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Notes on the Chernoff estimate

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*Dedicated to Igor Volovich, friend, colleague and coauthor,
on the occasion of his 75th birthday.*

The purpose of the present notes is to examine the following issues related to the the Chernoff estimate: (1) For contractions on a Banach space we modify the \sqrt{n} -estimate and apply it in the proof of the Chernoff product formula for C_0 -semigroups in the *strong* operator topology. (2) We use the idea of a probabilistic approach, proving the Chernoff estimate in the strong operator topology, to uplift it to the *operator-norm* estimate for *quasi-sectorial* contraction semigroups. (3) The operator-norm Chernoff estimate is applied to quasi-sectorial contraction semigroups for proving the operator-norm convergence of the *Dunford-Segal* approximants.

Keywords: Semigroup theory; Chernoff lemma; Quasi-sectorial contractions; Product formulae.

1. Introduction

The Chernoff \sqrt{n} -Lemma, see Lemma 2 in Ref. 5, is known as a key tool in the theory of semigroup approximations, see, for example, Ref. 8 (Chapter III, Section 5) and Ref. 24. A large variety of applications of the Chernoff approximation method (in the strong operator topology) one finds in a recent survey Ref. 3 (Section 2). For the reader convenience and for motivation of the present notes we show this lemma in the following below.

Lemma 1.1. *Let bounded operator C on a Banach space \mathfrak{X} ($C \in \mathcal{L}(\mathfrak{X})$) be a contraction, i.e., $\|C\| \leq 1$. Then $\{e^{t(C-\mathbf{1})}\}_{t \geq 0}$ is a norm-continuous contraction semigroup on \mathfrak{X} and one has the estimate*

$$\|(C^n - e^{n(C-\mathbf{1})})x\| \leq \sqrt{n} \|(C - \mathbf{1})x\| , \quad (1.1)$$

for all $x \in \mathfrak{X}$ and natural $n \in \mathbb{N}$.

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Proof. To prove the inequality (1.1) we use the representation

$$C^n - e^{n(C-1)} = e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} (C^m - C^n). \quad (1.2)$$

To proceed we insert

$$\|(C^n - C^m)x\| \leq \|(C^{|n-m|} - 1)x\| \leq |m - n| \|(C - 1)x\|, \quad (1.3)$$

into (1.2) to obtain by the Cauchy-Schwarz inequality the estimate:

$$\begin{aligned} \|(C^n - e^{n(C-1)})x\| &\leq \|(C - 1)x\| e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} |m - n| \leq \\ &\left\{ \sum_{m=0}^{\infty} e^{-n} \frac{n^m}{m!} |m - n|^2 \right\}^{1/2} \|(C - 1)x\|, \quad x \in \mathfrak{X}, \end{aligned} \quad (1.4)$$

Note that the sum in the right-hand side of (1.4) can be calculated explicitly. This gives the value n , which yields (1.1). \square

The aim of the present notes is to scrutinise the following issues related to the the Chernoff estimate.

First, we modify for contractions on a Banach space \mathfrak{X} the \sqrt{n} -estimate (1.1) and apply new estimates *à la* Chernoff (see Section 2 and Section 3) in the proof of the Chernoff product formula for *strongly* continuous semigroups (C_0 -semigroups) in the *strong* operator topology, cf. Ref. 3.

Second, we use the idea of the *probabilistic* approach^{4,22}, that proving the Chernoff estimate in the strong operator topology (Section 2), to uplift it to the *operator-norm* estimate for a special class of contractions: the *quasi-sectorial* contractions, see Section 4.

Finally, in Section 5 we use the operator-norm Chernoff estimate for illustration of its *direct* application in the approximation theory of holomorphic C_0 -semigroups for m -sectorial generators (that is, for *quasi-sectorial* contractions) in a Hilbert space. This allows to prove, besides the *Euler* approximation formula, the operator-norm convergence with optimal rate of the *Dunford-Segal* approximants⁹.

We warn the readers against a confusion between our probabilistic approach^{4,22} to alternative proof of the Chernoff estimate and a probabilistic approach to representation of C_0 -semigroups¹⁸ exploited in Ref. 10 for developing the approximation theory of operator semigroups.

2. The $\sqrt[3]{n}$ -Lemma and Product Formulæ

We start by a technical lemma. It is a *revised* version of the estimate (1.1). Our *variational* estimate (2.5) in $\sqrt[3]{n}$ -Lemma 2.1 and the *probabilistic* approach are, in a certain sense, more flexible than (1.1). Indeed, a revised scheme of the proof will be used later (Section 4) for *uplifting* the convergence of the Chernoff and the Lie-Trotter product formulæ to the *operator-norm* topology.

Lemma 2.1. ($\sqrt[3]{n}$ -Lemma) *Let C be a contraction on a Banach space \mathfrak{X} . Then $\{e^{t(C-\mathbf{1})}\}_{t \geq 0}$ is a norm-continuous contraction semigroup on \mathfrak{X} and one has the estimate*

$$\|(C^n - e^{n(C-\mathbf{1})})x\| \leq \frac{n}{\epsilon_n^2} 2 \|x\| + \epsilon_n \|(\mathbf{1} - C)x\|, \quad n \in \mathbb{N}, \quad (2.5)$$

for all $x \in \mathfrak{X}$ and $\epsilon_n > 0$. For the optimal value of the parameter ϵ_n :

$$\epsilon_n^* := \left(\frac{4n \|x\|}{\|(\mathbf{1} - C)x\|} \right)^{1/3}, \quad (2.6)$$

in the right-hand side of (2.5) we obtain estimate

$$\|(C^n - e^{n(C-\mathbf{1})})\| \leq \frac{3}{2} \sqrt[3]{n} \|2(\mathbf{1} - C)\|^{2/3}, \quad (2.7)$$

which we call the $\sqrt[3]{n}$ -Lemma.

Proof. Since operator C is bounded and $\|C\| \leq 1$, the operator $(\mathbf{1} - C)$ is generator of a norm-continuous contraction semigroup since:

$$\|e^{-t(\mathbf{1}-C)}\| \leq e^{-t} \left\| \sum_{m=0}^{\infty} \frac{t^m}{m!} C^m \right\| \leq 1, \quad t \geq 0. \quad (2.8)$$

For proving the estimate (2.5) we use representation

$$C^n - e^{n(C-\mathbf{1})} = e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} (C^n - C^m). \quad (2.9)$$

Then we *split* the sum (2.9) into two parts: the *central* part for $|m-n| \leq \epsilon_n$ and the *tails* for $|m-n| > \epsilon_n$. Optimisation of the *splitting* parameter ϵ_n in (2.5) yields the best estimate (2.7).

