|) + C_\varepsilon, \quad \forall s \in \mathbb{R}. \quad (4.6)$$

Let  $F \in L^2(\Omega)$  and let us define

$$B_R := \left\{ z \in L^\infty(\Omega) ; \|z\|_{L^\infty(\Omega)} \leq R \right\},$$

for  $R$  sufficiently large which will be defined later.

For each  $z \in B_R$ , we consider the linear system

$$\begin{cases} -\Delta y + g(z)y = F + h1_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.7)$$

According to Theorem 1.3, there exists a  $(y_{g(z)}, h_{g(z)}) \in [H^2 \cap H_0^1(\Omega)] \times L^2(\omega)$  such that  $y_{g(z)}$  satisfies the elliptic equation (4.7) and

$$\|y_{g(z)}\|_{H^2(\Omega)} + \|h_{g(z)}\|_{L^2(\omega)} \leq \exp\left(C\left(1 + K\left(\|g(z)\|_{L^\infty(\Omega)}\right)\right)\right) \|F\|_{L^2(\Omega)}. \quad (4.8)$$

We then introduce the map  $\mathcal{T}_R : B_R \rightarrow H^2(\Omega) \cap H_0^1(\Omega)$ , which to  $z \in B_R$  associates the solution  $y_{g(z)} \in H^2(\Omega) \cap H_0^1(\Omega)$  given by Theorem 1.3.

We claim that the map  $\mathcal{T}_R$  satisfies the assumptions of Schauder's fixed point theorem, provided  $R$  is chosen large enough, as we will check next.

First,  $B_R$  is a closed convex set for the topology of  $L^\infty(\Omega)$ .

Second, from (4.8), using the embedding of  $H^2(\Omega)$  into  $L^\infty(\Omega)$ , we have that for all  $z \in B_R$ ,

$$\begin{aligned} \|y_{g(z)}\|_{L^\infty(\Omega)} &\leq C \|y_{g(z)}\|_{H^2(\Omega)} \leq C \exp\left(C\left(1 + K\left(\|g(z)\|_{L^\infty(\Omega)}\right)\right)\right) \|F\|_{L^2(\Omega)} \\ &\leq C e^{C(1+\varepsilon \log(2+R)+C_\varepsilon)} \|F\|_{L^2(\Omega)} \quad (\text{by (4.6)}) \\ &\leq C e^{C_\varepsilon} (2+R)^{\varepsilon C} \|F\|_{L^2(\Omega)}. \end{aligned} \quad (4.9)$$

Therefore, by taking  $\varepsilon$  sufficiently small such that  $\varepsilon C = 1/2$ , we deduce from (4.9) that  $\mathcal{T}_R$  maps  $B_R$  into itself for  $R$  sufficiently large. From now on, we fix  $R$  such that  $\mathcal{T}_R$  maps  $B_R$  into itself.

We also have from (4.9) that  $\mathcal{T}_R(B_R)$  is included in a ball of  $H^2(\Omega)$ . Therefore, by Rellich's compact embedding theorem,  $\mathcal{T}_R(B_R)$  is a compact subset of  $B_R$ .

It remains to check that  $\mathcal{T}_R$  is continuous on  $B_R$  for the  $L^\infty(\Omega)$  topology. Let  $(z_k)_{k \in \mathbb{N}}$  be a convergent sequence of  $B_R$  in  $L^\infty(\Omega)$ , and denote its limit by  $z$ . Then, since  $g$  is continuous in  $\mathbb{R}$  (recall that  $f$  is  $C^1$  in  $\mathbb{R}$ ),  $g$  is uniformly continuous on  $[-R, R]$ , so that  $(g(z_k))_{k \in \mathbb{N}}$  strongly converges to  $g(z)$  in  $L^\infty(\Omega)$ . According to Theorem 1.3, we deduce that  $(y_{g(z_k)})_{k \in \mathbb{N}}$  weakly converges to  $y_{g(z)}$  in  $H^2(\Omega)$ . Using again Rellich's compact embedding theorem, the sequence  $(y_{g(z_k)})$  strongly converges in  $L^\infty(\Omega)$ . The map  $\mathcal{T}_R$  is thus continuous on  $B_R$  for the topology of  $L^\infty(\Omega)$ .

We can then apply Schauder's fixed point theorem to  $\mathcal{T}_R$ . There exists a fixed point  $y \in B_R$  of  $\mathcal{T}_R$ , and this fixed point  $y$  belongs to  $H^2(\Omega) \cap H_0^1(\Omega)$ , and satisfies, for some  $h = h_{g(y)}$ , the equation

$$\begin{cases} -\Delta y + g(y)y = F + h1_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Recalling that  $g(y)y = f(y)$  by construction, we have solved the semi-linear control problem (1.15).  $\square$

#### 4.2.2 Proof of the negative result of Theorem 1.5

The goal of this part is to prove the negative result of Theorem 1.5. The proof is very similar to the one of [FCZ00, Theorem 1.1].

*Proof.* Below we set  $p > 2$  and we define the function  $f$  by  $f(s) = \int_0^{|s|} \log^p(1 + \sigma) d\sigma$  for  $s \in \mathbb{R}$ .

