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► **To cite this version:**

Uwe Franz, Amaury Freslon, Adam Skalski. Connectedness and Gaussian Parts for Compact Quantum Groups. *Journal of Geometry and Physics*, 2023, 184, pp.104710. 10.1016/j.geomphys.2022.104710 . hal-03616091

HAL Id: hal-03616091

<https://hal.science/hal-03616091v1>

Submitted on 10 Sep 2024

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CONNECTEDNESS AND GAUSSIAN PARTS FOR COMPACT QUANTUM GROUPS

UWE FRANZ, AMAURY FRESLON, AND ADAM SKALSKI

ABSTRACT. We introduce the Gaussian part of a compact quantum group \mathbb{G} , namely the largest quantum subgroup of \mathbb{G} supporting all the Gaussian functionals of \mathbb{G} . We prove that the Gaussian part is always contained in the Kac part, and characterise Gaussian parts of classical compact groups, duals of classical discrete groups and q -deformations of compact Lie groups. The notion turns out to be related to a new concept of “strong connectedness” and we exhibit several examples of both strongly connected and totally strongly disconnected compact quantum groups.

1. INTRODUCTION

Lévy processes, i.e. stochastic processes with independent and identically distributed increments, form one of the most studied classes of stochastic processes ([Sat99]); and among these the Gaussian processes, and in particular the Brownian motion are crucial examples. They were initially studied in the Euclidean space, but it quickly became clear that the convolution product for probability measures afforded by the (locally compact) group structure gives a very natural framework for generalisations ([Hey77], [Lia04]). In the end of 1980s, motivated by the development of quantum probability theory, the algebraic language was used by Accardi, Schürmann and von Waldenfels to define in [ASvW88] abstract *quantum Lévy processes*, understood as certain families of quantum random variables living on a $*$ -bialgebra. Soon after, Schürmann introduced in [Sch90] an important subclass of quantum Lévy processes, namely *quantum Gaussian processes*, see also [Sch93, Section 5]. Schürmann’s definition is phrased in terms of *Gaussian generating functionals*. These should be viewed as quantum counterparts of ‘second-order’ or ‘quadratic’ generators of convolution semigroups of measures, and naturally lead to the concept of *Gaussian states on (compact) quantum groups*. The idea of Schürmann opened a path to studying quantum versions of Lévy-Khintchine decompositions ([DFKS18] and references therein) or to classification of Gaussian generators on concrete quantum groups (see [SS98]).

The last cited paper exhibited a curious phenomenon: all Gaussian generating functionals on Woronowicz’s $SU_q(2)$ (with $q \in (-1, 0) \cup (0, 1)$) are supported on the circle \mathbb{T} , the largest classical subgroup of $SU_q(2)$. In the same spirit, it is known that the free permutation do not support any non-trivial Gaussian generating functionals whatsoever ([FKS16]). Meanwhile, some examples show that in general Gaussian generating functionals need not be supported

2010 *Mathematics Subject Classification*. 16T05; 20G42.

Key words and phrases. Compact quantum groups; Gaussian generating functionals; Hopf $*$ -algebras.

U.F. and A.F. were partially supported by the ANR grant “Noncommutative analysis on groups and quantum groups” (ANR-19-CE40-0002). A.F. was also partially supported by the ANR grant “Operator algebras and dynamics on groups” (ANR-19-CE40-0008). A.S. was partially supported by the National Science Center (NCN) grant no. 2020/39/I/ST1/01566.

We thank Jacek Krajczok for useful comments.

on classical subgroups (see [DFKS18] or the results in Sections 5 and 6 below). This motivates the main question studied in this paper, which has natural connections for example with the notion of topological generation for compact quantum groups (see the end of Section 6.3):

- Given a compact quantum group \mathbb{G} , what is the smallest quantum subgroup of \mathbb{G} which supports all Gaussian generating functionals (equivalently, all Gaussian states) of \mathbb{G} ?

We will call the quantum subgroup as above (whose existence is easy to deduce) the *Gaussian part* of \mathbb{G} and denote it $\text{Gauss}(\mathbb{G})$, by analogy with the *classical part* (i.e. the largest classical subgroup) of \mathbb{G} and the *Kac part* (the largest quantum subgroup of Kac type) of \mathbb{G} . The analysis of the classical case shows that there the notion of the Gaussian part coincides with the connected component of the identity. This motivates the introduction of a natural quantum counterpart of classical connectedness. To distinguish this notion from a strictly weaker concept defined in [Wan09], we call it *strong connectedness*; roughly speaking a compact quantum group \mathbb{G} is strongly connected if the intersection of a certain family of ideals in $\mathcal{O}(\mathbb{G})$ trivialises. With this notion at hand – which appears to be of independent interest – we can state the first main result of this work.

Theorem A. *For any compact quantum group \mathbb{G} the Gaussian part of \mathbb{G} is contained in the strongly connected component of identity of \mathbb{G} , which in turn is contained in the Kac part of \mathbb{G} :*

$$\text{Gauss}(\mathbb{G}) \subset \mathbb{G}^{00} \subset \text{Kac}(\mathbb{G}).$$

Apart from the structural result above we compute the Gaussian parts of many compact quantum groups. We summarise the main statements obtained in the following theorem.

Theorem B. *The following hold:*

- *If G is a classical compact group, then its Gaussian part coincides with its (strongly) connected component of the identity: $\text{Gauss}(G) = G^0$;*
- *If Γ is a finitely generated discrete group, then $\text{Gauss}(\widehat{\Gamma}) = \Gamma/\sqrt{\widehat{\gamma_3(\Gamma)}}$;*
- *Gaussian parts of (pro-)finite quantum groups and of quantum permutation groups are trivial;*
- *If G is a simply connected semisimple compact Lie group and $q \in (0, 1)$ then $\text{Gauss}(\mathbb{G}_q)$ is the maximal torus $\mathbf{T} \subset \mathbb{G}_q$;*
- $\text{Gauss}(O_N^*) = SO(N)$;
- $\text{Gauss}(O_N^+)$ (for $N \geq 4$) and $\text{Gauss}(U_N^+)$ (for $N \geq 2$) are neither classical nor dual to discrete groups.

It is worth noting that we do not know whether the Gaussian part of the free orthogonal group O_N^+ is O_N^+ itself (similarly for the free unitary group). The problem seems to have subtle connections with the questions regarding topological generation inside free quantum groups.

The detailed plan of the paper is as follows: in Section 2 we recall the basic facts concerning generating functionals on $*$ -bialgebras, including the notion of Gaussian generating functionals and the subclass of drifts, together with several related characterizations, and establish a Wick-type formula which is useful for computations. Section 3 recalls basic facts of the theory of compact quantum groups, introduces the Gaussian part and establishes its basic properties. We also prove a that point that Gaussian parts satisfy the Kac property, and the proof relies on a new characterization of the maximal Kac quotient which we believe

to be of independent interest. Section 4 is devoted to the concept of strong connectedness; there we also complete the proof of Theorem A. Eventually, the last two sections contain computations of the Gaussian part in several examples. In Section 5 we determine Gaussian (and drift) parts for classical compact groups and duals of discrete groups, and in Section 6 we do the same for q -deformations and half-liberated orthogonal groups and discuss partial results we obtained for free quantum groups. This completes the proof of Theorem B, which is a combination of Corollary 5.3, Theorem 5.5, Proposition 4.10, Proposition 6.1, Proposition 6.3 and Proposition 6.5.

2. GAUSSIAN FUNCTIONALS

In this section we recall the notion of generating functionals on $*$ -bialgebras, focusing on the class of Gaussian functionals and drifts for which we provide several characterizations. We also show that Gaussian generating functionals satisfy a version of the Wick property.

2.1. Definition and properties. Let \mathcal{B} be an involutive bialgebra with unit $\mathbf{1}$ and counit ε . In all cases of interest afterwards, \mathcal{B} will be the Hopf $*$ -algebra $\mathcal{O}(\mathbb{G})$ of a compact quantum group \mathbb{G} , but we stay at a general level for the moment. A *generating functional* on \mathcal{B} is a linear functional $\phi : \mathcal{B} \rightarrow \mathbb{C}$ with the following three properties:

- (1) $\phi(\mathbf{1}) = 0$ (normalization);
- (2) $\phi(b^*) = \overline{\phi(b)}$ for all $b \in \mathcal{B}$ (hermiticity);
- (3) $\phi(b^*b) \geq 0$ for all $b \in \ker \varepsilon$ (conditional positivity).

We are interested in generating functionals, because it follows from [Sch93, Section 3.2] that they are in one-to-one correspondence with convolution semigroups of states.

Proposition 2.1. *Let \mathcal{B} be an involutive bialgebra and let $\phi : \mathcal{B} \rightarrow \mathbb{C}$ a linear functional. Set, for $t \in \mathbb{R}_+$,*

$$\varphi_t = \exp_*(t\phi) := \sum_{n=0}^{+\infty} \frac{(t\phi)^{\star n}}{n!},$$

with the convention $\phi^{\star 0} = \varepsilon$. Then, the following are equivalent:

- (1) ϕ is a generating functional;
- (2) φ_t is a state for all $t \geq 0$, in the sense that
 - (a) $\varphi_t(\mathbf{1}) = 1$;
 - (b) $\varphi_t(b^*b) \geq 0$ for all $b \in \mathcal{B}$.

By a GNS-type construction one can associate a so-called *Schürmann triple* to a generating functional, cf. [Sch93, Subsection 2.3]. Let us recall how this works. For an inner product space D (pre-Hilbert space), we denote by

$$\mathcal{L}(D) = \{X : D \rightarrow D \text{ linear} \mid \exists X^* : D \rightarrow D \text{ s.t. } \forall u, v \in D, \langle u, Xv \rangle = \langle X^*u, v \rangle\}$$

the $*$ -algebra of adjointable linear maps on D .