For evaluation the *tails* we use the *Tchebychëv inequality*. Let $X_n \in \mathbb{N}_0$ be the *Poisson random variable* with the *rate* parameter $n \in \mathbb{N}$, that is, with the probability distribution $\mathbb{P}\{X_n = m\} = n^m e^{-n}/m!$. Then one gets for

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expectation: $\mathbb{E}(X_n) = n$, and for variance: $\text{Var}(X_n) := \mathbb{E}((X_n - \mathbb{E}(X_n))^2) = n$. That being so, the Tchebychëv inequality yields

$$\mathbb{P}\{|X_n - \mathbb{E}(X_n)| > \epsilon\} \leq \frac{\text{Var}(X_n)}{\epsilon_n^2}, \quad \text{for any } \epsilon_n > 0. \quad (2.10)$$

Note that although for any $x \in \mathfrak{X}$ there is an evident bound: $\|(C^n - C^m)x\| \leq 2\|x\|$, when estimating (2.9) we shall also use below inequalities

$$\begin{aligned} \|(C^n - C^m)x\| &= \|C^{n-k}(C^k - C^{m-n+k})x\| \\ &\leq |m-n| \|C^{n-k}(\mathbf{1} - C)x\|, \quad k = 0, 1, \dots, n, \end{aligned} \quad (2.11)$$

that keep *difference*: $(\mathbf{1} - C)x$. Then by $\|C\| \leq 1$ and by the Tchebychëv inequality (2.10) we obtain the estimate for *tails*:

$$\begin{aligned} e^{-n} \sum_{|m-n| > \epsilon_n} \frac{n^m}{m!} \|(C^n - C^m)x\| &\leq e^{-n} \sum_{|m-n| > \epsilon_n} \frac{n^m}{m!} \cdot 2\|x\| \\ &= \mathbb{P}\{|X_n - \mathbb{E}(X_n)| > \epsilon_n\} \cdot 2\|x\| \leq \frac{n}{\epsilon_n^2} 2\|x\|. \end{aligned} \quad (2.12)$$

To evaluate the *central* part of the sum (2.9), when $|m-n| \leq \epsilon_n$, note that by virtue of (2.11):

$$\begin{aligned} \|(C^n - C^m)x\| &\leq |m-n| \|C^{n-[\epsilon_n]}(\mathbf{1} - C)x\| \\ &\leq \epsilon_n \|(\mathbf{1} - C)x\|. \end{aligned} \quad (2.13)$$

Then we obtain:

$$e^{-n} \sum_{|m-n| \leq \epsilon_n} \frac{n^m}{m!} \|(C^n - C^m)x\| \leq \epsilon_n \|(\mathbf{1} - C)x\|, \quad x \in \mathfrak{X}, \quad (2.14)$$

for $n \in \mathbb{N}$. Estimate (2.14), together with (2.12), yield (2.5) for all $u \in \mathfrak{X}$ and $\epsilon_n > 0$.

Minimising the estimate (2.5) with respect to parameter $\epsilon_n > 0$ one gets for ϵ_n the optimal value (2.6) and

$$\frac{n}{\epsilon_n^*} 2\|x\| + \epsilon_n^* \|(\mathbf{1} - C)x\| = \frac{3}{2} \sqrt[3]{n} (4\|x\|)^{1/3} \|(\mathbf{1} - C)x\|^{2/3}, \quad (2.15)$$

for all $x \in \mathfrak{X}$ and $n \in \mathbb{N}$. As a consequence, (2.5) and (2.15) yield (2.7), which is the $\sqrt[3]{n}$ -Lemma. \square

Theorem 2.1. (Chernoff product formula^{5,6}) *Let $\Phi : t \mapsto \Phi(t)$ be a function from \mathbb{R}_0^+ to contractions on \mathfrak{X} such that $\Phi(0) = \mathbf{1}$. Let $\{U_A(t)\}_{t \geq 0}$ be a contraction C_0 -semigroup, and let domain $D \subset \text{dom}(A)$ be a core of related generator A .*

If the function $\Phi(t)$ has a strong right-derivative $\Phi'(+0)$ at $t = 0$ (that is, $\Phi'(+0)x$ exists for any $x \in \text{dom}(\Phi'(+0))$) and if

$$\Phi'(+0)x := \lim_{t \rightarrow +0} \frac{1}{t}(\Phi(t) - \mathbf{1})x = -Ax, \quad (2.16)$$

for all $x \in D$, then

$$\lim_{n \rightarrow \infty} [\Phi(t/n)]^n x = U_A(t)x, \quad (2.17)$$

for all $t \in \mathbb{R}_0^+$ and $x \in \mathfrak{X}$.

Proof. Consider the bounded approximations $\{A_n(s)\}_{n \geq 1}$ of generator A :

$$A_n(s) := \frac{\mathbf{1} - \Phi(s/n)}{s/n}, \quad s \in \mathbb{R}^+, \quad n \in \mathbb{N}. \quad (2.18)$$

Note that these operators are m -accretive: $\|(A_n(s) + \zeta \mathbf{1})^{-1}\| \leq (\text{Re}(\zeta))^{-1}$ for $\text{Re}(\zeta) > 0$ and for any $n \in \mathbb{N}$. By $\|\Phi(t)\| \leq 1$ together with (2.18) and condition (2.16) we obtain $\|e^{-t A_n(s)}\| \leq 1$, but also

$$\lim_{n \rightarrow \infty} A_n(s)x = Ax, \quad (2.19)$$

for all $x \in D$ and any $s \in \mathbb{R}^+$. Then, given that $D = \text{core}(A)$, by virtue of the Trotter-Neveu-Kato generalised strong convergence theorem (see, e.g., Ref. 7 (Theorem 3.17), or Ref. 8 (Chapter III, Theorem 4.9)) one obtains

$$\lim_{n \rightarrow \infty} e^{-t A_n(s)} x = U_A(t)x, \quad x \in \mathfrak{X}, \quad s > 0, \quad t \in \mathbb{R}_0^+. \quad (2.20)$$

So, (2.20) is the strong and *uniform* in t and in s convergence of contractive approximants $\{e^{-t A_n(s)}\}_{n \geq 1}$ for $t \in [0, \tau]$ and $s \in (0, s_0]$.

Now, by Lemma 2.1 (2.7) we obtain for contraction $C := \Phi(t/n)$:

$$\begin{aligned} \|[\Phi(t/n)]^n x - e^{-t A_n(t)} x\| &= \|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbf{1})}) x\| \\ &\leq \frac{3}{2} \sqrt[3]{n} (4 \|x\|)^{1/3} \|(\mathbf{1} - \Phi(t/n))x\|^{2/3}, \quad x \in \mathfrak{X}. \end{aligned} \quad (2.21)$$

Since by (2.19) one gets for any $x \in D$ and uniformly on $(0, t_0]$:

$$\lim_{n \rightarrow \infty} \sqrt[3]{n} \|(\mathbf{1} - \Phi(t/n))x\|^{2/3} = \lim_{n \rightarrow \infty} t^{2/3} n^{-1/3} \|A_n(t)x\|^{2/3} = 0, \quad (2.22)$$

equations (2.21) and (2.22) provide uniformly on $(0, t_0]$

$$\lim_{n \rightarrow \infty} \|[\Phi(t/n)]^n x - e^{-t A_n(t)} x\| = 0, \quad x \in D. \quad (2.23)$$

Then (2.20) and (2.23) yield uniformly in $t \in [0, t_0]$ the limit:

$$\lim_{n \rightarrow \infty} [\Phi(t/n)]^n x = U_A(t)x, \quad x \in D. \quad (2.24)$$

Note that by *density* of D and by the *uniform* estimate $\| [\Phi(t/n)]^n x - e^{-tA_n(t)} x \| \leq 2 \|x\|$ the convergence in (2.23) can be extended to all $x \in \mathfrak{X}$. Indeed, it is known that on the *bounded* subsets of $\mathcal{L}(\mathfrak{X})$ the topology of *point-wise* convergence on a *dense* subset $D \subset \mathfrak{X}$ coincides with the *strong* operator topology, see, e.g., Ref. 14 (Chapter III, Lemma 3.5). As a consequence, the limit (2.23), when being extended to $x \in \mathfrak{X}$, and limit (2.20) yield the extension of (2.24) to (2.17). \square

The limit (2.17) that involves derivative (2.16) is called the *Chernoff product formula* for contractive C_0 -semigroup $\{U_A(t)\}_{t \geq 0}$ in the *strong* operator topology, cf. Ref. 8 (Chapter III, Section 5a.).