Let us introduce a non-negative function  $\rho \in C_c^\infty(\Omega)$  such that  $\rho = 0$  in  $\omega$  and  $\int_\Omega \rho dx = 1$ . Assume that  $h \in L^2(\omega)$  together with  $y \in H^2(\Omega) \cap H_0^1(\Omega)$  satisfy (1.15). We multiply (1.15) by  $\rho$  and integrate over  $\Omega$ , we obtain that

$$-\int_\Omega \Delta y \rho dx + \int_\Omega f(y) \rho dx = \int_\Omega F \rho dx.$$

We integrate by parts and use the parity of  $f$ , we find

$$-\int_\Omega y \Delta \rho dx + \int_\Omega f(|y|) \rho dx = \int_\Omega F \rho dx. \quad (4.10)$$

Taking into account that  $f$  is convex, we can introduce its conjugate  $f^*$ . From Young's inequality, we have

$$\left| \int_\Omega y \Delta \rho dx \right| \leq \int_\Omega \rho \left| \frac{\Delta \rho}{\rho} \right| |y| dx \leq \frac{1}{2} \int_\Omega \rho f^*(2\Delta \rho / \rho) dx + \frac{1}{2} \int_\Omega f(|y|) \rho dx.$$

We then deduce from (4.10) that

$$\int_{\Omega} F\rho \, dx \geq \frac{1}{2} \int_{\Omega} f(|y|)\rho \, dx - \frac{1}{2} \int_{\Omega} \rho f^*(2\Delta\rho/\rho) \, dx \geq -\frac{1}{2} \int_{\Omega} \rho f^*(2\Delta\rho/\rho) \, dx. \quad (4.11)$$

If we can construct  $\rho$  such that

$$\rho f^*(2\Delta\rho/\rho) \in L^1(\Omega), \quad (4.12)$$

from (4.11), we would obtain that for some positive constant  $C > 0$ ,

$$\int_{\Omega} F\rho \, dx \geq -C,$$

and taking  $F = -2C$  would give a contradiction.

Let us then show that there exists  $\rho \in C_c^\infty(\Omega)$  such that  $\rho = 0$  in  $\omega$ ,  $\int_{\Omega} \rho \, dx = 1$  and (4.12) holds.

We first claim that

$$f^*(s) \sim p|s|^{1-1/p} \exp(|s|^{1/p}), \quad |s| \rightarrow +\infty. \quad (4.13)$$

Indeed, we have by definition that  $f^*(s) = \sup_{a \in \mathbb{R}} (as - f(a))$ , and this supremum is achieved at a critical point  $a$  such that  $s - f'(a) = 0$ , i.e.  $a = (f')^{-1}(s)$ . Thus,  $f^*(s) = s(f')^{-1}(s) - f((f')^{-1}(s))$ . Moreover, from the definition of  $f$ , we have  $f'(s) = \text{sign}(s) \log^p(1 + |s|)$ , so we deduce that

$$f^*(s) = s(\exp(s^{1/p}) - 1) - \int_0^{\exp(s^{1/p})-1} \log^p(1 + \sigma) d\sigma. \quad (4.14)$$

It is then straightforward to deduce (4.13) from (4.14).

Choose  $x_0 \in \Omega$  and  $r > 0$  such that  $B(x_0, r)$  is contained in  $\Omega \setminus \bar{\omega}$ , and then take for  $m > 0$ ,

$$\rho(x) = \begin{cases} c \exp(-(r - |x - x_0|)^{-m}), & \text{for } |x - x_0| \leq r, \\ 0, & \text{for } |x - x_0| > r. \end{cases}$$

Easy computations lead to

$$\frac{|\Delta\rho|}{\rho} \underset{|x-x_0| \rightarrow r^-}{\sim} m^2(r - |x - x_0|)^{-(2m+2)}.$$

So

$$f^*\left(\frac{|\Delta\rho|}{\rho}\right) \underset{|x-x_0| \rightarrow r^-}{\sim} pm^{2(1-1/p)}(r - |x - x_0|)^{-(2m+2)(p-1)/p} \exp(m^{2/p}(r - |x - x_0|)^{-(2m+2)/p}).$$

By using  $p > 2$ , the integrability condition (4.12) is satisfied for  $m > (2m + 2)/p$ , equivalently  $m > 2/(p - 2)$ . Therefore, with  $m > 2/(p - 2)$ , we have constructed  $\rho \in C_c^\infty(\Omega)$  such that  $\rho = 0$  in  $\omega$ ,  $\int_{\Omega} \rho \, dx = 1$  and (4.12) holds. According to the above mentioned argument, this implies the existence of source terms  $F \in L^2(\Omega)$  such that for every  $h \in L^2(\omega)$ , the elliptic equation (1.15) has no solution  $y \in H^2(\Omega) \cap H_0^1(\Omega)$ .  $\square$

**Remark 4.3.** When there is no control, it is not difficult to check that, as soon as the semi-linear term  $f$  satisfies  $f(s) = f(-s)$ ,  $f(s) \geq 0$  for all  $s \geq 0$ ,  $f(0) = 0$ , and  $\lim_{|s| \rightarrow \infty} (f(s)/s) = +\infty$ , that there exist source terms  $F \in L^2(\Omega)$  such that there is no solution  $y \in H^2(\Omega) \cap H_0^1(\Omega)$  of  $-\Delta y + f(y) = F$  in  $\Omega$ .