Definition 2.2. Let $(\mathcal{B}, \varepsilon)$ be a pair consisting of a $*$ -algebra \mathcal{B} and a $*$ -homomorphism $\varepsilon : \mathcal{B} \rightarrow \mathbb{C}$ (an augmented $*$ -algebra), and let D be a pre-Hilbert space. A Schürmann triple on $(\mathcal{B}, \varepsilon)$ over D is a family of three linear maps $(\rho : \mathcal{B} \rightarrow \mathcal{L}(D), \eta : \mathcal{B} \rightarrow D, \phi : \mathcal{B} \rightarrow \mathbb{C})$ such that

- (1) ρ is a unital $*$ -homomorphism;

(2) η satisfies

$$(2.1) \quad \eta(ab) = \rho(a)\eta(b) + \eta(a)\varepsilon(b)$$

for all $a, b \in \mathcal{B}$,

(3) ϕ is hermitian and satisfies

$$(2.2) \quad \phi(a^*b) - \overline{\varepsilon(a)}\phi(b) - \overline{\phi(a)}\varepsilon(b) = \langle \eta(a), \eta(b) \rangle$$

for all $a, b \in \mathcal{B}$.

Relations (2.1) and (2.2) imply $\eta(\mathbf{1}) = 0$ and $\phi(\mathbf{1}) = 0$. Relation (2.2) furthermore shows that ϕ is positive on the kernel of ε , hence ϕ is a generating functional.

We call two Schürmann triples (ρ, η, ϕ) and (ρ', η', ϕ') on the same augmented $*$ -algebra $(\mathcal{B}, \varepsilon)$ and over pre-Hilbert spaces D and D' *equivalent*, if there exists a surjective isometry $V : D \rightarrow D'$ s.t.

$$V\eta(b) = \eta'(b) \quad \text{and} \quad V\rho(b) = \rho'(b)V$$

for all $b \in \mathcal{B}$. Note that we thus get one-to-one correspondences between convolution semi-groups of states, generating functionals, and Schürmann triples with surjective cocycle (up to equivalence) on a given involutive bialgebra \mathcal{B} . Schürmann [Sch93] proved that these three families of objects are also in one-to-one correspondence with *Lévy processes* on \mathcal{B} (up to stochastic equivalence).

We now introduce a family of ideals which will play a crucial rôle in this work. We set $K_1(\mathcal{B}) = \ker(\varepsilon)$ and $K_n(\mathcal{B}) = K_1(\mathcal{B})^n$ for $n \geq 1$, or more explicitly

$$K_n(\mathcal{B}) = \text{Span}\{b_1 \cdots b_n \mid b_1, \dots, b_n \in \ker(\varepsilon)\}$$

We also set

$$K_\infty(\mathcal{B}) = \bigcap_{n \geq 1} K_n(\mathcal{B}).$$

The family $(K_n(\mathcal{B}))_{n=1}^{+\infty}$ is decreasing, and the containments can be proper, as we will see later on. In order to lighten notations, we will simply write K_n as soon as there is no ambiguity concerning the algebra \mathcal{B} .

If $\mathcal{B} = \mathcal{O}(\mathbb{G})$, then K_1 is also a coideal and therefore a Hopf $*$ -ideal, as it is the kernel of a Hopf $*$ -algebra homomorphism. For $n \geq 2$ however, K_n is in general not a coideal but nevertheless defines a filtration since

$$\Delta(K_n) \subseteq \sum_{\ell=0}^n K_\ell \otimes K_{n-\ell} \subseteq K_{\lfloor \frac{n}{2} \rfloor} \otimes \mathcal{B} + \mathcal{B} \otimes K_{\lfloor \frac{n}{2} \rfloor}$$

(where $K_0 = \mathcal{B}$ by convention). It turns out that if \mathcal{B} is a Hopf algebra, then K_∞ is a Hopf ideal, see Proposition 4.1.

Example 2.3. Consider the $*$ -algebra $\mathcal{B} = \mathbb{C}[x]$ of polynomials in one self-adjoint variable (i.e. $x^* = x$) and the augmentation map determined by $\varepsilon(x) = 0$. Then the ideal K_n consists exactly of the polynomials that have a zero of order at least n at the origin, and $K_\infty = \{0\}$.

We are now ready for the definition of Gaussian processes, which are the main subject of this work.

Definition 2.4. A generating functional $\phi : \mathcal{B} \rightarrow \mathbb{C}$ on an augmented $*$ -algebra \mathcal{B} is called *Gaussian* (or *quadratic*, [Sch93, Section 5.1]), if $\phi|_{K_3} = 0$.

A state on \mathcal{B} is called Gaussian, if it is of the form $\varphi = \exp_{\star}(t\phi)$ for some $t \geq 0$ and ϕ a Gaussian generating functional. A cocycle $\eta : \mathcal{B} \rightarrow H$ is called Gaussian, if it is a derivation in the sense that $\eta(ab) = \varepsilon(a)\eta(b) + \eta(a)\varepsilon(b)$ for all $a, b \in \mathcal{B}$.

The connection between the last definition and the first two ones is not obvious and relies on the following result.

Proposition 2.5. [Sch93, Proposition 5.1.1] *Let \mathcal{B} be an augmented \ast -algebra and let (ρ, η, ϕ) be a Schürmann triple on \mathcal{B} over some pre-Hilbert space H with surjective cocycle η . Then, the following are equivalent:*

- (1) $\phi|_{K_3} = 0$,
- (2) $\phi(a^*a) = 0$ for all $a \in K_2$,
- (3) $\rho|_{K_1} = 0$,
- (4) $\rho(b) = \varepsilon(b)\text{id}_H$ for all $b \in \mathcal{B}$,
- (5) $\eta|_{K_2} = 0$,
- (6) $\eta(ab) = \varepsilon(a)\eta(b) + \eta(a)\varepsilon(b)$ for all $a, b \in \mathcal{B}$.

Note that the first property translates into the following condition, valid for all $a, b, c \in \mathcal{B}$:

$$\phi(abc) = \phi(ab)\varepsilon(c) + \phi(ac)\varepsilon(b) + \phi(bc)\varepsilon(a) - \phi(a)\varepsilon(bc) - \phi(b)\varepsilon(ac) - \phi(c)\varepsilon(ab).$$

This gives an inductive algorithm to compute ϕ which leads to a Wick-type formula, see Subsection 2.2.

Remark. Assume that \mathcal{B} is an augmented \ast -algebra, that $X \subset \mathcal{B}$ generates \mathcal{B} as a \ast -algebra, and let $\phi : \mathcal{B} \rightarrow \mathbb{C}$ be a Gaussian generating functional with associated cocycle $\eta : \mathcal{B} \rightarrow D$. Then, the following conditions are equivalent:

- (i) $\phi(xy) = \phi(yx)$ for all $x, y \in X$;
- (ii) $\langle \eta(x^*), \eta(y) \rangle = \langle \eta(y^*), \eta(x) \rangle$ for all $x, y \in X$;
- (iii) ϕ is a trace: $\phi(ab) = \phi(ba)$ for all $a, b \in \mathcal{B}$;
- (iv) ϕ factors through the commutator ideal of \mathcal{B} .

Indeed, (i) and (ii) are equivalent by (2.2). Then, it suffices to observe that if (i) holds then we can prove first (iii) and then (iv) using the formula displayed before the remark; the other implications are trivial. Even though we will not need nor use the terminology hereafter, such functionals may be called *classical* since they factor through a commutative algebra.

Looking at the definition of a Gaussian generating functional, one may wonder at the definition obtained by strengthening the condition to $\phi|_{K_2} = 0$. A functional satisfying this condition called a *drift* (or *degenerate quadratic*, [Sch93, Section 5.1]). Drifts can be characterized through their Schürmann triples similarly to Gaussian functionals.

Proposition 2.6. *Under the same assumptions as in Proposition 2.5, the following are equivalent:*

- (1) $\phi|_{K_2} = 0$;
- (2) ϕ is a hermitian derivation, i.e. $\phi(a^*) = \overline{\phi(a)}$ and

$$\phi(ab) = \varepsilon(a)\phi(b) + \phi(a)\varepsilon(b)$$

for all $a, b \in \mathcal{B}$;

- (3) $\phi(a^*a) = 0$ for all $a \in K_1$.

Remark. A generating functional ϕ is a drift if and only if $\varphi_t = \exp_*(t\phi)$ is a character for all $t \in \mathbb{R}_+$. One way is easy: if φ_t is a character for all $t \geq 0$, then for all $a, b \in \mathcal{B}$ we have

$$\phi(ab) = \lim_{t \rightarrow 0} \frac{\varphi_t(ab) - \varepsilon(ab)}{t} = \lim_{t \rightarrow 0} \frac{\varphi_t(a)\varphi_t(b) - \varepsilon(a)\varepsilon(b)}{t} = \phi(a)\varepsilon(b) + \phi(b)\varepsilon(a)$$

and combining this with the hermitian property which comes from the hermitian property of φ_t , we conclude by the second point of the above proposition. Conversely, if ϕ is a drift then a straightforward induction shows that for all $a, b \in \mathcal{B}$ and all $k \geq 0$,

$$\phi^{*k}(ab) = \sum_{p=0}^k \binom{k}{p} \phi^{*p}(a)\phi^{*(k-p)}(b)$$

with $\phi^{*0} = \varepsilon$. Multiplicativity of φ_t then follows from the equalities

$$\begin{aligned} \varphi_t(a)\varphi_t(b) &= \sum_{k,k'=0}^{+\infty} \frac{t^{k+k'}}{k!k'} \phi^{*k}(a)\phi^{*k'}(b) \\ &= \sum_{r=0}^{+\infty} \frac{t^r}{r!} \sum_{i=0}^r \frac{r!}{i!(r-i)!} \phi^{*i}(a)\phi^{*(r-i)}(b) \\ &= \sum_{r=0}^{+\infty} \frac{t^r}{r!} \phi^{*r}(ab) = \varphi_t(ab). \end{aligned}$$

2.2. A Wick-type formula for Gaussian generating functionals. The defining property of Gaussian generating functionals gives a way to compute their value on a product of elements by centering them and then applying the property recursively. This can be turned into a closed formula which is reminiscent of the Wick formula for operators on the full Fock space.

