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Hanspeter Kraft, M Zaidenberg

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ALGEBRAICALLY GENERATED GROUPS AND THEIR LIE ALGEBRAS

HANSPETER KRAFT AND MIKHAIL ZAIDENBERG

ABSTRACT. The automorphism group $\text{Aut}(X)$ of an affine variety X is an ind-group. Its Lie algebra is canonically embedded into the Lie algebra $\text{Vec}(X)$ of vector fields on X . We study the relations between subgroups of $\text{Aut}(X)$ and Lie subalgebras of $\text{Vec}(X)$.

We show that a subgroup $G \subseteq \text{Aut}(X)$ generated by a family of connected algebraic subgroups G_i of $\text{Aut}(X)$ is algebraic if and only if the Lie algebras $\text{Lie } G_i \subseteq \text{Vec}(X)$ generate a finite dimensional Lie subalgebra of $\text{Vec}(X)$.

Extending a result by COHEN-DRAISMA [CD03] we prove that a locally finite Lie algebra $L \subseteq \text{Vec}(X)$ generated by locally nilpotent vector fields is algebraic, i.e. $L = \text{Lie } G$ for an algebraic subgroup $G \subseteq \text{Aut}(X)$.

Along the same lines we prove that if a subgroup $G \subseteq \text{Aut}(X)$ generated by finitely many connected algebraic groups is solvable, then it is an algebraic group.

We also show that a unipotent algebraic subgroup $U \subseteq \text{Aut}(X)$ has derived length $\leq \dim X$. This result is based on the following triangulation theorem: *Every unipotent algebraic subgroup of $\text{Aut}(\mathbb{A}^n)$ with a dense orbit in \mathbb{A}^n is conjugate to a subgroup of the de Jonquières subgroup.*

Furthermore, we give an example of a free subgroup $F \subseteq \text{Aut}(\mathbb{A}^2)$ generated by two algebraic elements such that the Zariski closure \bar{F} is a free product of two nested commutative closed unipotent ind-subgroups.

To any affine ind-group \mathfrak{G} one can associate a canonical ideal $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$. It is linearly generated by the tangent spaces $T_e X$ for all algebraic subsets $X \subseteq \mathfrak{G}$ which are smooth in e . It has the important property that for a surjective homomorphism $\varphi: \mathfrak{G} \rightarrow \mathfrak{H}$ the induced homomorphism $d\varphi_e: L_{\mathfrak{G}} \rightarrow L_{\mathfrak{H}}$ is surjective as well. Moreover, if $\mathfrak{H} \subseteq \mathfrak{G}$ is a subnormal closed ind-subgroup of finite codimension, then $L_{\mathfrak{H}}$ has finite codimension in $L_{\mathfrak{G}}$.

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1. INTRODUCTION AND MAIN RESULTS

The introduction contains the necessary preliminaries and definitions, gives some background material and describes the main results of the paper. The details then follow in the subsequent sections.

1.1. Notation. Our base field \mathbb{k} is algebraically closed and of characteristic zero. \mathbb{A}^n stands for affine n -space over \mathbb{k} , and $\mathbb{G}_a := \mathbb{k}^+$ and $\mathbb{G}_m := \mathbb{k}^* := \mathbb{k} \setminus \{0\}$ denote the *additive* and the *multiplicative* groups of \mathbb{k} . An algebraic group is always an *affine algebraic group*, and a *G -variety* X is an affine variety with an action of G such that the corresponding map $G \times X \rightarrow X$ is a morphism.

For every G -variety X there is a canonical (anti-) homomorphism of Lie algebras $\xi: \text{Lie } G \rightarrow \text{Vec}(X)$, where $\text{Vec}(X)$ stands for the Lie algebra of (algebraic) vector fields on X . The construction of this anti-homomorphism goes as follows. For any $x \in X$ let $\theta_x: G \rightarrow X$ be the orbit map $g \mapsto gx$, and denote by

$$(*) \quad d\theta_x: \text{Lie } G := T_e G \rightarrow T_x X$$

its differential in $e \in G$. If $A \in \text{Lie } G$, then the corresponding vector field ξ_A is given by $\xi_A(x) := d\theta_x(A) \in T_x X$. If $L \subseteq \text{Vec}(X)$ denotes the image of $\text{Lie } G$, then one has $T_x Gx = L(x)$ for any $x \in X$. Moreover, the differential $(d\theta_x)_g: T_g G \rightarrow T_{gx} Gx$ of the orbit map θ_x is surjective in every $g \in G$. (For a reference one might look at [Kra16, Appendix A.4].)

For a G -variety X we have linear actions of G on the coordinate ring $\mathcal{O}(X)$ and on the vector fields $\text{Vec}(X)$, and these representations are *locally finite and rational*, i.e. every element is contained in a finite dimensional G -invariant subspace and the linear action of G on any finite dimensional G -invariant subspace is rational. Moreover, the homomorphism $\xi: \text{Lie } G \rightarrow \text{Vec}(X)$ is G -equivariant, see [FK18, Sect. 7.3].

1.2. Automorphism groups and vector fields. For an affine variety X the group of regular automorphisms $\text{Aut}(X)$ is an *affine ind-group*. We refer to the paper [FK18] for an introduction to ind-varieties and ind-groups and for basic concepts, cf. [Sha81, Kum02]. For an affine ind-group \mathfrak{G} the tangent space $T_e \mathfrak{G}$ carries a natural structure of a Lie algebra which will be denoted by $\text{Lie } \mathfrak{G}$. In case of $\text{Aut}(X)$ there is a canonical embedding $\xi: \text{Lie } \text{Aut}(X) \hookrightarrow \text{Vec}(X)$ which is an anti-homomorphism of Lie algebras. It is constructed in a similar way as explained above for an algebraic group G , see [FK18, Proposition 7.2.4]. It has the following property:

If $A \in \text{Lie Aut}(X)$ and $\xi_A \in \text{Vec}(X)$ the corresponding vector field, then one has, for every $x \in X$,

$$\xi_A(x) = d(\theta_x)_e(A) \quad \text{where } \theta_x: \text{Aut}(X) \rightarrow X \text{ is the orbit map in } x.$$

The group \mathfrak{G} acts on $\text{Lie } \mathfrak{G}$ by the adjoint action: $\text{Ad}(g) := d(\text{Int } g)_e$. For $\mathfrak{G} = \text{Aut}(X)$ this action is induced by the action on all vector fields $\text{Vec}(X)$ which is given by

$$\text{Ad } g(\delta)(gx) := d\mu_g \delta(x), \quad \text{i.e. } \text{Ad } g(\delta) = d\mu_g \circ \delta \circ \mu_g^{-1}$$

where $\mu_g: X \xrightarrow{\sim} X$ denotes the multiplication map $x \mapsto gx$. Here we consider a vector field $\delta \in \text{Vec}(X)$ as a section of the ‘‘tangent bundle’’ $\mathcal{T}_X \rightarrow X$, and then $\text{Ad } g(\delta)$ is defined by the following diagram:

$$\begin{array}{ccc} \mathcal{T}_X & \xleftarrow{d\mu_g} & \mathcal{T}_X \\ \text{Ad } g(\delta) \uparrow & & \uparrow \delta \\ X & \xrightarrow{\mu_g^{-1}} & X \end{array}$$

It is easy to see that the embedding $\xi: \text{Lie Aut}(X) \hookrightarrow \text{Vec}(X)$ defined above is $\text{Aut}(X)$ -equivariant:

$$\xi_{\text{Ad}(g)A} = \text{Ad } g(\xi_A) \quad \text{or} \quad \xi_{\text{Ad}(g)A}(gx) = d\mu_g \xi_A(x).$$

Ind-varieties and ind-groups carry a natural topology, called *Zariski-topology* or *ind-topology*. All topological notation in this paper will be with respect to this topology.

A subset $Y \subseteq \mathfrak{V}$ of an ind-variety $\mathfrak{V} = \bigcup_k \mathfrak{V}_k$ is called *bounded* if it is contained in \mathfrak{V}_k for some k . It is called *algebraic* if it is bounded and locally closed. If X is a G -variety, then the canonical map $G \rightarrow \text{Aut}(X)$ is a homomorphism of ind-groups, and the image of G is closed and algebraic, see [FK18, Proposition 2.7.1]. In particular, every algebraic subgroup $G \subseteq \text{Aut}(X)$ is closed and thus a linear algebraic group. It follows that $\text{Lie } G \subseteq \text{Lie Aut}(X)$ is a Lie subalgebra and that $\text{Lie } G$ is canonically embedded into the vector fields $\text{Vec}(X)$.

Convention. In the following our ind-groups and ind-varieties will always be *affine* ind-groups and *affine* ind-varieties. By abuse of notation, we will constantly identify, for an algebraic group $G \subseteq \text{Aut}(X)$, the Lie algebra $\text{Lie } G$ with its image $\xi(\text{Lie } G) \subseteq \text{Vec}(X)$, although the map ξ is an anti-homomorphism.

1.3. Algebraically generated groups. A first result showing a very strong relation between the Lie algebra of an ind-group and the group itself is the following, see [FK18, Proposition 2.2.1(3) and (4)].

Proposition 1.3.1. *Let \mathfrak{G} be an ind-group. Then the connected component \mathfrak{G}° is an algebraic group if and only if $\text{Lie } \mathfrak{G}$ is finite dimensional.*

We will prove a similar statement for so-called algebraically generated groups, see [AFK13]. Let $(G_i)_{i \in I}$ be a family of connected algebraic subgroups of an ind-group \mathfrak{G} . The subgroup $G \subseteq \mathfrak{G}$ generated by the G_i will be called *algebraically generated* (Definition 2.1.1).

Let $L(G) \subseteq \text{Lie } \mathfrak{G}$ be the Lie subalgebra generated by the Lie algebras $\text{Lie } G_i$. We will see in Theorem 2.3.1 that $L(G)$ is invariant under the action of G and of the closure \overline{G} . This implies that $L(G) \subseteq \text{Lie } \overline{G}$ is an ideal.

Question 1. *Do we have $L(G) = \text{Lie } \overline{G}$?*

Our result in this direction is the following, see Theorem 2.5.1.

Theorem A. *The subgroup $G \subseteq \mathfrak{G}$ is an algebraic group if and only if $L(G)$ is finite dimensional, and in this case we have $\text{Lie } G = L(G)$.*

See also Theorem 2.3.1. Our example in section 6 gives an infinite dimensional L where Question 1 above has a positive answer, see Remark 6.4.3(1).

Theorem A above is an important ingredient in the proof of the following result.

Theorem B. *Let $G \subseteq \mathfrak{G}$ be a solvable subgroup generated by a family of connected algebraic subgroups. If the family is finite, then G is an algebraic group. In general, G is nested, i.e. a filtered union of algebraic subgroups, and is of the form $U_G \rtimes T$ where T is a torus and U_G a nested unipotent group.*

In particular, if G is generated by a family of unipotent algebraic groups, then $G = U_G$ is a nested unipotent group.

The proof of Theorem B will be given in Section 5.1, see Theorem 5.1.1.

Remarks 1.3.2. (1) Clearly, a connected nested ind-group is algebraically generated.

(2) If G is an algebraic group and $H_1, \dots, H_m \subseteq G$ a finite set of connected closed subgroups, then the subgroup H generated by the H_i is closed. (Indeed, there is a sequence i_1, i_2, \dots, i_N such that the product $P := H_{i_1} H_{i_2} \cdots H_{i_N}$ is dense (and constructible) in \overline{H} . It then follows that $PP = \overline{H}$, hence $H = \overline{H}$.)

As a consequence we see that a subgroup of a nested ind-group \mathfrak{G} generated by finitely many connected algebraic groups is an algebraic group.

Note that this statement does not hold for non-connected subgroups. E.g. the two matrices $A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$ have both order 4, and the product $AB = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ has infinite order. Thus $\overline{\langle A, B \rangle} = N$, the normaliser of the diagonal torus, but $\langle A, B \rangle \subseteq \mathrm{SL}_2(\mathbb{Q})$. It follows that the subgroup of $\mathrm{SL}_2(\mathbb{k})$ generated by the finite groups $\langle A \rangle$ and $\langle B \rangle$ is not closed.

- (3) If the ind-group \mathfrak{G} verifies TITS' alternative, then the assumption of solvability of G in Theorem B can be replaced by the weaker assumption that G has no nonabelian free subgroup; see e.g. [AZ21] and the literature therein for a discussion of TITS' alternative in ind-groups of type $\mathrm{Aut}(X)$.

1.4. Solvability and triangulation. The de Jonquières subgroup $\mathrm{Jonq}(n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ consists of the automorphisms of the form

$$\varphi = (a_1x_1 + f_1, a_2x_2 + f_2(x_1), \dots, a_nx_n + f_n(x_1, \dots, x_{n-1}))$$

where $a_i \in \mathbb{k}^*$ and $f_i \in \mathbb{k}[x_1, \dots, x_{i-1}]$. It is known that the unipotent elements of $\mathrm{Jonq}(n)$ form a solvable subgroup of derived length n which is not nilpotent for $n > 1$ (see Remarks 5.2.1 and 5.3.3). More generally, we have the following result.

Theorem C. *A nested unipotent subgroup $U \subseteq \mathrm{Aut}(X)$ is solvable of derived length $\leq \max\{\dim Ux \mid x \in X\} \leq \dim X$.*

It is known that the derived length of a nilpotent (respectively, a solvable) connected Lie group G acting faithfully on a Hausdorff topological space M is bounded above by $\dim M$ (respectively, by $\dim M + 1$), see [ET79, Theorem 1.2]. Up to passing to a finite index subgroup, the former estimate works as well in the case of a finitely generated nilpotent group acting faithfully on a quasi-projective variety X defined over a field of characteristic zero [Abb22, Theorem B].

The proof of Theorem C will be given in Section 5.3, see Theorem 5.3.1. It is based on the following important triangulation result, see Theorem 5.2.2.

Theorem D. *Assume that a nested unipotent subgroup $U \subseteq \mathrm{Aut}(\mathbb{A}^n)$ has a dense orbit in \mathbb{A}^n . Then U acts transitively on \mathbb{A}^n and U is triangulable, i.e., it is conjugate to a subgroup of the de Jonquières subgroup.*

In [Bas84] BASS gave an example of a \mathbb{G}_a -action on \mathbb{A}^3 which is not triangulable, i.e. the image $A \subseteq \text{Aut}(\mathbb{A}^3)$ of \mathbb{G}_a is not conjugate to a subgroup of $\text{Jonq}(3)$. The reason is that the fixed points set $(\mathbb{A}^3)^{\mathbb{G}_a}$ is a hypersurface with an isolated singularity. This is not possible for a triangulable action, because for such an action the fixed point set has the form $X \times \mathbb{A}^1$. With the same idea one can construct non-triangulable \mathbb{G}_a -actions in any dimension, see [Pop87].

The \mathbb{G}_a -action of BASS corresponds to the locally nilpotent vector field $\delta := (xz + y^2)(-2y\partial_x + z\partial_y)$, and the image A of \mathbb{G}_a contains the famous Nagata automorphism

$$\eta = (x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z).$$

Due to the celebrated SHESTAKOV-UMIRBAEV Theorem ([SU03]) this \mathbb{G}_a -subgroup is not contained in the tame subgroup of $\text{Aut}(\mathbb{A}^3)$. However, it becomes tame in \mathbb{A}^4 , see [Smi89], but it is still non-triangulable in \mathbb{A}^4 , see [Fre06, Lemma 3.36]. The question arises if this action is *stably triangulable*, i.e. becomes triangulable in $\mathbb{A}^3 \times \mathbb{A}^d$ for a suitable $d \geq 1$. In this context we have the following negative answer.

Proposition 1.4.1. *Consider a \mathbb{G}_a -action on \mathbb{A}^n and assume that the fixed point set is a hypersurface with an isolated singularity. Then the action is not stably triangulable.*

Proof. (a) Denote by $F \subseteq \mathbb{A}^n$ the fixed point set and by $p \in F$ the isolated singularity. For the quotient morphism $\pi: \mathbb{A}^n \rightarrow \mathbb{A}^n // \mathbb{G}_a$ the fiber $\pi^{-1}(\pi(p))$ has dimension ≥ 1 and thus is not contained in the singularities of F . If we extend the action to $\mathbb{A}^n \times \mathbb{A}^d$ where \mathbb{G}_a acts trivially on \mathbb{A}^d , then $(\mathbb{A}^n \times \mathbb{A}^d) // \mathbb{G}_a = \mathbb{A}^n // \mathbb{G}_a \times \mathbb{A}^d$ and the quotient morphism is equal to $\tilde{\pi} := \pi \times \text{id}$. Moreover, the fixed point set is $\tilde{F} := F \times \mathbb{A}^d$, and the singularities of \tilde{F} are $\tilde{F}_{\text{sing}} := \{p\} \times \mathbb{A}^d$. It follows that $\tilde{\pi}^{-1}(\tilde{\pi}(\tilde{F}_{\text{sing}}))$ is not contained in \tilde{F}_{sing} .

(b) Now assume that the action on $\mathbb{A}^n \times \mathbb{A}^d$ is triangulable, so that the corresponding vector field is equivalent to one of the form

$$\tilde{\delta} = f_1\partial_{x_1} + f_2\partial_{x_2} + \cdots + f_m\partial_{x_m}$$

where $m := n + d$ and $f_i \in \mathbb{k}[x_1, \dots, x_{i-1}]$. The fixed point set \tilde{F} is an irreducible hypersurface defined by an invariant (irreducible) function h , and so h must divide all the f_i . Since the singular points of \tilde{F} form a subvariety of dimension d this implies that $f_1 = \cdots = f_n = 0$. In fact, if h would depend on less than n variables, then the zero set $\{h = 0\} \subseteq \mathbb{A}^m$ would have the form $X \times \mathbb{A}^{d+1}$ and so the singular set of \tilde{F} would have at least dimension $d + 1$.