Indeed, choose  $\rho$  as the eigenvector of the operator  $-\Delta$  with homogeneous Dirichlet boundary conditions on  $\partial\Omega$  which corresponds to the first eigenvalue of the Dirichlet Laplacian (suitably normalized to get  $\int_{\Omega} \rho(x) \, dx = 1$ ). Then we know that  $\rho \geq 0$  in  $\Omega$ ,  $-\Delta\rho = \lambda_1\rho$  in  $\Omega$ , and  $\rho$  vanishes only on  $\partial\Omega$ . Then doing as above, we can obtain (4.10) and (4.11). But  $\rho f^*(2\Delta\rho/\rho) = \rho f^*(2\lambda_1)$  clearly is in  $L^2(\Omega)$  since  $f^*(2\lambda_1)$  is finite under our assumptions. Accordingly, we can conclude as before that if  $F$  satisfies

$$\int_{\Omega} F\rho \, dx < -\frac{1}{2}f^*(2\lambda_1),$$

there is no solution  $y \in H^2(\Omega) \cap H_0^1(\Omega)$  of  $-\Delta y + f(y) = F$  in  $\Omega$ .

## 5 Complementary results

### 5.1 Observability inequality in the one-dimensional case

The goal of this part is to show that we can obtain the expected observability estimate (1.4) in the one-dimensional case, in fact even when  $V$  is complex-valued.

**Theorem 5.1.** *Let  $\Omega$  be an open interval of  $\mathbb{R}$ ,  $\omega$  be a non-empty open subset of  $\Omega$  and  $V \in L^\infty(\Omega; \mathbb{C})$ .*

*Then there exists a constant  $C = C(\Omega, \omega) > 0$  such that for every  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ ,*

$$\|u\|_{L^2(\Omega)} \leq C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2}\right) \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)}\right). \quad (5.1)$$

*Proof.* Without loss of generality, we assume that  $\omega$  strictly contains an interval of the form  $[a, b]$  for some  $a > 0$ , and the interval  $\Omega$  is of the form  $(A, B)$  with  $A < a < b < B$ .

We then choose a function  $\alpha$  in  $C^2([A, B]; \mathbb{R}_+)$  such that

$$\alpha(A) = \alpha(B) = 0, \quad \alpha'(x) = 1 \text{ for } x \in [A, a], \quad \alpha'(x) = -1 \text{ for } x \in [b, B].$$

Now, for  $u \in H^2(A, B) \cap H_0^1(A, B)$  and  $s \geq 1$ , we set  $v = e^{s\alpha}u$  in  $(A, B)$ . Then  $v$  satisfies:

$$\begin{cases} (\partial_x - s\partial_x\alpha)^2 v = e^{s\alpha}\partial_{xx}u, & \text{in } (A, B), \\ v(A) = v(B) = 0. \end{cases}$$

In particular, introducing  $w = (\partial_x - s\partial_x\alpha)v$ , we have

$$\begin{cases} (\partial_x - s\partial_x\alpha)v = w, & \text{in } (A, B), \\ (\partial_x - s\partial_x\alpha)w = e^{s\alpha}\partial_{xx}u, & \text{in } (A, B), \\ v(A) = v(B) = 0. \end{cases} \quad (5.2)$$

Taking the  $L^2(A, B)$  norm of each side of the second equation of (5.2), we get

$$\begin{aligned} \|e^{s\alpha}\partial_{xx}u\|_{L^2(A, B)}^2 &= \int_A^B \left(|\partial_x w|^2 + s^2(\partial_x\alpha)^2|w|^2 + s\partial_{xx}\alpha|w|^2\right) dx - s\partial_x\alpha(B)|w(B)|^2 + s\partial_x\alpha(A)|w(A)|^2 \\ &\geq \int_A^B \left(|\partial_x w|^2 + s^2|w|^2\right) dx - Cs^2 \int_a^b |w|^2 dx, \end{aligned}$$

for  $C$  independent of  $s \geq 1$ .

Similarly, taking the  $L^2(A, B)$  norm of each side of the first equation of (5.2), we obtain

$$\|w\|_{L^2(A, B)}^2 \geq \int_A^B \left(|\partial_x v|^2 + s^2|v|^2\right) dx - Cs^2 \int_a^b |v|^2 dx,$$

again with  $C$  independent of  $s \geq 1$ . Combining these two estimates, we obtain

$$\int_A^B \left(s^2|\partial_x v|^2 + s^4|v|^2\right) \leq \|e^{s\alpha}\partial_{xx}u\|_{L^2(A, B)}^2 + Cs^4 \int_a^b |v|^2 dx + Cs^2 \int_a^b |w|^2 dx,$$

for some  $C$  independent of  $s$ . According to the definition of  $w$ , this yields, for some  $C$  independent of  $s \geq 1$ , that

$$\int_A^B \left(s^2|\partial_x v|^2 + s^4|v|^2\right) dx \leq \|e^{s\alpha}\partial_{xx}u\|_{L^2(A, B)}^2 + Cs^4 \int_a^b |v|^2 dx + Cs^2 \int_a^b |\partial_x v|^2 dx,$$

Using  $v = ue^{s\alpha}$  and Cacciopoli's inequality (recall  $[a, b] \subset\subset \omega$ ), we finally get the existence of a constant  $C > 0$  such that for all  $s \geq 1$  and  $u \in H^2(A, B) \cap H_0^1(A, B)$ ,

$$s^4 \int_A^B e^{2s\alpha}|u|^2 dx \leq C \int_A^B e^{2s\alpha}|\partial_{xx}u|^2 dx + Cs^4 \int_\omega e^{2s\alpha}|u|^2 dx.$$