Proposition 2.7. *Let ϕ be a Gaussian generating functional on \mathcal{B} . Then we have for any $n \geq 2$ and any $a_1, \dots, a_n \in \mathcal{B}$,*

$$\begin{aligned} \phi(a_1 \cdots a_n) &= \sum_{1 \leq j < k \leq n} \phi(a_j a_k) \varepsilon(a_1 \cdots \check{a}_j \cdots \check{a}_k \cdots a_n) \\ &\quad - (n-2) \sum_{1 \leq j \leq n} \phi(a_j) \varepsilon(a_1 \cdots \check{a}_j \cdots a_n), \end{aligned}$$

where \check{a}_j means that this factor is omitted from the product.

Proof. For $n = 2$ this is trivially true, and for $n = 3$ it is one of the equivalent characterisations in Proposition 2.5.

The general case follows by induction. Let us first observe that for any $j, n \in \mathbb{N}$,

$$\begin{aligned} \phi(a_j a_n a_{n+1}) &= \phi(a_j a_n) \varepsilon(a_{n+1}) + \phi(a_j a_{n+1}) \varepsilon(a_n) + \phi(a_n a_{n+1}) \varepsilon(a_j) \\ &\quad - \phi(a_j) \varepsilon(a_n a_{n+1}) - \phi(a_n) \varepsilon(a_j a_{n+1}) - \phi(a_{n+1}) \varepsilon(a_j a_n). \end{aligned}$$

Now, let $n \geq 3$ and $a_1, \dots, a_{n+1} \in \mathcal{B}$. Then we have

$$\begin{aligned}
\phi(a_1 \cdots (a_n a_{n+1})) &= \sum_{1 \leq j < k < n} \phi(a_j a_k) \varepsilon(a_1 \cdots \check{a}_j \cdots \check{a}_k \cdots a_n a_{n+1}) \\
&+ \sum_{1 \leq j < n} \underbrace{\phi(a_j a_n a_{n+1})}_{\phi(a_j a_n a_{n+1})} \varepsilon(a_1 \cdots \check{a}_j \cdots a_{n-1}) \\
&- (n-2) \sum_{1 \leq j < n} \phi(a_j) \varepsilon(a_1 \cdots \check{a}_j \cdots a_n a_{n+1}) \\
&- (n-2) \phi(a_n a_{n+1}) \varepsilon(a_1 \cdots a_{n-1}) \\
&= \sum_{1 \leq j < k \leq n+1} \phi(a_j a_k) \varepsilon(a_1 \cdots \check{a}_j \cdots \check{a}_k \cdots a_{n+1}) \\
&- (n-1) \sum_{1 \leq j \leq n+1} \phi(a_j) \varepsilon(a_1 \cdots \check{a}_j \cdots a_{n+1}).
\end{aligned}$$

□

This implies immediately the following lemma which will be useful to determine conditions guaranteeing that Gaussian generating functionals vanish on certain ideals.

Corollary 2.8. *Assume that we have two subsets $X = \{a_1, \dots, a_n\} \subseteq \mathcal{B}$ and $Y = \{b_1, \dots, b_m\} \subseteq \ker(\varepsilon)$ such that*

- (1) X generates \mathcal{B} as an algebra;
- (2) $0 = \phi(b_k) = \phi(a_j b_k) = \phi(b_k a_j)$ for all $j \in \{1, \dots, n\}$ and all $k \in \{1, \dots, m\}$.

Then ϕ vanishes on the ideal generated by Y .

Proof. It suffices to show that $\phi(a_{j_1} \cdots a_{j_s} b_k a_{j_{s+1}} \cdots a_{j_{s+t}})$ vanishes for all $s, t \in \mathbb{N} \cup \{0\}$, $j_1, \dots, j_{s+t} \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$. Proposition 2.7 allows to do that by reducing this value to a linear combination of terms of the form appearing in Condition (2). □

3. THE GAUSSIAN PART OF A COMPACT QUANTUM GROUP

In this section we restrict the context of our study to Hopf $*$ -algebras related to compact quantum groups. We introduce the notion of the Gaussian part of a compact quantum group, discuss its basic properties, and prove that it is necessarily a quantum group of Kac type.

3.1. Compact quantum groups. In this work, we are interested in Gaussian processes on compact quantum groups. We will therefore briefly introduce these objects. We refer the reader to [Tim08] and [NT13] for detailed treatments of the theory. It is known since the work of M. Dijkhuizen and T. Koornwinder [DK94] that compact quantum groups can be treated algebraically through the following notion of a CQG-algebra.

Definition 3.1. A *CQG-algebra* is a Hopf $*$ -algebra which is spanned by the coefficients of its finite-dimensional unitary corepresentations.

If G is a compact group, then its algebra of regular functions $\mathcal{O}(G)$ is a CQG-algebra. Based on that example, and in an attempt to retain the intuition coming from the classical setting, we will denote a general CQG-algebra by $\mathcal{O}(\mathbb{G})$ and say that it corresponds to the *compact quantum group* \mathbb{G} . If Γ is a discrete group and $\mathbb{C}[\Gamma]$ denotes its group algebra, it is easy to endow it with a Hopf $*$ -algebra structure with coproduct given by $\Delta(g) = g \otimes g$ for all $g \in \Gamma$. Since this turns each $g \in \Gamma \subset \mathbb{C}[\Gamma]$ into a one-dimensional co-representation, it yields

a CQG-algebra. The resulting compact quantum group is called the *dual* of Γ and is denoted by $\hat{\Gamma}$.

We will at some point use arguments involving representation theory of compact quantum groups, which is just another point of view on the corepresentation theory of the corresponding CQG-algebra. Let us give a definition to make this clear.

Definition 3.2. An n -dimensional representation of a compact quantum group \mathbb{G} is an element $v \in M_n(\mathcal{O}(\mathbb{G}))$ which is invertible and such that for all $1 \leq i, j \leq n$,

$$\Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj}.$$

It is said to be *unitary* if it is unitary as an element of $M_n(\mathcal{O}(\mathbb{G}))$.

Given two representations v and w , one can form their direct sum by considering a block diagonal matrix with blocks v and w respectively, and their tensor product by considering the matrix with coefficients

$$(v \otimes w)_{(i,k),(j,\ell)} = v_{ij} w_{k\ell}.$$

In this setting, an intertwiner between two representations v and w of dimension respectively n and m will be a linear map $T : \mathbb{C}^n \rightarrow \mathbb{C}^m$ such that $Tv = wT$ (we are here identifying $M_n(\mathbb{C})$ with $M_n(\mathbb{C} \cdot 1_{\mathcal{O}(\mathbb{G})}) \subset M_n(\mathcal{O}(\mathbb{G}))$). The set of all intertwiners between v and w will be denoted by $\text{Mor}_{\mathbb{G}}(v, w)$.

If T is injective, then v is said to be a *subrepresentation* of w , and if w admits no non-zero subrepresentation apart from itself, then it is said to be *irreducible*. One of the fundamental results in the representation theory of compact quantum groups is due to S.L. Woronowicz in [Wor98] and can be summarized as follows:

Theorem 3.3 (Woronowicz). *Any finite-dimensional representation of a compact quantum group splits as a direct sum of irreducible ones, and any irreducible representation is equivalent to a unitary one.*

3.2. Definition and basic properties. Recall that if we are given two compact quantum groups \mathbb{G} and \mathbb{H} , then we say that \mathbb{H} is a (closed quantum) subgroup of \mathbb{G} if there is a surjective Hopf *-algebra morphism

$$q_{\mathbb{H}} : \mathcal{O}(\mathbb{G}) \rightarrow \mathcal{O}(\mathbb{H}).$$

In that case, $\mathcal{O}(\mathbb{H})$ is naturally a Hopf-quotient of $\mathcal{O}(\mathbb{G})$; we will sometimes denote the corresponding Hopf *-ideal by $\mathcal{I}_{\mathbb{H}}$. As we are dealing with CQG-algebras, each Hopf *-ideal \mathcal{I} of $\mathcal{O}(\mathbb{G})$ in fact determines a compact quantum group \mathbb{H} which is a subgroup of \mathbb{G} such that $\mathcal{I}_{\mathbb{H}} = \mathcal{I}$. Further note that as $q_{\mathbb{H}}$ preserves in particular the respective counits, it is easily checked that $q_{\mathbb{H}}(K_n(\mathcal{O}(\mathbb{G}))) = K_n(\mathcal{O}(\mathbb{H}))$ for all $n \in \mathbb{N} \cup \{\infty\}$.

Given a family $(\mathbb{H}_i)_{i \in I}$ of quantum subgroups of \mathbb{G} , we define its intersection $\bigwedge_{i \in I} \mathbb{H}_i$ as the quantum subgroup corresponding to the Hopf *-ideal generated by all the $\mathcal{I}_{\mathbb{H}_i}$, $i \in I$ (which is nothing but their algebraic sum). Conversely the subgroup generated by a given family $(\mathbb{H}_i)_{i \in I}$ of quantum subgroups of \mathbb{G} is defined as the quantum subgroup corresponding to the largest Hopf *-ideal contained in the intersection of the Hopf *-ideals $\mathcal{I}_{\mathbb{H}_i}$, $i \in I$ (note that an intersection of Hopf ideals is not Hopf in general), and will be denoted $\bigvee_{i \in I} \mathbb{H}_i$. All this is discussed in detail for instance in [CHK17].

Let us now move to generating functionals.

Definition 3.4. Let \mathbb{G} be a compact quantum group with a quantum subgroup \mathbb{H} . We say that a generating functional $\phi : \mathcal{O}(\mathbb{G}) \rightarrow \mathbb{C}$ *factors through* \mathbb{H} if there exists a functional $\phi_{\mathbb{H}} : \mathcal{O}(\mathbb{H}) \rightarrow \mathbb{C}$ such that $\phi_{\mathbb{H}} = \phi \circ q_{\mathbb{H}}$.

Assume that a generating functional $\phi : \mathcal{O}(\mathbb{G}) \rightarrow \mathbb{C}$ factors through \mathbb{H} . Then $\phi_{\mathbb{H}}$ as above is automatically a generating functional itself (this follows from the facts stated earlier) and moreover the whole Schürmann triple of $\phi_{\mathbb{H}}$ factors through \mathbb{H} , by which we mean that that if $(\rho_{\mathbb{H}}, \eta_{\mathbb{H}}, \phi_{\mathbb{H}})$ is a (surjective) Schürmann triple for $\phi_{\mathbb{H}}$, then $(\rho_{\mathbb{H}} \circ q_{\mathbb{H}}, \eta_{\mathbb{H}} \circ q_{\mathbb{H}}, \phi)$ is a (surjective) Schürmann triple for ϕ . Further observe that if ϕ as above factors through \mathbb{H} and is Gaussian (respectively, a drift) then $\phi_{\mathbb{H}}$ is also Gaussian (respectively, a drift).