Proposition 2.1. (Lie-Trotter product formula^{6,20}) *Let A, B and C be generators of contraction C_0 -semigroups on \mathfrak{X} . Suppose that algebraic sum*

$$Cx = Ax + Bx, \tag{2.25}$$

is valid for all $x \in D$, where domain $D = \text{core}(C)$. Then the semigroup $\{U_C(t)\}_{t \geq 0}$ can be approximated on \mathfrak{X} in the strong operator topology by the Lie-Trotter product formula:

$$e^{-tC} x = \lim_{n \rightarrow \infty} (e^{-tA/n} e^{-tB/n})^n x, \quad x \in \mathfrak{X}, \tag{2.26}$$

for all $t \in \mathbb{R}_0^+$ and $C := \overline{(A + B)}$ is closure of the operator-sum in (2.25).

Proof. Let us define the contraction $\mathbb{R}_0^+ \ni t \mapsto \Phi(t)$, $\Phi(0) = \mathbb{1}$, by

$$\Phi(t) := e^{-tA} e^{-tB}. \tag{2.27}$$

Note that if $x \in D$, then derivative

$$\Phi'(+0)x = \lim_{t \rightarrow +0} \frac{1}{t} (\Phi(t) - \mathbb{1}) x = -(A + B) x. \tag{2.28}$$

Now we are in position to apply Theorem 2.1. This yields (2.26) for generator $C := \overline{(A + B)}$. \square

Corollary 2.1. *Extensions of the strongly convergent Lie-Trotter product formula of Proposition 2.1 to quasi-bounded and holomorphic semigroups follows through verbatim.*

3. More Chernoff's Estimates

In this section we show a one more Chernoff-type estimate (see (3.29)), which is of a *different* nature than variational estimate (2.5) ($\sqrt[3]{n}$ -Lemma

2.1). In fact, it is a kind of *improvement* of the original Chernoff estimate (1.1) (\sqrt{n} -Lemma 1.1), which is still restricted to convergence in the strong operator topology.

Theorem 3.1. *Let $C \in \mathcal{L}(\mathfrak{X})$ be contraction on a Banach space \mathfrak{X} . Then $\{e^{t(C-\mathbf{1})}\}_{t \geq 0}$ is a norm-continuous contractive semigroup on \mathfrak{X} and the following estimate*

$$\|(C^n - e^{n(C-\mathbf{1})})x\| \leq \frac{n}{2} \left(\|(C - \mathbf{1})^2 x\| + \frac{e^2}{3} \|(C - \mathbf{1})^3 x\| \right), \quad (3.29)$$

holds for all $n \in \mathbb{N}$ and $x \in \mathfrak{X}$.

Proof. The first assertion is proven in Lemma 2.1, see (2.8).

To prove inequality (3.29) we use the *telescopic* representation:

$$C^n - e^{n(C-\mathbf{1})} = \sum_{k=0}^{n-1} C^{n-k-1} (C - e^{(C-\mathbf{1})}) e^{k(C-\mathbf{1})}. \quad (3.30)$$

To proceed we exploit that operator $C \in \mathcal{L}(\mathfrak{X})$ is bounded and therefore

$$C - e^{(C-\mathbf{1})} = -\frac{1}{2} (\mathbf{1} - C)^2 - (\mathbf{1} - C)^3 \sum_{m=3}^{\infty} \frac{(-1)^m}{m!} (\mathbf{1} - C)^{m-3}, \quad (3.31)$$

for the operator-norm convergent series. Hence, owing to $\|C\| \leq 1$ one gets estimate

$$\left\| \sum_{m=3}^{\infty} \frac{1}{m!} (\mathbf{1} - C)^{m-3} \right\| \leq \frac{1}{6} e^{\|\mathbf{1}-C\|} \leq \frac{e^2}{6}. \quad (3.32)$$

Then on account of (3.30) - (3.32) and (2.8) we obtain inequality (3.29). \square

Corollary 3.1. (Chernoff product formula) *Let $\Phi : t \mapsto \Phi(t)$ be a function from \mathbb{R}_0^+ to contractions on \mathfrak{X} such that $\Phi(0) = \mathbf{1}$, which satisfies conditions of Theorem 2.1. Then*

$$\lim_{n \rightarrow \infty} \|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbf{1})})x\| = 0, \quad x \in \mathfrak{X}, \quad (3.33)$$

and as a result one gets the product formula (2.17).

Proof. On account of Theorem 3.1 we obtain by (3.29) the estimate

$$\begin{aligned} \|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbf{1})})x\| &\leq \\ \frac{t^2}{2n} \left(\left\| \frac{n^2}{t^2} (\mathbf{1} - \Phi(t/n))^2 x \right\| + \frac{2e^2}{3} \left\| \frac{n^2}{t^2} (\mathbf{1} - \Phi(t/n))^2 x \right\| \right), & \quad x \in \mathfrak{X}. \end{aligned} \quad (3.34)$$

Note that by (2.19) for any $t \in \mathbb{R}^+$ we have on the dense set $D = \text{core}(A)$:

$$\lim_{n \rightarrow \infty} \frac{n}{t} (\mathbb{1} - \Phi(t/n)) x = Ax, \quad x \in D. \quad (3.35)$$

Given that generator A of contractive C_0 -semigroup is *accretive*, the range of resolvent: $\text{ran}((A + \zeta \mathbb{1})^{-1}) = \mathfrak{X}$, for $\text{Re}(\zeta) > 0$. As a consequence Ref. 14 (Chapter III, Problem 2.9 and Chapter IX, §1.2), domain $\text{dom}(A^2) \subset \text{dom}(A)$ is *dense* in \mathfrak{X} and limit (3.35) provides

$$\lim_{n \rightarrow \infty} (A_n(t))^2 x = A^2 x, \quad x \in D \subset \text{dom}(A^2), \quad (3.36)$$

where $A_n(t) := (t/n)^{-1} (\mathbb{1} - \Phi(t/n))$, cf. (2.18), and $D = \text{core}(A)$.