Accordingly, if  $V \in L^\infty(A, B)$ , taking  $s^4 \geq 4C\|V\|_{L^\infty(A, B)}^2$ , for all  $u \in H^2(A, B) \cap H_0^1(A, B)$ , we get

$$\frac{s^4}{2} \int_A^B e^{2s\alpha}|u|^2 dx \leq 2C \int_A^B e^{2s\alpha} |-\partial_{xx}u + Vu|^2 dx + Cs^4 \int_\omega e^{2s\alpha}|u|^2 dx.$$

Thus, fixing  $s = 2(\sqrt{C}\|V\|_{L^\infty(A, B)})^{1/2}$  and bounding the weight function  $e^{s\alpha}$  from below by 1 and from above by  $\exp(s\|\alpha\|_\infty)$ , we obtain (5.1).  $\square$

## 5.2 Observability inequality for positive solutions

The goal of this part is to underline that positivity induces nice observability estimates even in higher dimension, as it was already noticed in the work [LB20] by the second author.

**Theorem 5.2.** *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^N$ , and  $\omega$  a non-empty open subset of  $\Omega$ .*

*Then there exists a constant  $C = C(\Omega, \omega) > 0$  such that for every  $V \in L^\infty(\Omega; \mathbb{R})$  and for every  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  which satisfies  $u \geq 0$  in  $\Omega$ ,*

$$\|u\|_{L^2(\Omega)} \leq C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2}\right) \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)}\right). \quad (5.3)$$

*Proof.* Let us take  $\beta \in C^2(\bar{\Omega}; \mathbb{R}_+)$  such that

$$\beta = 0 \text{ on } \partial\Omega, \quad \beta > 0 \text{ in } \Omega, \quad \inf_{\Omega \setminus \omega} |\nabla\beta| > 0.$$

Such a function is known to exist, see for instance [TW09, Theorem 9.4.3].

Let  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  be non-negative in  $\Omega$ , and set  $v = ue^{s\beta}$  for some free parameter  $s \geq 1$ . We have that

$$\Delta v + s^2|\nabla\beta|^2v - 2s\nabla\beta \cdot \nabla v - s\Delta\beta v - Vv = -(-\Delta u + Vu)e^{s\beta} \quad \text{in } \Omega. \quad (5.4)$$

Let us multiply (5.4) by  $\beta$ , integrate in  $\Omega$  and do integration by parts recalling that  $\beta = 0$  on  $\partial\Omega$ , we obtain

$$\int_{\Omega} s^2|\nabla\beta|^2v\beta \, dx + 2 \int_{\Omega} s|\nabla\beta|^2v \, dx = - \int_{\Omega} v\Delta\beta \, dx - \int_{\Omega} s\beta\Delta\beta v \, dx - \int_{\Omega} (-\Delta u + Vu)e^{s\beta}\beta \, dx + \int_{\Omega} Vv\beta \, dx.$$

By using the properties of the weights and the fact that  $u$ , hence  $v$ , is non-negative, we deduce that for some  $C > 0$  depending on  $\Omega$ ,  $\omega$  and  $\beta$ ,

$$\begin{aligned} & \int_{\Omega} s^2\beta v \, dx + \int_{\Omega} sv \, dx \\ & \leq C \left( \int_{\omega} s^2v\beta \, dx + \int_{\omega} sv \, dx + \int_{\Omega} v \, dx + \int_{\Omega} s\beta v \, dx + \int_{\Omega} |V|v\beta \, dx + \int_{\Omega} |-\Delta u + Vu|e^{s\beta}\beta \, dx \right). \end{aligned}$$

The third and fourth terms of the right hand-side can be absorbed by the left hand-side of the inequality by taking  $s$  sufficiently large,  $s \geq C$ ; the fifth term in the right hand-side can be absorbed by the first one in the left hand side term by assuming  $s \geq C(1 + \|V\|_{L^\infty}^{1/2})$ . Doing the choice  $s = C(1 + \|V\|_{L^\infty}^{1/2})$ , and bounding the weight  $e^{s\beta}$  from below by 1, and from above by  $\exp(C(1 + \|V\|_{L^\infty}^{1/2}))$ , we obtain

$$\|u\|_{L^1(\Omega)} \leq C \exp\left(C\|V\|_{L^\infty(\Omega)}^{1/2}\right) \left(\|-\Delta u + Vu\|_{L^1(\Omega)} + \|u\|_{L^1(\omega)}\right). \quad (5.5)$$

Classical estimates on elliptic equation then enable to obtain (5.3) from (5.5). Details are left to the reader.  $\square$

## A Appendix

### A.1 Weak maximum principle

The following result is a weak maximum principle based on de Giorgi's iteration.

**Lemma A.1.** *Let  $\Omega$  be a smooth bounded open set, and  $O$  be a smooth open set such that  $O \subset \Omega$ . Take  $g \in L^2(O)$ ,  $\varphi \in L^\infty(O)$  such that  $1/2 \leq \varphi \leq 3/2$  in  $O$ . Let  $u \in H_0^1(O)$  be the weak solution to  $\nabla \cdot (\varphi^2 \nabla u) = g$ . Then there exists a constant  $C > 0$  depending on  $\Omega$  but independent of  $O$  such that*

$$\|u\|_{L^\infty(O)} \leq C \|g\|_{L^2(O)}.$$

*Proof.* In the proof, the constants  $C$  may vary from line to line, but they are only allowed to depend on  $\Omega$ , not on  $O$ .