Note that if ϕ as above factors through \mathbb{H} , then it also factors through any quantum subgroup of \mathbb{G} containing \mathbb{H} .

Definition 3.5. Let \mathbb{G} be a compact quantum group. We define the Gaussian part of \mathbb{G} to be intersection of all quantum subgroups of \mathbb{G} through which all Gaussian functionals factor. Note that the intersection is not empty since all functionals factor through \mathbb{G} itself. The drift part is defined similarly.

The Gaussian (respectively, drift) part of \mathbb{G} is determined by the largest Hopf *-ideal of $\mathcal{O}(\mathbb{G})$ contained in the intersection of all $\text{Ker}(\phi)$ with ϕ Gaussian (respectively, a drift). In other words, it is the smallest quantum subgroup \mathbb{H} of \mathbb{G} such that all Gaussian functionals (respectively, drifts) factor through \mathbb{H} . The next result tells us about the behaviour of this construction when we move between a quantum group and its subgroup.

Proposition 3.6. *Let \mathbb{G} be a compact quantum group and let \mathbb{H} be a quantum subgroup of \mathbb{G} . Then, the Gaussian part of \mathbb{G} contains the Gaussian part of \mathbb{H} . The same result holds for the drift parts.*

Proof. Let us denote the respective Gaussian parts by \mathbb{K} and \mathbb{K}' . The latter is viewed as a subgroup of \mathbb{H} – hence corresponds to a Hopf *-ideal of $\mathcal{O}(\mathbb{H})$ that we denote by $\mathcal{I}_{\mathbb{K}'}$. But we can also view it as a subgroup of \mathbb{G} , and then the corresponding Hopf *-ideal is $q_{\mathbb{H}}^{-1}(\mathcal{I}_{\mathbb{K}'})$, as can easily be checked. Thus our claim is equivalent to the inclusion

$$\mathcal{I}_{\mathbb{K}} \subset q_{\mathbb{H}}^{-1}(\mathcal{I}_{\mathbb{K}'}).$$

But we have already mentioned that

$$\bigcap_{\phi \text{ Gaussian}} \text{Ker}(\phi) \subset q_{\mathbb{H}}^{-1} \left(\bigcap_{\phi' \text{ Gaussian}} \text{Ker}(\phi') \right).$$

This ends the proof of the first part, if we note that as $q_{\mathbb{H}}$ is a surjective Hopf *-morphism, for any set $X \subset \mathcal{O}(\mathbb{H})$ if \mathcal{J} is the largest Hopf *-ideal contained in X , then $q_{\mathbb{H}}^{-1}(\mathcal{J})$ is the largest Hopf *-ideal contained in $q_{\mathbb{H}}^{-1}(X)$.

The proof for the drift part is similar. □

Definition 3.7. For a quantum group \mathbb{G} , we denote its Gaussian part by $\text{Gauss}(\mathbb{G})$ and for brevity we call \mathbb{G} Gaussian if $\mathbb{G} = \text{Gauss}(\mathbb{G})$.

Corollary 3.8. *If \mathbb{G} is a compact quantum group with a quantum subgroup \mathbb{H} , then $\text{Gauss}(\mathbb{H}) \subset \text{Gauss}(\mathbb{G})$. On the other hand if $\text{Gauss}(\mathbb{G})$ is contained in \mathbb{H} , then $\text{Gauss}(\mathbb{H}) = \text{Gauss}(\mathbb{G})$. Further if \mathbb{G} is generated by its Gaussian quantum subgroups, it is Gaussian itself.*

Proof. Follows immediately from Proposition 3.6 and respective definitions. □

3.3. Kac property of the Gaussian part of a compact quantum group. A compact quantum group is said to be *of Kac type* if (among many other equivalent characterizations, see for instance [NT13, Prop 1.7.9]), its antipode is involutive, i.e. $S^2 = \text{id}$. It turns out that this condition is closely linked to gaussianity. To express this, we will use the notion of maximal Kac type quantum subgroup introduced in [Sol05] but slightly revisited. Recall (see for instance [NT13, Sec 1.4]) that to any irreducible unitary representation $U = (u_{ij})_{i,j=1}^n$ of \mathbb{G} one can associate a unique positive invertible operator Q such that $\text{Tr}(Q) = \text{Tr}(Q^{-1})$ and $Q\bar{U}Q^{-1}$ is unitary. It is easy to see that conjugating U by a unitary scalar matrix again yields a unitary representation, hence we may, and will, assume that Q is diagonal. Let $(q_i)_{1 \leq i \leq n}$ be its diagonal coefficients, i.e., its eigenvalues. Then the formula

$$\tau_t(u_{ij}) = \left(\frac{q_i}{q_j} \right)^{-it} u_{ij}, \quad t \in \mathbb{R}, i, j = 1, \dots, n,$$

defines consistently a one-parameter group of *-homomorphisms of $\mathcal{O}(\mathbb{G})$, called the *scaling group* of \mathbb{G} (see for instance [NT13, Prop 1.7.6] for a proof).

This can be used to give a new description of the maximal Kac type quantum subgroup of a compact quantum group \mathbb{G} .

Lemma 3.9. *Let \mathbb{G} be a compact quantum group and let I_{Kac} be the ideal generated by the range of $S^2 - \text{id}$. Then, I_{Kac} is a Hopf *-ideal and the compact quantum group \mathbb{G}_{Kac} defined by $\mathcal{O}(\mathbb{H}) = \mathcal{O}(\mathbb{G})/I_{\text{Kac}}$ is the maximal Kac type quantum subgroup of \mathbb{G} . Moreover we also have that I_{Kac} is the ideal generated by the union of the ranges of all $\tau_s - \text{id}$, $s \in \mathbb{R}$.*

Proof. Note that the image of $S^2 - \text{id}$ is a selfadjoint subspace of $\mathcal{O}(\mathbb{G})$, as follows from the formula $S^2 \circ * = * \circ S^{-2}$. Moreover if $x \in \mathcal{O}(\mathbb{G})$, then using Sweedler's notation we have

$$\begin{aligned} \Delta(S^2(x) - x) &= (S^2 \otimes S^2 - \text{id}) \circ \Delta(x) \\ &= \sum S^2(x_{(1)}) \otimes S^2(x_{(2)}) - \sum x_{(1)} \otimes x_{(2)} \\ &= \sum (S^2(x_1) - x_1) \otimes S^2(x_2) + \sum x_{(1)} \otimes (S^2(x_2) - x_2) \end{aligned}$$

In other words, $\text{Im}(S^2 - \text{id})$ is a *-coideal.

Let now \mathbb{K} be a quantum subgroup of \mathbb{G} given by $\pi : \mathcal{O}(\mathbb{G}) \rightarrow \mathcal{O}(\mathbb{K})$. Then \mathbb{K} is of Kac type if and only if $(S^2 - \text{id}) \circ \pi = 0$. Because, π is a Hopf algebra homomorphism, this is equivalent to $\pi \circ (S^2 - \text{id}) = 0$. In other words, \mathbb{K} is of Kac type if and only if the range of $S^2 - \text{id}$ is contained in $\ker(\pi)$. As a consequence, the intersection of all Hopf *-ideals containing the range of $S^2 - \text{id}$ gives rise to the maximal Kac type quantum subgroup of \mathbb{G} , and that intersection is exactly the ideal I_{Kac} .

As for the last two statements, we can simply repeat the proof above, using the following facts: for any $t \in \mathbb{R}$ we have that τ_t is *-preserving and $\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$, and \mathbb{K} is of Kac type if and only if $(\tau_s - \text{id}) \circ \pi = 0$ for all $s \in \mathbb{R}$. \square

We will need hereafter a variant of that characterization, involving an explicit description of the map S^2 . More precisely, the family $(\tau_t)_{t \in \mathbb{R}}$ has a unique holomorphic extension $(\tau_z)_{z \in \mathbb{C}}$ which is given exactly by the same formula, and we have the equality $S^2 = \tau_{-i}$.

Corollary 3.10. *Let $\text{Irr}(\mathbb{G})$ be a complete set of pairwise inequivalent irreducible unitary representations of \mathbb{G} and fix for each $U \in \text{Irr}(\mathbb{G})$ an eigenbasis of the corresponding matrix Q .*

Then, with the notations above, the ideal I_{Kac} is generated by the set

$$\{u_{ij} \mid U = (u_{ij})_{i,j=1}^n \in \text{Irr}(\mathbb{G}), i, j = 1, \dots, n, q_i \neq q_j\}.$$

Proof. We know by Lemma 3.9 that I_{Kac} is generated by the range of $S^2 - \text{id}$. Because the coefficients of the elements of $\text{Irr}(\mathbb{G})$ form a basis of $\mathcal{O}(\mathbb{G})$, the range of $S^2 - \text{id}$ is the span of the images of all these coefficients. If $q_i = q_j$, then $S^2(u_{ij}) = u_{ij}$ while if $q_i \neq q_j$, then

$$u_{ij} = \left(\frac{q_i}{q_j} - 1\right)^{-1} (S^2(u_{ij}) - u_{ij}) \in \text{Im}(S^2 - \text{id}),$$

hence the result. \square

With this we can establish the link between Gaussian processes and the maximal Kac type quantum subgroup.

Theorem 3.11. *Let \mathbb{G} be a compact matrix quantum group. Any Gaussian process on \mathbb{G} factors through the maximal Kac type quantum subgroup of \mathbb{G} .*

Proof. By Lemma 3.9, we have to prove that ϕ vanishes on I_{Kac} . It is in fact enough to prove that it vanishes on $X = \text{Im}(S^2 - \text{id})$, i.e. that ϕ is S^2 -invariant, thanks to Corollary 2.8.