Let f_{r+1} be the first non-zero coefficient of $\tilde{\delta}$. Then $r \geq n$ and x_1, x_2, \dots, x_r are invariants. Since h divides f_{r+1} we see that h only depends on the variables x_1, \dots, x_r . It follows that the linear \mathbb{G}_a -invariant morphism $\varphi: \mathbb{A}^m \rightarrow \mathbb{A}^r$, $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_r)$, maps the fixed point set \tilde{F} to the hypersurface $F_r := \{h = 0\} \subseteq \mathbb{A}^r$, and $\tilde{F} = F_r \times \mathbb{A}^{m-r}$. This shows that $\varphi^{-1}(\varphi(\tilde{F}_{\text{sing}})) = \tilde{F}_{\text{sing}}$. Since the invariant map φ factors through the quotient map $\tilde{\pi}$ this contradicts what we have seen in (a). \square

Remarks 1.4.2.

- (1) The non-triangulable \mathbb{G}_a -actions on \mathbb{A}^n constructed in [Pop87] are not stably triangulable.
- (2) The Nagata automorphism η is not contained in a unipotent subgroup $U \subseteq \text{Aut}(\mathbb{A}^3)$ which has a dense orbit in \mathbb{A}^3 , by our Theorem D.
- (3) If we consider the diagonal action on $\mathbb{A}^3 \times \mathbb{A}^1$ where \mathbb{G}_a acts on \mathbb{A}^3 as in BASS' example and on \mathbb{A}^1 by translation, then this new action is triangulable. Indeed, we have the following result.

Let U be a unipotent group acting on \mathbb{A}^n . Then the diagonal action on $\mathbb{A}^n \times U$ where U acts by left multiplication on U is triangulable.

Indeed, consider the isomorphism $\varphi: \mathbb{A}^n \times U \xrightarrow{\sim} \mathbb{A}^n \times U$ given by $(x, u) \mapsto (u^{-1}x, u)$. This morphism is U -equivariant with respect to the diagonal action on the left hand side and the action on U by left multiplication (and the trivial action on \mathbb{A}^n) on the right hand side. Since the action on U is triangulable by our Theorem D above the claim follows.

Likewise, given any \mathbb{k} -algebra R and a locally nilpotent derivation ∂ of R , its extension $\bar{\partial}$ to $R[x]$ by letting $\bar{\partial}(x) = 1$ is conjugate to the locally nilpotent R -derivation δ of $R[x]$ defined by $\delta(x) = 1$.

1.5. Locally finite subsets. Let V be a vector space over \mathbb{k} . We denote by $\mathcal{L}(V)$ the algebra of linear endomorphisms of V . An endomorphism $\lambda \in \mathcal{L}(V)$ is called *locally finite* if the linear span $\langle \lambda^j(v) \mid j \in \mathbb{N} \rangle$ is finite dimensional for any $v \in V$. It is called *semisimple* if there is a basis of eigenvectors, and *locally nilpotent* if for any $v \in V$ there is an $m \in \mathbb{N}$ such that $\lambda^m(v) = 0$. Every locally finite endomorphism λ has a uniquely defined *additive Jordan-decomposition* $\lambda = \lambda_s + \lambda_n$ where λ_s is semisimple, λ_n locally nilpotent, and $\lambda_s \circ \lambda_n = \lambda_n \circ \lambda_s$.

A subset $S \subseteq \mathcal{L}(V)$ is called *locally finite* if every $v \in V$ is contained in a finite dimensional subspace $W \subseteq V$ which is invariant under all elements from S . If $S \subseteq \mathrm{GL}(V)$, then S is locally finite if and only if the group $\langle S \rangle$ generated by S is locally finite.

Note that a locally finite subspace $A \subseteq \mathcal{L}(V)$ is not necessarily finite dimensional. In fact, let (e_1, e_2, \dots) be a basis of $V := \mathbb{k}^\infty$ and define $\lambda_k \in \mathcal{L}(V)$ by $\lambda_k(e_j) := \delta_{kj}e_j$. Then $A := \bigoplus_k \mathbb{k}\lambda_k$ is an infinite dimensional locally finite subspace.

Definition 1.5.1. Let X be an affine variety. A morphism $\varphi: X \rightarrow X$ is called *locally finite* if the induced endomorphism $\varphi^* \in \mathcal{L}(\mathcal{O}(X))$ is locally finite. It is called *semisimple* if φ^* is semisimple, and *locally nilpotent* if φ^* is locally nilpotent.

A subgroup $H \subseteq \mathrm{Aut}(X)$ is called *locally finite* if the image of H in $\mathcal{L}(\mathcal{O}(X))$ is locally finite.

Note that a subgroup $H \subseteq \mathrm{Aut}(X)$ is locally finite if and only if the closure $\overline{H} \subseteq \mathrm{Aut}(X)$ is an algebraic group. If $g \in \mathrm{Aut}(X)$ is locally finite, then the closed commutative subgroup $\overline{\langle g \rangle}$ has the form $\mathbb{G}_m^p \times \mathbb{G}_a^q \times F$ where $p \in \mathbb{Z}_{\geq 0}$, $q \in \{0, 1\}$ and F is a finite cyclic group.

There are many examples of subgroups $G \subseteq \mathrm{Aut}(X)$ that are not locally finite while being generated by locally finite elements. We will discuss such an example in Section 6. We also refer to the interesting discussions in [FK18, 9.4.3–9.4.5] and [PR22] of subgroups $G \subseteq \mathrm{Aut}(X)$ which consist of locally finite elements.

1.6. Locally finite vector fields. Recall that the vector fields $\mathrm{Vec}(X)$ are the derivations $\mathrm{Der}(\mathcal{O}(X))$ of $\mathcal{O}(X)$. Since $\mathrm{Der}(\mathcal{O}(X)) \subseteq \mathcal{L}(\mathcal{O}(X))$ we can talk about *locally finite vector fields* and *locally finite subspaces* of $\mathrm{Vec}(X)$.

Example 1.6.1. If $G \subseteq \mathrm{Aut}(X)$ is an algebraic subgroup, then its Lie algebra $\mathrm{Lie}(G) \subseteq \mathrm{Vec}(X)$ is locally finite. These Lie algebras are called *algebraic*.

In contrast to the general situation we have the following finiteness result.

Lemma 1.6.2. *Let $L \subseteq \mathrm{Vec}(X)$ be a locally finite subspace. Then L is finite dimensional.*

Proof. If $f_1, \dots, f_m \in \mathcal{O}(X)$ is a set of generators, then every vector field δ is determined by the values $\delta(f_1), \dots, \delta(f_m)$. It follows that for any subspace $L \subseteq \mathrm{Vec}(X)$ the linear map $L \rightarrow \mathcal{O}(X)^m$, $\delta \mapsto (\delta(f_1), \dots, \delta(f_m))$ is injective. If L is locally finite, then the image of

this map is contained in the finite dimensional subspace $\bigoplus_{i=1}^m Lf_i$, and the claim follows. \square

The fact that a vector field is determined by its values on a finite generating set of $\mathcal{O}(X)$ has the following important consequences.

Lemma 1.6.3. *Let $V \subseteq \mathcal{O}(X)$ be a finite dimensional subspace generating $\mathcal{O}(X)$, and let $\delta \in \text{Vec}(X)$ be a vector field. If V is δ -invariant, then the following holds.*

- (1) δ is locally finite.
- (2) δ is semisimple if and only if $\delta|_V \in \mathfrak{gl}(V)$ is semisimple.
- (3) δ is locally nilpotent if and only if $\delta|_V \in \mathfrak{gl}(V)$ is nilpotent.
- (4) If $\delta = \delta_s + \delta_n$ is the Jordan decomposition in $\mathcal{L}(\mathcal{O}(X))$, then $\delta_s, \delta_n \in \text{Vec}(X)$.
- (5) V is δ_s - and δ_n -invariant, and $\delta|_V = \delta_s|_V + \delta_n|_V \in \mathfrak{gl}(V)$ is the usual Jordan decomposition in $\mathfrak{gl}(V)$.

Proof. Denote by $V^{(n)} \subseteq \mathcal{O}(X)$ the linear subspace generated by the n -fold products of elements from V . Then the following is obvious: (a) $V^{(n)}$ is δ -invariant; (b) if $\delta|_V$ is semisimple resp. nilpotent, then so is $\delta|_{V^{(n)}}$. This proves (1)–(3). The claim (4) follows from [FK18, Proposition 7.6.1], and (5) follows from (2) and (3). \square

Question 2. *Let $L_i \subseteq \text{Vec}(X)$ ($i \in I$) be a family of locally finite Lie subalgebras, and denote by $L \subseteq \text{Vec}(X)$ the Lie subalgebra generated by the L_i . Is L locally finite in case L is finite dimensional?*

In this direction we have the following consequence of Theorem A.

Corollary. *Let $L \subseteq \text{Vec}(X)$ be the Lie subalgebra generated by a family of locally finite Lie subalgebras $L_i \subseteq \text{Vec}(X)$, $i \in I$. Assume that each L_i is algebraic, i.e. $L_i = \text{Lie } G_i$ for a connected algebraic group $G_i \subseteq \text{Aut}(X)$. Then L is locally finite if and only if L is finite dimensional. In this case the subgroup G generated by the G_i is algebraic and $L = \text{Lie } G$. In particular, L is algebraic.*

Proof. If L is locally finite, then it is finite dimensional, by Lemma 1.6.2. If L is finite dimensional, then the claim follows from Theorem A. \square

Remark 1.6.4. For a locally finite vector field $\delta \in \text{Vec}(X)$ there is a uniquely determined minimal algebraic group $H \subseteq \text{Aut}(X)$ such that $\delta \in \text{Lie } H$. Moreover, H is commutative and connected, and H is a torus if δ is semisimple and $H \simeq \mathbb{G}_a$ if δ is locally nilpotent and non-zero, see [FK18, Proposition 7.6.1].

1.7. The adjoint action on $\text{Vec}(X)$. For any vector field δ we have the *adjoint action* $\text{ad } \delta$ on $\text{Vec}(X)$ defined in the usual way:

$$\text{ad } \delta(\eta) := [\delta, \eta] = \delta \circ \eta - \eta \circ \delta.$$

Lemma 1.7.1. *Let $\delta \in \text{Vec}(X)$ be a locally finite vector field with Jordan decomposition $\delta = \delta_s + \delta_n$. Then $\text{ad } \delta$ acting on $\text{Vec}(X)$ is locally finite, and $\text{ad } \delta = \text{ad } \delta_s + \text{ad } \delta_n$ is the Jordan decomposition.*

Proof. Let $V \subseteq \mathcal{O}(X)$ be a finite dimensional δ -invariant subspace which generates $\mathcal{O}(X)$. Then we have an inclusion

$$\text{Vec}(X) \hookrightarrow \mathcal{L}(V, \mathcal{O}(X)), \quad \mu \mapsto \mu|_V.$$

Given $\mu \in \text{Vec}(X)$ we choose a finite dimensional δ -invariant subspace W which contains $\mu(V)$. In $\mathcal{L}(\mathcal{O}(X))$ we have

$$(\text{ad } \delta)^m \mu = \sum_{i=0}^m (-1)^i \binom{m}{i} \delta^{m-i} \mu \delta^i.$$

Hence $(\text{ad } \delta)^m \mu$ sends V to W for all $m \geq 0$, and so the linear span $\langle (\text{ad } \delta)^m \mu \mid m \geq 0 \rangle \subseteq \text{Vec}(X)$ is finite dimensional. This shows that $\text{ad } \delta$ is locally finite.

If δ is locally nilpotent, then $(\delta|_V)^m = 0$ and $(\delta|_W)^m = 0$ for a suitable $m > 0$. The above formula implies that $(\text{ad } \delta)^{2m-1} \mu = 0$, hence $\text{ad } \delta$ is locally nilpotent.

Next, assume that δ is semisimple. Then we have eigenspace decompositions $V = \bigoplus_a V_a$ and $\mathcal{O}(X) = \bigoplus_b \mathcal{O}(X)_b$. As a consequence, we get a decomposition

$$\mathcal{L}(V, \mathcal{O}(X)) = \bigoplus_{a,b} \mathcal{L}(V_a, \mathcal{O}(X)_b)$$

into eigenspaces of $\text{ad } \delta$, and so $\text{ad } \delta$ is semisimple.

Finally, $\text{ad } \delta_s$ and $\text{ad } \delta_n$ commute, because $[\delta_s, \delta_n] = 0$, and thus $\text{ad } \delta = \text{ad } \delta_s + \text{ad } \delta_n$ is the Jordan decomposition. \square

1.8. Toral subalgebras.

Definition 1.8.1. A finite dimensional Lie subalgebra $S \subseteq \text{Vec}(X)$ is called *toral* if it consists of semisimple elements. It follows that S is commutative [Hum72, 8.1, Lemma] and thus locally finite. In particular, we obtain a *weight decomposition* $\mathcal{O}(X) = \bigoplus_{\alpha \in S^*} \mathcal{O}(X)_\alpha$ where

$$\mathcal{O}(X)_\alpha := \{f \in \mathcal{O}(X) \mid \tau f = \alpha(\tau) \cdot f \text{ for } \tau \in S\}.$$

If $T \subseteq \text{Aut}(X)$ is a torus, then $\text{Lie } T \subseteq \text{Vec}(X)$ is a toral subalgebra, but it is not true that every toral subalgebra is of this form. Indeed, if $T \subseteq \text{Aut}(X)$ is a two dimensional torus, then the one dimensional tori $S \subseteq T$ are the kernels of certain characters $\chi \in X(T) \simeq \mathbb{Z}^2$ whereas every one dimensional subspace of $\text{Lie } T$ is a toral subalgebra. The best we can get is the following result.

Lemma 1.8.2. *For a toral Lie subalgebra $S \subseteq \text{Vec}(X)$ there exists a uniquely defined smallest torus $T \subseteq \text{Aut}(X)$ such that $\text{Lie } T \supseteq S$. Moreover, if a subspace $M \subseteq \text{Vec}(X)$ is invariant under the adjoint action of S , then M is invariant under T .*

Proof. (a) One easily verifies that the weight decomposition of $\mathcal{O}(X)$ has the following properties, cf. [FK18, Proposition 7.6.1 and its proof]:

- $\mathcal{O}(X)_\alpha \cdot \mathcal{O}(X)_\beta \subseteq \mathcal{O}(X)_{\alpha+\beta}$;
- the set of weights $\Lambda := \{\gamma \mid \mathcal{O}(X)_\gamma \neq 0\} \subseteq S^*$ generates a free subgroup $\mathbb{Z}\Lambda = \bigoplus_{i=1}^k \mathbb{Z}\mu_i$.

This defines a faithful action of the torus $T := (\mathbb{G}_m)^k$ on $\mathcal{O}(X)$ in the following way: For $\alpha = \sum_i n_i \mu_i \in \Lambda$, $f \in \mathcal{O}(X)_\alpha$ and $t = (t_1, \dots, t_k) \in T$ we put

$$t f := t_1^{n_1} \cdots t_k^{n_k} \cdot f.$$

The corresponding action of $c = (c_1, \dots, c_k) \in \text{Lie } T = \mathbb{k}^m$ is then given by the vector field δ_c where

$$\delta_c|_{\mathcal{O}(X)_\alpha} = \text{scalar multiplication with } \sum_i n_i c_i \text{ for } \alpha = \sum_i n_i \mu_i \in \Lambda.$$

It follows that $\text{Lie } T$ contains S and that every torus with this property contains T , see [FK18, proof of Proposition 7.6.1(2)].

(b) With respect to the adjoint action of S on $\text{Vec}(X)$ we also have a weight space decomposition $\text{Vec}(X) = \bigoplus_{\gamma \in S^*} \text{Vec}(X)_\gamma$ where

$$\text{Vec}(X)_\gamma := \{\delta \in \text{Vec}(X) \mid (\text{ad } \tau)\delta = \gamma(\tau) \cdot \delta \text{ for } \tau \in S\},$$

see Lemma 1.7.1. Equivalently, we have

$$\text{Vec}(X)_\gamma := \{\delta \in \text{Vec}(X) \mid \delta(\mathcal{O}(X)_\alpha) \subseteq \mathcal{O}(X)_{\alpha+\gamma} \text{ for } \alpha \in S^*\}.$$

It follows that if $\text{Vec}(X)_\gamma \neq 0$, then $\gamma \in \mathbb{Z}\Lambda$. In fact, if $\delta \in \text{Vec}(X)_\gamma$, $\delta \neq 0$, then there is an $\alpha \in \Lambda$ and a non-zero $f \in \mathcal{O}(X)_\alpha$ such that $\delta f \neq 0$. Thus α and $\alpha + \gamma$ belong to Λ , and so $\gamma \in \mathbb{Z}\Lambda$. Hence, the subspaces $\text{Vec}(X)_\gamma$ are weight spaces of the action of T on $\text{Vec}(X)$, and thus every S -invariant subspace of $\text{Vec}(X)$ is invariant under T . \square

1.9. Integration of Lie algebras. The first part of the following result is due to COHEN and DRAISMA, see [CD03, Theorem 1].

Theorem E. *Let $L \subseteq \text{Vec}(X)$ be a Lie subalgebra. Then $L \subseteq \text{Lie } G$ for some algebraic subgroup $G \subseteq \text{Aut}(X)$ if and only if L is locally finite. If L is locally finite and generated by locally nilpotent elements, then L is algebraic, i.e. there is an algebraic group $G \subseteq \text{Aut}(X)$ such that $L = \text{Lie } G$.*

The proof will be given in Section 3.1, see Theorem 3.1.1. A weaker form of Question 2 is the following.

Question 3. *Let $\xi, \eta \in \text{Vec}(X)$ be locally finite vector fields. If the Lie subalgebra $L := \langle \xi, \eta \rangle_{\text{Lie}}$ generated by ξ and η is finite dimensional, does it follow that L is locally finite?*

From Theorem E we get the following result in this direction.

Corollary. *Let $\{\eta_i \mid i \in I\}$ be a family of locally nilpotent vector fields, and denote by $L := \langle \eta_i \mid i \in I \rangle_{\text{Lie}}$ the Lie subalgebra generated by the η_i . If L is finite dimensional, then L is locally finite and algebraic.*

1.10. A free subgroup of $\text{Aut}(\mathbb{A}^2)$ and its closure. In Section 6 we provide an example of a nonabelian free subgroup $F \simeq \mathbb{F}_2$ of $\text{Aut}(\mathbb{A}^2)$ generated by two locally finite elements whose product is not locally finite. In particular, F is not locally finite. We compute its closure $\mathfrak{F} := \overline{F} \subseteq \text{Aut}(\mathbb{A}^2)$ and describe the Lie algebra $\text{Lie } \mathfrak{F}$. It occurs that \mathfrak{F} coincides with the double centralizer of F (Lemma 6.1.1) and is a free product of two abelian nested unipotent ind-subgroups \mathfrak{J} and \mathfrak{J}^- (Theorem 6.4.2(3)). Furthermore, any algebraic subgroup G of \mathfrak{F} is abelian and unipotent and is conjugate to a subgroup of \mathfrak{J} or \mathfrak{J}^- (Theorem 6.4.2(4)).