By virtue of estimate (3.34) and (3.36) we obtain

$$\lim_{n \rightarrow \infty} \|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbb{1})}) x\| = 0, \quad x \in D. \quad (3.37)$$

Then similarly to concluding arguments in Theorem 2.1, saying that on the *bounded* subsets of $\mathcal{L}(\mathfrak{X})$ the topology of *point-wise* convergence on a *dense* subset $D \subset \mathfrak{X}$ coincides with the *strong* operator topology, the limit (3.37) can be extended to $x \in \mathfrak{X}$.

Now, given that $D = \text{core}(A)$, by virtue of the *Trotter-Neveu-Kato* theorem we obtain the limit (2.20), and owing to (3.37) for $x \in \mathfrak{X}$, we deduce the *Chernoff product formula* (2.17). \square

Remark 3.1. Resuming the Chernoff \sqrt{n} -estimate (1.1), and its varieties: (2.5) and (3.29), we conclude that due to the terms with difference $\|(C - \mathbb{1})x\|$ all of them control only the *strong* convergence of the product formulae. By definition (2.18) the *rates*: $R_n(t)$, of these converges *conditioned* to $x \in D$ have the following asymptotic form for $t > 0$ and large $n \in \mathbb{N}$:

- (a) For (1.1): $R_n(t) = \|A_n(t)x\|/\sqrt{n}$.
- (b) For (2.5): $R_n(t) = \|A_n(t)x\|^{2/3}/\sqrt[3]{n}$.
- (c) For (3.29): $R_n(t) = \|A_n(t)^2 x\|/n$.

Remark 3.2. None of these three methods has an evident straightforward extension that could ensure the *operator-norm* convergence of the Chernoff product formula²⁴. In the next Section 4 we show that only a relatively more sophisticated method (cf.(b)) based on the Tchebychëv inequality (Section 2) is, *a fortiori*, sufficiently accurate. Indeed, it allows an *uplifting* of convergence the Chernoff product formula to the operator-norm topology for *quasi-sectorial* contractions on a Hilbert space.

4. Operator-Norm Chernoff Estimate

Definition 4.1. ^{4,21} Contraction C on the Hilbert space \mathfrak{H} is called *quasi-sectorial* for semi-angle $\alpha \in [0, \pi/2)$ with respect to the vertex at $z = 1$, if its numerical range $W(C) \subseteq D_\alpha$. Here the subset of complex plane

$$D_\alpha := \{z \in \mathbb{C} : |z| \leq \sin \alpha\} \cup \{z \in \mathbb{C} : |\arg(1 - z)| \leq \alpha \text{ and } |z - 1| \leq \cos \alpha\}. \quad (4.38)$$

We comment that $D_\alpha \subset D_{\pi/2} = \mathbb{D}$ (unit disc) and recall that a *general* contraction C satisfies a weaker condition: $W(C) \subseteq \mathbb{D}$.

Note that if operator C is a quasi-sectorial contraction, then $\mathbb{1} - C$ is an m -sectorial operator with vertex $z = 0$ and semi-angle α . Consequently, the numerical range: $W(\mathbb{1} - C) \subset \overline{S}_\alpha$, for the closure of sector

$$S_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\}.$$

Then for operator C the limits: $\alpha = 0$ and $\alpha = \pi/2$, correspond respectively to self-adjoint and to contraction operators, whereas for $\mathbb{1} - C$ they give a non-negative self-adjoint and an m -accretive (bounded) operators.

The *resolvent* of an m -sectorial operator A , with semi-angle $\alpha \in [0, \alpha_0]$, for some $\alpha_0 < \pi/2$ and vertex at $z = 0$, that is, for $W(A) \subset \overline{S}_\alpha$, provides the first non-trivial example of a quasi-sectorial contraction. Indeed, the following assertion holds.

Proposition 4.1. *Let A be m -sectorial operator with semi-angle $\alpha \in [0, \pi/4]$ and vertex at $z = 0$. Then $\{F(t) := (\mathbb{1} + tA)^{-1}\}_{t>0}$ is a family of quasi-sectorial contractions, such that numerical ranges $W(F(t)) \subset D_\alpha$ for all $t > 0$.*

Proof. Seeing that the spectrum $\sigma(A)$ is a subset of the closure $\overline{W(A)}$ of numerical range $W(A) \subset \overline{S}_\alpha$, by the estimate of resolvent: $\|(A - z\mathbb{1})^{-1}\| \leq (\text{dist}(z, \overline{W(A)}))^{-1}$ and by $\overline{W(A)} \subseteq \overline{S}_\alpha$ we obtain the operator-norm bound

$$\|F(t)\| \leq \frac{1}{t \text{ dist}(1/t, -\overline{S}_\alpha)} = 1, \quad t > 0. \quad (4.39)$$

As a consequence the family of operators $\{F(t)\}_{t \geq 0}$ consists of contractions with numerical ranges $W(F(t)) \subset \mathbb{D}$.

Next, for any $u \in \mathfrak{H}$ ($\|u\| = 1$) one gets $(u, F(t)u) = (v_t, v_t) + t(Av_t, v_t) \in \overline{S}_\alpha$, where $v_t := F(t)u$. So, for all $t > 0$ numerical range $W(F(t)) \subseteq \overline{S}_\alpha$.

Similarly, one finds that $(u, (\mathbf{1} - F(t))u) = t(v_t, Av_t) + t^2(Av_t, Av_t) \in \overline{S}_\alpha$, that is, $W(\mathbf{1} - F(t)) \subseteq \overline{S}_\alpha$, or $W(F(t)) \subseteq (1 - \overline{S}_\alpha)$. Then for all $t > 0$:

$$W(F(t)) \subseteq (\overline{S}_\alpha \cap (1 - \overline{S}_\alpha)) \subset \mathbb{D}.$$

Moreover, by Definition 4.1 the condition $\alpha \leq \pi/4$ yields that $(\overline{S}_\alpha \cap (1 - \overline{S}_\alpha)) \subset D_\alpha$. Hence, for $\alpha \in [0, \pi/4]$ the operators $\{F(t)\}_{t \geq 0}$ are quasi-sectorial contractions with numerical range in D_α . \square

Note that the upper bound $\alpha \leq \pi/4$ is stemming from Definition 4.1 and observation that $(\overline{S}_\alpha \cap (1 - \overline{S}_\alpha)) \not\subseteq D_\alpha$ for $\alpha > \pi/4$, cf. (4.38).