We first claim that for any Gaussian cocycle η we have $\eta \circ (S + \text{id}) = 0$. Indeed, for any $x \in \mathcal{O}(\mathbb{G})$,

$$\begin{aligned} 0 &= \eta(\varepsilon(x)1) \\ &= \eta(x_{(1)}S(x_{(2)})) \\ &= \eta(x_{(1)})\varepsilon(S(x_{(2)})) + \varepsilon(x_{(1)})\eta(S(x_{(2)})) \\ &= \eta(x_{(1)})\varepsilon(x_{(2)}) + \varepsilon(S(x_{(1)}))\eta(S(x_{(2)})) \\ &= \eta(x) + \eta(S(x)) \end{aligned}$$

where the last step uses the fact that X is a $*$ -coideal. This implies $\eta \circ S^2(x) = \eta(x)$ for all $x \in \mathcal{O}(\mathbb{G})$, hence η vanishes on X . As a consequence, for any $a \in X$,

$$\begin{aligned} 0 &= \phi(\varepsilon(a)1) = \phi(a_{(1)}S(a_{(2)})) \\ &= \phi(\varepsilon(a_{(1)})S(a_{(2)})) + \phi(a_{(1)}\varepsilon(S(a_{(2)}))) + \langle \eta(a_{(1)}^*), \eta(S(a_{(2)})) \rangle \\ &= \phi(S(a)) + \phi(a) - \langle \eta(a_{(1)}^*), \eta(a_{(2)}) \rangle \\ &= \phi(S(a)) + \phi(a). \end{aligned}$$

This implies that for any $a \in X$, $\phi(S^2(a)) = a$. Let us now consider an irreducible unitary representation $U = (u_{ij})_{i,j=1}^n$ and consider $i, j = 1, \dots, n$ such that $q_i \neq q_j$. Then $u_{ij} \in X$ by Corollary 3.10, so that

$$\phi(u_{ij}) = \phi(S^2(u_{ij})) = \frac{q_i}{q_j} \phi(u_{ij}).$$

This implies $\phi(u_{ij}) = 0$, hence the result. \square

4. STRONG CONNECTEDNESS

Before turning to examples, it will be useful to have at hand a condition which is necessary for a compact quantum group to be Gaussian. To introduce it, recall that by definition the intersection of the kernels of all Gaussian functionals contains K_3 . Thus, any Hopf $*$ -ideal contained in K_3 is contained in $\mathcal{I}_{\text{Gauss}}$. In particular, since K_∞ is such an ideal (see below),

it must be trivial as soon as \mathbb{G} is Gaussian. Our purpose in this section is to explore the condition $K_\infty = \{0\}$.

4.1. The definition. Before going further, let us give a proof of the coideal property for K_∞ which was alluded to in the beginning. It does in fact follow from a more general result.

Proposition 4.1. *Let A be a Hopf algebra and let I be a Hopf ideal. Then,*

$$I^\infty = \bigcap_{k \geq 1} I^k$$

is a Hopf ideal.

Proof. First observe that I^∞ is by construction an ideal which is contained in $I \subset \ker(\varepsilon)$ and invariant under the antipode. Moreover for any $k \in \mathbb{N}$ we have

$$\Delta(I^{2k}) \subset \sum_{n=0}^{2k} I^n \otimes I^{2k-n} \subset I^k \otimes A + A \otimes I^k.$$

As a consequence, we have the inclusion

$$(I^k)^\perp \cdot (I^k)^\perp \subset (I^{2k})^\perp$$

of vector subspaces of the dual algebra A^* (endowed with the convolution product). This implies that the increasing union

$$\bigcup_{k \geq 1} (I^k)^\perp$$

is a subalgebra of A^* and it then follows (from instance from [Abe04, Thm 2.3.6 (i)]) that

$$J = \left(\bigcup_{k \geq 1} (I^k)^\perp \right)^\perp = \left\{ x \in A \mid f(x) = 0 \text{ for all } f \in A^* \text{ such that } \exists_{k \in \mathbb{N}} f|_{I^k} = 0 \right\}$$

is a coideal of A . We now claim that $I^\infty = J$, which is enough to conclude.

Let us first consider $x \in I^\infty$. Then, if $f : A \rightarrow \mathbb{C}$ is a linear map which vanishes on I^{k_0} for some $k_0 \in \mathbb{N}$, it vanishes on I^∞ , hence in particular on x . In other words, $x \in J$. Conversely, let $x \notin I^\infty$. Then, there exists $k_0 \in \mathbb{N}$ such that $x \notin I^{k_0}$. By taking a basis of I^{k_0} and completing it into a basis of A containing x , we see that there exist a linear map $f : A \rightarrow \mathbb{C}$ such that $f(x) = 1$ and $f|_{I^{k_0}} = 0$. As a consequence, x is not in the kernel of all the elements of $\bigcup_{k \geq 1} (I^k)^\perp$, which means that it is not in J , concluding the proof. \square

Our goal in this section is to investigate compact quantum groups with the property that $K_\infty = \{0\}$. To get a better intuition for the meaning of that condition, let us consider the classical case.

Lemma 4.2. *Let G be a classical compact group. Then $K_\infty = \{0\}$ if and only if G is connected.*

Proof. It follows from standard Gelfand duality arguments that the closure of K_∞ in the C^* -algebra $C(G)$ of continuous complex-valued functions on G is the ideal of functions vanishing outside some open subgroup $H \subset G$. Since any open subgroup in a topological group is also closed, it follows that H is a union of connected components of G . Let now Z be a connected component of G not containing the neutral element. Then, the indicator function $p = \mathbf{1}_Z$ is a continuous function on G and moreover belongs to K_1 . Thus, $p = p^n \in K_n$ for all $n \in \mathbb{N}$,

i.e. $p \in K_\infty$ so that Z is not contained in H . As a conclusion, H is the connected component of the identity and the result follows from the fact that $K_\infty = \{0\}$ if and only if its closure equals $\{0\}$. \square

In view of this result, we might want to call a compact quantum group \mathbb{G} connected if $K_\infty = \{0\}$. However, there is already a notion of connectedness in the literature, introduced by Wang in [Wan09] and studied in detail in [CDPR14]: a compact quantum group is said to be *connected* if $\mathcal{O}(\mathbb{G})$ does not contain any finite-dimensional Hopf $*$ -subalgebra. It turns out that vanishing of K_∞ is stronger than this (see Proposition 4.5 below), hence we choose the following terminology.

Definition 4.3. A compact quantum group \mathbb{G} is said to be *strongly connected* if $K_\infty = \{0\}$.

To see the link with the aforementioned definition of connectedness, we need to generalize the idea concerning projections used in the proof of Lemma 4.2.

Lemma 4.4. *Let \mathbb{G} be a compact quantum group and let $p \in \mathcal{O}(\mathbb{G})$ be a non-trivial projection. Then, $p - \varepsilon(p)1 \in K_\infty(\mathcal{O}(\mathbb{G})) \neq \{0\}$.*

Proof. Because the counit ε is a $*$ -homomorphism, $\varepsilon(p)$ is an idempotent in \mathbb{C} , hence equals 0 or 1. In both cases, $q = p - \varepsilon(p) \in K_1$ and satisfies $q^2 \in \{q, -q\}$. Thus, $q = \pm q^n \in K_n$ for all n , hence $q \in K_\infty$. \square

We are now ready to state and prove several fundamental properties of strong connectedness. Note that for duals of discrete groups, this has already been studied as the residual nilpotency of the augmentation ideal of the group ring, and the monograph [Pas79] gives a comprehensive survey of the known results to which we will refer.

Proposition 4.5. *Let \mathbb{G} be a compact quantum group. The following hold:*

- (1) *If \mathbb{G} is the dual of a discrete group Γ , then it is strongly connected if and only if Γ is residually ‘torsion-free nilpotent’ (recall that it means that for any non-identity element $\gamma \in \Gamma$, there is a normal subgroup N of Γ such that $\gamma \notin N$ and Γ/N is torsion-free nilpotent);*
- (2) *If \mathbb{G} is strongly connected, then it is connected but the converse does not hold in general;*
- (3) *If \mathbb{G} is topologically generated (in the sense of [Chi20]) by strongly connected quantum subgroups, then it is strongly connected.*

Proof. (1) This is the contents of [Pas79, Thm VI.2.26].

- (2) Assume that \mathbb{G} is not connected and $\mathcal{O}(\mathbb{H}) \subset \mathcal{O}(\mathbb{G})$ is a finite-dimensional Hopf $*$ -subalgebra. As $\mathcal{O}(\mathbb{H})$ is a finite-dimensional C^* -algebra, it contains non-zero projections, hence the result follows from Lemma 4.4.

For the dual of a discrete group, connectedness is equivalent to torsion-freeness. Hence, the dual of any torsion-free group which is not residually nilpotent is connected but not strongly connected.

- (3) Assume that \mathbb{G} is topologically generated by quantum subgroups $(\mathbb{H}_i)_{i \in I}$ and let us consider the corresponding surjective $*$ -homomorphisms $\pi_i : \mathcal{O}(\mathbb{G}) \rightarrow \mathcal{O}(\mathbb{H}_i)$. Obviously, $\pi_i(K_\infty(\mathcal{O}(\mathbb{G}))) \subset K_\infty(\mathcal{O}(\mathbb{H}_i))$ so that under the assumption of the statement,

$$K_\infty(\mathcal{O}(\mathbb{G})) \subset \bigcap_{i \in I} \ker(\pi_i).$$

By definition of topological generation, the right-hand side does not contain any non-zero Hopf *-ideal, hence the result. \square

Remark. The first point in the previous proposition suggests that there may be a connection between strong connectedness of a compact quantum group and torsion-freeness of its dual discrete quantum groups. There is however no relationship with torsion-freeness in the sense of Meyer [Mey08]. Indeed,

- (1) $SO(N)$ is connected but not simply connected, hence it is strongly connected but its dual is not torsion free (it has a projective representation coming from its universal covering which yields an ergodic finite-dimensional action not equivariantly Morita equivalent to the trivial one);
- (2) If Γ is a torsion-free group which is not residually nilpotent, then it is torsion-free while its dual is not strongly connected.

The topological generation criterion can prove useful to provide examples which are neither commutative nor cocommutative.