By definition of a weak solution,  $u$  satisfies

$$\int_O \varphi^2 \nabla u \cdot \nabla v \, dx = \int_O gv \, dx, \quad \forall v \in H_0^1(O).$$

For  $k > 0$  a free parameter, we choose  $v = (u - k)_+$  in the above identity (note that  $v \in H_0^1(O)$ ), and using the pointwise estimates on  $\varphi$ , we get

$$\int_O |\nabla v|^2 dx \leq 4 \int_O |gv| dx. \quad (\text{A.1})$$

Since  $v \in H_0^1(O)$ , extending  $v$  by 0 in  $\Omega \setminus O$  and still denoting this extension by  $v$ ,  $v$  can also be seen as an element of  $H_0^1(\Omega)$ . Since  $H_0^1(\Omega)$  embeds into  $L^p(\Omega)$  for any  $p < \infty$ , we get, for some constant  $C_p(\Omega)$  depending on  $p$  and  $\Omega$ :

$$\left( \int_O |v|^p dx \right)^{2/p} \leq C_p(\Omega)^2 \int_O |\nabla v|^2 dx.$$

For simplicity, we choose  $p = 6$  (in fact, any  $p > 4$  would allow to conclude), so that we get:

$$\left( \int_O |v|^6 dx \right)^{1/3} \leq C \int_O |\nabla v|^2 dx. \quad (\text{A.2})$$

Using then (A.1) and (A.2), we get

$$\left( \int_O |v|^6 dx \right)^{1/3} \leq C \int_O |gv| dx.$$

In other words, introducing the set

$$A(k) := \{x \in O; u(x) > k\},$$

we have

$$\left( \int_{A(k)} |v|^6 dx \right)^{1/3} \leq C \int_{A(k)} |gv| dx. \quad (\text{A.3})$$

Using Hölder's inequality

$$\int_{A(k)} |gv| dx \leq \left( \int_{A(k)} |v|^6 dx \right)^{1/6} \left( \int_{A(k)} |g|^{6/5} dx \right)^{5/6},$$

so with (A.3) and Hölder's estimate, we obtain

$$\left( \int_{A(k)} |v|^6 dx \right)^{1/6} \leq C \left( \int_{A(k)} |g|^{6/5} dx \right)^{5/6} \leq C \|g\|_{L^2(O)} |A(k)|^{1/3}. \quad (\text{A.4})$$

For  $h > k$ ,  $A(h) \subset A(k)$  and  $v \geq h - k$  on  $A(h)$ , so that for  $h > k$ ,

$$\int_{A(k)} |v|^6 dx \geq \int_{A(h)} |v|^6 dx \geq (h - k)^6 |A(h)|. \quad (\text{A.5})$$

According to (A.4) and (A.5) together with Hölder's estimate, we obtain that for all  $h > k > 0$

$$(h - k) |A(h)|^{1/6} \leq C \|g\|_{L^2(O)} |A(k)|^{1/3},$$

i.e.

$$|A(h)| \leq \left( \frac{C \|g\|_{L^2(O)}}{h - k} \right)^6 |A(k)|^2, \quad \forall h > k > 0.$$

Setting

$$M = 4C \|g\|_{L^2(O)} (2|\Omega|)^{1/6},$$

and introducing the sequence given by  $k_n = M(1 - 1/2^n)$  for  $n \in \mathbb{N}$ , we get, for all  $n \in \mathbb{N}$

$$|A(k_{n+1})| \leq \left( \frac{C \|g\|_{L^2(O)} 2^{(n+1)}}{M} \right)^6 |A(k_n)|^2 \leq \frac{2^{6(n-1)}}{2|\Omega|} |A(k_n)|^2.$$

Accordingly, for all  $n \in \mathbb{N}$ ,

$$\frac{2^{6(n+1)} |A(k_{n+1})|}{2|\Omega|} \leq \left( \frac{2^{6n} |A(k_n)|}{2|\Omega|} \right)^2.$$

Since  $|A(k_0)| \leq |\Omega|$ , we easily check that for all  $n \in \mathbb{N}$ ,

$$\frac{2^{6n} |A(k_n)|}{2|\Omega|} \leq \frac{1}{2^n}.$$

Letting  $n \rightarrow +\infty$ , we deduce that  $|A(M)| = 0$ . This proves that  $u \leq M$  in  $O$ . The other sense of the inequality can be done in the same way.  $\square$

## A.2 Harnack's inequality

The goal of this part is to state an Harnack's inequality for harmonic functions defined in an annulus.

**Theorem A.2** (Harnack's inequality). *There exists  $C > 0$  such that for all  $r > 0$ , and  $u \in L^2(B(0, 8r) \setminus B(0, r))$  satisfying for some  $\gamma > 0$ ,*

$$-\Delta u = 0, \quad \text{and} \quad u \geq -\gamma \text{ or } u \leq \gamma \quad \text{in } B(0, 8r) \setminus B(0, r),$$

we have

$$\max_{B(0, 7r) \setminus B(0, 2r)} |u| \leq C \left( \min_{B(0, 7r) \setminus B(0, 2r)} |u| + \gamma \right), \quad (\text{A.6})$$

and

$$r \|\nabla u\|_{L^\infty(B(0, 4r) \setminus B(0, 3r))} \leq C \left( \min_{B(0, 7r) \setminus B(0, 2r)} |u| + \gamma \right). \quad (\text{A.7})$$

*Proof.* Set  $\tilde{u}(x) = u(rx)$ , then  $\tilde{u} \in L^2(B(0, 8) \setminus B(0, 1))$  is harmonic and satisfies  $\tilde{u} \geq -\gamma$  or  $\tilde{u} \leq \gamma$  on  $B(0, 8) \setminus B(0, 1)$ . We then apply the standard Harnack's inequality, [GT83, Section 8.8, Corollary 8.21] to  $\tilde{u} + \gamma$  or  $\tilde{u} - \gamma$  to get

$$\max_{B(0, 7) \setminus B(0, 2)} |\tilde{u}| \leq C \left( \min_{B(0, 7) \setminus B(0, 2)} |\tilde{u}| + \gamma \right),$$

then (A.6) by scaling.