Corollary 4.1. *Let A be an m -sectorial operator with semi-angle $\alpha \in [0, \pi/4]$ and with vertex at $z = 0$. Then $\{e^{-tA}\}_{t \geq 0}$ is a holomorphic quasi-sectorial contraction semigroup with numerical ranges $W(e^{-tA}) \subset D_\alpha$ for all $t > 0$ and one has the strongly convergent Euler limit:*

$$s - \lim_{n \rightarrow \infty} (\mathbf{1} + tA)^{-n} = e^{-tA}, \quad t \geq 0. \quad (4.40)$$

Remark 4.1. (Sketch of the proof.) We comment that holomorphic property of $\{e^{-zA}\}_{z \in S_{\pi/2-\alpha}}$ follows from conditions on generator A . Since A is m -sectorial with vertex at $z = 0$, it is *a fortiori* accretive. Then by standard arguments for construction of C_0 -semigroups (see Ref. 14, Chapter IX) yield, due to (4.39) for approximants $\{(\mathbf{1} + tA/n)^{-n}\}_{t \geq 0, n \in \mathbb{N}}$, the strongly convergent Euler formula (4.40). Note that although by Proposition 4.1 the family $\{(\mathbf{1} + tA)^{-1}\}_{t > 0}$ for $\alpha \in [0, \pi/4]$ consists of quasi-sectorial contractions with numerical ranges in D_α , a proof of the claim about *inheritance* of this property by approximants $\{(\mathbf{1} + tA/n)^{-n}\}_{t > 0, n \in \mathbb{N}}$ and by the limit $\{e^{-tA}\}_{t > 0}$ demands additional reasoning²¹. It is heavily based on the *Kato numerical range mapping* theorem Ref. 12.

We also note that *extension* of Corollary 4.1 to semi-angle $\alpha \in [0, \pi/2]$ needs merely a more refined arguments, which were developed in Ref. 1.

Proposition 4.2. *Let operator C on a Hilbert space \mathfrak{H} be a quasi-sectorial contraction with semi-angle $0 \leq \alpha < \pi/2$. Then for $\alpha < \alpha' < \pi/2$*

$$\|C^n(\mathbf{1} - C)\| \leq \frac{K_{\alpha, \alpha'}}{n+1}, \quad n \in \mathbb{N}, \quad (4.41)$$

where $K_{\alpha, \alpha'}$ is given by (4.45).

Proof. Since operator C is a quasi-sectorial contraction, the spectrum $\sigma(C)$ is a subset of closure $\overline{W(C)}$ of the numerical range $W(C) \subset D_\alpha$.

So, taking $\alpha < \alpha' < \pi/2$ one gets by definition (4.38): $D_{\alpha'} \supset D_\alpha$. Hence, contour $\partial D_{\alpha'}$ is outside of D_α , but inside the unit disc \mathbb{D} , and all of them have only one common point $z = 1$. Then the *Riesz-Dunford* functional calculus provides the following representation of operator in (4.41):

$$C^n(\mathbb{1} - C) = \frac{1}{2\pi i} \int_{\partial D_{\alpha'}} dz \frac{z^n(1-z)}{z\mathbb{1} - C}. \quad (4.42)$$

If z belongs to resolvent set $\rho(C)$ of C , then evaluation of the norm of resolvent: $\|(C - z\mathbb{1})^{-1}\| \leq \text{dist}(z, \overline{W(C)})^{-1}$, yields $\|(z\mathbb{1} - C)^{-1}\| \leq \text{dist}(z, D_\alpha)^{-1}$ for $z \in \partial D_{\alpha'} \subset \rho(C)$ in (4.42). We consider the following parametrisation of the positively oriented contour $\partial D_{\alpha'}$:

- for the arc (A, B) with end-points at $A = e^{i(\pi/2 - \alpha')} \sin \alpha'$ and at $B = e^{i(3\pi/2 + \alpha')} \sin \alpha'$, we take $z(t) = e^{it} \sin \alpha'$ with $\pi/2 - \alpha' \leq t \leq 3\pi/2 + \alpha'$;

- for the straight lines $(1, A)$ and $(B, 1)$, we take correspondingly $z_-(s) = 1 - se^{-i\alpha'}$ with $s \in [0, \cos \alpha']$ and $z_+(s) = 1 - se^{+i\alpha'}$ with $s \in [\cos \alpha', 0]$.

As a consequence, by definition of the (*shortest*) distance from $z \in \partial D_{\alpha'}$ to D_α , denoted as $\text{dist}(z, D_\alpha)$, we obtain:

- $\|(z\mathbb{1} - C)^{-1}\| \leq (\cos \alpha' \sin(\alpha' - \alpha))^{-1}$ for $|\arg z| \geq \pi/2 - \alpha'$, where we used that $\text{dist}(z, D_\alpha) \in [\cos \alpha' \sin(\alpha' - \alpha), (\sin \alpha' - \sin \alpha)]$ for $z \in \text{arc}(A, B)$, that is, for $z \in \{e^{it} \sin \alpha'\}_{t \in [\pi/2 - \alpha', 3\pi/2 + \alpha']}$;

- $\|(z\mathbb{1} - C)^{-1}\| \leq (|1 - z| \sin(\alpha' - \alpha))^{-1}$ for $|\arg z| \leq \pi/2 - \alpha'$, that is, for $z \in \{1 - se^{\mp i\alpha'}\}_{s \in [0, \cos \alpha']}$.

Then operator-norm estimate of the left-hand side in representation (4.42) takes the form

$$\begin{aligned} \|C^n(\mathbb{1} - C)\| &\leq \frac{1}{2\pi} \int_{\frac{\pi}{2} - \alpha'}^{3\pi/2 + \alpha'} dt \frac{|\sin \alpha'|^{n+1} |1 - e^{it} \sin \alpha'|}{\cos \alpha' \sin(\alpha' - \alpha)} + \\ &\quad + \frac{1}{\pi} \int_0^{\cos \alpha'} ds \frac{|(1 - e^{i\alpha'} s)^n e^{i\alpha'} s|}{s \sin(\alpha' - \alpha)} \quad (4.43) \\ &\leq \frac{2(\sin \alpha')^{n+1}}{\cos \alpha' \sin(\alpha' - \alpha)} + \int_0^{\cos \alpha'} ds \frac{((1 - s \cos \alpha')^2 + s^2(\sin \alpha')^2)^{n/2}}{\pi \sin(\alpha' - \alpha)}. \end{aligned}$$

Taking into account convexity of the mapping: $s \mapsto (1 - s \cos \alpha')^2 + s^2(\sin \alpha')^2$, for $s \in [0, \cos \alpha']$, one gets that

$$(1 - s \cos \alpha')^2 + s^2(\sin \alpha')^2 \leq 1 - s \cos \alpha',$$

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which leads to inequality:

$$\begin{aligned} & \int_0^{\cos \alpha'} ds \left((1 - s \cos \alpha')^2 + s^2 (\sin \alpha')^2 \right)^{n/2} \leq \int_0^{\cos \alpha'} ds (1 - s \cos \alpha')^{n/2} \\ & = \int_{(\sin \alpha')^2}^1 du \frac{u^{n/2}}{\cos \alpha'} \leq \frac{1 - (\sin \alpha')^{n+2}}{(n/2 + 1) \cos \alpha'}. \end{aligned}$$

Therefore, by (4.43) we obtain the estimate

$$\begin{aligned} \|C^n(\mathbf{1} - C)\| & \leq \frac{2 (\sin \alpha')^{n+1}}{\cos \alpha' \sin(\alpha' - \alpha)} + 2 \frac{1 - (\sin \alpha')^{n+2}}{\pi (n+2) \cos \alpha' \sin(\alpha' - \alpha)} \leq \\ & \frac{2}{(n+1) \cos \alpha' \sin(\alpha' - \alpha)} \left(\frac{1}{\pi} + (n+1) (\sin \alpha')^{n+1} \right). \end{aligned} \quad (4.44)$$