The construction shows that \mathfrak{F} contains two algebraic subgroups U and V , both isomorphic to \mathbb{G}_a , with the following properties:

$$\langle U, V \rangle = U * V, \quad \mathfrak{F} = \overline{\langle U, V \rangle} \quad \text{and} \quad \text{Lie } \mathfrak{F} = \langle \text{Lie } U, \text{Lie } V \rangle_{\text{Lie}},$$

see Lemma 6.2.1. This is an instance where Question 1 has a positive answer.

Yet another instance is the following. Consider the ind-group $\mathfrak{G} := \text{Aut}_{\mathcal{K}}(\mathcal{K}[x, y])$ where $\mathcal{K} := \mathbb{k}[z]$, and let $\mathfrak{G}^t \subseteq \mathfrak{G}$ be the subgroup consisting of tame automorphisms of $\mathbb{k}[x, y, z]$. Then \mathfrak{G} is connected and closed in $\text{Aut}_{\mathbb{k}}(\mathbb{k}[x, y, z])$, $\mathfrak{G}^t \subsetneq \mathfrak{G}$ is closed and $\text{Lie } \mathfrak{G}^t = \text{Lie } \mathfrak{G}$, see [FK18, Theorem 17.3.1]. The group \mathfrak{G}^t is generated by the affine and the de Jonquières transformations which fix z (cf. [FK18, Remark 17.3.10]).

Since the de Jonquières group is nested (Remark 5.2.1), it is algebraically generated (Remark 1.3.2(1)), and so \mathfrak{G}^t is algebraically generated. By the UMIRBAEV-SHESTAKOV Theorem, the one dimensional unipotent subgroup U of \mathfrak{G} corresponding to the locally nilpotent Nagata derivation is not contained in \mathfrak{G}^t . However, since the algebraically generated group $\mathfrak{H} := \langle \mathfrak{G}^t, U \rangle$ is contained in \mathfrak{G} and $\text{Lie } U \subseteq \text{Lie } \mathfrak{G} = \text{Lie } \mathfrak{G}^t$ we have

$$\text{Lie } \overline{\mathfrak{H}} = L_{\mathfrak{H}} := \langle \text{Lie } \mathfrak{G}^t, \text{Lie } U \rangle = \text{Lie } \mathfrak{G}.$$

1.11. Ind-group actions and path-connected subsets. The theory of algebraic group actions on affine varieties can be generalized, to some extent, to ind-groups acting on affine ind-varieties, see [FK18]. If \mathfrak{G} is an ind-group, then a \mathfrak{G} -variety X is a variety with a \mathfrak{G} -action such that the map $\mathfrak{G} \times X \rightarrow X$ is a morphism of ind-varieties.

A subset $M \subseteq \mathfrak{X}$ of an ind-variety \mathfrak{X} is called *path-connected* if for any two points $x, y \in M$ there is an irreducible variety Y and a morphism $\varphi: Y \rightarrow M$ such that $x, y \in \varphi(Y) \subseteq \mathfrak{X}$, cf. [Ram64, p. 26]. One can always assume that Y is an irreducible curve, because any two points in an irreducible variety are contained in an irreducible curve, cf. [FK18, Section 1.6]. A connected ind-variety is not necessarily path-connected ([FK18, Example 1.6.5]), but a connected ind-group is path-connected ([FK18, Proposition 2.2.1]).

Note that a path-connected subset of an algebraic variety is not necessarily closed; take \mathbb{C}^2 and remove a ball around the origin. However, we will see that a path-connected subgroup of an algebraic group is closed, cf. Corollary 2.4.2.

In contrast to the finite-dimensional case, a strict closed subgroup of a path-connected ind-group may have the same Lie algebra, see [FK18, Theorem 17.3.1] and the example in Section 1.10. This disproves [Sha81, Theorems 1 and 2]. However, a homomorphism of a path-connected ind-group to an ind-group is uniquely determined by the induced homomorphism of their Lie algebras, see [FK18, Proposition 7.4.7].

In the last section 7 we define a canonical Lie subalgebra $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$ for an ind-group \mathfrak{G} and show that for a surjective homomorphism $\mathfrak{G} \rightarrow \mathfrak{H}$ the corresponding homomorphism $L_{\mathfrak{G}} \rightarrow L_{\mathfrak{H}}$ is also surjective. We do not know if this holds for $\text{Lie } \mathfrak{G} \rightarrow \text{Lie } \mathfrak{H}$. We also study subgroups $\mathfrak{H} \subseteq \mathfrak{G}$ of finite codimension and show that under additional assumptions $L_{\mathfrak{H}}$ has finite codimension in $L_{\mathfrak{G}}$.

2. ALGEBRAICALLY GENERATED SUBGROUPS

2.1. Orbits of algebraically generated subgroups. Let $\mathfrak{G} = \bigcup_k \mathfrak{G}_k$ be an ind-group. The following concept was introduced in [AFK13].

Definition 2.1.1. A subgroup $G \subseteq \mathfrak{G}$ is called *algebraically generated* if there is a family $(G_i)_{i \in I}$ of connected algebraic subgroups $G_i \subseteq \mathfrak{G}$ generating G as an abstract group: $G = \langle G_i \mid i \in I \rangle$.

It is clear that an algebraically generated subgroup is path-connected (section 1.11), hence connected. The following result can be found in [AFK13, Prop. 1.2 and 1.3]. For the reader's convenience we provide a short argument.

Proposition 2.1.2. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ be an algebraically generated subgroup. For every $x \in X$ the G -orbit $Gx \subseteq X$ is open in its closure \overline{Gx} . Moreover, there is a finite sequence $j_1, \dots, j_n \in I$ such that $Gx = G_{j_1} \cdots G_{j_n}x$ for all $x \in X$.*

Proof. (a) Choose a sequence i_1, \dots, i_m such that $\dim \overline{G_{i_1} \cdots G_{i_m}x}$ is maximal. Since $\overline{G_{i_1} \cdots G_{i_m}x}$ is irreducible, it follows that $\overline{G_{j_1} \cdots G_{j_\ell}x} \subseteq \overline{G_{i_1} \cdots G_{i_m}x}$ for every sequence $j_1, \dots, j_\ell \in I$. Hence $\overline{G_{i_1} \cdots G_{i_m}x} = \overline{Gx}$, and there is an open dense set $U \subseteq \overline{Gx}$ contained in $G_{i_1} \cdots G_{i_m}x$. This implies that $Gx = GU$ is open in \overline{Gx} .

(b) For the second claim we first remark that for every $x \in X$ with $\dim Gx$ maximal there is an open set $U_x \subseteq X$ containing x and a sequence $\mathbf{i}_x = (i_1, \dots, i_m)$, depending on x , such that $\overline{G_{i_1} \cdots G_{i_m}y} = \overline{Gy}$ for all $y \in U_x$. Indeed, if $d := \dim Gx$ is maximal and $Gx \subseteq \overline{G_{i_1} \cdots G_{i_m}x}$, then $\dim \overline{G_{i_1} \cdots G_{i_m}y} = d$ for all y in an open neighborhood of x . Now the union $U := \bigcup_x U_x$ is a G -invariant open set, hence covered by finitely many U_{x_j} . By joining the finitely many sequences \mathbf{i}_{x_j} corresponding to the points x_j , we obtain a sequence (j_1, \dots, j_ℓ) such that $\overline{G_{j_1} \cdots G_{j_\ell}y} = \overline{Gy}$ for all $y \in U$. A standard argument implies that $G_{j_\ell}G_{j_{\ell-1}} \cdots G_{j_2}G_{j_1}G_{j_2} \cdots G_{j_\ell}y = Gy$ for all $y \in U$. Since $X \setminus U$ is a closed G -invariant subvariety of smaller dimension, the claim follows by induction. \square

Remark 2.1.3. A similar result holds for the action of a connected ind-group \mathfrak{G} on an affine variety X , see [FK18, Proposition 7.1.2].

2.2. The multiplication map. In the following lemma we use the adjoint action Ad of \mathfrak{G} on its Lie algebra $\text{Lie } \mathfrak{G}$, cf. [FK18]. Moreover, for $g \in \mathfrak{G}$, we denote by $\rho_g: \mathfrak{G} \xrightarrow{\sim} \mathfrak{G}$ the right multiplication with g .

Lemma 2.2.1. *Let $G_1, \dots, G_m \subseteq \mathfrak{G}$ be connected algebraic subgroups and consider the “multiplication map” $\varphi: G_1 \times \dots \times G_m \rightarrow \mathfrak{G}$. Fix $g_i \in G_i$, $i = 1, \dots, m$, and set $g := g_1 \cdots g_m$. Then the image of the composition*

$$d\rho_{g^{-1}} \circ d\varphi_{(g_1, \dots, g_m)}: T_{g_1}G_1 \oplus \dots \oplus T_{g_m}G_m \rightarrow \text{Lie } \mathfrak{G}$$

is given by $\sum_i \text{Ad } h_i(\text{Lie } G_i)$ where $h_1 = e$ and $h_i := g_1 \cdots g_{i-1}$ for $i > 1$.

Proof. The map $\rho_{g^{-1}} \circ \varphi$ restricted to G_i is given

$$g \mapsto g_1 \cdots g_{i-1} g g_{i+1} \cdots g_m \cdot g^{-1} = h_i g h_i^{-1},$$

and the claim follows. \square

In case of an automorphism group $\mathfrak{G} = \text{Aut}(X)$ we can reformulate the lemma above in terms of vector fields. Recall that the action of $\text{Aut}(X)$ on the vector fields is given by $\text{Ad } g(\delta)(gx) = d\mu_g \delta(x)$, see section 1.2.

If $L \subseteq \text{Vec}(X)$ is a subspace, then $L(x) \subseteq T_x X$ denotes the image of L under the evaluation map $\text{ev}_x: \text{Vec}(X) \rightarrow T_x X$.

Lemma 2.2.2. *Let $G_1, \dots, G_m \subseteq \text{Aut}(X)$ be connected algebraic subgroups. Set $L_i := \text{Lie } G_i \subseteq \text{Vec}(X)$ and consider the “orbit map”*

$$\theta_x: G_1 \times \dots \times G_m \rightarrow X, \quad \mathbf{g} := (g_1, \dots, g_m) \mapsto \mathbf{g}x := g_1 \cdots g_m x.$$

Then the image of the differential $(d\theta_x)_{\mathbf{g}}: T_{g_1}G_1 \oplus \dots \oplus T_{g_m}G_m \rightarrow T_{\mathbf{g}x}X$ is given by $\sum_i d\mu_{h_i} L_i(g_i \cdots g_m x)$.

Proof. By Lemma 2.2.1 above the image of the differential $(d\theta_x)_{\mathbf{g}}$ is given by $\sum_i \text{Ad } h_i(L_i)(\mathbf{g}x)$. Since $\text{Ad } h_i(L_i)(\mathbf{g}x) = d\mu_{h_i} L_i(g_i \cdots g_m x)$ the claim follows. \square

2.3. Lie algebras of algebraically generated groups. For an algebraically generated subgroup $G = \langle G_i \mid i \in I \rangle \subseteq \mathfrak{G}$ of the ind-group \mathfrak{G} we define

$$L(G) := \langle \text{Lie } G_i \mid i \in I \rangle_{\text{Lie}} \subseteq \text{Lie } \mathfrak{G}$$

to be the Lie subalgebra generated by the $\text{Lie } G_i$. If $Z \subseteq \mathfrak{G}$ is an algebraic subset and $g \in Z$ we can use the right multiplication with g^{-1} to send $T_g Z$ into $T_e \mathfrak{G} = \text{Lie } \mathfrak{G}$: $d\rho_{g^{-1}}: T_g Z \hookrightarrow \text{Lie } \mathfrak{G}$. Note that for an algebraic subgroup $G \subseteq \mathfrak{G}$ this image of $T_g G$ is equal to $\text{Lie } G$ for any $g \in G$.

Theorem 2.3.1. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \mathfrak{G}$ be an algebraically generated subgroup.*

- (1) $L(G)$ is stable under the adjoint actions of G and of $\overline{G} \subseteq \mathfrak{G}$. In particular, $L(G)$ is an ideal in $\text{Lie } \overline{G}$.

Assume in addition that the index set I is countable and that the base field \mathbb{k} is uncountable.

- (2) If $Z \subseteq \mathfrak{G}$ is an algebraic subset contained in G , then the image of $T_g Z$ under $d\rho_{g^{-1}}$ is contained in $L(G)$ for any g in an open dense subset of Z .
- (3) If $H \subseteq \mathfrak{G}$ is an algebraic subgroup contained in G , then $\text{Lie } H \subseteq L(G)$. In particular, $L(G)$ depends only on G and not on the generating subgroups G_i .

Remark 2.3.2. We do not know if the additional assumptions for the statements (2) and (3) are necessary.

Proof. (1) It suffices to show that L is invariant under all the G_i . Since G_i is connected, we are reduced to prove that L is invariant under the adjoint action of $\text{Lie } G_i$ (see [FK18, Proposition 6.3.4]) which holds by construction.

(2) We first show that $Z \subseteq G_{i_1} \cdots G_{i_m}$ for a suitable sequence (i_1, \dots, i_m) . We can clearly assume that Z is irreducible. Since the index set I is countable the group G is the union of countably many products $G_{i_1} \cdots G_{i_m}$, and so Z is the union of countably many constructible subsets $Z \cap G_{i_1} \cdots G_{i_m}$. Since the base field \mathbb{k} is uncountable we get $Z = Z \cap G_{i_1} \cdots G_{i_m} \subseteq G_{i_1} \cdots G_{i_m}$ for a suitable sequence (i_1, \dots, i_m) ([FK18, Lemma 1.3.1]).

Now consider the multiplication map $\varphi: G_{i_1} \times \cdots \times G_{i_m} \rightarrow \mathfrak{G}$. The induced morphism $\varphi': \varphi^{-1}(Z) \rightarrow Z$ is smooth on a non-empty open set $U \subseteq \varphi^{-1}(Z)$ where we can assume that $\varphi(U) \subseteq Z_{\text{reg}}$. Since $d\varphi_u: T_u \varphi^{-1}(Z) \rightarrow T_{\varphi(u)} Z$ is surjective for all $u \in U$ it follows from Lemma 2.2.1 that the image of $T_{\varphi(u)} Z$ in $\text{Lie } \mathfrak{G}$ is contained in $\sum_i \text{Ad } h_{i_k} \cdot \text{Lie } G_{i_k} \subseteq L(G)$ where the latter inclusion follows from (1).

(3) This is an immediate consequence of (2). □

Now consider the case where where $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ and so $L(G) \subseteq \text{Lie } \text{Aut}(X) \subseteq \text{Vec}(X)$. If $Y \subseteq X$ is a G -stable closed subset, then $\text{Lie } G_i|_Y \subseteq \text{Vec}(Y)$ for all i and so $L(G)|_Y \subseteq \text{Vec}(Y)$. Choosing $Y := \overline{Gx}$ we get $L(G)(x) \subseteq T_x Gx$ for any $x \in X$.

Proposition 2.3.3. *We have $T_x Gx = L(G)(x)$ for all $x \in X$. In particular, any vector field in $L(G)$ is tangent to the G -orbits.*

Proof. We already know that $L(G)(x) \subseteq T_x Gx$. Now we choose a surjective ‘‘orbit map’’

$$\theta_x: G_{j_1} \times \cdots \times G_{j_n} \rightarrow Gx \subseteq X,$$

see Proposition 2.1.2. It follows that for a suitable $\mathbf{g} = (g_1, \dots, g_n)$ the differential

$$(d\theta_x)_{\mathbf{g}}: T_{g_1}G_{j_1} \oplus \cdots \oplus T_{g_n}G_{j_n} \rightarrow T_{\mathbf{g}x}Gx$$

is also surjective. Thus we get $T_{\mathbf{g}x}Gx = \sum_j d\mu_{h_j}L_j(g_j \cdots g_n x)$, by Lemma 2.2.2. By Theorem 2.3.1(1) L is invariant under the action of G , and so

$$d\mu_{h_j}L_j(g_j \cdots g_n x) \subseteq d\mu_{h_j}L(g_j \cdots g_n x) = L(\mathbf{g}x),$$

and the claim follows. \square

2.4. Ramanujam's alternative. The following result is due to RAMANUJAM ([Ram64, Theorem on p. 26]).

Proposition 2.4.1. *Let $G \subseteq \mathfrak{G}$ be a path-connected subgroup of an ind-group \mathfrak{G} . Then one of the following holds:*

- (i) G is an algebraic subgroup;
- (ii) G contains algebraic subsets of arbitrary large dimension.

Proof. Assume that we are not in case (ii), and let $Y \subseteq G$ be an irreducible algebraic subset of maximal dimension d . We may assume that $e \in Y$. We claim that $G \subseteq \overline{Y} = \overline{G}$.

Since G is path-connected we can find, for any $g \in G$, an irreducible variety C_g and a morphism $\varphi_g: C_g \rightarrow \mathfrak{G}$ such that $\varphi_g(C_g) \subseteq G$ and $e, g \in \varphi_g(C_g)$. Clearly, $\varphi_g(C_g) \cdot Y \subseteq G$, and the closure $\overline{\varphi_g(C_g) \cdot Y} \subseteq \mathfrak{G}$ is an irreducible subvariety which contains \overline{Y} and g . The constructible set $\varphi_g(C_g) \cdot Y$ contains a subset U_g which is open and dense in its closure $\overline{\varphi_g(C_g) \cdot Y}$. Thus $\dim U_g \leq d$, and so $\overline{U_g} = \overline{\varphi_g(C_g) \cdot Y} = \overline{Y}$. Hence, $G \subseteq \overline{Y} = \overline{G}$ as claimed.