For the gradient estimate, we use [GT83, Chapter 3, Section 3.4, Theorem 3.9] with  $\Omega = B(0, 7) \setminus B(0, 2)$  for  $f = 0$ ,  $d_x = d(x, \partial\Omega)$ ,

$$\sup_{x \in \Omega} d_x |\nabla \tilde{u}(x)| \leq C \sup_{x \in \Omega} |\tilde{u}(x)|.$$

Accordingly,

$$\|\nabla \tilde{u}\|_{L^\infty(B(0, 4) \setminus B(0, 3))} \leq C \sup_{x \in \Omega} |\tilde{u}(x)|.$$

Again, this provides (A.7) by recalling  $\tilde{u}(x) = u(rx)$  and (A.6).  $\square$

## A.3 Carleman estimates

The goal of this part is to prove the Carleman estimate stated in Proposition 2.20. Actually, we will prove a more general result.

**Proposition A.3.** *There exist constants  $C > 0$ ,  $\lambda \geq 1$ , and  $s_0 \geq 1$ , depending on  $r'$ ,  $R$ , such that for every  $s \geq s_0$  and every  $y \in H^2(B(0, R))$ ,  $\tilde{y} = ye^{s\alpha}$  satisfies*

$$\begin{aligned} & \int_{\Omega} |-\Delta \tilde{y} - s^2 |\nabla \alpha|^2 \tilde{y}|^2 dx + \int_{\Omega} |s \nabla \alpha \cdot \nabla \tilde{y}|^2 dx + s^3 \int_{\Omega} |\tilde{y}|^2 dx + s \int_{\Omega} |\nabla \tilde{y}|^2 dx + s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma \\ & + s \int_{\partial\Omega} |\partial_n \tilde{y}|^2 d\sigma \leq C \left( \|(\Delta y) e^{s\alpha}\|_{L^2(\Omega)}^2 + s^3 \int_{\omega'} |\tilde{y}|^2 dx + s \int_{\partial\Omega} |\partial_\theta \tilde{y}|^2 d\sigma \right). \end{aligned}$$

*Proof of Proposition A.3.* For  $y \in H^2(\Omega)$ ,  $\tilde{y} = ye^{s\alpha}$  satisfies

$$-\Delta \tilde{y} - s^2 |\nabla \alpha|^2 \tilde{y} + 2s \nabla \alpha \cdot \nabla \tilde{y} + s \Delta \alpha \tilde{y} = (-\Delta y) e^{s\alpha}, \quad \text{in } \Omega. \quad (\text{A.8})$$

We then write

$$S \tilde{y} = -\Delta \tilde{y} - s^2 |\nabla \alpha|^2 \tilde{y}, \quad A \tilde{y} = 2s \nabla \alpha \cdot \nabla \tilde{y} + 2s \Delta \alpha \tilde{y}, \quad (\text{A.9})$$

and we get that

$$\|S \tilde{y}\|_{L^2(\Omega)}^2 + \|A \tilde{y}\|_{L^2(\Omega)}^2 + 2 \int_{\Omega} A \tilde{y} S \tilde{y} dx \leq 2 \|(-\Delta y) e^{s\alpha}\|_{L^2(\Omega)}^2 + 2s^2 \|\Delta \alpha\|_{L^\infty(\Omega)} \|\tilde{y}\|_{L^2(\Omega)}^2. \quad (\text{A.10})$$

We then compute the cross product  $\int_{\Omega} A\tilde{y} S\tilde{y} dx$ :

$$\begin{aligned}\int_{\Omega} (-\Delta\tilde{y})(2s\nabla\alpha \cdot \nabla\tilde{y}) dx &= s \int_{\Omega} (2D^2\alpha(\nabla\tilde{y}, \nabla\tilde{y}) - \Delta\alpha|\nabla\tilde{y}|^2) dx - 2s \int_{\partial\Omega} \partial_n\tilde{y}\nabla\alpha \cdot \nabla\tilde{y} d\sigma + s \int_{\partial\Omega} \partial_n\alpha|\nabla\tilde{y}|^2 d\sigma, \\ \int_{\Omega} (-\Delta\tilde{y})(2s\Delta\alpha\tilde{y}) dx &= 2s \int_{\Omega} \Delta\alpha|\nabla\tilde{y}|^2 dx - s \int_{\Omega} \Delta^2\alpha|\tilde{y}|^2 dx - 2s \int_{\partial\Omega} \partial_n\tilde{y}\Delta\alpha\tilde{y} d\sigma + s \int_{\partial\Omega} \partial_n\Delta\alpha|\tilde{y}|^2 d\sigma, \\ \int_{\Omega} (-s^2|\nabla\alpha|^2\tilde{y})(2s\nabla\alpha \cdot \nabla\tilde{y}) dx &= s^3 \int_{\Omega} \nabla \cdot (|\nabla\alpha|^2\nabla\alpha)|\tilde{y}|^2 dx - s^3 \int_{\partial\Omega} \partial_n\alpha|\nabla\alpha|^2|\tilde{y}|^2 d\sigma, \\ \int_{\Omega} (-s^2|\nabla\alpha|^2\tilde{y})(2s\Delta\alpha\tilde{y}) dx &= -2s^3 \int_{\Omega} \Delta\alpha|\nabla\alpha|^2|\tilde{y}|^2 dx.\end{aligned}$$