After *optimisation* of the last factor in the right-hand side of (4.44) with respect to $n \in \mathbb{N}$, we infer (4.41) for

$$K_{\alpha, \alpha'} := \frac{2}{\cos \alpha' \sin(\alpha' - \alpha)} \left(\frac{1}{\pi} - \frac{1}{e \ln(\sin \alpha')} \right), \quad (4.45)$$

where $\alpha < \alpha' < \pi/2$. \square

The property (4.41) implies that the quasi-sectorial contractions belong to the class of so-called *Ritt's operators*¹⁹. This allows to go beyond the $\sqrt[3]{n}$ -Lemma 2.1 to the $(\sqrt[3]{n})^{-1}$ -Theorem and also from estimates in the strong operator topology to the operator-norm topology. The first step is the operator-norm Chernoff estimate (cf. (2.5)):

Theorem 4.1. ($(\sqrt[3]{n})^{-1}$ -Theorem) *Let C be a quasi-sectorial contraction on \mathfrak{H} with numerical range $W(C) \subseteq D_\alpha$, $0 \leq \alpha < \pi/2$. Then*

$$\|C^n - e^{n(C-1)}\| \leq \frac{L_\alpha}{n^{1/3}}, \quad n \in \mathbb{N}, \quad (4.46)$$

where $L_\alpha = 2K_\alpha + 2$ and $K_\alpha := \min_{\alpha' \in (\alpha, \pi/2)} K_{\alpha, \alpha'}$, is defined by (4.45).

Proof. With help of inequality (4.41) we can improve the estimate of the *central* part of the sum (2.9) in Lemma 2.1. Note that on account of (2.11) we obtain by (4.41) and $\|C\| \leq 1$:

$$\|C^n - C^m\| \leq |m - n| \|C^{m - [\epsilon_n]}(\mathbf{1} - C)\| \leq \epsilon_n \frac{K_\alpha}{n - [\epsilon_n] + 1}, \quad (4.47)$$

cf. (2.13). Here $\epsilon_n := n^{\delta+1/2}$ for $\delta < 1/2$, which makes sense for the estimate (2.12) of *tails*, and $[\epsilon_n]$ is the *integer part* of $\epsilon_n \geq |m - n|$. Then

owing to (4.47) the *central* part has estimate

$$e^{-n} \sum_{|m-n| \leq \epsilon_n} \frac{n^m}{m!} \|(C^m - C^n)x\| \leq \epsilon_n \frac{K_\alpha}{n - [\epsilon_n] + 1} \|x\|, \quad x \in \mathfrak{X}, \quad n \in \mathbb{N}. \quad (4.48)$$

As a consequence, (2.12) and (4.48) yield instead of (2.7) (or (1.1)) the *operator-norm* estimate:

$$\|C^n - e^{n(C-1)}\| \leq \frac{2}{n^{2\delta}} + \epsilon_n \frac{K_\alpha}{n - [\epsilon_n] + 1}, \quad n \in \mathbb{N}. \quad (4.49)$$

Let $n_0 \in \mathbb{N}$ satisfy inequality: $n_0 \geq 2([\epsilon_n] - 1)$. Then (4.49) yields

$$\|C^n - e^{n(C-1)}\| \leq \frac{2}{n^{2\delta}} + \frac{2K_\alpha}{n^{1/2-\delta}}, \quad n > n_0. \quad (4.50)$$

Then estimate $M_\alpha/n^{1/3}$ of the Theorem 4.1 results from the *optimal* choice in (4.50) of the value: $\delta = 1/6$. \square

Corollary 4.2. *If in (4.50) no estimate of the convergence rate is required, then the operator-norm convergence to zero in (4.46) follows directly from the Riesz-Dunford representation of $C^n - e^{n(C-1)}$ as the operator-valued integral along the contour $\partial D_{\alpha'}$ for $\alpha < \alpha' < \pi/2$:*

$$C^n - e^{n(C-1)} = \frac{1}{2\pi i} \int_{\partial D_{\alpha'}} dz \frac{z^n - e^{n(z-1)}}{z - C}, \quad n \in \mathbb{N},$$

which provides for $n \in \mathbb{N}$ inequalities:

$$\begin{aligned} \|C^n - e^{n(C-1)}\| &\leq \frac{1}{2\pi} \int_{\frac{\pi}{2}-\alpha'}^{\frac{3\pi}{2}+\alpha'} dt \sin \alpha' \frac{|(e^{it} \sin \alpha')^n - \exp\{n(e^{it} \sin \alpha' - 1)\}|}{\cos \alpha' \sin(\alpha' - \alpha)} \\ &\quad + \frac{1}{\pi} \int_0^{\cos \alpha'} ds \frac{|(1 - e^{-ia's})^n - \exp(-ns e^{-ia'})|}{s \sin(\alpha' - \alpha)} \leq \\ &\leq \sin \alpha' \frac{(\sin \alpha')^n + e^{n(\sin \alpha' - 1)}}{\cos \alpha' \sin(\alpha' - \alpha)} + \\ &\quad + \int_0^{\cos \alpha'} dr \frac{|(1 - e^{-ia'r/n})^n - \exp(-r e^{-ia'})|}{r \sin(\alpha' - \alpha)}. \end{aligned} \quad (4.51)$$

(For parametrisation of integrands in representation (4.51) see notes in the proof of Proposition 4.2.)

Then $\lim_{n \rightarrow \infty} \|C^n - e^{n(C-1)}\| = 0$ issues by conditions $\alpha < \alpha' < \pi/2$ and the Lebesgue dominated convergence theorem applied to the last integral in the right-hand side of inequalities (4.51).

Remark 4.2. Recall⁴ that if the quasi-sectorial contraction C is *self-adjoint* (i.e., $\alpha = 0$), then one obtains for the rate of convergence optimal estimates:

$$\|C^n(\mathbb{1} - C)\| \leq \frac{1}{n+1} \quad \text{and} \quad \|C^n - e^{n(C-1)}\| \leq \frac{e^{-1}}{n}, \quad n \in \mathbb{N}, \quad (4.52)$$

directly from the spectral representation of C .

In a full similarity with $(\sqrt[3]{n})$ -Lemma for the *strong* operator approximation, the $(\sqrt[3]{n})^{-1}$ -Theorem is only the first step in developing the *operator-norm* approximation formula à la Chernoff. To this end one needs an operator-norm analogue of Theorem 2.1. The preceding includes the Trotter-Neveu-Kato *strong* convergence theorem. On that account, now we need the *operator-norm* extension of this assertion for quasi-sectorial contractions.

Proposition 4.3. (Refs. 4,24) *Let $\{X(s)\}_{s>0}$ be a family of m -sectorial operators in a Hilbert space \mathfrak{H} such that for some $0 < \alpha < \pi/2$ and any $s > 0$ the numerical range $W(X(s)) \subseteq S_\alpha$. Let X_0 be an m -sectorial operator defined in \mathfrak{H} , with $W(X_0) \subseteq S_\alpha$. Then the two following assertions are equivalent :*

- (a) $\lim_{s \rightarrow +0} \|(\zeta\mathbb{1} + X(s))^{-1} - (\zeta\mathbb{1} + X_0)^{-1}\| = 0$, for $\zeta \in S_{\pi-\alpha}$,
- (b) $\lim_{s \rightarrow +0} \|e^{-tX(s)} - e^{-tX_0}\| = 0$, for $t > 0$.