As a consequence, $\overline{G} \subseteq \mathfrak{G}$ is an algebraic subgroup. Since $Y \subseteq G$ is open and dense in \overline{G} it follows that $\overline{G} = Y^{-1} \cdot Y \subseteq G$, and so (i) holds. \square

Corollary 2.4.2. *A path-connected subgroup of an algebraic group is a closed subgroup.*

2.5. The finite dimensional case. The next result shows that an algebraically generated subgroup of an ind-group \mathfrak{G} is an algebraic group if and only if the Lie algebra $L(G)$ is finite dimensional. It also proves Theorem A from Section 1.3 of the introduction.

Theorem 2.5.1. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \mathfrak{G}$ be an algebraically generated group. Then the following two statements are equivalent.*

- (i) G is an algebraic subgroup of \mathfrak{G} ;
- (ii) The Lie algebra $L(G)$ is finite dimensional.

In this case $\text{Lie } G = L(G)$. In particular, $L(G)$ is locally finite and algebraic.

Proof. (a) Assume that G is an algebraic group. Since $G_i \subseteq G$ is a closed subgroup for any i , we have $\text{Lie } G_i \subseteq \text{Lie } G$, and so $L(G) \subseteq \text{Lie } G$ is finite dimensional. Moreover, there is a finite sequence (i_1, \dots, i_m) such that the multiplication map $\varphi: G_{i_1} \times \cdots \times G_{i_m} \rightarrow G$ is surjective. Then Lemma 2.2.1 applied to $g_1 = g_2 = \cdots = g_m = e$ shows that $\text{Lie } G \subseteq \sum_i \text{Lie } G_i$, and so $L(G) = \text{Lie } G$.

(b) Now assume that $L(G)$ is finite dimensional, and consider the multiplication map $\varphi: G_{i_1} \times \cdots \times G_{i_m} \rightarrow \mathfrak{G}$. There is an open dense subset $U \subseteq G_{i_1} \times \cdots \times G_{i_m}$ such that $\varphi(U) \subseteq \overline{G_{i_1} \cdots G_{i_m}} \subseteq \mathfrak{G}$ is open and contained in $(\overline{G_{i_1} \cdots G_{i_m}})_{\text{reg}}$, and that $\varphi|_U: U \rightarrow \varphi(U)$ is smooth. It follows from Lemma 2.2.1 that $T_g \varphi(U) \subseteq L(G)$ for all $g \in \varphi(U)$ and so $\dim \overline{G_{i_1} \cdots G_{i_m}} \leq \dim L(G)$. Hence there is a sequence (i_1, \dots, i_m) such that $\dim \overline{G_{i_1} \cdots G_{i_m}}$ is maximal which implies that all G_i are contained in $\overline{G_{i_1} \cdots G_{i_m}}$, and so G is an algebraic subgroup. \square

3. INTEGRATION OF VECTOR FIELDS

3.1. Integration of locally finite Lie algebras. The next result is our Theorem E from the introduction, see Section 1.9.

Theorem 3.1.1. *Let $L \subseteq \text{Vec}(X)$ be a Lie subalgebra.*

- (1) *If L is finite dimensional and generated by locally nilpotent vector fields, then L is algebraic, i.e. there is a connected algebraic subgroup $G \subseteq \text{Aut}(X)$ such that $L = \text{Lie } G$.*
- (2) *If L is locally finite, then there exists a unique minimal connected algebraic subgroup $G \subseteq \text{Aut}(X)$ such that $L \subseteq \text{Lie } G$.*

Proof of (1). Assume that L is finite dimensional and generated by locally nilpotent vector fields $\eta_1, \eta_2, \dots, \eta_m$. By [FK18, Proposition 7.6.1] there are (uniquely determined) algebraic subgroups $U_i \subseteq \text{Aut}(X)$, $U_i \simeq \mathbb{G}_a$, such that $\text{Lie } U_i = \mathbb{k}\eta_i$. Now Theorem 2.5.1 implies that $G := \langle U_i \mid i = 1, \dots, m \rangle \subseteq \text{Aut}(X)$ is an algebraic group with $\text{Lie } G = L$. \square

Example 3.1.2. Assume that $L \subseteq \text{Vec}(X)$ is a locally finite semisimple Lie subalgebra. Then there is a semisimple group $G \subseteq \text{Aut}(X)$ such that $L = \xi(\text{Lie } G)$. In particular, L is algebraic.

In fact, the semisimple Lie algebra L is generated by the root subspaces with respect to some Cartan subalgebra (see [Hum72, Proposition 8.4(f)]), hence by nilpotent elements. Lemma 1.6.3(3) shows that these elements are locally nilpotent in $\text{Vec}(X)$, and thus L is generated by locally nilpotent vector fields.

The proof of (2) needs some preparation. It will be given at the end of section 3.2. Let $L = L_0 \ltimes R$ be the Levi decomposition where L_0 is semisimple and $R := \text{rad } L$ is the solvable radical (see [Pro07, 10.1.6, pp. 305–306]). The example above shows that $L_0 = \text{Lie } G_0$ for a semisimple subgroup $G_0 \subseteq \text{Aut}(X)$. It remains to find a solvable subgroup $Q \subseteq \text{Aut}(X)$ normalized by G_0 such that $\text{Lie } Q \supseteq R$. Then the product $G := G_0 Q$ has the required property.

Lemma 3.1.3. *Let $M \subseteq \text{Vec}(X)$ be a locally finite solvable Lie subalgebra. Assume that M contains with every element μ its semisimple and locally nilpotent parts μ_s and μ_n . Then the locally nilpotent elements from M form a Lie subalgebra M_n , and $M = S \ltimes M_n$ where S is any maximal toral subalgebra of M .*

Proof. Since M is locally finite we may assume that $M \subseteq \mathfrak{gl}(V)$ where $V \subseteq \mathcal{O}(X)$ is a finite dimensional subspace generating $\mathcal{O}(X)$. Since M is solvable there is a basis of V such that M is contained in the upper triangular matrices $\mathfrak{b}_d \subseteq \mathfrak{gl}_d$ where $d := \dim V$ (Lie's Theorem). Clearly, the nilpotent elements from M belong to the subalgebra $\mathfrak{n}_d \subseteq \mathfrak{b}_d$ of upper triangular matrices with zeros along the diagonal, and so $M_n := M \cap \mathfrak{n}_d$ is the ideal of M consisting of the nilpotent elements. By Lemma 1.6.3 every nilpotent element of M is also locally nilpotent viewed as a vector field on X , and vice versa. If $M = M_n$ then $S = 0$ and $M = S \ltimes M_n$, as desired.

In the general case, choose a maximal toral subalgebra $S \subseteq M$, and consider the corresponding weight space decomposition $M = \bigoplus_{\alpha \in S^*} M_\alpha$ where S^* is the group of characters of S and

$$M_\alpha := \{\mu \in M \mid \text{ad } \tau(\mu) = [\tau, \mu] = \alpha(\tau) \cdot \mu \text{ for } \tau \in S\}.$$

Similarly, we get a decomposition $\mathfrak{gl}_d = \bigoplus_{\alpha \in S^*} (\mathfrak{gl}_d)_\alpha$. Using the identity $[s, a \cdot b] = [s, a] \cdot b + a \cdot [s, b]$ we get $(\mathfrak{gl}_d)_\alpha \cdot (\mathfrak{gl}_d)_\beta \subseteq (\mathfrak{gl}_d)_{\alpha+\beta}$ which implies that M_α consists of nilpotent elements for $\alpha \neq 0$. Hence, $\bigoplus_{\alpha \neq 0} M_\alpha \subseteq \mathfrak{n}_d$ is a solvable subalgebra consisting of nilpotent elements, while $M_0 = \text{Cent}_M(S) \supseteq S$.

Let $\mu \in M_0$ have Jordan decomposition $\mu = \mu_s + \mu_n$. Since S commutes with μ it also commutes with μ_s and μ_n , and so $\mu_s \in M_0$ and $\mu_n \in M_0 \cap \mathfrak{n}_d$. Since S is a maximal toral subalgebra, we have $\mu_s \in S$, and so $M_0 = S \oplus (M_0 \cap \mathfrak{n}_d)$. As a consequence we see that

$M_n = (M_0 \cap \mathfrak{n}_d) \oplus \bigoplus_{\alpha \neq 0} M_\alpha$ is a subalgebra consisting of nilpotent elements, and that $M = S \ltimes M_n$. \square

From the first part of the proof we get the following result.

Corollary 3.1.4. *If a locally finite solvable Lie subalgebra $M \subseteq \text{Vec}(X)$ is generated by locally nilpotent elements, then it consists of locally nilpotent elements.*

3.2. Jordan-saturated subspaces. The assumption in Lemma 3.1.3 above leads to the following definition. A locally finite subspace $W \subseteq \text{Vec}(X)$ is called *Jordan-saturated* (shortly *J-saturated*) if it contains with every element η the semisimple and the locally nilpotent parts η_s and η_n . Clearly, for any algebraic subgroup $G \subseteq \text{Aut}(X)$ the Lie algebra $\text{Lie } G \subseteq \text{Vec}(X)$ is *J-saturated*.

Lemma 3.2.1. *Let $R \subseteq \text{Vec}(X)$ be a locally finite and solvable Lie subalgebra. Then the smallest J-saturated Lie subalgebra $\tilde{R} \subseteq \text{Vec}(X)$ containing R has the following properties:*

- (1) \tilde{R} is locally finite and solvable.
- (2) If a subspace $V \subseteq \mathcal{O}(X)$ is R -invariant, then it is invariant under \tilde{R} .
- (3) If an automorphism $\varphi \in \text{Aut}(X)$ normalizes R , then it normalizes \tilde{R} .

Notice that any algebraic Lie subalgebra of $\text{Vec}(X)$ which contains R also contains \tilde{R} .

Proof. As before, we fix a finite dimensional R -invariant subspace $V \subseteq \mathcal{O}(X)$ which generates $\mathcal{O}(X)$. Choosing a suitable basis of V we may assume that the image of R in $\mathfrak{gl}(V) = \mathfrak{gl}_d$ is contained in the upper triangular matrices \mathfrak{b}_d .

If $\eta = \eta_s + \eta_n$ is the Jordan-decomposition of an element $\eta \in R$, then every finite dimensional R -invariant subspace $V \subseteq \mathcal{O}(X)$ is invariant under η_s and η_n , hence invariant under the Lie subalgebra R_1 generated by $R + \mathbb{k}\eta_n$. Thus R_1 is locally finite. The Jordan-decomposition of η carries over to the image of η in $\mathfrak{gl}(V) = \mathfrak{gl}_d$. Hence the images of η_s and η_n also belong to \mathfrak{b}_d which shows that the image of R_1 is contained in \mathfrak{b}_d .

This procedure of adding the locally nilpotent part and forming the Lie algebra has to end with a locally finite and solvable Lie subalgebra \tilde{R} containing R with the properties (1) and (2).

Finally, assume that $\varphi \in \text{Aut}(X)$ stabilizes R , and let $\eta \in R$. Then $\varphi(\eta) = \varphi(\eta_s) + \varphi(\eta_n)$ is the Jordan decomposition of $\varphi(\eta) \in R$, hence

$\varphi(\eta_s)$ and $\varphi(\eta_n)$ belong to \tilde{R} . Now it follows from the construction that \tilde{R} is invariant under φ . \square

Proof of Theorem 3.1.1(2). Let $L = L_0 \ltimes R$ be the Levi decomposition where L_0 is semisimple and R is the solvable radical of L . By part (1) and Example 3.1.2 there is a semisimple group $G_0 \subseteq \text{Aut}(X)$ with $\text{Lie } G_0 = L_0$. Clearly, R is G_0 -invariant, as well as \tilde{R} , the J -saturation of R , see Lemma 3.2.1. By Lemma 3.1.3 we have $\tilde{R} = S \ltimes \tilde{R}_n$ where S is toral and \tilde{R}_n is solvable and consists of locally nilpotent elements. Thus, again by part (1), there is a unipotent group $U \subseteq \text{Aut}(X)$ such that $\text{Lie } U = \tilde{R}_n$.

It follows from Lemma 1.8.2 that there is a well-defined minimal torus $T \subseteq \text{Aut}(X)$ such that $\text{Lie } T \supseteq S$, and that T normalizes R_n . By Lemmas 1.8.2 and 3.2.1, T normalizes U and \tilde{R}_n , and so the product $Q := TU = UT \subseteq \text{Aut}(X)$ is a connected solvable subgroup with $\text{Lie } Q \supseteq \tilde{R}$. By construction, Q is the smallest subgroup with this property. It remains to show that G_0 normalizes Q . Indeed, gQg^{-1} is a closed subgroup for any $g \in G_0$, and $\text{Lie } gQg^{-1} \supseteq g\tilde{R}g^{-1} = \tilde{R}$.

The uniqueness statement follows from the fact that $\text{Lie}(G \cap G') = \text{Lie } G \cap \text{Lie } G'$ for any algebraic subgroups $G, G' \subseteq \text{Aut}(X)$, see [FK18, Proposition 7.7.1]. \square

4. NESTED IND-GROUPS AND NESTED SUBGROUPS

4.1. Nested groups and nested Lie algebras. For the following concept see [FK18, Section 9.4] and [KPZ17].

Definition 4.1.1. An ind-group \mathfrak{G} is called *nested* if it admits an admissible filtration consisting of algebraic subgroups. A Lie algebra L is called *nested* if there exists a sequence $L_1 \subseteq L_2 \subseteq \cdots \subseteq L$ of finite dimensional Lie algebras L_i such that $L = \bigcup_i L_i$.

Lemma 4.1.2. (1) *The Lie algebra $\text{Lie } \mathfrak{G}$ of a nested ind-group \mathfrak{G} is nested.*

- (2) *Every filtration of a nested ind-group by algebraic groups is admissible.*
- (3) *A connected nested ind-group admits a filtration by connected algebraic subgroups. In particular, it is algebraically generated (Definition 2.1.1).*
- (4) *Assume that the base field \mathbb{k} is uncountable. If an ind-group \mathfrak{G} has a filtration by algebraic subgroups, then \mathfrak{G} is nested.*

Proof. (1) This is clear.

(2) Let $\mathfrak{G} = \bigcup_k \mathfrak{G}_k$ be a nested ind-group, i.e. the \mathfrak{G}_k are algebraic subgroups, and let $G_1 \subseteq G_2 \subseteq \cdots \subseteq \mathfrak{G}$ be a sequence of algebraic subgroups such that $\mathfrak{G} = \bigcup_i G_i$. Then each G_i is contained in some \mathfrak{G}_k , and $\mathfrak{G}_k = \bigcup_i (\mathfrak{G}_k \cap G_i)$ is a countable union of closed subgroups. It follows from Lemma 4.1.3 below that $\mathfrak{G}_k = \mathfrak{G}_k \cap G_i$ for some i and so $\mathfrak{G}_k \subseteq G_i$.

(3) If $\mathfrak{G} = \bigcup_k \mathfrak{G}_k$ is connected, then $\mathfrak{G} = \bigcup_k \mathfrak{G}_k^\circ$ by [FK18, Proposition 2.2.1].

(4) If \mathbb{k} is uncountable, then every filtration by closed algebraic subsets is admissible, see [FK18, Theorem 1.3.3]. \square

The following lemma is well-known.

Lemma 4.1.3. *For any algebraic group G there exists a finite set $\{g_1, g_2, \dots, g_n\} \subseteq G$ such that $G = \overline{\langle g_1, g_2, \dots, g_n \rangle}$.*

Outline of Proof. The statement is clear for a unipotent group U since $\langle u \rangle \simeq \mathbb{G}_a$ for a unipotent element $u \neq e$. It is also clear for a torus T since there is always an element $t \in T$ such that $T = \overline{\langle t \rangle}$. Finally, for a semisimple group G we fix a maximal torus T , choose a “dense” element $t \in T$ and unipotent elements in every root subgroup of G . \square

Definition 4.1.4. A subgroup G of an ind-group \mathfrak{G} is called *quasi-nested* if there exists a sequence $G_1 \subseteq G_2 \subseteq \cdots \subseteq \mathfrak{G}$ of bounded subgroups of \mathfrak{G} (see section 1.2) such that $G = \bigcup_i G_i$. It is called *nested* if the G_i are algebraic subgroups.

The situation here is subtle. Every nested subgroup G of an ind-group \mathfrak{G} has a natural structure of a (nested) ind-group given by the filtration by algebraic subgroups, and thus $\text{Lie } G$ is well defined and is a subalgebra of $\text{Lie } \mathfrak{G}$. But in general G is not a closed ind-subgroup as shown by the following example.

Example 4.1.5. The subgroup $\mu_2 \subseteq \mathbb{G}_m$ of elements of order a power of 2 is nested by the finite subgroups $\simeq \mathbb{Z}/2^n\mathbb{Z}$. But it is not closed.

Remarks 4.1.6. (a) A quasi-nested subgroup $G = \bigcup_i G_i \subseteq \mathfrak{G}$ defines a nested subgroup, namely $\bigcup_i \overline{G_i}$.

(b) Every subgroup of a nested ind-group is quasi-nested.

(c) Let $\varphi: \mathfrak{G} \rightarrow \mathfrak{H}$ be a homomorphism of ind-groups. If \mathfrak{G} is nested, then the image $\varphi(\mathfrak{G}) \subseteq \mathfrak{H}$ is a nested subgroup.

(d) If the subgroup $G \subseteq \mathfrak{G}$ is quasi-nested, $G = \bigcup_i G_i$, then one can always assume that the subgroups $G_i \subseteq G$ are closed in G .

Proposition 4.1.7. *Assume that the base field \mathbb{k} is uncountable. Then a path-connected quasi-nested subgroup $G \subseteq \mathfrak{G}$ is nested.*

Proof. Let $G = \bigcup G_i \subseteq \mathfrak{G}$ where the G_i are closed in G , see Remark 4.1.6(d). Define $G_i^\circ \subseteq G_i$ to be the path-connected component of e , i.e. the set of elements of G_i which can be connected to $e \in G_i$. Then G_i° is a path-connected bounded subgroup, hence an algebraic group by Corollary 2.4.2.