Combining all these computations, we obtain:

$$\begin{aligned}\int_{\Omega} A\tilde{y} S\tilde{y} dx &= \int_{\Omega} |\tilde{y}|^2 (2s^3 D^2\alpha(\nabla\alpha, \nabla\alpha) - s^3 \Delta\alpha|\nabla\alpha|^2 - s\Delta^2\alpha) dx \\ &\quad + \int_{\Omega} (2sD^2\alpha(\nabla\tilde{y}, \nabla\tilde{y}) + s\Delta\alpha|\nabla\tilde{y}|^2) dx - 2s \int_{\partial\Omega} \partial_n\tilde{y}\nabla\alpha \cdot \nabla\tilde{y} d\sigma + s \int_{\partial\Omega} \partial_n\alpha|\nabla\tilde{y}|^2 d\sigma \\ &\quad - 2s \int_{\partial\Omega} \partial_n\tilde{y}\Delta\alpha\tilde{y} d\sigma + s \int_{\partial\Omega} \partial_n\Delta\alpha|\tilde{y}|^2 d\sigma - s^3 \int_{\partial\Omega} \partial_n\alpha|\nabla\alpha|^2|\tilde{y}|^2 d\sigma.\end{aligned}\tag{A.11}$$

Next, we specify these computations to the case  $\Omega = B(0, R)$ ,  $\omega'_1 = B(0, r'/2)$ ,  $\omega' = B(0, r')$ , and  $\alpha$  as in (2.33). Easy computations show that

$$\begin{aligned}\nabla\alpha &= \lambda\alpha\nabla\beta, \\ D^2\alpha &= \lambda^2\alpha\nabla\beta(\nabla\beta)^t + \lambda\alpha D^2\beta, \\ \Delta\alpha &= \lambda^2\alpha|\nabla\beta|^2 + \lambda\alpha\Delta\beta, \\ 2D^2\alpha(\nabla\alpha, \nabla\alpha) - \Delta\alpha|\nabla\alpha|^2 &= \lambda^4\alpha^3|\nabla\beta|^4 + \lambda^3\alpha^3(2D^2\beta(\nabla\beta, \nabla\beta) - \Delta\beta|\nabla\beta|^2).\end{aligned}$$

Accordingly, there exist  $\lambda \geq 1$  and a positive constant  $c_0 > 0$  such that for all  $x \in \Omega \setminus \omega'_1$  and  $\xi \in \mathbb{R}^2$ ,

$$2D^2\alpha(x)(\xi, \xi) + \Delta\alpha(x)|\xi|^2 \geq c_0|\xi|^2, \quad \text{and} \quad 2D^2\alpha(x)(\nabla\alpha(x), \nabla\alpha(x)) - \Delta\alpha(x)|\nabla\alpha(x)|^2 \geq c_0,$$

and, on  $\partial\Omega$ ,

$$\nabla\alpha = \partial_n\alpha\vec{n}, \quad \text{with } \partial_n\alpha \leq -c_0, \quad \text{and} \quad \partial_n\alpha|\nabla\alpha|^2 \leq -c_0.$$

Also, there exists a constant  $C > 0$  such that

$$\begin{aligned}\|2D^2\alpha(\nabla\alpha, \nabla\alpha) - \Delta\alpha|\nabla\alpha|^2\|_{L^\infty(\omega'_1)} + \|\Delta^2\alpha\|_{L^\infty(\Omega)} + \|2D^2\alpha + \Delta\alpha Id\|_{L^\infty(\Omega)} \\ + \|\partial_n\alpha\|_{L^\infty(\partial\Omega)} + \|\partial_n\Delta\alpha\|_{L^\infty(\partial\Omega)} + \|\Delta\alpha\|_{L^\infty(\partial\Omega)} \leq C.\end{aligned}$$

Using these estimates in the identity (A.11), we obtain:

$$\begin{aligned}\int_{\Omega} A\tilde{y} S\tilde{y} dx &\geq c_0s^3 \int_{\Omega} |\tilde{y}|^2 dx + c_0s \int_{\Omega} |\nabla\tilde{y}|^2 dx + c_0s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma + c_0s \int_{\partial\Omega} |\partial_n\tilde{y}|^2 d\sigma \\ &\quad - Cs^3 \int_{\omega'_1} |\tilde{y}|^2 dx - Cs \int_{\omega'_1} |\nabla\tilde{y}|^2 dx - Cs \int_{\partial\Omega} \left| \frac{\partial_{\theta}\tilde{y}}{R} \right|^2 d\sigma \\ &\quad - Cs \int_{\Omega} |\tilde{y}|^2 dx - Cs \int_{\partial\Omega} |\partial_n\tilde{y}|\tilde{y}| d\sigma - Cs \int_{\partial\Omega} |\tilde{y}|^2 d\sigma.\end{aligned}$$

In the third line, the part of the first term outside  $\omega'_1$  can be absorbed by the first one of the first line by taking  $s$  large enough, the part in  $\omega'_1$  can be included in the first term of the second line. The second and the third terms of the third line can be absorbed similarly by the two last terms of the first line by taking  $s$  large enough.