Here $S_\alpha = \{z \in \mathbb{C} : |\arg(z)| < \alpha\}$ is a sector in complex plane \mathbb{C} with semi-angle α and vertex at $z = 0$.

Now $(\sqrt[3]{n})^{-1}$ -Theorem 4.1 (or Corollary 4.2) and the Trotter-Neveu-Kato theorem (Proposition 4.3) yield a desired generalisation of the Chernoff product formula (cf. (2.17)) for the operator-norm convergence.

Proposition 4.4. (Refs. 4,21) *Let $\{\Phi(s)\}_{s \geq 0}$ be a strongly measurable family of uniformly quasi-sectorial contractions on a Hilbert space \mathfrak{H} , such that $\Phi(0) = \mathbb{1}$ and $W(\Phi(s)) \subset D_\alpha$ for all $s > 0$, where $0 \leq \alpha < \pi/2$. Let*

$$X(s) := (\mathbb{1} - \Phi(s))/s, \quad s > 0, \quad (4.53)$$

and let X_0 be a closed operator with non-empty resolvent set, defined in \mathfrak{H} . Then the family $\{X(s)\}_{s>0}$ converges, when $s \rightarrow +0$, in the uniform resolvent sense to the operator X_0 (cf. Proposition 4.3), if and only if

$$\lim_{n \rightarrow \infty} \|\Phi(t/n)^n - e^{-tX_0}\| = 0, \quad \text{for } t > 0. \quad (4.54)$$

Corollary 4.3. (operator-norm Euler formula) *If A is an m -sectorial operator in Hilbert space \mathfrak{H} , with semi-angle $\alpha \in [0, \pi/2)$ and vertex at $z = 0$, then*

$$\lim_{n \rightarrow \infty} \|(\mathbb{1} + tA/n)^{-n} - e^{-tA}\| = 0, \quad t \in S_{\pi/2-\alpha}, \quad (4.55)$$

for $n \in \mathbb{N}$.

Proof. Since by condition the numerical range $W(A) \subset S_\alpha$, on account of Proposition 4.1 and Remark 4.1, $\{\Phi(t) := (\mathbb{1} + tA)^{-1}\}_{t>0}$ is a family of quasi-sectorial contractions with $W(\Phi(t)) \subset D_\alpha$, $\alpha \in [0, \pi/2)$.

Let $X(s) := (\mathbb{1} - \Phi(s))/s$, $s > 0$, and $X_0 := A$. Then for $\zeta \in S_{\pi-\alpha}$ on account of estimate:

$$\left\| \frac{A}{\zeta\mathbb{1} + A + \zeta sA} \cdot \frac{A}{\zeta\mathbb{1} + A} \right\| \leq \left(1 + \frac{|\zeta|}{\text{dist}(\zeta(1 + s\zeta)^{-1}, -S_\alpha)} \right) \left(1 + \frac{|\zeta|}{\text{dist}(\zeta, -S_\alpha)} \right),$$

the family $\{X(s)\}_{s>0}$ converges, when $s \rightarrow +0$, to X_0 in the resolvent uniform (i.e., operator-norm) sense with asymptotic:

$$\|(\zeta\mathbb{1} + X(s))^{-1} - (\zeta\mathbb{1} + X_0)^{-1}\| = s \left\| \frac{A}{\zeta\mathbb{1} + A + \zeta sA} \cdot \frac{A}{\zeta\mathbb{1} + A} \right\| = O(s).$$

As a consequence of Proposition 4.3, the family $\{\Phi(t)\}_{t \geq 0}$ satisfies the conditions of Proposition 4.4. Then seeing that for $t > 0$ and $n \in \mathbb{N}$ one has estimate

$$\|\Phi(t/n)^n - e^{-tX_0}\| \leq \|\Phi(t/n)^n - e^{-tX(t/n)}\| + \|e^{-tX(t/n)} - e^{-tX_0}\|. \quad (4.56)$$

This provides by (4.46) for $C = \Phi(t/n)$ and Proposition 4.3 (b) the operator-norm approximation (4.54), which is the *Euler formula* (4.55). \square

According to (4.56) the *rate* of operator-norm convergence of the *Chernoff* product formula (4.54) is determined by convergence rate of the *Chernoff* estimate (4.46) of $\|\Phi(t/n)^n - e^{-tX(t/n)}\|$ along with the rate of convergence in the *Trotter-Neveu-Kato* theorem for $\|e^{-tX(t/n)} - e^{-tX_0}\|$, Proposition 4.3 (b). Then for the *Euler formula* (4.55) the accuracy of this *two-step* estimate is limited by the order $O(1/n^{1/3})$ because of the Chernoff estimate (4.46) on the the *first* step in the right-hand side of (4.56).

Remark 4.3. Note that the rate $O(1/n^{1/3})$ is far from to be *optimal*, which *a fortiori* is known as $O(1/n)$. Indeed, in Ref. 4 (Theorem 5.1) by a

direct *one-step* telescopic estimate it was shown that the rate of convergence in (4.55) is at least of the order $O(\ln(n)/n)$.

Corollary 4.4. (operator-norm Trotter product formula) *If contractions $\{\Phi(t) := e^{-tA}e^{-tB}\}_{t \geq 0}$ (2.27) satisfy conditions Proposition 4.4, then by necessity part of this assertion and (4.56) the limit $n \rightarrow \infty$ yields the Trotter product formula (2.26) in the operator-norm topology, cf. Proposition 2.1. The rate of convergence (if any) is determined by the first and the second steps in the right-hand side of (4.56), cf. Theorem 5.3 in Ref. 4.*

Remark 4.4. Note that in contrast to the problem of semigroup approximation (Corollary 4.3), that needs assumptions only on generator A , the approximation by the Trotter product formula requests a condition on a couple of generators $\{A, B\}$ (Proposition 2.1 and Corollary 4.4). Since the Trotter product approximants $\{\Phi(t/n)^n\}_{n \geq 1, t \geq 0}$ involve a *couple* of generators $\{A, B\}$, the proof must take into account a *subordination* of generators A and B . A variety of the *one-* and *two-step* methods (including the Chernoff estimate), as well as of the corresponding conditions, that ensure the convergence of the Trotter product formula, is quite large, see, for example, Chapters 5.1-5.3 in Ref. 23. They determine the sense of the algebraic sum $(A+B)$ and the accuracy of the operator-norm convergence estimates, compare, e.g., Ref. 4 and Refs. 11,16.

5. Chernoff Estimate and Dunford-Segal Approximation

Here we continue with more comments about *optimal* results for the rate of convergence for operator-norm approximants of C_0 -semigroups mentioned in Remark 4.3. This approximation theory was advanced in Ref. 9 (and improved later in Ref. 10) for the *Yosida*, the *Dunford-Segal* and the *Euler* approximations of C_0 -semigroups. In addition to standard analysis of approximations in the strong operator topology (cf. Ref. 3) the paper Ref. 9 proposed to study a *vector-dependent* estimates of the convergence rate for approximants, see Remark 3.1. For holomorphic C_0 -semigroups this estimates of convergence can be uplifted to the operator-norm topology.