We claim that $G = \bigcup_i G_i^\circ$. For any $g \in G$ there is an irreducible variety Y and a morphism $\varphi: Y \rightarrow \mathfrak{G}$ such that $e, g \in \varphi(Y) \subseteq G$. Since the subgroups $G_i \subseteq G$ are closed, the inverse images $\varphi^{-1}(G_i) \subseteq Y$ are closed, and $\bigcup_i \varphi^{-1}(G_i) = Y$. Since \mathbb{k} is uncountable there is an i such that $Y = \varphi^{-1}(G_i)$. Hence $\varphi(Y) \subseteq G_i^\circ$ and so $g \in \bigcup_i G_i^\circ$. \square

Question 4. *Let $G \subseteq \mathfrak{G}$ be a path-connected nested subgroup, e.g. a unipotent nested subgroup. Is it true that G is closed?*

In order to give a positive answer to this question it suffices to show that for every closed algebraic subset $A \subseteq \mathfrak{G}$ the intersection $A \cap G$ is contained in an algebraic subgroup of G . In fact, if $A \cap G \subseteq G_i$, then $A \cap G = A \cap G_i$ is closed in G_i .

The next result is an immediate consequence of Proposition 2.1.2 and Lemma 4.1.2(3).

Corollary 4.1.8. *Let $G = \bigcup_k G_k \subseteq \text{Aut}(X)$ be a path-connected nested subgroup. Then there is a k_0 such that for $k \geq k_0$ the groups G and G_k have the same orbits on X .*

4.2. Integration of nested Lie algebras. If $G \subseteq \text{Aut}(X)$ is a nested subgroup, then $\text{Lie } G \subseteq \text{Vec}(X)$ is a nested Lie algebra filtered by locally finite subalgebras. The converse of this is also true.

Proposition 4.2.1. *Let $L = \bigcup_i L_i \subseteq \text{Vec}(X)$ be a nested Lie subalgebra. Then the following hold.*

- (1) *If L is generated by locally nilpotent vector fields, then there exists a nested unipotent subgroup $G = \bigcup_i G_i \subseteq \text{Aut}(X)$ such that $L = \bigcup_i \text{Lie } G_i$.*
- (2) *Assume that all L_i are locally finite. Then there exists a smallest nested subgroup $G \subseteq \text{Aut}(X)$ such that $L \subseteq \text{Lie } G \subseteq \text{Vec}(X)$. Moreover, a closed ind-subgroup $\mathfrak{G} \subseteq \text{Aut}(X)$ contains G if and only if $L \subseteq \text{Lie } \mathfrak{G}$.*

Proof. (1) Any finite collection $\mathcal{S} \subseteq L$ of locally nilpotent vector fields generates a Lie subalgebra, say, $M_{\mathcal{S}}$ of some L_j . According to Theorem 3.1.1(1), $M_{\mathcal{S}} = \text{Lie } G_{\mathcal{S}}$ for an algebraic subgroup $G_{\mathcal{S}}$ of $\text{Aut}(X)$. On the other hand, L_i is contained in some $M_i := M_{\mathcal{S}_i}$, hence $L =$

$\bigcup_i M_i$. We may suppose that $M_i \subseteq M_{i+1}$, and so $G_i \subseteq G_{i+1}$ where $G_i := G_{S_i}$. The union $G := \bigcup_i G_i$ is a nested subgroup of $\text{Aut } X$ such that $L = \bigcup_i \text{Lie } G_i$.

(2) Let $G_i \subseteq \text{Aut}(X)$ be the smallest algebraic subgroup with $\text{Lie } G_i \supseteq L_i$ (see Theorem 3.1.1(2)). Then $G_i \subseteq G_{i+1}$ for all i and so $G := \bigcup_i G_i$ is a nested subgroup with the required properties.

Now assume that $\text{Lie } \mathfrak{G} \supseteq L$ and define $G'_i := \mathfrak{G} \cap G_i$. Then $\text{Lie } G'_i = \text{Lie } \mathfrak{G} \cap \text{Lie } G_i \supseteq L_i$, hence $G'_i = G_i$ and the claim follows. \square

5. SOLVABILITY AND TRIANGULATION

5.1. Solvable subgroups. This section is devoted to the proof of our Theorem B from the introduction.

Theorem 5.1.1. *Let $G = \langle H_i \mid i \in I \rangle$ be a solvable algebraically generated group. If the family I is finite, then G is a solvable algebraic group with Lie algebra $\text{Lie } G = L(G) = \langle \text{Lie } H_i \mid i \in I \rangle$. In general, G is nested and is of the form $G = U_G \rtimes T$ where $T \subseteq \text{Aut}(X)$ is a torus and U_G is a nested unipotent group consisting of the unipotent elements of G .*

Corollary 5.1.2. *A solvable subgroup $G \subseteq \text{Aut}(X)$ generated by unipotent elements is quasi-nested and consists of unipotent elements.*

Proof of Corollary. If G is generated by the unipotent elements $(u_i)_{i \in I}$, then the closure $\overline{G} \subseteq \overline{\text{Aut}(X)}$ contains the subgroup \tilde{G} generated by the unipotent subgroups $\langle u_i \rangle$. We will see in the following Lemma 5.1.3(2) that \overline{G} is also solvable. It then follows from the theorem above that $\tilde{G} = U_{\tilde{G}} \rtimes T$ is nested, hence G is quasi-nested and is contained in $U_{\tilde{G}}$. \square

The proof of Theorem 5.1.1 needs some preparation. Given a solvable group G we denote by $G^{(i)}$ the members of the derived series of G and by $d(G)$ the derived length:

$$G^{(0)} := G \supseteq G^{(1)} := (G, G) \supseteq G^{(2)} := (G^{(1)}, G^{(1)}) \supseteq \cdots \supseteq G^{(d)} = \{e\}.$$

If G is nilpotent, then G_i stands for the members of the lower central series of G and $n(G)$ for its nilpotency class:

$$G_0 := G \supseteq G_1 := (G, G) \supseteq G_2 := (G, G_1) \supseteq \cdots \supseteq G_n = \{e\}.$$

In the following lemma we collect some important facts about closures of solvable and nilpotent groups. Statement (2) can be found in [FP18, Lemma 2.3(3)].

Lemma 5.1.3. *Let $G, H \subseteq \mathfrak{G}$ be subgroups of an ind-group \mathfrak{G} .*

- (1) We have $(\overline{G}, \overline{H}) \subseteq \overline{(G, H)}$, $\overline{G}^{(i)} \subseteq \overline{G^{(i)}}$ and $\overline{G}_i \subseteq \overline{G}_i$.
- (2) If G is solvable, then so is \overline{G} , and $d(\overline{G}) = d(G)$.
- (3) If G is nilpotent, then so is \overline{G} , and $n(\overline{G}) = n(G)$.

Proof. (1) Let $\gamma: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$ be the commutator map, $(g, h) \mapsto ghg^{-1}h^{-1}$. Then $\gamma(\overline{G} \times \overline{H}) \subseteq \overline{\gamma(G \times H)}$, and so $(\overline{G}, \overline{H}) \subseteq \overline{\langle \gamma(G \times H) \rangle} \subseteq \overline{(G, H)}$. Now the remaining inclusions follow.

(2) By (1) we have $G^{(i)} \subseteq \overline{G}^{(i)} \subseteq \overline{G^{(i)}}$. Hence $\overline{G}^{(d(G))} = \{e\}$ and $\overline{G}^{(i)} \neq \{e\}$ for $i < d(G)$.

(3) By (1) we have $G_i \subseteq \overline{G}_i \subseteq \overline{G}_i$. Hence $\overline{G}_{n(G)} = \{e\}$ and $\overline{G}_i \neq \{e\}$ for $i < n(G)$. \square

Lemma 5.1.4. *Let \mathfrak{G} be an ind-group and \mathfrak{H} be a closed subgroup of \mathfrak{G} . Then the following hold.*

- (1) $[\text{Lie } \mathfrak{H}, \text{Lie } \mathfrak{G}] \subseteq \overline{\text{Lie } (\mathfrak{H}, \mathfrak{G})}$.
- (2) If \mathfrak{G} is solvable, then so is $\text{Lie } \mathfrak{G}$.
- (3) If \mathfrak{G} is nilpotent, then so is $\text{Lie } \mathfrak{G}$.

Proof. (Cf. [FK18, Section 7.5])

- (1) For a fixed $h \in \mathfrak{H}$ consider the morphism

$$\gamma_h: \mathfrak{G} \rightarrow (\mathfrak{H}, \mathfrak{G}), \quad \gamma_h(g) := hgh^{-1}g^{-1}.$$

The differential $(d\gamma_h)_e: \text{Lie } \mathfrak{G} \rightarrow \text{Lie } \overline{(\mathfrak{H}, \mathfrak{G})}$ is given by $(d\gamma_h)_e = \text{Ad}(h) - \text{id}$. Now fix $A \in \text{Lie } \mathfrak{G}$ and consider the morphism

$$\alpha_A: \mathfrak{H} \rightarrow \text{Lie } \mathfrak{G}, \quad \alpha_A(h) := \text{Ad}(h)A - A = d\gamma_h(A).$$

The differential at e of this map is $-\text{ad}(A)$; it sends $\text{Lie } \mathfrak{H}$ onto $[\text{Lie } \mathfrak{H}, A]$. By the preceding, $(d\gamma_h)_e(A) \in \text{Lie } \overline{(\mathfrak{H}, \mathfrak{G})}$ for any $h \in \mathfrak{H}$ and $A \in \text{Lie } \mathfrak{G}$. Hence, $[\text{Lie } \mathfrak{H}, A] \subseteq \overline{\text{Lie } (\mathfrak{H}, \mathfrak{G})}$ for any $A \in \text{Lie } \mathfrak{G}$. This yields (1).

(2) Now assume that \mathfrak{G} is solvable. From the derived series for \mathfrak{G} we get the normal series

$$\mathfrak{G}^{(0)} = \mathfrak{G} \supseteq \overline{\mathfrak{G}^{(1)}} \supseteq \overline{\mathfrak{G}^{(2)}} \supseteq \dots \supseteq \overline{\mathfrak{G}^{(n)}} = \{e\}$$

of closed subgroups with the property that the factor groups are all commutative (see Lemma 5.1.3(1)). Passing to the series

$$\text{Lie } \mathfrak{G} \supseteq \text{Lie } \overline{\mathfrak{G}^{(1)}} \supseteq \text{Lie } \overline{\mathfrak{G}^{(2)}} \supseteq \dots \supseteq \text{Lie } \overline{\mathfrak{G}^{(n)}} = 0$$

we get from (1)

$$\text{Lie } \overline{\mathfrak{G}^{(i+1)}} = \text{Lie } \overline{(\mathfrak{G}^{(i)}, \mathfrak{G}^{(i)})} = \overline{\text{Lie } (\mathfrak{G}^{(i)}, \mathfrak{G}^{(i)})} \supseteq [\text{Lie } \overline{\mathfrak{G}^{(i)}}, \text{Lie } \overline{\mathfrak{G}^{(i)}}].$$

Thus the factors $\text{Lie } \overline{\mathfrak{G}}^{(i)} / \text{Lie } \overline{\mathfrak{G}}^{(i+1)}$ are all commutative, which implies that $\text{Lie } \mathfrak{G}$ is solvable.

(3) Assume now that \mathfrak{G} is nilpotent. From the lower central series for \mathfrak{G} we get the series of closed subgroups

$$\mathfrak{G}_0 = \mathfrak{G} \supseteq \overline{\mathfrak{G}}_1 \supseteq \overline{\mathfrak{G}}_2 \supseteq \cdots \supseteq \overline{\mathfrak{G}}_n = \{e\}$$

and the corresponding series

$$(*) \quad \text{Lie } \mathfrak{G} \supseteq \text{Lie } \overline{\mathfrak{G}}_1 \supseteq \text{Lie } \overline{\mathfrak{G}}_2 \supseteq \cdots \supseteq \text{Lie } \overline{\mathfrak{G}}_n = \{0\}.$$

By virtue of (1) we have

$$\text{Lie } \overline{\mathfrak{G}}_{i+1} = \text{Lie } (\overline{\mathfrak{G}}, \overline{\mathfrak{G}}_i) = \text{Lie } (\overline{\mathfrak{G}}, \overline{\mathfrak{G}}_i) \supseteq [\text{Lie } \overline{\mathfrak{G}}, \text{Lie } \overline{\mathfrak{G}}_i].$$

Thus (*) is a central series and so $\text{Lie } \mathfrak{G}$ is nilpotent. \square

Question 5. *Let \mathfrak{G} be a path connected ind-group. Is it true that \mathfrak{G} is solvable (resp. nilpotent) if $\text{Lie } \mathfrak{G}$ is?*

Remark 5.1.5. It is known that a path-connected ind-group \mathfrak{G} is commutative if $\text{Lie } \mathfrak{G}$ is, see [FK18, Corollary 7.5.3].

Lemma 5.1.6. *Let L be a finitely generated Lie algebra. If L is solvable, then L is finite dimensional.*

Proof. In the derived series

$$L^{(0)} := L \supseteq L^{(1)} := [L^{(0)}, L^{(0)}] \supseteq L^{(2)} := [L^{(1)}, L^{(1)}] \supseteq \cdots \supseteq L^{(n)} = \{0\}$$

every member $L^{(i)}$ is finitely generated. Therefore $L^{(i)} / L^{(i+1)}$ is a commutative and finitely generated Lie algebra, hence finite dimensional, and the claim follows. \square

Proof of Theorem 5.1.1. (a) We first consider the case where I is finite. The closure \overline{G} is solvable (Lemma 5.1.3(2)) and so $\text{Lie } \overline{G}$ is solvable (Lemma 5.1.4(2)). It follows that $L(G) = \langle \text{Lie } H_i \mid i = 1, \dots, m \rangle_{\text{Lie}}$ is also solvable (Theorem 2.3.1(1)), hence finite dimensional (Lemma 5.1.6). Now the claim follows from Theorem 2.5.1.

(b) In general, it follows from (a) that for any finite subset $F \subseteq I$ the subgroup $G_F := \langle H_i \mid i \in F \rangle$ is a connected solvable algebraic subgroup of \mathfrak{G} . If I is countable we have an ascending filtration $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq I$ by finite subsets, and so $G = \bigcup_j G_{F_j}$ is nested.

(c) For a general I we first remark that the Lie algebra $L \subseteq \text{Vec}(X)$ generated by the $\text{Lie } H_i$ has countable dimension, and so $L = \langle \text{Lie } H_j \mid j \in J \rangle_{\text{Lie}}$ for a countable subset $J \subseteq I$. We claim that G is generated by the H_j , $j \in J$, which implies by (b) that G is nested. In fact, for any H_i there are finitely many H_{j_1}, \dots, H_{j_m} with $j_k \in J$ such that

$\mathrm{Lie} H_i \subseteq \langle \mathrm{Lie} H_{j_k} \mid k = 1, \dots, m \rangle_{\mathrm{Lie}}$. Thus $\mathrm{Lie} H_i \subseteq \mathrm{Lie} \tilde{H}_i$ where \tilde{H}_i is the solvable algebraic subgroup generated by the H_{j_k} , and so $H_i \subseteq \tilde{H}_i$ by [FK18, Remark 17.3.3].

(d) If the nested subgroup $G = \bigcup_k G_k$ does not contain semisimple elements, then $G = U_G$ is a nested unipotent group. Otherwise we choose a torus $T \subseteq G$ of maximal dimension. We can assume that T is contained in all G_k , hence $G_k = U_{G_k} \rtimes T$ for all k , and $U_{G_k} \subseteq U_{G_{k+1}}$. Thus $U_G := \bigcup_k U_{G_k}$ is a nested unipotent group containing all unipotent elements of G , and $G = U_G \rtimes T$. \square

5.2. Triangulation in $\mathrm{Aut}(\mathbb{A}^n)$. By definition, an automorphism φ of \mathbb{A}^n belongs to the de Jonquières subgroup $\mathrm{Jonq}(n)$ (see Section 1.4) if and only if the comorphism φ^* stabilizes the flag

$$\mathbb{k}[x_1, \dots, x_{n-1}, x_n] \supseteq \mathbb{k}[x_1, \dots, x_{n-1}] \supseteq \dots \supseteq \mathbb{k}[x_1, x_2] \supseteq \mathbb{k}[x_1].$$

Equivalently, φ stabilizes the coflag

$$\mathcal{F}: \mathbb{A}^n \rightarrow \mathbb{A}^{n-1} \rightarrow \mathbb{A}^{n-2} \rightarrow \dots \rightarrow \mathbb{A}^2 \rightarrow \mathbb{A}^1$$

which means that φ induce an automorphism on each \mathbb{A}^k such that the projections $\mathbb{A}^k \rightarrow \mathbb{A}^{k-1}$, $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-1})$ are φ -equivariant.

Remark 5.2.1. $\mathrm{Jonq}(n)$ is a connected, nested solvable group of the form $\mathrm{Jonq}(n) = \mathrm{Jonq}(n)_u \rtimes T$ where $T := \mathbb{k}^{*n}$ is the standard maximal torus and $\mathrm{Jonq}(n)_u$ is the subgroup of unipotent elements, cf. [FK18, Section 15.1]. $\mathrm{Jonq}(n)$ has derived length $d(\mathrm{Jonq}(n)) = n + 1$ ([FP18, Lemma 3.2]) and thus $d(\mathrm{Jonq}(n)_u) = n$, because $(\mathrm{Jonq}(n), \mathrm{Jonq}(n)) = \mathrm{Jonq}(n)_u$.

Theorem 5.2.2. *Let $U \subseteq \mathrm{Aut}(\mathbb{A}^n)$ be a nested unipotent subgroup. If U has a dense orbit on \mathbb{A}^n , then U acts transitively on \mathbb{A}^n , and U is conjugate to a subgroup of $\mathrm{Jonq}(n)_u$.*

Proof. The first statement follows Corollary 4.1.8 and the fact, that U -orbits are closed.

For the second claim we have to construct a U -stable coflag $\mathbb{A}^n \rightarrow \mathbb{A}^{n-1} \rightarrow \dots \rightarrow \mathbb{A}^1$ where the projections are of the form $\mathbb{A}^k \simeq \mathbb{A}^{k-1} \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}} \mathbb{A}^{k-1}$.