Proposition 4.6. *The free unitary quantum group U_N^+ is strongly connected for all $N \in \mathbb{N}$.*

Proof. For $N = 1$, U_N^+ is just the circle, which is strongly connected because it is connected. For $N \geq 2$, the result immediately follows from the following two facts:

- U_N^+ is topologically generated by $U(N)$ and \mathbb{F}_N by [Chi20];
- $K_\infty(\mathcal{O}(U(N))) = \{0\}$ by connectedness while $K_\infty(\mathbb{C}[\mathbb{F}_N]) = \{0\}$ as \mathbb{F}_N is residually ‘torsion-free nilpotent’ by a result of Magnus [Mag35].

\square

For O_N^+ one cannot apply directly the same strategy, because the diagonal quotient is \mathbb{Z}_2^{*N} , which is not torsion free. To get a better insight, let us deal with the very special case $N = 2$.

Lemma 4.7. *The strongly connected component of the identity of O_2^+ is the circle \mathbb{T} .*

Proof. Recall that O_2^+ is isomorphic to $SU_{-1}(2)$, so that $\mathcal{O}(O_2^+)$ is generated by two elements α and γ such that γ is normal, anti-commutes with α and

$$\alpha\alpha^* + \gamma\gamma^* = 1.$$

Moreover, $\gamma \in K_1(\mathcal{O}(O_2^+))$ and $\beta = \alpha - 1 \in K_1(\mathcal{O}(O_2^+))$. Now, the equation above yields

$$\beta + \beta^* = \beta^*\beta + \gamma^*\gamma \in K_2$$

while the anti-commutation relations translate into

$$\gamma(\beta + \beta^*) = -(\beta + \beta^*)\gamma - 4\gamma$$

so that

$$\gamma = \frac{-1}{4}(\gamma(\beta + \beta^*) + (\beta + \beta^*)\gamma).$$

As a consequence, if $\gamma \in K_n$ then $\gamma \in K_{n+1}$. Since $\gamma \in K_1$, it follows by a straightforward induction that $\gamma \in K_\infty$. Thus, $\langle \gamma \rangle \subset K_\infty$. On the other hand, if π denotes the surjection onto $\mathcal{O}(\mathbb{T}) = \mathcal{O}(O_2^+)/\langle \gamma \rangle$, we have

$$\pi(K_\infty(\mathcal{O}(O_2^+))) \subset K_\infty(\mathcal{O}(\mathbb{T})) = \{0\},$$

so that $K_\infty \subset \langle \gamma \rangle$, concluding the proof. \square

4.2. Totally strongly disconnected quantum groups. It is quite natural to consider the Hopf *-algebra $\mathcal{O}(\mathbb{G})/K_\infty$ to be the function algebra on the strongly connected component of the identity. We will therefore denote it by $\mathcal{O}(\mathbb{G}^{00})$, the double 0 exponent being meant to distinguish it from the connected component of the identity \mathbb{G}^0 in the sense of [CDPR14]. Of course, if G is classical then this coincides with the usual connected component of the identity. Another extreme case, in contrast to being strongly connected, is when \mathbb{G}^{00} reduces to the trivial group.

Definition 4.8. A compact quantum group is said to be *totally strongly disconnected* if \mathbb{G}^{00} is trivial, i.e. if $K_\infty = K_1$.

Remark. One may wonder whether the property of being totally strongly disconnected can be expressed in terms of representation theory. We do not know, but we can at least mention that it is not a property of the corresponding C*-tensor category since it is not preserved under monoidal equivalence. Indeed, we will see below that S_N^+ is totally strongly disconnected, while the quantum automorphism group of $(M_N(\mathbb{C}), \text{tr})$, which is monoidally equivalent to $S_{N^2}^+$, is strongly connected (because U_N^+ is).

By definition, \mathbb{G}^{00} is strongly connected, hence connected. As a consequence, it is a quantum subgroup of the connected component of the identity \mathbb{G}^0 in the sense of [CDPR14, Def 4.11]. This implies that if \mathbb{G} is totally disconnected in the sense of [CDPR14], then it is also totally strongly disconnected.

Before going further, let us describe that property for duals of discrete groups. Interestingly, there turns out to be a simple characterization. We will denote by $\gamma_2(\Gamma)$ the subgroup of Γ generated by commutators and by $\sqrt{\gamma_2(\Gamma)}$ the group of all elements of Γ of which a finite power lies in $\gamma_2(\Gamma)$.

Proposition 4.9. *Let Γ be a discrete group. Then $\widehat{\Gamma}$ is totally strongly disconnected if and only if its abelianization is torsion.*

Proof. Obviously, $K_\infty = K_1$ if and only if $K_2 = K_1$ if and only if $K_2 + 1 = \mathbb{C}[\Gamma]$. Taking intersections with Γ in the last equality then yields by [Pas79, Thm IV.1.5] the equivalent condition $\sqrt{\gamma_2(\Gamma)} = \Gamma$, which in turn means that the abelianization of Γ is torsion. \square

Remark. A group has torsion abelianization if and only if it has no non-zero homomorphism to \mathbb{Q} . This can in turn be restated in a homological way by saying that the first Betti number of Γ vanishes, or that $H_1(\Gamma, \mathbb{Q}) = \{0\}$.

Totally strongly disconnected compact quantum groups are interesting to us because they have, by definition, trivial Gaussian part. Here is a sufficient criterion for total strong disconnectedness, which will yield our first examples of computation of Gaussian parts.

Proposition 4.10. *If \mathbb{G} is a compact quantum group such that $\mathcal{O}(\mathbb{G})$ is generated by projections, then it is strongly totally disconnected.*

Proof. Assume that $\mathcal{O}(\mathbb{G})$ is generated by projections $(p_i)_{i \in I}$. By Lemma 4.4, $p_i - \varepsilon(p_i) \in K_\infty$ for all $i \in I$ and since these elements generate K_1 , the proof is complete. \square

Remark. The quantum permutation group S_N^+ is totally strongly disconnected in our sense while it is connected in the sense of [CDPR14]. This shows that strong connectedness is strictly stronger than connectedness.

Some straightforward examples to which the previous statement applies are the following:

- Finite quantum groups, i.e. those for which the corresponding Hopf algebra is finite-dimensional;
- Quantum permutation groups, i.e. quantum subgroups of S_N^+ for $N \in \mathbb{N}$ (this includes for instance the quantum reflection groups H_N^{s+} for all $1 \leq s < +\infty$);
- Profinite compact quantum groups in the sense of [CDPR14], or in the terms of [CS19], duals of locally finite discrete quantum groups. Note also that [CDPR14] gives examples of duals of discrete groups which are totally disconnected as compact quantum groups, hence totally strongly disconnected, but not profinite.

Remark. It is natural to wonder whether when \mathbb{G} is totally strongly disconnected, then $\mathcal{O}(\mathbb{G})$ is generated by projections. A negative answer can be provided by an example of a discrete group Γ with torsion abelianization such that $\mathbb{C}[\Gamma]$ does not contain any non-trivial projection. For instance, the group whose existence is given in [GKO16, Cor 3.2] is torsion-free (because it is orderable) and amenable (because it is locally solvable) hence satisfies the Kadison-Kaplansky conjecture (there is no non-trivial idempotent in $\mathbb{C}[\Gamma]$) but has trivial abelianization since it is perfect.

4.3. Link to Kac type quantum groups. Even though the definition of strong connectedness is very general, it turns out that it entails a strong restriction on compact quantum groups. More precisely, a strongly connected compact quantum group must be of Kac type.

Proposition 4.11. *A strongly connected compact quantum group is of Kac type.*

Proof. Let \mathbb{G} be a compact quantum group. Following the notations of Corollary 3.10, let $U \in \text{Irr}(\mathbb{G})$ be an n -dimensional unitary representation of \mathbb{G} .

For $j, k = 1, \dots, n$, set $\hat{u}_{jk} = u_{jk} - \delta_{jk}1 \in K_1$. Then if $j \neq k$ we have

$$0 = \sum_{\ell=1}^n u_{j\ell} u_{k\ell}^* = \sum_{\ell=1}^n \hat{u}_{j\ell} \hat{u}_{k\ell}^* + u_{jk} + u_{kj}^*$$

and

$$0 = \sum_{\ell=1}^n \frac{q_k}{q_\ell} u_{\ell k} u_{j\ell}^* = \sum_{\ell=1}^n \frac{q_k}{q_\ell} \hat{u}_{\ell k} \hat{u}_{j\ell}^* + \frac{q_k}{q_j} u_{jk} + u_{kj}^*.$$

Taking the difference, we get

$$\left(\frac{q_k}{q_j} - 1 \right) u_{jk} = \sum_{\ell=1}^n \hat{u}_{j\ell} \hat{u}_{k\ell}^* - \sum_{\ell=1}^n \frac{q_k}{q_\ell} \hat{u}_{\ell k} \hat{u}_{j\ell}^*.$$

This shows that if $q_j \neq q_k$ then $u_{jk} = \hat{u}_{jk} \in K_2$, and by taking adjoints, $u_{jk}^* = \hat{u}_{jk}^* \in K_2$. If we now look again at the expression above, still assuming $q_j \neq q_k$, we see that each element of the sum on the right hand side is a product of elements in K_1 and K_2 (this follows, as whenever $q_j \neq q_k$ we necessarily have for any $\ell = 1, \dots, n$ that either $q_\ell \neq q_j$ or $q_\ell \neq q_k$). Thus in fact $u_{jk} = \hat{u}_{jk} \in K_3$, and similarly $u_{jk}^* = \hat{u}_{jk}^* \in K_3$. By a straightforward induction, we conclude that $u_{jk}, u_{jk}^* \in K_\infty$ whenever $q_j \neq q_k$.

By Corollary 3.10, the elements above generate I_{Kac} as an ideal, hence $I_{\text{Kac}} \subset K_\infty$. \square

As noted in the beginning of the section, because $K_\infty \subset K_3$, any Gaussian functional factors through the strongly connected part. Combining that observation with the previous statement yields Theorem A of the introduction.

5. EXAMPLES I

In this section we fully characterise the Gaussian parts of classical compact groups and duals of classical discrete groups.