This last aspect will be considered in the present section versus the Chernoff operator-norm estimate. We start by citation of the optimal $O(1/n)$ result for the rate of the operator-norm convergence for a simple case of the Euler approximants (*Euler formula* (4.55)), which also shows a sectorial dependence of the upper bound.

Proposition 5.1. (Ref. 1) *Let A be an m -sectorial operator in Hilbert*

space \mathfrak{H} , with semi-angle $\alpha \in [0, \pi/2)$ and vertex at $z = 0$. Then $\{e^{-tA}\}_{t \geq 0}$ is a holomorphic quasi-sectorial contraction semigroup and one infers that

$$\left\| (\mathbf{1} + tA/n)^{-n} - e^{-tA} \right\| \leq \frac{M_\alpha}{(\cos \alpha)^2 n}, \quad t \geq 0, \quad n \in \mathbb{N}, \quad (5.57)$$

where

$$\frac{\pi \sin \alpha}{2\alpha} \leq M_\alpha \leq \min \left(\frac{\pi - \alpha}{\alpha}, M \right) \quad \text{and} \quad \frac{\pi}{2} \leq M \leq 2 + \frac{2}{\sqrt{3}}. \quad (5.58)$$

Note that estimate (5.58) of the coefficient M_α in (5.57) is not *optimal*. To this aim we elucidate (5.58) for the case of a non-negative self-adjoint operator A , that is, for $\alpha = 0$, cf. Remark 4.2. As a result one obtains by the spectral calculus:

$$\left\| (\mathbf{1} + tA/n)^{-n} - e^{-tA} \right\| \leq \frac{e^{-1}}{n}, \quad t \geq 0, \quad n \in \mathbb{N}, \quad (5.59)$$

cf. one of the error bound in (4.52). In Ref. 9 (Corollary 1.6 c) the Proposition 5.1 was extended to a Banach space and any bounded holomorphic C_0 -semigroup.

We conclude this section by demonstration that the Chernoff estimate (4.46) *itself*, i.e., without the two-step construction (4.56), yields for holomorphic C_0 -semigroups the *Dunford-Segal* approximation introduced in Ref. 9, Theorem 1.1 b).

Theorem 5.1. *If A is an m -sectorial operator in Hilbert space \mathfrak{H} , with semi-angle $\alpha \in [0, \pi/2)$ and vertex at $z = 0$, then*

$$\left\| e^{-n(\mathbf{1} - e^{-tA/n})} - e^{-tA} \right\| \leq \frac{N}{(\cos \alpha)^2 n}, \quad t \geq 0, \quad n \in \mathbb{N}, \quad (5.60)$$

for some bounded $N > 0$.

Proof. By virtue of Proposition 5.1 one gets that $\{e^{-tA/n}\}_{n \geq 1}$ for $t \geq 0$ and each $n \in \mathbb{N}$ is a holomorphic quasi-sectorial contraction C_0 -semigroup $\{e^{-tA/n}\}_{t \geq 0}$. Let $C := e^{-tA/n}$. Then by the operator-norm Chernoff estimate (Theorem 4.1) we obtain

$$\left\| e^{-tA} - e^{-n(\mathbf{1} - e^{-tA/n})} \right\| \leq \frac{L_\alpha}{n^{1/3}}, \quad t \geq 0, \quad n \in \mathbb{N}. \quad (5.61)$$

This yields the operator-norm approximation of quasi-sectorial contraction semigroup $\{e^{-tA}\}_{t \geq 0}$ by the *Dunford-Segal* approximants $\{e^{-n(\mathbf{1} - e^{-tA/n})}\}_{n \geq 1}$ with the rate $O(1/n^{1/3})$ on a Hilbert space \mathfrak{H} .

The uplifting the estimate (5.61) to sectorial dependent optimal rate (5.60) follows *verbatim* the arguments in the proof of Proposition 5.1. \square

Note that our definition of the *Dunford-Segal* approximants is (slightly) different from that in Refs. 9 (Corollary 1.6 b), where assertion was extended to a Banach space for any bounded holomorphic C_0 -semigroup.

6. Concluding remarks

1. Summarising, we infer that for quasi-sectorial contractions (Definition 4.1) one obtains, instead of *divergent* (for $n \rightarrow \infty$) Chernoff's estimate (1.1), the estimate (4.50), which converges for $n \rightarrow \infty$ to zero in the *operator-norm* topology. Note that the rate $O(1/n^{1/3})$, (4.46), of this convergence is obtained with help of the Poisson representation and the Tchebychëv inequality in the spirit of the proof of Lemma 2.1, and that it is not optimal.
2. The estimate $M_\alpha/n^{1/3}$ in the $(\sqrt[3]{n})^{-1}$ -Theorem 4.1 can be improved by a more refined lines of reasoning. For example, scrutinising our probabilistic arguments in Section 2 one can find a more precise Tchebychëv-type bound for estimate of *tails*. This improves the estimate (4.50) and provides the rate $O(\sqrt{\ln(n)/n})$, see Ref. 17 (Theorem 1). Although again it is possible only for *quasi-sectorial* contractions ensuring due to Proposition 4.2 the *operator-norm* contrôl (4.48) of the *central* part.
3. The very same improvement permits also to amend the estimate for the rate of convergence in the *Euler product formula* (4.55) from $O(\ln(n)/n)$ (Ref. 4 (Theorem 5.1)) to the optimal $O(1/n)$, Ref. 17 (Theorem 4), cf. the proof of the estimate $O(1/n)$ in Ref. 2 (Theorem 1.3).

A careful analysis of the *numerical range* localisation for quasi-sectorial contractions^{1,21}, which are generated in a Hilbert space \mathfrak{H} by m -sectorial operators with a semi-angle $\alpha \in [0, \pi/2)$, allows to uplift the operator-norm estimate for the rate of convergence of the *Euler formula* (5.57) to the ultimate optimal α -dependent rate $O(1/n)$, see Ref. 1 (Theorem 4.1).

4. We note that in case of *self-adjoint* contractions C (that is, for $\alpha = 0$) with help of the spectral representation one can easily obtain the optimal rate $O(1/n)$ for the *Ritt property* (4.41), for the *Chernoff estimate* (4.52), as well as for the *Euler formula* (5.59).

5. Theorem 5.1 is an illustration of a *direct* application of the Chernoff estimate in the approximation theory of holomorphic C_0 -semigroups for m -sectorial generators in Hilbert space for a particular case of the *Dunford-Segal* approximants.

Note that in Ref. 9 and Ref. 10 a generalisation on Banach space was developed for approximants involving the *Bernstein functions*, see (1.4) in Ref. 10. We remark that a similar generalisation of approximants is known

since Ref. 13 as the *Kato functions* for the Trotter-Kato product formulæ, see Ref. 23 (Appendix C) for details.

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