Let $U_x \subseteq U$ be the isotropy group of some point $x \in \mathbb{A}^n$. By Lemma 5.2.3 below we can find an element $a \in \mathrm{Norm}_U U_x \setminus U_x$. Then the subgroup $A := \overline{\langle a \rangle}$ is isomorphic to \mathbb{G}_a and normalizes U_x . Then there is a right action of A on $U/U_x \xrightarrow{\sim} \mathbb{A}^n$ which commutes with the U -action. Thus we get an action of U on the quotient $\mathbb{A}^n // A := \mathrm{Spec} \mathcal{O}(\mathbb{A}^n)^A$ such

that the quotient map $p: \mathbb{A}^n \rightarrow \mathbb{A}^n // A$ is U -equivariant. It follows that U has a dense orbit in $\mathbb{A}^n // A$ and so $\mathbb{A}^n // A \simeq \mathbb{A}^{n-1}$, because orbits of unipotent groups are closed and isomorphic to affine spaces, cf. [FK18, Theorem 11.1.1].

Moreover, the A -action on \mathbb{A}^n admits a local slice, hence a local section of p , see [Fre06, Section 1.4, Principle 1(c)]. Since U acts transitively on \mathbb{A}^n and p is U -equivariant there exist local sections of p in any $z \in \mathbb{A}^n$ and thus $p: \mathbb{A}^n \rightarrow \mathbb{A}^n // A$ is a principal A -bundle. By a theorem of Serre [Ser58, Section 5.1] we have $H^1(X, \mathbb{G}_a) = 0$ for any affine variety X , and hence this bundle is trivial:

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{\simeq} & \mathbb{A}^{n-1} \times \mathbb{A}^1 \\ \downarrow p & & \downarrow \text{pr} \\ \mathbb{A}^n // A & \xrightarrow{\simeq} & \mathbb{A}^{n-1} \end{array}$$

Thus, we have constructed a U -equivariant projection $\mathbb{A}^n \rightarrow \mathbb{A}^{n-1}$, and the claim follows by induction. \square

Lemma 5.2.3. *Let $U \subseteq \text{Aut}(X)$ be a nested unipotent subgroup. For any point $x \in X \setminus X^U$ the normalizer $\text{Norm}_U U_x$ strictly contains the isotropy group U_x .*

Proof. The claim of the lemma is well-known for a unipotent algebraic group U since such a group is nilpotent. In general, $U = \bigcup_k U_k$ with closed unipotent algebraic subgroups U_k such that $U_k \subseteq U_{k+1}$, and, similarly, $U_x = \bigcup_k (U_k)_x$ with closed unipotent algebraic subgroups $(U_k)_x = U_x \cap U_k$. Corollary 4.1.8 implies that we can find a k_0 such that $(U_k)_x$ has the same orbits on X as U_x for all $k \geq k_0$.

Now we use the following general fact. If a group G acts on a space X , and if $x \in X$, then $g \in G$ belongs to the normaliser of the isotropy group G_x if and only if the isotropy group of gx is equal to G_x , i.e. if and only if $gx \in X^{G_x}$.

Assume that $g \in U_k$ belongs to the normaliser of $(U_k)_x$. Then $gx \in X^{(U_k)_x} = X^{U_x}$, and so g is in the normaliser of U_x . If $g \notin (U_k)_x = U_k \cap U_x$, then $g \notin U_x$, and the claim follows. \square

It is a basic fact in algebraic transformation groups that every affine G -variety X admits a closed G -equivariant embedding $X \hookrightarrow V$ where V is G -module. The corresponding statement for ind-groups does not hold, see [FK18, Proposition 2.6.5].

Question 6. *Let $G \subseteq \text{Aut}(X)$ be a solvable or nilpotent connected subgroup. Does there exist a closed embedding $X \hookrightarrow \mathbb{A}^n$ such that G*

extends to a subgroup of the de Jonquères group $\text{Jonq}(n)$? Is this true if G is nested?

5.3. Nested unipotent subgroups of $\text{Aut}(X)$ are solvable. We know from Theorem B that a solvable subgroup G of $\text{Aut}(X)$ generated by a family of unipotent algebraic groups is a nested unipotent group. The next result shows that the converse holds as well.

Theorem 5.3.1. *A nested unipotent subgroup $U \subseteq \text{Aut}(X)$ is solvable of derived length $\leq \max\{\dim Ux \mid x \in X\} \leq \dim X$.*

Proof. It suffices to consider the case of a unipotent algebraic subgroup $U \subseteq \text{Aut}(X)$. Then every orbit $O = Ux$ is closed and isomorphic to an affine space. Therefore, by Theorem 5.2.2 and Remark 5.2.1, the image of U in $\text{Aut}(O)$ has derived length $\leq \dim O$. If $m := \max\{\dim Ux \mid x \in X\}$, then the m th member $U^{(m)}$ of the derived series of U acts trivially on every orbit, hence on X , and so $U^{(m)} = \{e\}$. \square

Corollary 5.3.2. (1) *A solvable algebraically generated subgroup $G \subseteq \text{Aut}(X)$ is of derived length $\leq \dim X + 1$.*
 (2) *Let $\mathfrak{G} = \bigcup_k \mathfrak{G}_k \subseteq \text{Aut}(X)$ be a connected nested subgroup where the \mathfrak{G}_k are solvable algebraic groups. Then \mathfrak{G} is solvable of derived length $\leq \dim X + 1$.*

Proof. (1) By Theorem 5.1.1, $G = U_G \rtimes T$ where $T \subseteq G$ is a maximal torus and U_G is a nested unipotent subgroup. By Theorem 5.3.1, G is solvable of derived length $\leq \dim X + 1$.

(2) By Lemma 4.1.2(3) we may suppose that all the \mathfrak{G}_k are connected. Thus, $\mathfrak{G}_k = U_k \rtimes T_k$ where $U_k = R_u(\mathfrak{G}_k)$ and T_k is the maximal torus of \mathfrak{G}_k . Clearly, we have $U_k \subseteq U_{k+1}$, and we can arrange that $T_k \subseteq T_{k+1}$ for all k . Then again $\mathfrak{G} = U \rtimes T$ where $U := \bigcup_k U_k$ and $T := \bigcup_k T_k$ is a maximal torus of \mathfrak{G} of dimension $\leq \dim(X)$. Now the assertion follows from Theorem 5.3.1 above. \square

Remark 5.3.3. A unipotent algebraic group is nilpotent, but the nilpotency class of unipotent subgroups of $\text{Aut}(X)$ might not be bounded, and thus a nested unipotent subgroup is not necessarily nilpotent.

For example, $\mathcal{J}_n := \text{Jonq}(n)_u$ is a closed nested unipotent subgroup of $\text{Aut}(\mathbb{A}^n)$. For $n \geq 2$ its Lie algebra $\text{Lie } \mathcal{J}_n$ is not nilpotent, and so \mathcal{J}_n is neither, by Lemma 5.1.4(3). Consider, for instance, the case $n = 2$, and define

$$L_d := \langle \partial/\partial x_1, x_1^d \partial/\partial x_2 \rangle \subseteq \text{Lie } \mathcal{J}_2.$$

It is easily seen that the d th member $(L_d)_d$ of the lower central series of L_d does not vanish, and so $(\text{Lie } \mathcal{J}_2)_d \neq 0$ for any $d \geq 1$.

6. AN INTERESTING EXAMPLE

In this section we study the subgroup $F \subseteq \text{Aut}(\mathbb{A}^2)$ generated by $\mathbf{u} := (x+y^2, y)$ and $\mathbf{v} := (x, y+x^2)$. We will show that F is a free group in two generators and we will describe the closure $\mathfrak{F} := \overline{F}$ and its Lie algebra $\text{Lie } \mathfrak{F}$. It turns out that the following holds (see Theorem 6.4.2).

- \mathfrak{F} is a free product of two nested closed abelian unipotent ind-subgroups \mathfrak{J} and $\mathfrak{J}^- \subseteq \text{Aut}(\mathbb{A}^2)$. In particular, \mathfrak{F} is torsion free.
- As a Lie algebra, $\text{Lie } \mathfrak{F}$ is generated by $\text{Lie } U$ and $\text{Lie } V$ where $U := \langle \mathbf{u} \rangle \simeq \mathbb{G}_a$ and $V := \langle \mathbf{v} \rangle \simeq \mathbb{G}_a$.
- Any algebraic subgroup of \mathfrak{F} is abelian and unipotent and is conjugate to a subgroup of \mathfrak{J} or \mathfrak{J}^- .

6.1. **Notation and first results.** Let

$$T := \{(ax, by) \mid a, b \in \mathbb{k}^*\} \subseteq \text{GL}_2(\mathbb{k}) \subseteq \text{Aut}(\mathbb{A}^2)$$

denote the standard 2-dimensional torus, and let

$$\text{SAut}(\mathbb{A}^2) := \{\mathbf{g} \in \text{Aut}(X) \mid \text{jac}(\mathbf{g}) = 1\}$$

where the *Jacobian* of $\mathbf{g} = (f, h)$ is defined in the usual way: $\text{jac}(\mathbf{g}) = \det \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix}$. Furthermore, we set

$$\text{Aut}_0(\mathbb{A}^2) := \{\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \mid \mathbf{g}(0) = 0\},$$

the group of automorphisms fixing the origin, and

$$\text{SAut}_0(\mathbb{A}^2) := \text{SAut}(\mathbb{A}^2) \cap \text{Aut}_0(\mathbb{A}^2).$$

There is a canonical homomorphism $D: \text{Aut}_0(\mathbb{A}^2) \rightarrow \text{GL}_2(\mathbb{k})$ defined by $D(\mathbf{g}) := (d\mathbf{g})_0$. If $\mathbf{g} = (f, h)$, then $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ if and only if $f(0) = h(0) = 0$, i.e. f and h have no constant terms, and $D(\mathbf{g}) = (f_1, h_1)$, the endomorphism given by the linear terms of $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$. We will be interested in the kernel of D :

$$\begin{aligned} \mathfrak{A} &:= \ker D = \{\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2) \mid (d\mathbf{g})_0 = \text{id}\} \\ &= \{(f, h) \in \text{Aut}_0(\mathbb{A}^2) \mid (f_1, h_1) = (x, y)\}. \end{aligned}$$

We get a split exact sequence of ind-groups

$$(*) \quad 1 \longrightarrow \mathfrak{A} \longrightarrow \text{Aut}_0(\mathbb{A}^2) \xrightarrow{D} \text{GL}_2(\mathbb{k}) \longrightarrow 1$$

Thus \mathfrak{A} is a closed path-connected ind-subgroup of $\text{Aut}(\mathbb{A}^2)$ which contains $\mathbf{u} = (x+y^2, y)$ and $\mathbf{v} = (x, y+x^2)$, and so $\mathfrak{F} \subseteq \mathfrak{A}$.

Let $S := \{(\zeta x, \zeta^2 y) \mid \zeta^3 = 1\} \subseteq T \cap \text{SL}_2 \subseteq \text{SAut}(\mathbb{A}^2)$ be the cyclic diagonal subgroup of order 3. The elements of S commute with \mathbf{u} and \mathbf{v} and thus with all elements from F and hence from $\mathfrak{F} = \overline{F}$.

Lemma 6.1.1. *One has $S = \text{Cent}_{\text{Aut}(\mathbb{A}^2)}(F)$.*

Proof. Note that any $\mathbf{g} \in \text{Aut}(\mathbb{A}^2)$ commuting with \mathbf{u} and \mathbf{v} , and therefore with U and V , fixes the origin which is the unique common fixed point of U and V . It also leaves invariant the set of orbits of U and of V , i.e. the pencils of horizontal and vertical lines. Hence, $\mathbf{g} \in T$, and an easy calculation shows that, Indeed, $\mathbf{g} \in S$. \square

Considering the action of S on $\text{Aut}(\mathbb{A}^2)$ by conjugation, one sees that $\text{SAut}(\mathbb{A}^2)$ and \mathfrak{A} are stable under this action, and thus

$$\mathfrak{F} \subseteq \mathfrak{A}^S := \{\mathbf{g} \in \mathfrak{A} \mid \mathbf{g} \circ s = s \circ \mathbf{g} \text{ for all } s \in S\}.$$

Moreover, S also acts on the Lie algebras $\text{Lie Aut}(\mathbb{A}^2)$ and $\text{Lie } \mathfrak{A}$, and so

$$(1) \quad \text{Lie } \mathfrak{F} \subseteq \text{Lie } \mathfrak{A}^S \subseteq (\text{Lie } \mathfrak{A})^S.$$

6.2. The Lie algebra of \mathfrak{F} . The *divergence* div of a vector field $\delta = f \frac{\partial}{\partial x} + h \frac{\partial}{\partial y}$ is defined as $\text{div } \delta = \frac{\partial f}{\partial x} + \frac{\partial h}{\partial y}$. For the Lie algebras of $\text{Aut}(\mathbb{A}^2)$, $\text{SAut}(\mathbb{A}^2)$ and $\text{Aut}_0(\mathbb{A}^2)$ we have the following description:

- $\text{Lie Aut}(\mathbb{A}^2) \xrightarrow{\sim} \text{Vec}^c(\mathbb{A}^2) := \{\delta \in \text{Vec}(\mathbb{A}^2) \mid \text{div } \delta \in \mathbb{k}\},$
- $\text{Lie SAut}(\mathbb{A}^2) \xrightarrow{\sim} \text{Vec}^0(\mathbb{A}^2) := \{\delta \in \text{Vec}(\mathbb{A}^2) \mid \text{div } \delta = 0\},$
- $\text{Lie Aut}_0(\mathbb{A}^2) \xrightarrow{\sim} \text{Vec}_0(\mathbb{A}^2) := \{\delta \in \text{Vec}^c(\mathbb{A}^2) \mid \delta(0) = 0\},$

The first two are given in [FK18, Proposition 15.7.2], and the last is an immediate consequence. The split exact sequence $(*)$ from the previous section 6.1 implies that

$$\begin{aligned} \text{Lie } \mathfrak{A} &= \ker(dD_e: \text{Vec}_0(\mathbb{A}^2) \rightarrow \mathfrak{gl}_2) \\ &= \{\delta = f \frac{\partial}{\partial x} + h \frac{\partial}{\partial y} \in \text{Vec}_0(\mathbb{A}^2) \mid f_1 = h_1 = 0\}. \end{aligned}$$

Now consider the following vector fields from $\text{Vec}^0(\mathbb{A}^2)$,

$$\partial_{i,j} := (j+1)x^{i+1}y^j \frac{\partial}{\partial x} - (i+1)x^i y^{j+1} \frac{\partial}{\partial y}$$

where

$$(i, j) \in \Lambda := \{(k, l) \in \mathbb{Z}^2 \mid k, l \geq -1, k+l \geq 0\}.$$

The torus T acts by conjugation on $\text{Aut}(\mathbb{A}^2)$ and on $\text{Vec}(\mathbb{A}^2)$, and the $\partial_{i,j}$ are eigenvectors of weight (i, j) . Moreover, the action of T on $\text{Vec}^0(\mathbb{A}^2)$ is multiplicity-free, and we have the weight decompositions

$$\text{Vec}^0(\mathbb{A}^2) = \bigoplus_{(i,j) \in \Lambda} \mathbb{k} \partial_{i,j} \quad \text{and} \quad \text{Lie } \mathfrak{A} = \bigoplus_{(i,j) \in \Lambda, i+j > 0} \mathbb{k} \partial_{i,j}.$$

Note that $\partial_{i,j}$ is fixed by the finite subgroup $S \subseteq T$ if and only if $i - j \equiv 0 \pmod{3}$. Setting $\Lambda_0 := \{(k, l) \in \Lambda \mid k - l \equiv 0 \pmod{3}, (k, l) \neq (0, 0)\}$ we get

$$(2) \quad \text{Lie } \mathfrak{A}^S \subseteq (\text{Lie } \mathfrak{A})^S = \bigoplus_{(i,j) \in \Lambda_0} \mathbb{k} \partial_{i,j}.$$

The next lemma shows that $\text{Lie } \mathfrak{F} = \text{Lie } \mathfrak{A}^S = (\text{Lie } \mathfrak{A})^S$. In contrast to the situation of algebraic groups, this does not automatically imply that $\mathfrak{F} = \mathfrak{A}^S$. In fact, FURTER-KRAFT give an example of a closed ind-subgroup of a path-connected ind-group which has the same Lie algebra, but is strictly smaller (see [FK18, Theorem 17.3.1]). In order to show that $\mathfrak{F} = \mathfrak{A}^S$ (see Theorem 6.4.2) we therefore need an additional argument.

Lemma 6.2.1. *The Lie algebra $(\text{Lie } \mathfrak{A})^S$ is generated by $\text{Lie } U$ and $\text{Lie } V$, hence $\text{Lie } \mathfrak{F} = \text{Lie } \mathfrak{A}^S = (\text{Lie } \mathfrak{A})^S$.*

Proof. We have $\text{Lie } U = \mathbb{k} \partial_{-1,2}$ and $\text{Lie } V = \mathbb{k} \partial_{2,-1}$. An easy calculation shows that

$$\text{ad } \partial_{-1,2}(\partial_{i,j}) = a \partial_{i-1,j+2} \ (i \geq 0) \text{ and } \text{ad } \partial_{2,-1}(\partial_{i,j}) = b \partial_{i+2,j-1} \ (j \geq 0)$$

with non-zero constants $a, b \in \mathbb{k}$. This implies that the Lie algebra generated by $\partial_{-1,2}$ and $\partial_{2,-1}$ contains all $\partial_{i,j}$ with $(i, j) \in \Lambda_0$, hence $\langle \text{Lie } U, \text{Lie } V \rangle_{\text{Lie}} \supseteq (\text{Lie } \mathfrak{A})^S$, see formula (2). Since $\text{Lie } U, \text{Lie } V \subseteq \text{Lie } \mathfrak{F}$ the claim follows from the inclusions in (1). \square

It is easy to see that the vector field $\partial_{i,j}$ is locally nilpotent if and only if $i = -1$ or $j = -1$. In these cases, there are unique algebraic subgroups $U_{-1,k} \simeq \mathbb{G}_a$ and $U_{k,-1} \simeq \mathbb{G}_a$ with $\text{Lie } U_{-1,k} = \mathbb{k} \partial_{-1,k}$ and $\text{Lie } U_{k,-1} = \mathbb{k} \partial_{k,-1}$. Note that $U_{-1,2} = U = \overline{\langle \mathbf{u} \rangle}$ and $U_{2,-1} = V = \overline{\langle \mathbf{v} \rangle}$. Moreover, the subgroups $U_{-1,k}$ and $U_{-1,l}$ commute as well as the vector fields $\partial_{-1,k}$ and $\partial_{-1,l}$.