5.1. Classical compact groups. We will now determine the Gaussian part of any classical compact group. We will in fact show a stronger result concerning the drift part of any compact quantum group. This will be done in two steps, the first being a reduction to the classical case. To do so, let us say that for a compact quantum group \mathbb{G} , its *classical version* is the quantum subgroup corresponding to the Hopf $*$ -ideal of commutators (see for instance [Daw16]).

Proposition 5.1. *Let \mathbb{G} be a compact quantum group. Then, its drift part is the closed subgroup of its classical version generated by all continuous one-parameter subgroups.*

Proof. We start by observing that since $[a, b] = [a - \varepsilon(a), b - \varepsilon(b)]$, the Hopf $*$ -ideal of commutators of $\mathcal{O}(\mathbb{G})$ is contained in K_2 . Hence, any drift vanishes on it so that if \mathbb{H} denotes the drift part of \mathbb{G} , then \mathbb{H} is contained in the quantum group obtained by quotienting $\mathcal{O}(\mathbb{G})$ with the commutators, i.e. in the classical version of \mathbb{G} .

Because characters on $\mathcal{O}(G)$ correspond to elements of G , Remark 2.1 implies that drifts are in one-to-one correspondence with continuous one-parameter subgroups. Now if ϕ is a drift and H is a closed subgroup of G , then as soon as ϕ factors through the restriction map $\mathcal{O}(G) \rightarrow \mathcal{O}(H)$, the corresponding one-parameter subgroup lives in H . Thus, any closed subgroup through which all drifts factor must contain the closed subgroup K generated by all continuous one-parameter subgroups. Since K is obviously its own drift part, the proof is complete. \square

Remark. There is one subtlety in the statement, which is that the drift part is by definition a closed subgroup. Hence, the result states that the subgroup generated by one-parameter subgroups in a connected compact group is dense, but not that it is itself equal to the whole group. Indeed, solenoids such as the Pontryagin dual of the additive group of rationals have a unique one-parameter subgroup which is not closed but which is dense.

We are now left with characterizing the drift part of a classical compact group, and this is easily done using the notion of generalized Lie algebra.

Proposition 5.2. *The drift part of a compact group G is the connected component of the identity.*

Proof. We will prove the equivalent statement that G equals its drift part if and only if it is connected. One way is clear: the subgroup generated by one-parameter subgroups is path-connected, hence also connected, so that its closure is connected. As for the other direction, recall that any connected compact group is a projective limit of connected compact Lie groups. As a consequence, these are LP-groups in the sense of [Las57, Def 3.1]. It then follows from [Las57, Thm 3.5] that the connected component of the identity is the closure of the range of the exponential map on the generalized Lie algebra and is therefore contained in the drift part. \square

Corollary 5.3. *The Gaussian part of a classical compact group is the connected component of the identity.*

Proof. It follows from Proposition 5.2 that if G is connected, then it equals its drift part, which is itself contained in the Gaussian part, so that any connected compact group is Gaussian. Conversely, if G is Gaussian then it is strongly connected, hence connected by Lemma 4.2. \square

5.2. Duals of discrete groups. The next case to consider is that of duals of discrete groups. Let us recall that for a group Γ , one defines its canonical lower central series by setting $\gamma_1(\Gamma) = \Gamma$ and for any $n \in \mathbb{N}$ putting $\gamma_{n+1}(\Gamma)$ to be the subgroup generated by $[\gamma_n(\Gamma), \Gamma]$. Moreover, for a subgroup $\Lambda \subset \Gamma$, we write $\sqrt{\Lambda}$ for the subgroup of all elements $\gamma \in \Gamma$ such that there exists $n \in \mathbb{N}$ satisfying $\gamma^n \in \Lambda$.

Before proving the main result of this section, we need to clarify the connection between group commutators and the structure of the group algebra.

Lemma 5.4. *Let Γ be a discrete group and let $g, h, k \in \Gamma$. Then,*

$$[[g, h], k] - 1 \in K_3(\mathbb{C}[\Gamma]).$$

Proof. For clarity, we will denote by $[\cdot, \cdot]_\Gamma$ the group commutator and by $[\cdot, \cdot]_{\mathbb{C}[\Gamma]}$ the algebra commutator. Start by noticing that for $g, h \in \Gamma$,

$$[g, h]_\Gamma - 1 = [g, h]_{\mathbb{C}[\Gamma]} g^{-1} h^{-1}.$$

It then follows that

$$\begin{aligned} [[g, h], k]_\Gamma - 1 &= [[g, h]_\Gamma, k]_{\mathbb{C}[\Gamma]} [g, h]_\Gamma^{-1} k^{-1} \\ &= [[g, h]_{\mathbb{C}[\Gamma]} g^{-1} h^{-1}, k]_{\mathbb{C}[\Gamma]} [g, h]_\Gamma^{-1} k^{-1}. \end{aligned}$$

Now observe that for any $a, b \in \mathbb{C}[\Gamma]$,

$$[a, b]_{\mathbb{C}[\Gamma]} = [a - \varepsilon(a), b - \varepsilon(b)]_{\mathbb{C}[\Gamma]}$$

so that in particular all algebra commutators are in K_2 and if $a \in K_2$, then the commutator is in K_3 . The result then follows. \square

We are now ready for the characterization of Gaussianity for duals of discrete groups. Recall that a group is said to be *nilpotent of class 2* if all its commutators are central.

Theorem 5.5. *Let Γ be a finitely generated discrete group. Then,*

$$\text{Gauss}(\widehat{\Gamma}) = \Gamma / \sqrt{\widehat{\gamma_3(\Gamma)}}.$$

In particular, $\widehat{\Gamma}$ is Gaussian if and only if Γ is torsion-free nilpotent of class 2.

Proof. Assume first that $\widehat{\Gamma}$ is Gaussian. The subgroup $\gamma_3(\Gamma)$ being normal, the set $\mathbb{C}[\gamma_3(\Gamma)] - 1$ is a Hopf *-ideal in $\mathbb{C}[\Gamma]$. Because it is contained in K_3 by Lemma 5.4, all Gaussian functionals vanish on it by definition so that they factor through $\Gamma / \widehat{\gamma_3(\Gamma)}$. Thus, $\gamma_3(\Gamma)$ must be trivial so that Γ is nilpotent of class 2. Let furthermore $g \in \Gamma \setminus \{e\}$ and $n \in \mathbb{N}$ be such that $g^n = e$. Then,

$$p_g = \frac{1}{n} \sum_{k=0}^{n-1} g^k$$

is a non-trivial projection in $\mathbb{C}[\Gamma]$, hence $\widehat{\Gamma}$ is not strongly connected by Lemma 4.4, contradicting Gaussianity. Therefore, we have proven that Γ is torsion-free nilpotent of class 2.

This means that for an arbitrary discrete group Γ , the Gaussian part of $\widehat{\Gamma}$ is of the form $\widehat{\Lambda}$ with Λ a torsion-free nilpotent of class 2 quotient of Γ . There is one such quotient which is maximal in the sense that all the other ones factor through it, namely $\Gamma / \sqrt{\widehat{\gamma_3(\Gamma)}}$. To conclude it is enough to prove that the duals of all such groups are Gaussian. To do this, note first

that $\Lambda/Z(\Lambda)$ is an abelian finitely generated group. By [Mal49], $\Lambda/Z(\Lambda)$ is also torsion free, thus

$$\widehat{\Lambda/Z(\Lambda)} \cong \mathbb{T}^n$$

for some $n \in \mathbb{N}_0$ and $\widehat{\Lambda/Z(\Lambda)}$ is Gaussian because it is a connected compact group. By Corollary 3.8, this means that $\text{Gauss}(\widehat{\Lambda})$ contains $\widehat{\Lambda/Z(\Lambda)}$, or in other words that $\text{Gauss}(\widehat{\Lambda}) = \widehat{\Lambda/\Theta}$ where $\Theta < Z(\Lambda)$.

It remains then to show that for every $\gamma_0 \in Z(\Lambda) \setminus \{e\}$ there is a Gaussian generating functional on $\mathbb{C}[\Lambda]$ – i.e. conditionally positive-definite function ϕ on Λ of Gaussian type – such that $\phi(\gamma_0) \neq 0$. But once again we have $Z(\Lambda) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}_0$ and using the usual Laplacian on \mathbb{T}^m we first get a function $\phi_0 : Z(\Lambda) \rightarrow \mathbb{C}$ as above by setting

$$\phi_0(k_1, \dots, k_m) = \sum_{i=1}^m k_i^2,$$

and then extend it to Λ as follows:

$$\phi(\gamma) = \begin{cases} \phi_0(\gamma) & \gamma \in Z(\Lambda) \\ 0 & \gamma \notin Z(\Lambda) \end{cases}.$$

An elementary check shows that ϕ is a conditionally positive-definite function of Gaussian type. \square

Remark. In the case of group algebras, it turns out that K_3 is a Hopf ideal, and it follows from [Pas79, Thm IV.1.5] that it equals $\mathbb{C}[\sqrt{\gamma_3(\Gamma)}] - 1$. We could have used this (involved) result to prove directly that a discrete group dual is Gaussian only if it is torsion-free nilpotent of class 2. We have chosen however to give a direct and self-contained proof which is furthermore completely elementary.

Let us conclude this section with a word on free wreath products of a discrete group by the quantum permutation group S_N^+ . We refer the reader to [Bic04] for the definition of these objects, whose Gaussian part can be easily expressed in terms of the building discrete group.

Proposition 5.6. *Let $N \in \mathbb{N}$ and let Γ be a discrete group. Then*

$$\text{Gauss}(\widehat{\Gamma} \wr_* S_N^+) = \text{Gauss}(\widehat{\Gamma}^{*N}).$$

Proof. Recall that we have a generating family of N -dimensional representations $\{u(g) : g \in \Gamma\}$ whose coefficients satisfy in particular the relations

$$u_{ij}(g)u_{ik}(h) = \delta_{jk}u_{ij}(gh),$$

and

$$\varepsilon(u(g)_{ij}) = \delta_{ij}$$

for all $g, h \in \Gamma$ and all $1 \leq i, j \leq N$. It follows that for any $g \in \Gamma$ and $i, j = 1, \dots, n, i \neq j$,

$$u_{ij}(g) = u_{ij}(g)(u_{ii}(g) - 1) \in K_2$$

and by induction, all the off-diagonal coefficients are in K_∞ . As a consequence, the Gaussian part factors through the diagonal quantum subgroup, which is the dual of Γ^{*N} . \square

In particular, the Gaussian part of $H_N^{\infty+}$ is the free residually “torsion-free nilpotent” group of rank N .