Therefore, there exists  $s_0 \geq 1$  such that for all  $s \geq s_0$ ,

$$\begin{aligned}\int_{\Omega} A\tilde{y} S\tilde{y} dx &\geq \frac{c_0}{2}s^3 \int_{\Omega} |\tilde{y}|^2 dx + c_0s \int_{\Omega} |\nabla\tilde{y}|^2 dx + \frac{c_0}{2}s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma + \frac{c_0}{2}s \int_{\partial\Omega} |\partial_n\tilde{y}|^2 d\sigma \\ &\quad - 2Cs^3 \int_{\omega'_1} |\tilde{y}|^2 dx - Cs \int_{\omega'_1} |\nabla\tilde{y}|^2 dx - Cs \int_{\partial\Omega} \left| \frac{\partial_{\theta}\tilde{y}}{R} \right|^2 d\sigma.\end{aligned}$$

Plugging this estimate in (A.10), we obtain that there exists a constant  $C > 0$  such that for all  $s \geq s_0$ ,

$$\begin{aligned} & \|S\tilde{y}\|_{L^2(\Omega)}^2 + \|A\tilde{y}\|_{L^2(\Omega)}^2 + c_0s^3 \int_{\Omega} |\tilde{y}|^2 dx + c_0s \int_{\Omega} |\nabla\tilde{y}|^2 dx + c_0s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma + c_0s \int_{\partial\Omega} |\partial_n\tilde{y}|^2 d\sigma \\ & \leq C\|(\Delta y)e^{s\alpha}\|_{L^2(\Omega)}^2 + Cs^2\|\tilde{y}\|_{L^2(\Omega)}^2 + Cs^3 \int_{\omega'_1} |\tilde{y}|^2 dx + Cs \int_{\omega'_1} |\nabla\tilde{y}|^2 dx + Cs \int_{\partial\Omega} \left| \frac{\partial_{\theta}\tilde{y}}{R} \right|^2 d\sigma. \end{aligned}$$

Recalling then the definition of  $A\tilde{y}$  in (A.9), we easily get that

$$\|s\nabla\alpha \cdot \nabla\tilde{y}\|_{L^2(\Omega)}^2 \leq \|A\tilde{y}\|_{L^2(\Omega)}^2 + Cs^2\|\tilde{y}\|_{L^2(\Omega)}^2.$$

Accordingly, there exist  $s_0 \geq 1$  and a constant  $C > 0$  such that for all  $s \geq s_0$ ,

$$\begin{aligned} & \|S\tilde{y}\|_{L^2(\Omega)}^2 + \|s\nabla\alpha \cdot \nabla\tilde{y}\|_{L^2(\Omega)}^2 + c_0s^3 \int_{\Omega} |\tilde{y}|^2 dx + c_0s \int_{\Omega} |\nabla\tilde{y}|^2 dx + c_0s^3 \int_{\partial\Omega} |\tilde{y}|^2 d\sigma + c_0s \int_{\partial\Omega} |\partial_n\tilde{y}|^2 d\sigma \\ & \leq C\|(\Delta y)e^{s\alpha}\|_{L^2(\Omega)}^2 + Cs^3 \int_{\omega'_1} |\tilde{y}|^2 dx + Cs \int_{\omega'_1} |\nabla\tilde{y}|^2 dx + Cs \int_{\partial\Omega} \left| \frac{\partial_{\theta}\tilde{y}}{R} \right|^2 d\sigma. \end{aligned}$$

To finish the proof of Proposition 2.20, we simply have to show that there exists a constant  $C > 0$  such that for all  $s \geq s_0$ ,

$$s \int_{\omega'_1} |\nabla\tilde{y}|^2 dx \leq C \left( s^3 \int_{\omega'} |\tilde{y}|^2 dx + \|(\Delta y)e^{s\alpha}\|_{L^2(\Omega)}^2 \right).$$

In order to do that, we take a smooth cut-off function  $\chi$  with

$$\chi \in C_c^2(\omega'), \chi = 1 \text{ in } \omega'_1, 0 \leq \chi \leq 1,$$

and we multiply equation (A.8) by  $\chi\tilde{y}$  and do integration by parts:

$$\int_{\Omega} \chi |\nabla\tilde{y}|^2 dx = \int_{\Omega} (-\Delta y)e^{s\alpha}\chi\tilde{y} dx + \int_{\Omega} |\tilde{y}|^2 \left( \frac{1}{2}\Delta\chi + s^2|\nabla\alpha|^2 + s\nabla\chi \cdot \nabla\alpha \right).$$

Accordingly, due to the fact that  $\chi$  and its derivatives are supported in  $\omega'$ ,

$$\int_{\omega'_1} |\nabla\tilde{y}|^2 dx \leq \int_{\Omega} \chi |\nabla\tilde{y}|^2 dx \leq \frac{1}{s} \|(\Delta y)e^{s\alpha}\|_{L^2(\Omega)}^2 + Cs^2 \int_{\omega'} |\tilde{y}|^2 dx.$$

This concludes the proof of Proposition A.3. □

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