6.3. De Jonquières subgroups and amalgamated products. This and the subsequent sections are largely inspired by [AZ13, Section 4]. Consider the de Jonquières group

$$\text{Jonq} := \{(ax + h(y), cy + d) \mid a, c \in \mathbb{k}^*, d \in \mathbb{k}, h \in \mathbb{k}[y]\}$$

and its subgroups

$$\begin{aligned} \text{Jonq}_0 &:= \{(ax + yf(y), by) \mid a, b \in \mathbb{k}^*, f \in \mathbb{k}[y]\} = \text{Jonq} \cap \text{Aut}_0(\mathbb{A}^2), \\ \mathfrak{J}_0 &:= \{(x + yf(y), y) \mid f \in \mathbb{k}[y]\} \subseteq \text{Jonq}_0. \end{aligned}$$

\mathfrak{J}_0 is a commutative closed unipotent ind-subgroup, isomorphic to $\mathbb{k}[y]^+$. It admits a filtration by closed unipotent subgroups $\prod_{i=1}^d U_{-1,i} \simeq \mathbb{G}_a^d$, $d \geq 1$. Note that $\text{Jonq}_0 = \mathfrak{J}_0 \rtimes T$.

Lemma 6.3.1. *The group \mathfrak{A}^S is the free product $(\mathfrak{J}_0)^S * (\mathfrak{J}_0^-)^S$ where $\mathfrak{J}_0^- := \tau \circ \mathfrak{J}_0 \circ \tau$ and $\tau := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

Proof. (1) The amalgamated product structure $\text{Aut}(\mathbb{A}^2) = \text{Jonq} *_{\mathfrak{B}} \text{Aff}_2$ where $\mathfrak{B} := \text{Jonq} \cap \text{Aff}_2$ implies that every element $\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \setminus \mathfrak{B}$ can be expressed in one of the following 4 forms, called *the type* of \mathbf{g} ,

$$\begin{aligned} \mathbf{g} &= j_1 a_1 j_2 \cdots j_n, & \mathbf{g} &= j_1 a_1 j_2 \cdots j_n a_n, \\ \mathbf{g} &= a_1 j_1 a_2 \cdots j_n, & \mathbf{g} &= a_1 j_1 a_2 \cdots j_n a_{n+1} \end{aligned}$$

where $j_k \in \text{Jonq} \setminus \mathfrak{B}$ and $a_k \in \text{Aff}_2 \setminus \mathfrak{B}$. The length of the expression, its type and the degrees of the j_k are uniquely determined by \mathbf{g} .

(2) Let now $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$. Using $\text{Jonq} = \text{Jonq}_0 \rtimes \text{Trans}_2$ and $\text{Aff}_2 = \text{GL}_2 \rtimes \text{Trans}_2$ where $\text{Trans}_2 \simeq \mathbb{G}_a^2$ is the group of translations, one can assume that $j_k \in \text{Jonq}_0 \setminus B$ and $a_k \in \text{GL}_2 \setminus B$ where $B \subseteq \text{GL}_2$ is the Borel subgroup of upper triangular matrices. Since $\text{GL}_2 \setminus B = U\tau B = B\tau U$ where $U \subseteq B$ is the unipotent radical, we see that any $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ has a presentation in one of the 4 forms

$$\begin{aligned} \mathbf{g} &= j_1 \tau j_2 \cdots \tau j_n, & \mathbf{g} &= j_1 \tau j_2 \cdots j_n \tau, \\ \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n, & \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n \tau \end{aligned}$$

where $j_k \in \text{Jonq}_0 \setminus B$. Again, the length, the type and the degrees of the j_k are uniquely determined by \mathbf{g} .

(3) Now we remark that $\text{Jonq}_0 = \mathfrak{J}_0 \rtimes T$ and $T\tau = \tau T$. It follows that we can reach one of the following forms

$$\begin{aligned} \mathbf{g} &= j_1 \tau j_2 \cdots \tau j_n t, & \mathbf{g} &= j_1 \tau j_2 \cdots j_n \tau t, \\ \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n t, & \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n \tau t \end{aligned}$$

where $j_k \in \mathfrak{J}_0$ and $t \in T$. We claim that this form is uniquely determined by \mathbf{g} . Since the type and the length are given by \mathbf{g} , it suffices to compare the similar decompositions

$$\mathbf{g} = j_1 \tau j_2 \cdots j_n t = j'_1 \tau j'_2 \cdots j'_n t' \quad \text{and} \quad \mathbf{g} = j_1 \tau j_2 \cdots j_n \tau t = j'_1 \tau j'_2 \cdots j'_n \tau t'.$$

It follows that

$$\tau j_2 \cdots j_n t = j_1^{-1} j'_1 \tau j'_2 \cdots j'_n t' \quad \text{resp.} \quad \tau j_2 \cdots j_n \tau t = j_1^{-1} j'_1 \tau j'_2 \cdots j'_n \tau t',$$

hence $j'_1 = j_1$. Now the claim follows by induction.

(4) Using the equalities $st = ts$ and $s\tau s = \tau$ for $s \in S$ and $t \in T$ we find for an S -invariant $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ of the first form $\mathbf{g} = j_1\tau j_2 \cdots j_n t$ in (3) above that

$$\mathbf{g} = s\mathbf{g}s^{-1} = \begin{cases} (sj_1s^{-1})\tau(s^{-1}j_2s)\tau(sj_3s^{-1}) \cdots (sj_ns^{-1})t & \text{if } n \text{ is odd,} \\ (sj_1s^{-1})\tau(s^{-1}j_2s)\tau(sj_3s^{-1}) \cdots (s^{-1}j_ns^{-1})t & \text{if } n \text{ is even.} \end{cases}$$

In the first case, we get that each j_k is S -invariant. In the second case, we obtain $sj_ns = j_n$, a contradiction. Looking at the other forms of \mathbf{g} in (3) we see that for an S -invariant \mathbf{g} the number of τ 's must be even and each j_k is S -invariant, i.e. $j_k \in (\mathfrak{J}_0)^S \subseteq \mathfrak{A}^S$.

(5) Finally, assume that $\mathbf{g} \in \mathfrak{A}^S$. If \mathbf{g} is given in one of the forms of (3), then $t = \text{id}$, because $d\mathbf{g}_0 = \text{id}$ and $(dj_k)_0 = \text{id}$ for $j_k \in (\mathfrak{J}_0)^S$. Since the number of τ 's is even, we see that such an automorphism \mathbf{g} is a product of the form $\cdots j_k j'_k j_{k+1} j'_{k+1} \cdots$ where $j_k \in (\mathfrak{J}_0)^S$ and $j'_k \in (\mathfrak{J}_0^-)^S = (\tau\mathfrak{J}_0\tau)^S = \tau(\mathfrak{J}_0)^S\tau$. \square

We have seen in the proof above that every $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ has a unique presentation in one of the 4 forms

$$\begin{aligned} \mathbf{g} &= j_1\tau j_2 \cdots \tau j_n, & \mathbf{g} &= j_1\tau j_2 \cdots j_n\tau, \\ \mathbf{g} &= \tau j_1\tau \cdots \tau j_n, & \mathbf{g} &= \tau j_1\tau \cdots \tau j_n\tau \end{aligned}$$

where $j_k \in \text{Jonq}_0 \setminus B$. As usual, we define the *degree* of an element $(f, h) \in \text{Aut}(\mathbb{A}^2)$ as $\deg(f, h) := \max\{\deg f, \deg h\}$. The following lemma is known, see e.g. [Kam79, Lemma 4.1].

Lemma 6.3.2. *If \mathbf{g} is as above, then $\deg \mathbf{g} = \deg j_1 \cdot \deg j_2 \cdots \deg j_n$.*

Proof. It suffices to consider the cases $\mathbf{g} = \tau j_1\tau \cdots \tau j_n = (f_n, h_n)$. We now prove by induction that $\deg f_k > \deg h_k$ and that $\deg f_{k+1} = \deg f_k \cdot \deg j_{k+1}$. Clearly, $\deg f_1 > \deg h_1 = 1$, because $j_1 \notin B$. If $j_{k+1} = (x, y + p(x))$, then

$$(f_{k+1}, h_{k+1}) = \tau \circ j_{k+1} \circ (f_k, h_k) = (h_k + p(f_k), f_k).$$

Since $\deg j_{k+1} = \deg p > 1$ and $\deg h_k < \deg f_k$ we get $\deg f_{k+1} = \deg p \cdot \deg f_k > \deg f_k$, hence the claim. \square

6.4. The fixed points under S . Recall that $\mathfrak{F} \subseteq \text{Aut}(\mathbb{A}^2)$ denotes the closure of the free group F generated by \mathbf{u} and \mathbf{v} .

Proposition 6.4.1. $(\mathfrak{J}_0)^S \subseteq \mathfrak{F}$. In particular, $\mathfrak{F} = \mathfrak{A}^S = (\mathfrak{J}_0)^S * (\mathfrak{J}_0^-)^S$.

Proof. We have seen in Section 6.3 that $\mathfrak{J}_0 \simeq \mathbb{k}[y]^+$ admits a filtration by the closed unipotent subgroups $\prod_{i=2}^d U_{-1,i}$, $d \geq 2$. It follows that $\mathfrak{J}_0^S = \{(x + y^2 f(y^3), y) \mid f \in \mathbb{k}[y]\} \xrightarrow{\sim} \mathbb{k}[y]^+$, and that \mathfrak{J}_0^S admits a filtration by the closed unipotent subgroups $\prod_{j=1}^k U_{-1,3j-1}$. Note that this product contains any automorphism of \mathbb{A}^2 of the form $(x + \sum_{j=1}^k c_j y^{3j-1}, y)$. We now show by induction that $U_{-1,3j-1} \subseteq \mathfrak{F}$.

Assume that this holds for $j \leq k$. Set

$$\begin{aligned} \varphi &:= \mathbf{u} \circ \mathbf{v} = (x + y^2, y) \circ (x, y + x^2) = \\ &= (x + y^2 + 2x^2 y + x^4, y + x^2) \in F = \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

and consider the product

$$\varphi \circ (x - \sum_{j=1}^k c_j y^{3j-1}, y) =: (f_k, h_k) \in F \cdot \prod_{j=1}^k U_{-1,3j-1}.$$

Then

$$f_k(0, y) = -\sum_1^k c_j y^{3j-1} + y^2 + 2y(\sum_1^k c_j y^{3j-1})^2 + (\sum_1^k c_j y^{3j-1})^4.$$

In particular, the exponents of y in $f_k(0, y)$ are all $\equiv 2 \pmod{3}$. Setting $c_1 = 1$ the second power of y disappears and we get

$$f_k(0, y) = -\sum_2^k c_j y^{3j-1} + 2y(\sum_1^k c_j y^{3j-1})^2 + (\sum_1^k c_j y^{3j-1})^4.$$

Hence, for $j \leq k$, the coefficient d_j of y^{3j-1} has the form

$$d_j = -c_j + 2 \sum_{j_1+j_2=j} c_{j_1} c_{j_2} + \sum_{j_1+j_2+j_3+j_4=j+1} c_{j_1} c_{j_2} c_{j_3} c_{j_4}.$$

It follows that for any $j \leq k$ the system of equations $d_1 = d_2 = \dots = d_j = 0$ admits a unique solution in positive integers c_i which does not depend on k ($c_2 = 2, c_3 = 9, c_4 = 52, c_5 = 340, c_6 = 2394, \dots$). Moreover, we see that the coefficient d_{k+1} of y^{3k+2} is positive. But then

$$(t^{-3k-2}x, t^{-1}y) \circ (f_k, h_k) \circ (t^{3k+2}x, ty) \xrightarrow{t \rightarrow 0} (x + d_{k+1}y^{3k+2}, y) \in U_{-1,3k+2}.$$

Since \mathfrak{F} is closed and stable under the action of the torus T we get $U_{-1,3k+2} \subseteq \mathfrak{F}$, and the claim follows. \square

Summing up we have the following result.

Theorem 6.4.2. *Let $F \subseteq \text{Aut}(\mathbb{A}^2)$ be the subgroup generated by $\mathbf{u} := (x + y^2, y)$ and $\mathbf{v} := (x, y + x^2)$, and denote by $\mathfrak{F} := \overline{F}$ its closure in $\text{Aut}(\mathbb{A}^2)$. Furthermore, define $\mathfrak{A} := \{\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \mid \mathbf{g}(0) = 0, d\mathbf{g}_0 = \text{id}\}$.*

- (1) *F is a free group in the two generators \mathbf{u}, \mathbf{v} , containing elements which are not locally finite.*

- (2) $\mathfrak{F} = \mathfrak{A}^S = \text{Cent}_{\mathfrak{A}}(S)$ where $S := \{(\zeta x, \zeta^2 y) \mid \zeta^3 = 1\} \subseteq T$ is the centralizer of F in $\text{Aut}(\mathbb{A}^2)$.
- (3) \mathfrak{F} is a free product $\mathfrak{J} * \mathfrak{J}^-$ where $\mathfrak{J} := \{(x + y^2 f(y^3), y) \mid f \in \mathbb{k}[y]\} = (\mathfrak{J}_0)^S$ and $\mathfrak{J}^- = \{(x, y + x^2 f(x^3)) \mid f \in \mathbb{k}[x]\} = \tau \mathfrak{J} \tau$. In particular, \mathfrak{J} is a commutative nested unipotent closed ind-subgroup of $\text{Aut}(\mathbb{A}^2)$ isomorphic to $\mathbb{k}[y]^+$.
- (4) Every algebraic subgroup of \mathfrak{F} is commutative and unipotent and is conjugate to a subgroup of \mathfrak{J} or of \mathfrak{J}^- .

Proof. (1) The first part is clear. For the second we notice that $\deg(\mathbf{uv})^k$ tends to infinity for $k \rightarrow \infty$. Hence, $\mathbf{uv} \in \mathfrak{F}$ is not locally finite.

(2) & (3) This is Proposition 6.4.1 and Lemma 6.1.1.

(4) For an algebraic subset $X \subseteq \mathfrak{F}$ the degrees of the elements from X are bounded above. It follows from Lemma 6.3.2 that the elements of X have bounded length in \mathfrak{F} . Hence a famous result of SERRE's (see [Ser03, Theorem 8]) implies that an algebraic subgroup of the free product $\mathfrak{F} = \mathfrak{J} * \mathfrak{J}^-$ is conjugate to a subgroup of one of the factors. \square

Remarks 6.4.3. (1) According to Lemma 3.2.1 the subgroup $\mathfrak{F} \subseteq \text{Aut}(X)$ is the closure of the subgroup generated by U and V , and $\text{Lie } \mathfrak{F}$ is the Lie algebra generated by $\text{Lie } U$ and $\text{Lie } V$. Thus our example is a positive instance of our Question 1.

(2) The free product $U * V$ is strictly contained in \mathfrak{F} , because, by Lemma 6.3.2, the degree of an element from $U * V$ is a power of 2, and so $U * V$ cannot contain all the groups $U_{-1, 3j-1}$. Even more, by the same lemma, every algebraic subgroup of $U * V$ is of bounded length, hence conjugate to either U or V by SERRE's result [Ser03, Theorem 8]).

(3) The theorem above shows that there is an ind-structure on the free product of the two ind-groups \mathfrak{J} and \mathfrak{J}^- . However, this structure *is not universal*, i.e. it is not true that given two homomorphisms of ind-groups $\mathfrak{J} \rightarrow \mathfrak{G}$ and $\mathfrak{J}^- \rightarrow \mathfrak{G}$, then the induced homomorphism $\mathfrak{J} * \mathfrak{J}^- \rightarrow \mathfrak{G}$ is a homomorphism of ind-groups.

In order to see this consider the two projections $\mathfrak{J} \rightarrow U \subseteq \text{Aut}(\mathbb{A}^2)$ and $\mathfrak{J}^- \rightarrow V \subseteq \text{Aut}(\mathbb{A}^2)$. We claim that the induced homomorphism $\varphi: \mathfrak{J} * \mathfrak{J}^- \rightarrow \text{Aut}(\mathbb{A}^2)$ is not an ind-homomorphism. In fact, the image of φ is $U * V \subseteq \mathfrak{J} * \mathfrak{J}^-$ and φ induces the identity on $U * V$. Since $U * V$ is dense in $\mathfrak{J} * \mathfrak{J}^-$ it follows that φ is the identity on $\mathfrak{J} * \mathfrak{J}^-$, a contradiction.

7. A CANONICAL LIE ALGEBRA FOR AN IND-GROUP

For an ind-variety \mathfrak{V} the tangent space $T_x\mathfrak{V}$ in a point $x \in \mathfrak{V}$ can be defined as the limit $\varinjlim T_x X$ where X runs through all algebraic subsets of \mathfrak{V} containing x . There is an canonical subspace of $T_x\mathfrak{V}$ generated by those $T_x X$ where X is smooth in x :

$$T_x^{(s)}\mathfrak{V} := \text{span}(T_x X \mid X \subseteq \mathfrak{V} \text{ algebraic and } x \in X_{\text{reg}}).$$

If \mathfrak{V} is *strongly smooth in x* , i.e. there is an admissible filtration $\mathfrak{V} = \bigcup_k \mathfrak{V}_k$ such that each \mathfrak{V}_k is smooth in x , then $T_x^{(s)}\mathfrak{V} = T_x\mathfrak{V}$. In general, we could not prove any useful results for this subspace, but for ind-groups it turns out that it has very remarkable properties.

7.1. Definition and first properties.

Definition 7.1.1. For an ind-group \mathfrak{G} we define

$$L_{\mathfrak{G}} := \text{span}(T_e X \mid X \subseteq \mathfrak{G} \text{ algebraic and } e \in X_{\text{reg}}) \subseteq \text{Lie } \mathfrak{G}.$$

The next proposition shows that $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$ is a canonical Lie subalgebra.

Proposition 7.1.2.