6. EXAMPLES II

In this, last section we describe Gaussian parts of q -deformations, and discuss the case of free quantum groups.

6.1. q -deformations. Consider a simply connected semisimple compact Lie group G equipped with a standard Poisson structure and the deformation parameter $q \in (0, 1)$. The quantisation procedure due to Korogodski and Soibelman [KS98], see also [NT12], leads to a compact quantum group \mathbb{G}_q . The procedure is compatible with the deformation of Poisson subgroups of G ; in particular the maximal torus, i.e. a maximal abelian connected subgroup of G , which is unique up to a conjugation, remains (a classical) subgroup of \mathbb{G}_q .

Proposition 6.1. *For a simply connected semisimple compact Lie group G and $q \in (0, 1)$ the Gaussian part of the quantum group \mathbb{G}_q is the maximal torus $\mathbf{T} \subset \mathbb{G}_q$.*

Proof. Lemma 4.10 in [Tom07], based on the knowledge of the representation theory of $\mathcal{O}(\mathbb{G}_q)$, shows that the maximal torus \mathbf{T} coincides with the Kac part of \mathbb{G}_q . As \mathbf{T} is by definition connected, the proof is complete. \square

6.2. Deformations of the orthogonal group. We will now consider the *half-liberated quantum orthogonal group* O_N^* introduced in [BS09]. Recall that $\mathcal{O}(O_N^*)$ is defined to be the quotient of $\mathcal{O}(O_N^+)$ by the relations $abc = cba$ for all $a, b, c \in \{u_{jk} \mid 1 \leq j, k \leq N\}$. We will prove that its Gaussian part is that of the classical group O_N . This requires first characterizing Gaussian functionals on O_N^+ lying in its classical part. More precisely, any Gaussian functional on O_N^* yields a Gaussian functional on O_N^+ by composition with the quotient map, and it is easy to determine when such a Gaussian functional in fact factors through the abelianization.

Lemma 6.2. *Let $N \in \mathbb{N}$. A Gaussian triple on O_N^+ factors through O_N if and only if the corresponding cocycle η satisfies*

$$\langle \eta(a), \eta(b) \rangle \in \mathbb{R} \quad \forall a, b \in \{u_{jk} \mid 1 \leq j, k \leq N\}.$$

Proof. This follows immediately from the equivalence of (ii) and (iv) in Remark 2.1. \square

We can now elucidate the Gaussian part of O_N^* .

Proposition 6.3. *Let $N \in \mathbb{N}$. The Gaussian part of O_N^* is SO_N .*

Proof. First note that for any Gaussian generating functional we have

$$\begin{aligned} \phi(abc) &= \varepsilon(ab)\phi(c) + \varepsilon(ac)\phi(b) + \varepsilon(bc)\phi(a) \\ &\quad + \langle \eta(a^*), \eta(b) \rangle \varepsilon(c) + \langle \eta(a^*), \eta(c) \rangle \varepsilon(b) + \langle \eta(b^*), \eta(c) \rangle \varepsilon(a). \end{aligned}$$

Thus, if $j, k = 1, \dots, N$ and we have $\eta(u_{jk}) = v_{jk} \in \mathbb{C}$, then the condition $\phi(abc) = \phi(cba)$ for $a, b, c \in \{u_{jk} \mid 1 \leq j, k \leq N\}$ becomes

$$\delta_{jk} \overline{v_{\ell m}} v_{np} + \delta_{\ell m} \overline{v_{jk}} v_{np} + \delta_{np} \overline{v_{jk}} v_{\ell m} = \delta_{np} \overline{v_{\ell m}} v_{jk} + \delta_{\ell m} \overline{v_{np}} v_{jk} + \delta_{jk} \overline{v_{np}} v_{\ell m}$$

for all $j, k, \ell, m, n, p \in \{1, \dots, N\}$. Now, by [DFKS18, Proposition 3.7], $v = (v_{jk})_{j,k=1}^N$ has to be anti-symmetric, therefore $v_{jj} = 0$ for $j = 1, \dots, N$. We have several possibilities:

- If two or three of the index pairs contain twice the same index, e.g. $j = k$ and $\ell = m$, then the all terms vanish;
- If none of the index pairs contains twice the same index, i.e. if $j \neq k$, $\ell \neq m$, and $n \neq p$, then we also find $0 = 0$, thanks to the Kronecker symbols;

- The only non-trivial case occurs if only one index pair contains twice the same index. Without loss of generality we can assume that this is the first pair, i.e. $j = k$. In that case we find the condition

$$\overline{v_{\ell m}}v_{np} = \overline{v_{np}}v_{\ell m}$$

for all $\ell, m, n, p \in \{1, \dots, N\}$.

The last condition exactly means that $\langle \eta(u_{\ell m}), \eta(u_{np}) \rangle \in \mathbb{R}$, so that we can conclude by Lemma 6.2 that all Gaussian triples factor through O_N . As a consequence,

$$\text{Gauss}(O_N^*) = \text{Gauss}(O_N) = SO_N.$$

□

We can deal in a similar way with another variant of O_N^+ called the *twisted orthogonal quantum group* \overline{O}_N and introduced in [BBC07] as a type of q -deformation of orthogonal groups at $q = -1$. More precisely, we let $\mathcal{O}(\overline{O}_N)$ be the quotient of $\mathcal{O}(O_N^+)$ by the relations :

$$u_{ij}u_{kl} = \begin{cases} -u_{k\ell}u_{ij} & \text{if } i = k \ \& \ j \neq \ell \ \text{or } i \neq k \ \& \ j = \ell \\ u_{k\ell}u_{ij} & \text{otherwise} \end{cases}$$

In other words, the generators anti-commute if they are on the same row or column and commute otherwise.

Proposition 6.4. *The Gaussian part of \overline{O}_N is trivial for any $N \in \mathbb{N}$.*

Proof. The computations of Lemma 6.2 show that the condition for the second case is equivalent to $\overline{v_{ij}}v_{kl} \in \mathbb{R}$ for any $i, j, k, l = 1, \dots, N$ as soon as $i \neq k$ and $j \neq l$. As for the first case, observe that

$$\phi(u_{ij}u_{i\ell} + u_{i\ell}u_{ij}) = 2(\delta_{ij}\phi(u_{i\ell}) + \delta_{i\ell}\phi(u_{ij})) + 2\text{Re}(\overline{v_{ij}}v_{i\ell}).$$

Recalling that for any Gaussian functional on O_N^+ , v is anti-symmetric (see [DFKS18, Section 3]), we get for $\ell \neq j = i$

$$\phi(u_{i\ell}) = \text{Re}(\overline{v_{ii}}v_{i\ell}) = 0$$

so that ϕ factors through the quotient by the ideal generated by all the off-diagonal coefficients. The quotient is easily seen to be the dual of $\mathbb{Z}_2^{\times N}$, which has trivial Gaussian part because it is totally strongly disconnected, hence the result. □

6.3. Free quantum groups.

Proposition 6.5. *For $N \geq 4$, the Gaussian part of O_N^+ is neither classical nor dual to a discrete group. The same holds for U_N^+ with $N \geq 2$.*

Proof. Let us first note the Gaussian part of the dual of the free group \mathbb{F}_2 on two generators is not classical. Indeed, it arises from the quotient of \mathbb{F}_2 obtained by the relations making the commutator of the two generators central, which is easily seen to be isomorphic to the discrete Heisenberg group. As a consequence, O_4^+ contains, through the quotient $\mathcal{O}(O_4^+) \rightarrow \mathcal{O}(O_2^+ * O_2^+)$, the dual of the (non-abelian) Heisenberg group, which is Gaussian. It also contains the nonabelian group SO_4 , which is also Gaussian. It then follows from Corollary 3.8 that $\text{Gauss}(O_4^+)$ contains a classical non-abelian group and a dual of a non-commutative discrete group. For $N > 4$, simply observe that O_N^+ contains O_4^+ , hence a similar containment holds for their Gaussian parts, again by Corollary 3.8.

As for U_N^+ , it contains the dual of the free group \mathbb{F}_N , hence the free 2-nilpotent torsion-free group, as well as the connected compact group U_N . □

We do not know whether U_N^+ is Gaussian or not. Let us notice however that the problem can be reduced to the case $N = 2$ thanks to the following observation.

Lemma 6.6. *If \mathbb{G} is topologically generated by Gaussian quantum subgroups $(\mathbb{G}_i)_{i \in I}$, then it is Gaussian.*

Proof. Let I be a Hopf $*$ -ideal on which all Gaussian functionals of \mathbb{G} vanish. Given a Gaussian functional ϕ in $\mathcal{O}(\mathbb{G}_i)$, $\phi \circ \pi_i$ is a Gaussian functional on $\mathcal{O}(\mathbb{G})$, hence it vanishes on I . In other words, $\pi_i(I)$ is a Hopf $*$ -ideal annihilating all Gaussian functionals, hence $\pi_i(I) = 0$ by assumption. This is in turn equivalent to $I \subset \bigcap_{i \in I} \ker(\pi_i)$, which by topological generation forces $I = 0$. \square

Because U_N^+ is topologically generated by U_{N-1}^+ and U_N for all $N \geq 3$ by [Chi20], it follows by induction that if U_2^+ is Gaussian, then U_N^+ is Gaussian for all $N \geq 2$. Nevertheless, it seems difficult to understand all the Gaussian processes on U_2^+ . One strategy to prove Gaussianity would be to find enough Gaussian quantum subgroups to topologically generate everything. For instance U_2 and the dual of the Heisenberg group \mathbb{H}_3 are both Gaussian quantum subgroups of U_2^+ , leading to the question: is U_2^+ topologically generated by U_2 and the dual of \mathbb{H}_3 ? More generally one may ask: for which quotients Γ of \mathbb{F}_2 is U_2^+ topologically generated by U_2 and the dual of Γ ?

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