- (a) $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$ is stable under the adjoint action of \mathfrak{G} , i.e. $L_{\mathfrak{G}}$ is an ideal in $\text{Lie } \mathfrak{G}$.
- (b) For every algebraic subgroup $G \subseteq \mathfrak{G}$ we have $\text{Lie } G \subseteq L_{\mathfrak{G}}$.
- (c) $L_{\mathfrak{G}^\circ} = L_{\mathfrak{G}}$.
- (d) $L_{\mathfrak{G}}$ is finite dimensional if and only if \mathfrak{G}° is an algebraic group, and in this case we have $L_{\mathfrak{G}} = \text{Lie } \mathfrak{G}$.

Proof. (a) If $X \subseteq \mathfrak{G}$ is an algebraic subset with $e \in X_{\text{reg}}$ and if $g \in \mathfrak{G}$, then $X' := gXg^{-1}$ is an algebraic subset and $e \in X'_{\text{reg}}$. Moreover, $\text{Ad } g(T_e X) = T_e X' \subseteq L_{\mathfrak{G}}$, and the claim follows.

(b) and (c) are clear.

(d) If \mathfrak{G}° is an algebraic group, then the claims follow from (b) and (c). If $L_{\mathfrak{G}}$ is finite dimensional, then so is $L_{\mathfrak{G}^\circ}$ by (c). As a consequence, the dimension of an irreducible subvariety $X \subseteq \mathfrak{G}^\circ$ is bounded by $\dim L_{\mathfrak{G}^\circ} = \dim L_{\mathfrak{G}}$, and the claim follows from Proposition 2.4.1. \square

The following results show that one has $L_{\mathfrak{G}} = \text{Lie } \mathfrak{G}$ in many cases, but we have no example where $L_{\mathfrak{G}} \neq \text{Lie } \mathfrak{G}$.

Corollary 7.1.3. (1) *Let \mathfrak{G} be algebraically generated by a family $\{G_i\}_{i \in I}$ of connected algebraic subgroups. Then $L(\mathfrak{G}) \subseteq L_{\mathfrak{G}}$ is an ideal.*

- (2) If \mathfrak{G} is strongly smooth in e , then $L_{\mathfrak{G}} = \text{Lie}(G)$. This holds, in particular, for a nested ind-group \mathfrak{G} .

Recall that

$$\text{SAut}(\mathbb{A}^n) := \{\alpha \in \text{Aut}(\mathbb{A}^n) \mid \text{jac}(\alpha) = 1\}$$

is a normal subgroup of $\text{Aut}(\mathbb{A}^n)$ with Lie algebra $\text{Lie SAut}(\mathbb{A}^n) = \text{Vec}^0(\mathbb{A}^n)$, the algebra of polynomial vector fields on \mathbb{A}^n with zero divergence (see Section 6.2).

Corollary 7.1.4. *We have $L_{\text{SAut}(\mathbb{A}^n)} = \text{Lie SAut}(\mathbb{A}^n)$ and $L_{\text{Aut}(\mathbb{A}^n)} = \text{Lie Aut}(\mathbb{A}^n)$.*

Note that $\text{Aut}(\mathbb{A}^2)$ is not strongly smooth in e (see [FK18, Corollary 14.1.2]) and so we cannot apply Corollary 7.1.3(2).

Proof. By [Sha81, Lemma 3], the Lie algebra

$$\text{Lie SAut}(\mathbb{A}^n) = [\text{Lie Aut}(\mathbb{A}^n), \text{Lie Aut}(\mathbb{A}^n)]$$

is simple, and by Proposition 7.1.2(a) $L_{\text{SAut}(\mathbb{A}^n)}$ is a (non-zero) ideal in $\text{Lie SAut}(\mathbb{A}^n)$. This proves the first claim.

The second claim is a consequence of Proposition 7.1.2(b) due to the fact that

$$\text{Lie Aut}(\mathbb{A}^n) = \text{Lie SAut}(\mathbb{A}^n) \oplus \text{Lie } \mathbb{k}^*,$$

where $\mathbb{k}^* \subseteq \text{Aut}(\mathbb{A}^n)$ is realized as the subgroup of the scalar multiplications $\lambda \cdot \text{id}$, $\lambda \in \mathbb{k}^*$, cf. [FK18, Prop. 15.7.2]. \square

7.2. Functorial properties.

Proposition 7.2.1. *Let $\varphi: \mathfrak{H} \rightarrow \mathfrak{G}$ be a homomorphism of ind-groups.*

- (1) *Then $d\varphi_e(L_{\mathfrak{H}}) \subseteq L_{\mathfrak{G}}$. In particular, $L_{\mathfrak{G}}$ is invariant under every ind-group automorphism $\mathfrak{G} \xrightarrow{\sim} \mathfrak{G}$.*
- (2) *Assume that \mathbb{k} is uncountable. If φ is surjective, then the differential $d\varphi_e: L_{\mathfrak{H}} \rightarrow L_{\mathfrak{G}}$ is surjective.*

Proof. (a) Let $Y \subseteq \mathfrak{H}$ be an irreducible algebraic subset such that $e \in \overline{Y_{\text{reg}}}$. There is an open dense subset $U \subseteq Y_{\text{reg}}$ such that $\varphi(U) \subseteq \overline{\varphi(Y)_{\text{reg}}}$ and that $\varphi|_U: U \rightarrow \varphi(U)$ is smooth in every point $u \in U$. In particular, for all $u \in U$, the image $\varphi(Yu^{-1})$ is smooth in e , hence the image of $T_e Yu^{-1}$ is contained in $L_{\mathfrak{G}}$ for all $u \in U$. Now the claim follows from Lemma 7.2.3 below, because U is dense in Y_{reg} .

(b) Let $X \subseteq \mathfrak{G}$ be an irreducible algebraic subset such that $e \in X_{\text{reg}}$. We have to show that $T_e X \subseteq d\varphi_e(L_{\mathfrak{H}})$. Since \mathbb{k} is uncountable there exists an irreducible closed algebraic subset $Y \subseteq \mathfrak{H}$ such that $\varphi(Y) =$

X ([FK18, Prop. 1.3.2]). As in (a) we choose a dense open $U \subseteq Y$ such that $\varphi(U)$ is an open subset of X_{reg} and that $\varphi|_U: U \rightarrow \varphi(U)$ is smooth. Then $d\varphi_e(T_e Yu^{-1}) = T_e X \varphi(u)^{-1}$ for all $u \in U$. It follows that $T_e X v^{-1} \subseteq d\varphi_e(L_{\mathfrak{H}})$ for all $v \in \varphi(U)$, and hence for all $v \in X_{reg}$, again by Lemma 7.2.3 below. \square

Remark 7.2.2. The assumption that \mathbb{k} is uncountable is necessary, as seen from the following example. Take the bijective homomorphism $\mathbb{k}^+ \rightarrow \mathbb{G}_a$ where \mathbb{k}^+ is endowed with the discrete topology.

Lemma 7.2.3. *Let $Y \subseteq \mathfrak{H}$ be an irreducible algebraic subset. Then $\text{span}(T_e Y g^{-1} \mid g \in Y) \subseteq \text{Lie } \mathfrak{H}$ is finite dimensional, and*

$$\text{span}(T_e Y g^{-1} \mid g \in Y_{reg}) = \text{span}(T_e Y g^{-1} \mid g \in U)$$

for any dense subset $U \subseteq Y_{reg}$.

Proof. Set $X := \overline{Y Y^{-1}} \subseteq \mathfrak{G}$. Then $T_e Y g^{-1} \subseteq T_e X$ for all $g \in Y$ which implies the first claim.

For the second claim let $\text{Grass}_n T_e X$ denote the Grassmannian of n -dimensional subspaces of $T_e X$ where $n := \dim Y$, and consider the morphism $\mu: Y_{reg} \rightarrow \text{Grass}_n T_e X$ given by $g \mapsto T_e Y g^{-1}$. Since $\text{Grass}_n V \subseteq \text{Grass}_n T_e X$ is closed for any subspace $V \subseteq T_e X$ it follows that a dense subset of Y_{reg} has the same span as Y_{reg} . \square

Question 7. *Assume we have a surjective homomorphism $\mathfrak{G} \rightarrow G$ where \mathfrak{G} is a path-connected ind-group, G an algebraic group and \mathbb{k} uncountable. Does \mathfrak{G} contain an algebraic subgroup which is sent surjectively onto G ?*

7.3. Subgroups of finite codimension. Let $\mathfrak{G} = \bigcup_d \mathfrak{G}_d$ be an ind-group.

Definition 7.3.1. We say that a closed subgroup $\mathfrak{H} \subseteq \mathfrak{G}$ is of *finite codimension* if $\mathfrak{G}_d \cdot \mathfrak{H} = \mathfrak{G}$ for some $d \in \mathbb{N}$.

For instance, if \mathfrak{G} is connected and acts on an algebraic variety X , then the stabilizer \mathfrak{G}_x of any point $x \in X$ is of finite codimension. Indeed, by [FK18, Prop. 7.1.2] the orbit $\mathfrak{G}x$ is an irreducible subvariety of X and $\mathfrak{G}x = \mathfrak{G}_d x$ for some d . The latter implies $\mathfrak{G} = \mathfrak{G}_d \cdot \mathfrak{G}_x$.

We do not know if every subgroup \mathfrak{H} of finite codimension of a connected ind-group \mathfrak{G} is the stabilizer of a point of a \mathfrak{G} -variety, i.e. if $\mathfrak{G}/\mathfrak{H}$ has a canonical structure of an algebraic variety.

Example 7.3.2. The connected component \mathfrak{G}° has finite codimension in \mathfrak{G} if and only if it has finite index in \mathfrak{G} . Indeed, a closed algebraic

subset $Z \subseteq \mathfrak{G}$ can only meet finitely many of the components of \mathfrak{G} . Similarly, if \mathfrak{H} has finite codimension in \mathfrak{G} , then the image of \mathfrak{H} in $\mathfrak{G}/\mathfrak{G}^\circ$ has finite index.

One could expect that for a subgroup $\mathfrak{H} \subseteq \mathfrak{G}$ of finite codimension the Lie algebra $\text{Lie } \mathfrak{H} \subseteq \text{Lie } \mathfrak{G}$ has finite codimension as well. What we can show is the following result.

Proposition 7.3.3. *Assume that the base field \mathbb{k} is uncountable. Let \mathfrak{G} be an ind-group and $\mathfrak{H} \subseteq \mathfrak{G}$ a closed subgroup of finite codimension. If \mathfrak{H} is subnormal in \mathfrak{G} , i.e. there is a finite series $\mathfrak{H} = \mathfrak{H}_0 \subseteq \mathfrak{H}_1 \subseteq \cdots \subseteq \mathfrak{H}_\ell = \mathfrak{G}$ such that \mathfrak{H}_i is normal in \mathfrak{H}_{i+1} , then $L_{\mathfrak{H}}$ has finite codimension in $L_{\mathfrak{G}}$.*

The proof needs some preparation. For an algebraic subset $X \subseteq \mathfrak{G}$ we define

$$M_X := \text{span}(T_e Xg^{-1} \mid g \in X) \subseteq \text{Lie } \mathfrak{G}.$$

We have seen in Lemma 7.2.3 that M_X is finite dimensional. Moreover, if X is smooth, then $M_U = M_X$ for an open dense subset U of X , and $M_X \subseteq L_{\mathfrak{G}}$.

Lemma 7.3.4. *Assume that \mathbb{k} is uncountable. Let \mathfrak{G} be an ind-group and $\mathfrak{H} \subseteq \mathfrak{G}$ a closed normal subgroup. If there exists a closed irreducible algebraic subset $Z \subseteq \mathfrak{G}$ such that $Z \cdot \mathfrak{H} = \mathfrak{G}$, then $L_{\mathfrak{G}} \subseteq M_Z + L_{\mathfrak{H}}$. In particular, $L_{\mathfrak{H}}$ has finite codimension in $L_{\mathfrak{G}}$.*

Proof. (1) Let $X \subseteq \mathfrak{G}$ be a closed irreducible algebraic subset. Since \mathbb{k} is uncountable we can find an irreducible algebraic subset $Y \subseteq \mathfrak{H}$ such that $Z \cdot Y \supseteq X$. Thus X is contained in the image of the multiplication morphism $\mu: Z \times Y \rightarrow \mathfrak{G}$, $\mu(z, y) := z \cdot y$. For every $g := z \cdot y$ where $(z, y) \in Z \times Y$ we have the following commutative diagram

$$\begin{array}{ccc} Z \times Y & \xrightarrow{\mu} & \mathfrak{G} \\ \rho_{z^{-1}} \times \text{Int } z \circ \rho_{y^{-1}} \downarrow \simeq & & \simeq \downarrow \rho_{g^{-1}} \\ Zz^{-1} \times z(Yy^{-1})z^{-1} & \xrightarrow{\mu} & \mathfrak{G} \end{array}$$

which induced the commutative diagram of tangent maps

$$\begin{array}{ccc} T_z Z \oplus T_y Y & \xrightarrow{d\mu_{(z,y)}} & T_g \mathfrak{G} \\ d\rho_{z^{-1}} \times \text{Ad } z \circ d\rho_{y^{-1}} \downarrow \simeq & & \simeq \downarrow d\rho_{g^{-1}} \\ T_e(Zz^{-1}) \oplus T_e(z(Yy^{-1})z^{-1}) & \xrightarrow{d\mu_{(e,e)}} & T_e \mathfrak{G} \end{array}$$

(2) There exists an irreducible open subset $U \subseteq \mu^{-1}(X)_{\text{reg}}$ such that $\mu(U) \subseteq X_{\text{reg}}$ is open and dense, and that $\mu|_U: U \rightarrow X$ is smooth.

Replacing Y by $\overline{\text{pr}_Y(U)}$ if necessary we can assume that $\text{pr}_Y(U) \subseteq Y_{\text{reg}}$. For $g = z \cdot y$ where $(z, y) \in U$ it follows from the diagram above that

$$T_e Xg^{-1} \subseteq T_e Zz^{-1} + \text{Ad } z(T_e Yy^{-1}) \subseteq M_Z + \text{Ad } z(L_{\mathfrak{H}}),$$

hence $M_{X_{\text{reg}}} \subseteq M_Z + L_{\mathfrak{H}}$ by Lemma 7.2.3. Since the $M_{X_{\text{reg}}} \subseteq L_{\mathfrak{G}}$ generate $L_{\mathfrak{G}}$ the claim follows. \square

Proof of Proposition 7.3.3. We can clearly assume that \mathfrak{H} is normal in \mathfrak{G} .

(1) We first claim that $\mathfrak{H} \cap \mathfrak{G}^\circ$ has finite codimension in \mathfrak{G}° . The ind-group \mathfrak{G} is a countable disjoint union $\mathfrak{G} = \bigcup_{j \in J} g_j \mathfrak{G}^\circ = \bigcup_{j \in J} \mathfrak{G}^\circ g_j$ where the components are open, closed, and connected, cf. [FK18, Section 2.2]. Similarly we get a countable disjoint union of open and closed subsets of \mathfrak{H} in the form $\mathfrak{H} = \bigcup_{i \in I} h_i (\mathfrak{H} \cap \mathfrak{G}^\circ)$.

Assume that $Z \cdot \mathfrak{H} = \mathfrak{G}$ for some algebraic subset $Z \subseteq \mathfrak{G}$. Then $Z = \bigcup_{j \in J_0} Z_j g_j$ for a finite subset $J_0 \subseteq J$ and algebraic subsets $Z_j \subseteq \mathfrak{G}^\circ$. It follows that

$$\mathfrak{G} = Z \cdot \mathfrak{H} = \bigcup_{j \in J_0} Z_j g_j \cdot \mathfrak{H} \quad \text{and} \quad Z_j g_j \cdot \mathfrak{H} = \bigcup_{i \in I} Z_j g_j h_i \cdot (\mathfrak{H} \cap \mathfrak{G}^\circ).$$

Each $Z_j g_j h_i \cdot (\mathfrak{H} \cap \mathfrak{G}^\circ)$ belongs to a connected component of \mathfrak{G} , and $Z_j g_j h_i \cdot (\mathfrak{H} \cap \mathfrak{G}^\circ) \subseteq \mathfrak{G}^\circ$ if and only if $g_j h_i \in \mathfrak{G}^\circ$. Thus

$$\mathfrak{G}^\circ = \bigcup_{j \in J_0, g_j h_i \in \mathfrak{G}^\circ} Z_j g_j h_i \cdot (\mathfrak{H} \cap \mathfrak{G}^\circ) = \left(\bigcup_{j \in J_0, g_j h_i \in \mathfrak{G}^\circ} Z_j g_j h_i \right) \cdot (\mathfrak{H} \cap \mathfrak{G}^\circ).$$

By construction, $\bigcup_{j \in J_0, g_j h_i \in \mathfrak{G}^\circ} Z_j g_j h_i$ is a finite union of algebraic subsets and thus contained in \mathfrak{G}_k° for some k where $\mathfrak{G}^\circ = \bigcup_k \mathfrak{G}_k^\circ$ is an admissible filtration.

(2) Since \mathfrak{G}° is connected we can assume that the \mathfrak{G}_k° are irreducible ([FK18, Prop. 1.6.3 and Prop. 2.2.1(2)]) and so $Z' \cdot (\mathfrak{H} \cap \mathfrak{G}^\circ) = \mathfrak{G}^\circ$ with an irreducible algebraic subset $Z' \subseteq \mathfrak{G}^\circ$. Now Lemma 7.3.4 and Proposition 7.1.2(c) imply that $L_{\mathfrak{H}} = L_{\mathfrak{H} \cap \mathfrak{G}^\circ}$ has finite codimension in $L_{\mathfrak{G}} = L_{\mathfrak{G}^\circ}$. \square

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DEPARTEMENT MATHEMATIK UND INFORMATIK,
UNIVERSITÄT BASEL, SPIEGELGASSE 1, CH-4051 BASEL, SWITZERLAND
E-mail address: `hanspeter.kraft@unibas.ch`

INSTITUT FOURIER, UMR 5582, LABORATOIRE DE MATHÉMATIQUES,
UNIVERSITÉ GRENOBLE ALPES, CS 40700, 38058 GRENOBLE CEDEX 9, FRANCE
E-mail address: `mikhail.zaidenberg@univ-grenoble-alpes.fr`